Limit theorems for conditioned multitype Dawson-Watanabe processes and Feller diffusions

Nicolas Champagnat, Sylvie Roelly

Limit theorems for conditioned multitype Dawson-Watanabe processes and Feller diffusions

Nicolas Champagnat

Project-team TOSCA
INRIA Sophia Antipolis
2004 route des lucioles - BP 93
06902 Sophia Antipolis, France
e-mail : Nicolas.Champagnat@sophia.inria.fr

Sylvie Rœlly *

Institut für Mathematik
Universität Potsdam
Am Neuen Palais 10
14469 Potsdam, Germany
e-mail : roelly@math.uni-potsdam.de

December 9, 2008

Abstract

A multitype Dawson-Watanabe process is conditioned, in subcritical and critical cases, on non-extinction in the remote future. On every finite time interval, its distribution is absolutely continuous with respect to the law of the unconditioned process. A martingale problem characterization is also given. Several results on the long time behavior of the conditioned mass process—the conditioned multitype Feller branching diffusion—are then proved. The general case is considered first, where the mutation matrix which models the interaction between the types, is irreducible. Several two-type models with decomposable mutation matrices are also analyzed.

AMS 2000 Subject Classifications : 60J80, 60G57.
KEY-WORDS: multitype measure-valued branching processes; conditioned Dawson-Watanabe process; critical and subcritical Dawson-Watanabe process; conditioned Feller diffusion; remote survival; long time behavior.

*On leave of absence Centre de Mathématiques Appliquées, UMR C.N.R.S. 7641, École Polytechnique, 91128 Palaiseau Cédex, France.

Introduction

The paper focuses on some conditioning of the measure-valued process called multitype Dawson-Watanabe (MDW) process, and on its mass process, the well-known multitype Feller (MF) diffusion. We consider the critical and subcritical cases, in which, for any finite initial condition, the MF diffusion vanishes in finite time, that is the MDW process dies out a.s. In these cases, it is interesting to condition the processes to stay alive forever - an event which we call **remote survival**, see the exact definition in (3).
Such a study was initiated for the monotype Dawson-Watanabe process by A. Rouault and the second author in [28] (see also [10], [9] and [8] for the study of various aspects of conditioned monotype superprocesses). Their results were a generalization at the level of measure-valued processes of the pioneer work of Lamperti and Ney ([20], Section 2), who studied the same questions applied to Galton-Watson processes.
We are interested here in the **multitype** setting which is much different from the monotype one. The mutation matrix D introduced in (2), which measures the quantitative interaction between types, will play a crucial role.
We now briefly describe the contents of the paper. The model is precisely defined in the first section. In the second section we define the conditioned MDW process, express its law as a locally absolutely continuous measure with respect to the law of the unconditioned process, write explicitly the martingale problem it satisfies and give the form of its Laplace functional; all this in the case of an irreducible mutation matrix. Since D is irreducible, all the types communicate and conditioning by remote survival is equivalent to conditioning by the non-extinction of only one type (see Remark 2.5). The third section is devoted to the long time behavior of the mass of the conditioned MDW process, which is then a conditioned MF diffusion. First the monotype case is analyzed (it was not considered in [28]), and then the irreducible multitype case. We also prove that both limits interchange: the long time limit and the conditioning by long time survival (see Theorem 3.7). In the last section we treat the same questions as in Section 3 for various reducible 2-types models. Since D is decomposable, the two types can have very different behaviors, that also depend on the precise conditioning that is considered (see Section 4.1).

1 The model

In this paper, we will assume for simplicity that the (physical) space is $\mathbb{R}$. k is the number of types. Any k-dimensional vector $u \in \mathbb{R}^k$ is denoted by

$(u_1; \cdots; u_k)$. $\mathbf{1}$ will denote the vector $(1; \ldots; 1) \in \mathbb{R}^k$. $\|u\|$ is the euclidean norm of $u \in \mathbb{R}^k$ and (u, v) the scalar product between u and v in $\mathbb{R}^k$. If $u \in \mathbb{R}^k$, $|u|$ is the vector in $\mathbb{R}^k$ with coordinates $|u_i|, 1 \leq i \leq k$.
We will use the notations $u > v$ (resp. $u \geq v$) when u and v are vectors or matrices such that $u - v$ has positive (resp. non-negative) entries.
Let $C_b(\mathbb{R}, \mathbb{R}^k)$ denote the space of $\mathbb{R}^k$-valued continuous bounded functions on $\mathbb{R}$. By $C_b(\mathbb{R}, \mathbb{R}^k)_+$ we denote the set of non-negative elements of $C_b(\mathbb{R}, \mathbb{R}^k)$. $M(\mathbb{R})$ is the set of finite positive measures on $\mathbb{R}$, and $M(\mathbb{R})^k$ the set of k-dimensional vectors of finite positive measures.
The duality between measures and functions will be denoted by $\langle \cdot, \cdot \rangle$: $\langle \nu, f \rangle := \int f d\nu$ if $\nu \in M(\mathbb{R})$ and f is defined on $\mathbb{R}$, and in the vectorial case

$$\langle (\nu_1; \ldots; \nu_k), (f_1; \ldots; f_k) \rangle := \sum_{i=1}^{k} \int f_i d\nu_i = \big((\langle \nu_1, f_1 \rangle; \ldots; \langle \nu_k, f_k \rangle), \mathbf{1} \big)$$

for $\nu = (\nu_1; \ldots; \nu_k) \in M(\mathbb{R})^k$ and $f = (f_1; \ldots; f_k) \in C_b(\mathbb{R}, \mathbb{R}^k)$. For any $\lambda \in \mathbb{R}^k$, the constant function of $C_b(\mathbb{R}, \mathbb{R}^k)$ equal to λ will be also denoted by λ.

A multitype Dawson-Watanabe process with *mutation matrix* $D = (d_{ij})_{1 \leq i,j \leq k}$ is a continuous $M(\mathbb{R})^k$-valued Markov process whose law $\mathbb{P}$ on the canonical space $(\Omega := C(\mathbb{R}_+, M(\mathbb{R})^k), (X_t)_{t \geq 0}, (\mathcal{F}_t)_{t \geq 0})$ has as transition Laplace functional

$$\forall f \in C_b(\mathbb{R}, \mathbb{R}^k)_+, \quad \mathbb{E}(\exp -\langle X_t, f \rangle \mid X_0 = m) = \exp -\langle m, U_t f \rangle \tag{1}$$

where $U_t f \in C_b(\mathbb{R}, \mathbb{R}^k)_+$, the so-called cumulant semigroup, is the unique solution of the non-linear PDE

$$\begin{cases} \dfrac{\partial (U_t f)}{\partial t} &= \Delta U_t f + D U_t f - \dfrac{c}{2} (U_t f)^{\odot 2} \\ U_0 f &= f. \end{cases} \tag{2}$$

Here, $u \odot v$ denotes the componentwise product $(u_i v_i)_{1 \leq i \leq k}$ of two k-dimensional vectors u and v and $u^{\odot 2} = u \odot u$. To avoid heavy notation, when no confusion is possible, we do not write differently column and row vectors when multiplied by a matrix. In particular, in the previous equation, Du actually stands for Du'.
The MDW process arises as the diffusion limit of a sequence of particle systems $(\frac{1}{K} N^K)_K$, where N^K is an appropriate rescaled multitype branching Brownian particle system (see e.g. [15] and [16], or [32] for the monotype

model): after an exponential lifetime with parameter K, each Brownian particle splits or dies, in such a way that the number of offsprings of type j produced by a particle of type i has as (nonnegative) mean $\delta_{ij} + \frac{1}{K} d_{ij}$ and as second factorial moment c (δ_{ij} denotes the Kronecker function, equal to 1 if $i = j$ and to 0 otherwise). Therefore, the average number of offsprings of each particle is asymptotically one and the matrix D measures the (rescaled) discrepancy between the mean matrix and the identity matrix I, which corresponds to the pure critical case of independent types.

For general literature on DW processes we refer the reader e.g. to the lectures of D. Dawson [3] and E. Perkins [25] and the monographs [5] and [7].

Let us remark that we introduced a variance parameter c which is type-independent. In fact we could replace it by a vector $c = (c_1; \cdots ; c_k)$, where c_i corresponds to type i. If $\inf_{1 \le i \le k} c_i > 0$, then all the results of this paper are still true. We decided to take c independent of the type to simplify the notation.

When the mutation matrix $D = (d_{ij})_{1 \le i,j \le k}$ is not diagonal, it represents the interaction between the types, which justifies its name. Its non diagonal elements are non-negative. These matrices are sometimes called Metzler-Leontief matrices in financial mathematics (see [29] § 2.3 and the bibliography therein). Since there exists a positive constant α such that $D + \alpha I \ge 0$, it follows from Perron-Frobenius theory that D has a real eigenvalue μ such that no other eigenvalue of D has its real part exceeding μ. Moreover, the matrix D has a non negative right eigenvector associated to the eigenvalue μ (see e.g. [14], Satz 3 § 13.3 or [29] Exercise 2.11). The cases $\mu < 0$, $\mu = 0$ and $\mu > 0$ correspond respectively to a *subcritical*, *critical* and *supercritical* processes.

In the present paper, we only consider the case $\mu \le 0$, in which the MDW dies out a.s. (see Jirina [17]).

2 (Sub)critical irreducible MDW process conditioned by remote survival.

Let us recall the definition of irreducibility of a matrix.

Definition 2.1 *A square matrix D is called* irreducible *if there is no permutation matrix Q such that $Q^{-1}DQ$ is block triangular.*

In all this section and in the next one, the mutation matrix D is assumed to be irreducible. By Perron-Frobenius' theorem (see e.g. [29] Theorem 1.5 or [14], Satz 2 §13.2, based on [27] and [13]), the eigenspace associated to the maximal real eigenvalue μ of D is one-dimensional. We will always denote its generating right (resp. left) eigenvector by ξ (resp. by η) with the normalization conventions $(\xi, \mathbf{1}) = 1$ and $(\xi, \eta) = 1$. All the coordinates of both vectors ξ and η are positive.

2.1 The conditioned process as a h-process

The natural way to define the law $\mathbb{P}^*$ of the MDW process conditioned to never die out is by

$$\forall B \in \mathcal{F}_t, \quad \mathbb{P}^*(B) := \lim_{\theta \to \infty} \mathbb{P}(B \mid \langle X_{t+\theta}, \mathbf{1} \rangle > 0) \tag{3}$$

if this limit exists.

The following Theorem 2.2 proves that $\mathbb{P}^*$ is well-defined by (3) and is a probability measure on $\mathcal{F}_t$ absolutely continuous with respect to $\mathbb{P}$. Furthermore, the density is a martingale, so that $\mathbb{P}^*$ can be extended to $\vee_{t \geq 0} \mathcal{F}_t$, defining a Doob h-transform of $\mathbb{P}$ (see the seminal work [22] on h-transforms and [24] for applications to monotype DW processes).

Theorem 2.2 *Let $\mathbb{P}$ be the distribution of a critical or subcritical MDW process characterized by* (1)*, with an irreducible mutation matrix D and initial measure $m \in M(\mathbb{R})^k \setminus \{0\}$. Then, the limit in* (3) *exists and defines a probability measure $\mathbb{P}^*$ on $\vee_{t \geq 0} \mathcal{F}_t$ such that, for any $t > 0$,*

$$\mathbb{P}^* \big|_{\mathcal{F}_t} = \frac{\langle X_t, \xi \rangle}{\langle m, \xi \rangle} \, e^{-\mu t} \quad \mathbb{P} \big|_{\mathcal{F}_t} \tag{4}$$

where $\xi \in \mathbb{R}^k$ is the unitary right eigenvector associated to the maximal real eigenvalue μ of D.

Proof of Theorem 2.2 By definition, for $B \in \mathcal{F}_t$,

$$\mathbb{E}(\mathbb{1}_B \mid \langle X_{t+\theta}, \mathbf{1} \rangle > 0) = \frac{\mathbb{E}\big(\mathbb{1}_B (1 - \mathbb{P}(\langle X_{t+\theta}, \mathbf{1} \rangle = 0 \mid \mathcal{F}_t))\big)}{1 - \mathbb{P}(\langle X_{t+\theta}, \mathbf{1} \rangle = 0)}.$$

For any time $s > 0$, $x_s := (\langle X_{s,1}, 1 \rangle; \ldots; \langle X_{s,k}, 1 \rangle)$, the total mass at time s of the MDW process—a multitype Feller diffusion—is a continuous $\mathbb{R}_+^k$-valued process with initial value $x = (\langle m_1, 1 \rangle; \ldots; \langle m_k, 1 \rangle)$ characterized by its transition Laplace transform

$$\forall \lambda \in \mathbb{R}_+^k, \quad \mathbb{E}(e^{-(x_t, \lambda)} \mid x_0 = x) = e^{-(x, u_t^\lambda)}. \tag{5}$$

Here, $u_t^\lambda = (u_{t,1}^\lambda; \dots; u_{t,k}^\lambda) := U_t\lambda$ satisfies the non-linear differential system

$$\begin{cases} \dfrac{du_t^\lambda}{dt} &= Du_t^\lambda - \dfrac{c}{2}(u_t^\lambda)^{\odot 2} \\ u_0^\lambda &= \lambda, \end{cases} \tag{6}$$

or componentwise

$$\forall i \in \{1, \dots, k\}, \quad \frac{du_{t,i}^\lambda}{dt} = \sum_{j=1}^k d_{ij} u_{t,j}^\lambda - \frac{c}{2}(u_{t,i}^\lambda)^2, \quad u_{0,i}^\lambda = \lambda_i.$$

Then,

$$\mathbb{P}(\langle X_s, \mathbf{1}\rangle = 0) = \lim_{\lambda \hookrightarrow \infty} \mathbb{E}(e^{-\langle X_s, \lambda\rangle} \mid X_0 = m) = e^{-(x, \lim_{\lambda\hookrightarrow\infty} u_s^\lambda)}$$

where $\lambda \hookrightarrow \infty$ means that all coordinates of λ go to $+\infty$. Using the Markov property of the MDW process, one obtains

$$\mathbb{E}(\mathbb{1}_B \mid \langle X_{t+\theta}, \mathbf{1}\rangle > 0) = \frac{\mathbb{E}\Big(\mathbb{1}_B\big(1 - e^{-(x_t, \lim_{\lambda\hookrightarrow\infty} u_\theta^\lambda)}\big)\Big)}{1 - e^{-(x, \lim_{\lambda\hookrightarrow\infty} u_{t+\theta}^\lambda)}}. \tag{7}$$

In the monotype case ($k = 1$), u_t^λ can be computed explicitly (see Section 3.1), but this is not possible in the multitype case. Nevertheless, one can obtain upper and lower bounds for u_t^λ. This is the goal of the following two lemmas, the proofs of which are postponed after the end of the proof of Theorem 2.2.

Lemma 2.3 *Let $u_t^\lambda = (u_{t,1}^\lambda; \dots; u_{t,k}^\lambda)$ be the solution of* (6)*.*

(i) *For any $\lambda \in \mathbb{R}_+^k \setminus \{0\}$ and any $t > 0$, $u_t^\lambda > 0$.*

(ii) *Let $C_t^\lambda := \sup_{1\leq i\leq k} \dfrac{u_{t,i}^\lambda}{\xi_i}$ and $\underline{\xi} := \inf_i \xi_i$. For $t > 0$ and $\lambda \in \mathbb{R}_+^k$,*

- *in the critical case ($\mu = 0$)*

$$C_t^\lambda \leq \frac{C_0^\lambda}{1 + \frac{c\underline{\xi}}{2} C_0^\lambda\, t} \quad \textit{and therefore} \quad \sup_{\lambda\in\mathbb{R}_+^k} C_t^\lambda \leq \frac{2}{c\underline{\xi}\, t} \tag{8}$$

- *in the subcritical case ($\mu < 0$)*

$$C_t^\lambda \leq \frac{C_0^\lambda e^{\mu t}}{1 + \frac{c\underline{\xi}}{2|\mu|} C_0^\lambda (1 - e^{\mu t})} \quad \textit{and therefore} \quad \sup_{\lambda\in\mathbb{R}_+^k} C_t^\lambda \leq \frac{2|\mu| e^{\mu t}}{c\underline{\xi}(1 - e^{\mu t})} \tag{9}$$

(iii) *Let* $B_t^\lambda := \inf_{1\leq i\leq k} \frac{u_{t,i}^\lambda}{\xi_i}$ *and* $\bar\xi := \sup_i \xi_i$*. Then*

$$\forall t\geq 0,\ \lambda\in\mathbb{R}_+^k,\quad B_t^\lambda \geq \begin{cases} \dfrac{B_0^\lambda}{1+\frac{c\bar\xi}{2}B_0^\lambda\, t} & \text{if } \mu=0 \\ \dfrac{B_0^\lambda e^{\mu t}}{1+\frac{c\bar\xi}{2|\mu|}B_0^\lambda(1-e^{\mu t})} & \text{if } \mu<0. \end{cases} \tag{10}$$

(iv) *For any* $\lambda\in\mathbb{R}_+^k$ *and* $t\geq 0$*,*

$$u_t^\lambda \geq \begin{cases} \left(1+\frac{c\underline{\xi}}{2}C_0^\lambda\, t\right)^{-\bar\xi/\underline{\xi}} e^{Dt}\lambda & \text{if } \mu=0 \\ \left(1+\frac{c\underline{\xi}}{2|\mu|}C_0^\lambda(1-e^{\mu t})\right)^{-\bar\xi/\underline{\xi}} e^{Dt}\lambda & \text{if } \mu<0. \end{cases} \tag{11}$$

The main difficulty in the multitype setting comes from the non-commutativity of matrices. For example (6) can be expressed as $\frac{du_t^\lambda}{dt} = (D+A_t)u_t^\lambda$ where the matrix A_t is diagonal with i-th diagonal element $cu_{t,i}^\lambda/2$. However, since D and A_t do not commute, it is not possible to express u_t^λ in terms of the exponential of $\int_0^t (D+A_s)\,ds$. The following lemma gives the main tool we use to solve this difficulty.

Lemma 2.4 *Assume that* $t\mapsto f(t)\in\mathbb{R}$ *is a continuous function on* $\mathbb{R}_+$ *and* $t\mapsto u_t\in\mathbb{R}^k$ *is a differentiable function on* $\mathbb{R}_+$*. Then*

$$\frac{du_t}{dt}\geq (D+f(t)I)u_t,\quad \forall t\geq 0 \quad\Longrightarrow\quad u_t\geq \exp\Big(\int_0^t (D+f(s)I)\,ds\Big)u_0,\quad \forall t\geq 0$$

For any $1\leq i\leq k$, applying (5) with $x=e^i$ where $e^i_j=\delta_{ij}, 1\leq j\leq k$, one easily deduces the existence of a limit in $[0,\infty]$ of $u_{t,i}^\lambda$ when $\lambda\hookrightarrow\infty$. Moreover, by Lemma 2.3 (ii) and (iii), for any $t>0$,

$$0<\frac{2f(\theta)}{c\bar\xi}\leq \lim_{\lambda\hookrightarrow\infty} u_\theta^\lambda \leq \frac{2f(\theta)}{c\underline{\xi}}<+\infty$$

where $f(\theta)=1/\theta$ if $\mu=0$ or $f(\theta)=|\mu|e^{\mu\theta}/(1-e^{\mu\theta})$ if $\mu<0$. Therefore $\lim_{\theta\to\infty}\lim_{\lambda\hookrightarrow\infty} u_\theta^\lambda=0$ and, for sufficiently large θ,

$$\frac{1-e^{-(x_t,\lim_{\lambda\hookrightarrow\infty}u_\theta^\lambda)}}{1-e^{-(x,\lim_{\lambda\hookrightarrow\infty}u_{t+\theta}^\lambda)}}\leq K\frac{(x_t,\mathbf{1})}{(x,\mathbf{1})}$$

for some constant K that may depend on t but is independent of θ. Since $\mathbb{E}\langle X_t,\mathbf{1}\rangle<\infty$ for any $t\geq 0$ (see [15] or [16]), Lebesgue's dominated convergence theorem can be applied to make a first-order expansion in θ in (7).

This yields that the density with respect to $\mathbb{P}$ of $\mathbb{P}$ conditioned on the non-extinction at time $t+\theta$ on $\mathcal{F}_t$, converges in $L^1(\mathbb{P})$ when $\theta \to \infty$ to

$$\big(x_t, \lim_{\theta\to\infty} \frac{\lim_{\lambda\hookrightarrow\infty} u_\theta^\lambda}{(x, \lim_{\lambda\hookrightarrow\infty} u_{t+\theta}^\lambda)}\big) \tag{12}$$

if this limit exists.
We will actually prove that

$$\lim_{\theta\to\infty} \sup_{\lambda\neq 0} \| \frac{1}{(x, u_{t+\theta}^\lambda)} u_\theta^\lambda - \frac{e^{-\mu t}}{(x,\xi)} \xi \| = 0. \tag{13}$$

This will imply that the limits in θ and in λ can be exchanged in (12) and thus

$$\lim_{\theta\to\infty} \mathbb{E}(\mathbb{1}_B \mid \langle X_{t+\theta}, \mathbf{1}\rangle > 0) = e^{-\mu t}\, \mathbb{E}\Big(\mathbb{1}_B \frac{(x_t,\xi)}{(x,\xi)}\Big) = e^{-\mu t}\, \mathbb{E}\Big(\mathbb{1}_B \frac{\langle X_t, \xi\rangle}{\langle m,\xi\rangle}\Big),$$

completing the proof of Theorem 2.2.
Subcritical case: $\mu < 0$
As a preliminary result, observe that, since D has nonnegative nondiagonal entries, there exists $\alpha > 0$ such that $D + \alpha I \geq 0$, and then $\exp(Dt) \geq 0$.

Since $\dfrac{du_t^\lambda}{dt} \leq D u_t^\lambda$, we first remark by Lemma 2.4 (applied to $-u_t^\lambda$), that

$$\forall t \geq 0, \quad u_t^\lambda \leq e^{Dt}\lambda.$$

Second, it follows from Lemma 2.3 (iv) that

$$\begin{aligned} e^{Dt}\lambda - u_t^\lambda &\leq \Big(1 - \big(1 + \frac{c\underline{\xi}}{2|\mu|} C_0^\lambda (1 - e^{\mu t})\big)^{-\bar{\xi}/\underline{\xi}}\Big) e^{Dt}\lambda \\ &\leq \frac{c\bar{\xi}}{2|\mu|} C_0^\lambda (1 - e^{\mu t})\, e^{Dt}\lambda. \end{aligned}$$

Therefore, since $C_0^\lambda = \sup_i \lambda_i/\xi_i$, there exists a constant K independent of λ such that

$$\forall \lambda \geq 0,\ \forall t \geq 0, \quad e^{Dt}\lambda - u_t^\lambda \leq K\|\lambda\| e^{Dt}\lambda. \tag{14}$$

In particular,

$$\|\lambda\| \leq \frac{1}{2K} \quad \Rightarrow \quad u_t^\lambda \geq \frac{1}{2} e^{Dt}\lambda \quad \forall t \geq 0.$$

Third, it follows from Lemma 2.3 (ii) that there exists t_0 such that

$$\forall t \geq t_0,\ \forall \lambda \geq 0, \quad \|u_t^\lambda\| \leq \frac{1}{2K}.$$

Fourth, as a consequence of Perron-Frobenius' theorem, the exponential matrix e^{Dt} decreases like $e^{\mu t}$ for t large in the following sense: as $t \to \infty$,

$$\exists \gamma > 0, \quad e^{Dt} = e^{\mu t} P + O(e^{(\mu-\gamma)t}) \tag{15}$$

where $P := (\xi_i \eta_j)_{1 \le i,j \le k}$ (see [29] Theorem 2.7). Therefore, there exists θ_0 such that

$$\forall t \ge \theta_0, \quad \frac{1}{2} e^{\mu t} P \le e^{Dt} \le 2 e^{\mu t} P.$$

Last, there exists a positive constant K' such that

$$\forall u, v \in \mathbb{R}_+^k, \quad (v, Pu) = (u, \eta)(v, \xi) \ge \underline{\xi}\,\underline{\eta}\,(u, \mathbf{1})\,(v, \mathbf{1}) \ \ge K' \|u\| \|v\|.$$

Combining all the above inequalities, we get for any $a \in \mathbb{R}_+^k$, $b, \lambda \in \mathbb{R}_+^k \setminus \{0\}$ and for any $\theta \ge \theta_0$,

$$\begin{aligned}
&\left| \frac{(a, u^\lambda_{t_0+\theta})}{(b, u^\lambda_{t_0+\theta+t})} - \frac{(a, e^{D\theta} u^\lambda_{t_0})}{(b, e^{D(\theta+t)} u^\lambda_{t_0})} \right| \\
&\qquad \le \frac{(a, |u^\lambda_{t_0+\theta} - e^{D\theta} u^\lambda_{t_0}|)}{(b, u^\lambda_{t_0+\theta+t})} + \frac{(a, e^{D\theta} u^\lambda_{t_0})(b, |u^\lambda_{t_0+\theta+t} - e^{D(\theta+t)} u^\lambda_{t_0}|)}{(b, u^\lambda_{t_0+\theta+t})(b, e^{D(\theta+t)} u^\lambda_{t_0})} \\
&\qquad \le \frac{2K \|a\| \|u^\lambda_{t_0}\| \|e^{D\theta} u^\lambda_{t_0}\|}{(b, e^{D(\theta+t)} u^\lambda_{t_0})} + \frac{2K \|a\| \|e^{D\theta} u^\lambda_{t_0}\| \|b\| \|u^\lambda_{t_0}\| \|e^{D(\theta+t)} u^\lambda_{t_0}\|}{(b, e^{D(\theta+t)} u^\lambda_{t_0})^2} \\
&\qquad \le \bar{K} \|a\| \|u^\lambda_{t_0}\| e^{-\mu t} \left(\frac{\|P u^\lambda_{t_0}\|}{(b, P u^\lambda_{t_0})} + \frac{\|b\| \|P u^\lambda_{t_0}\|^2}{(b, P u^\lambda_{t_0})^2} \right) \\
&\qquad \le \bar{K} e^{-\mu t} \frac{\|a\|}{\|b\|} \|u^\lambda_{t_0}\|
\end{aligned} \tag{16}$$

where the constants $\bar{K}$ may vary from line to line, but are independent of λ and t_0.

Now, let $t_0(\theta)$ be an increasing function of θ larger than t_0 such that $t_0(\theta) \to \infty$ when $\theta \to \infty$. By Lemma 2.3 (ii), $\|u^\lambda_{t_0(\theta)}\| \to 0$ when $\theta \to \infty$, uniformly in $\lambda \ge 0$. Then, by (16), uniformly in $\lambda \ge 0$,

$$\begin{aligned}
\lim_{\theta \to \infty} \frac{(a, u^\lambda_{t_0(\theta)+\theta})}{(b, u^\lambda_{t_0(\theta)+\theta+t})} &= \lim_{\theta \to \infty} \frac{(a, e^{D\theta} u^\lambda_{t_0(\theta)})}{(b, e^{D(\theta+t)} u^\lambda_{t_0(\theta)})} \\
&= \lim_{\theta \to \infty} \frac{(a, e^{\mu\theta} P u^\lambda_{t_0(\theta)})}{(b, e^{\mu(\theta+t)} P u^\lambda_{t_0(\theta)})} \\
&= \lim_{\theta \to \infty} e^{-\mu t} \frac{(\eta, u^\lambda_{t_0(\theta)})(a, \xi)}{(\eta, u^\lambda_{t_0(\theta)})(b, \xi)} \\
&= e^{-\mu t} \frac{(a, \xi)}{(b, \xi)}
\end{aligned}$$

which completes the proof of Theorem 2.2 in the case $\mu < 0$.
Critical case: $\mu = 0$
The above computation has to be slightly modified. Inequality (14) becomes

$$|u_t^\lambda - e^{Dt}\lambda| \leq \left(1 - \left(1 + \frac{c\underline{\xi}}{2} C_0^\lambda\, t\right)^{-\bar{\xi}/\underline{\xi}}\right) e^{Dt}\lambda$$
$$\leq K\|\lambda\| t e^{Dt}\lambda. \tag{17}$$

Therefore, the right-hand side of (16) has to be replaced by

$$K\frac{\|a\|}{\|b\|}\|u_{t_0}^\lambda\|(\theta + t). \tag{18}$$

Now, using Lemma 2.3 (iii) again, it suffices to choose a function $t_0(\theta)$ in such a way that $\lim_{\theta\to\infty}\theta \sup_{\lambda\geq 0}\|u_{t_0(\theta)}^\lambda\| = 0$. One can now complete the proof of Theorem 2.2 as above. □

Proof of Lemma 2.3

(i) First, observe that, by (5), $u_t^\lambda \geq 0$ for any $t \geq 0$. Next, since D is nonnegative outside the diagonal,

$$\frac{du_{t,i}^\lambda}{dt} = \sum_{j=1}^k d_{ij}u_{t,j}^\lambda - \frac{c}{2}(u_{t,i}^\lambda)^2 \geq (d_{ii} - \frac{c}{2}u_{t,i}^\lambda)u_{t,i}^\lambda. \tag{19}$$

Therefore, for any i such that $\lambda_i > 0$, $u_{t,i}^\lambda > 0$ for any $t \geq 0$.
Let $I := \{i : \lambda_i > 0\}$ and $J := \{j : \lambda_j = 0\}$. By the irreducibility of the matrix D, there exist $i \in I$ and $j \in J$ such that $d_{ji} > 0$. Therefore, for sufficiently small $t > 0$,

$$\frac{du_{t,j}^\lambda}{dt} = \sum_{l=1}^k d_{jl}u_{t,l}^\lambda - \frac{c}{2}(u_{t,j}^\lambda)^2 > \frac{d_{ji}}{2}u_{t,i}^\lambda$$

and thus $u_{t,j}^\lambda > 0$ for $t > 0$ in a neighborhood of 0. Moreover, as long as $u_{t,i}^\lambda > 0$, for the same reason, $u_{t,j}^\lambda$ cannot reach 0.
Defining $I' = I \cup \{j\}$ and $J' = J \setminus \{j\}$, there exists $i' \in I'$ and $j' \in J'$ such that $d_{j'i'} > 0$. For sufficiently small $\varepsilon > 0$, $u_{\varepsilon,i'}^\lambda > 0$ and the previous argument shows that $u_{\varepsilon+t,j'}^\lambda > 0$ for $t > 0$ as long as $u_{\varepsilon+t,i'}^\lambda > 0$. Letting ε go to 0 yields that $u_{t,j'}^\lambda > 0$ for sufficiently small $t > 0$.
Applying the same argument inductively shows that $u_t^\lambda > 0$ for $t > 0$ in a neighborhood of 0. Using (19) again, this property can be extended to all $t > 0$.

(ii) and (iii) As the supremum of finitely many continuously differentiable functions, $t \mapsto C_t^\lambda$ is differentiable except at at most countably many points. Indeed, it is not differentiable at time t if and only if there exist two types i and j such that $u_{t,i}^\lambda/\xi_i = u_{t,j}^\lambda/\xi_j$ **and** $d(u_{t,i}^\lambda/\xi_i)/dt \neq d(u_{t,j}^\lambda/\xi_j)/dt$. For fixed i and j, such points are necessarily isolated, and hence are at most denumerable.
Fix a time t at which C_t^λ is differentiable and fix i such that $u_{t,i}^\lambda = C_t^\lambda \xi_i$. Then

$$\begin{aligned}\frac{dC_t^\lambda}{dt}\xi_i = \frac{du_{t,i}^\lambda}{dt} &= \sum_{j=1}^k d_{ij}u_{t,j}^\lambda - \frac{c}{2}(u_{t,i}^\lambda)^2 \\ &\leq C_t^\lambda \sum_{j\neq i} d_{ij}\xi_j + d_{ii}u_{t,i}^\lambda - \frac{c}{2}(u_{t,i}^\lambda)^2 \\ &= C_t^\lambda (D\xi)_i - \frac{c}{2}(u_{t,i}^\lambda)^2 = \mu C_t^\lambda \xi_i - \frac{c}{2}\xi_i^2 (C_t^\lambda)^2\end{aligned}$$

where the inequality comes from the fact that D is nonnegative outside of the diagonal and where the third line comes from the specific choice of the subscript i. Therefore,

$$\frac{dC_t^\lambda}{dt} \leq \mu C_t^\lambda - \frac{c}{2}\underline{\xi}\,(C_t^\lambda)^2. \tag{20}$$

Assume $\mu = 0$.
By Point (i), if $\lambda \neq 0$, $C_t^\lambda > 0$ for any $t \geq 0$ (the case $\lambda = 0$ is trivial). Then, for any $t \geq 0$, except at at most countably many points,

$$-\frac{dC_t^\lambda/dt}{(C_t^\lambda)^2} \geq \frac{c}{2}\underline{\xi}.$$

Integrating this inequality between 0 and t, we get

$$\frac{1}{C_t^\lambda} \geq \frac{1}{C_0^\lambda} + \frac{c}{2}\underline{\xi}t \quad \Rightarrow \quad C_t^\lambda \leq \frac{C_0^\lambda}{1 + \frac{cC_0^\lambda}{2}\underline{\xi}t}.$$

The proof of the case $\mu < 0$ can be done by the same argument applied to $t \mapsto e^{-\mu t}C_t^\lambda$. Inequalities (iii) are obtained in a similar way too.

(iv) By definition of C_t^λ, (6) implies that

$$\frac{du_t^\lambda}{dt} \geq \left(D - \frac{c}{2}\bar{\xi}\,C_t^\lambda I\right)u_t^\lambda.$$

Then, (iv) follows from (ii) and Lemma 2.4. $\square$

Proof of Lemma 2.4 Fix $\varepsilon > 0$ and let

$$u_t^{(\varepsilon)} := \exp\Big(\int_0^t (D + f(s)I)ds\Big)(u_0 - \varepsilon).$$

Then

$$\frac{du_t}{dt} - \frac{du_t^{(\varepsilon)}}{dt} \geq (D + f(t)I)(u_t - u_t^{(\varepsilon)}).$$

Let $t_0 := \inf\{t \geq 0 : \exists i \in \{1, \dots, k\}, u_{t,i} < u_{t,i}^{(\varepsilon)}\}$. For any $t \leq t_0$, since D is nonnegative outside of the diagonal,

$$\forall i \in \{1, \dots, k\}, \quad \frac{d}{dt}(u_{t,i} - u_{t,i}^{(\varepsilon)}) \geq (d_{ii} + f(t))(u_{t,i} - u_{t,i}^{(\varepsilon)}).$$

Since $u_0 > u_0^{(\varepsilon)}$, this implies that $u_t - u_t^{(\varepsilon)} > 0$ for any $t \leq t_0$ and thus $t_0 = +\infty$. Letting ε go to 0 completes the proof of Lemma 2.4. □

Remark 2.5 Since the limit in (13) is uniform in λ, one can choose in particular $\lambda = \lambda^i, 1 \leq i \leq k$, where $\lambda_j^i = 0$ for $j \neq i$. Thus, for each type i,

$$\lim_{\theta\to\infty} \frac{\lim_{\lambda_i^i\to\infty} u_\theta^{\lambda^i}}{(x, \lim_{\lambda_i^i\to\infty} u_{t+\theta}^{\lambda^i})} = \frac{e^{-\mu t}}{(x, \xi)}\,\xi$$

which implies as in (7) that, for $B \in \mathcal{F}_t$,

$$\lim_{\theta\to\infty} \mathbb{P}(B \mid \langle X_{t+\theta,i}, 1\rangle > 0) = \lim_{\theta\to\infty} \mathbb{P}(B \mid \langle X_{t+\theta}, \mathbf{1}\rangle > 0) = \mathbb{P}^*(B).$$

Therefore, Theorem 2.2 remains valid if the conditioning by the non-extinction of the whole population is replaced by the non-extinction of type i only. This property relies strongly on the irreducibility of the mutation matrix D. In Section 4, we will show that it does not always hold true when D is reducible (see for example Theorem 4.1 or Theorem 4.4). ♢

2.2 Laplace functional of $\mathbb{P}^*$ and Martingale Problem

To better understand the properties of $\mathbb{P}^*$, its Laplace functional provides a very useful tool.

Theorem 2.6 $\mathbb{P}^*$ *is characterized by:* $\forall f \in C_b(\mathbb{R}, \mathbb{R}^k)_+$

$$\mathbb{E}^*(\exp -\langle X_t, f\rangle \mid X_0 = m) = \frac{\langle m, V_t f\rangle}{\langle m, \xi\rangle} e^{-\mu t} e^{-\langle m, U_t f\rangle} \tag{21}$$

where the semigroup $V_t f$ *is the unique solution of the PDE*

$$\frac{\partial V_t f}{\partial t} = \Delta V_t f + D V_t f - c U_t f \odot V_t f, \quad V_0 f = \xi. \tag{22}$$

Proof From Theorem 2.2 and (1) we get

$$\begin{aligned}\mathbb{E}^*(e^{-\langle X_t,f\rangle} \mid X_0=m) &= \mathbb{E}\left(\frac{\langle X_t,\xi\rangle}{\langle m,\xi\rangle}e^{-\mu t}e^{-\langle X_t,f\rangle} \mid X_0=m\right)\\ &= \frac{e^{-\mu t}}{\langle m,\xi\rangle}\frac{\partial}{\partial\varepsilon}\mathbb{E}\left(e^{-\langle X_t,f+\varepsilon\xi\rangle}\right)\Big|_{\varepsilon=0}\\ &= \frac{e^{-\mu t}}{\langle m,\xi\rangle}e^{-\langle m,U_t f\rangle}\frac{\partial}{\partial\varepsilon}\langle m,U_t(f+\varepsilon\xi)\rangle\Big|_{\varepsilon=0}.\end{aligned}$$

Let $V_t f := \frac{\partial}{\partial\varepsilon}U_t(f+\varepsilon\xi)\big|_{\varepsilon=0}$. Then $V_t f$ is solution of

$$\begin{aligned}\frac{\partial V_t f}{\partial t} &= \frac{\partial}{\partial\varepsilon}\Big(\Delta U_f(f+\varepsilon\xi)+DU_t(f+\varepsilon\xi)-\frac{c}{2}U_t(f+\varepsilon\xi)^{\odot 2}\Big)\Big|_{\varepsilon=0}\\ &= (\Delta+D)V_t f - cU_t f\odot V_t f\end{aligned}$$

and $V_0 f = \frac{\partial}{\partial\varepsilon}(f+\varepsilon\xi)\big|_{\varepsilon=0}=\xi$. □

Comparing with the Laplace functional of $\mathbb{P}$, the multiplicative term $\frac{\langle m,V_t f\rangle}{\langle m,\xi\rangle}e^{-\mu t}$ appears in the Laplace functional of $\mathbb{P}^*$. In particular, the multi-type Feller diffusion x_t is characterized under $\mathbb{P}^*$ by

$$\mathbb{E}^*(\exp-(x_t,\lambda) \mid x_0=x) = \frac{(x,v_t^\lambda)}{(x,\xi)}e^{-\mu t}e^{-(x,u_t^\lambda)},\quad \lambda\in\mathbb{R}_+^k \tag{23}$$

where $v_t^\lambda := V_t\lambda$ satisfies the differential system

$$\frac{dv_t^\lambda}{dt} = Dv_t^\lambda - cu_t^\lambda\odot v_t^\lambda,\quad v_0^\lambda=\xi. \tag{24}$$

The following theorem gives the martingale problem satisfied by the conditioned MDW process. This formulation also allows one to interpret $\mathbb{P}^*$ as an unconditioned MDW process with immigration (see Remark 2.8 below). The term with Laplace functional $\frac{\langle m,V_t f\rangle}{\langle m,\xi\rangle}e^{-\mu t}$ that we just mentioned is another way to interpret this immigration.

Theorem 2.7 *$\mathbb{P}^*$ is the unique solution of the following martingale problem: for all $f\in C_b^2(\mathbb{R},\mathbb{R}^k)_+$,*

$$\begin{aligned}&\exp(-\langle X_t,f\rangle)-\exp(-\langle m,f\rangle)\\ &\quad+\int_0^t\Big(\langle X_s,(\Delta+D)f\rangle + c\frac{\langle X_s,f\odot\xi\rangle}{\langle X_s,\xi\rangle}-\frac{c}{2}\langle X_s,f^{\odot 2}\rangle\Big)\exp(-\langle X_s,f\rangle)\,ds\end{aligned} \tag{25}$$

is a $\mathbb{P}^*$-local martingale.

Proof According to [15] (see also [6] for the monotype case), $\mathbb{P}$ is the unique solution of the following martingale problem : for any function $F : M(\mathbb{R})^k \to \mathbb{R}$ of the form $\varphi(\langle \cdot, f\rangle)$ with $\varphi \in C^2(\mathbb{R},\mathbb{R})$ and $f \in C_b^2(\mathbb{R},\mathbb{R}^k)_+$,

$$F(X_t) - F(X_0) - \int_0^t \mathcal{A}F(X_s)\,ds \quad \text{is a } \mathbb{P}\text{-local martingale.} \tag{26}$$

Here the infinitesimal generator $\mathcal{A}$ is given by

$$\begin{aligned}\mathcal{A}F(m) &= \langle m, (\Delta + D)\frac{\partial F}{\partial m}\rangle + \frac{c}{2}\langle m, \partial^2 F/\partial m^2\rangle \\ &= \sum_{i=1}^k \langle m_i, \Delta \frac{\partial F}{\partial m_i} + \sum_{j=1}^k d_{ij}\frac{\partial F}{\partial m_j}\rangle + \frac{c}{2}\sum_{i=1}^k \langle m_i, \frac{\partial^2 F}{\partial m_i^2}\rangle.\end{aligned}$$

where we use the notation $\partial F/\partial m = (\partial F/\partial m_i)_{1\le i\le k}$ and $\partial^2 F/\partial m^2 = (\partial^2 F/\partial m_i^2)_{1\le i\le k}$ with

$$\frac{\partial F}{\partial m_i}(x) := \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\big(F(m_1,\ldots,m_i+\varepsilon\delta_x,\ldots,m_k) - F(m)\big), x\in\mathbb{R}.$$

Applying this to the time-dependent function

$$F(s,m) := \langle m,\xi\rangle e^{-\mu s}e^{-\langle m,f\rangle} \text{ with } f\in C_b^2(\mathbb{R},\mathbb{R}^k)_+$$

for which

$$\begin{aligned}&\frac{\partial F(s,m)}{\partial m}(x) = -f(x)F(s,m) + \xi e^{-\mu s - \langle m,f\rangle}\\ \text{and}\quad &\frac{\partial^2 F(s,m)}{\partial m^2}(x) = f^{\odot 2}(x)F(s,m) - 2f(x)\odot\xi e^{-\mu s-\langle m,f\rangle},\end{aligned}$$

one gets

$$\begin{aligned}&\frac{\partial F}{\partial s}(s,m) + \mathcal{A}(F(s,\cdot))(m)\\ &\qquad = -\langle m,(\Delta+D)f\rangle F + \frac{c}{2}\langle m, f^{\odot 2}\rangle F - c\frac{\langle m, f\odot\xi\rangle}{\langle m,\xi\rangle}F.\end{aligned}$$

Therefore,

$$\begin{aligned}&\langle X_t,\xi\rangle e^{-\mu t - \langle X_t,f\rangle} - \langle m,\xi\rangle e^{-\langle m,f\rangle}\\ &+\int_0^t \langle X_s,\xi\rangle e^{-\mu s}\Big(\langle X_s,(\Delta+D)f\rangle + c\frac{\langle X_s, f\odot\xi\rangle}{\langle X_s,\xi\rangle} - \frac{c}{2}\langle X_s, f^{\odot 2}\rangle\Big)e^{-\langle X_s,f\rangle}ds\end{aligned}$$

is a $\mathbb{P}$-local martingale, which implies that (25) is a $\mathbb{P}^*$-local martingale. The uniqueness of the solution $\mathbb{P}^*$ to the martingale problem (25) comes from the uniqueness of the solution of the martingale problem (26). □

Remark 2.8 Due to the form of the martingale problem (25), the probability measure $\mathbb{P}^*$ can be interpreted as the law of a MDW process with interactive immigration whose rate at time s, if conditioned by X_s, is a random measure with Laplace functional $\exp -c\frac{\langle X_s, f\odot\xi\rangle}{\langle X_s,\xi\rangle}$. Monotype DW processes with deterministic immigration rate were introduced by Dawson in [2]. The first interpretation of conditioned branching processes as branching processes with immigration goes back to Kawazu and Watanabe in [18], Example 2.1. See also [28] and [10] for further properties in the monotype case. ♢

3 Long time behavior of conditioned multi-type Feller diffusions

We are now interested in the long time behavior of the MDW process conditioned on non-extinction in the remote future. Unfortunately, because of the Laplacian term in (22), there is no hope to obtain a limit of X_t under $\mathbb{P}^*$ at the level of measure (however, see [10] for the long time behavior of critical monotype conditioned Dawson-Watanabe processes with *ergodic* spatial motion). Therefore, we will restrict our attention to the $\mathbb{R}^k$-valued multitype Feller diffusion x_t. As a first step in our study, we begin this section with the monotype case.

3.1 Monotype case

In this subsection, we first give asymptotic behavior of x_t under $\mathbb{P}^*$ (Proposition 3.1). This result is already known, but we give a proof that will be useful in the following section. We also give a new result about the exchange of limits (Proposition 3.3).

Let us first introduce some notation for the monotype case.
The matrix D is reduced to its eigenvalue μ, the vector ξ is equal to the number 1. Since we only consider the critical and subcritical cases, one has $\mu \leq 0$. The law $\mathbb{P}^*$ of the MDW process conditioned on non-extinction in the remote future is locally absolutely continuous with respect to $\mathbb{P}$ (monotype version of Theorem 2.2, already proved in [28], Proposition 1). More precisely

$$\mathbb{P}^*\big|_{\mathcal{F}_t} = \frac{\langle X_t, 1\rangle}{\langle m, 1\rangle} e^{-\mu t}\, \mathbb{P}\big|_{\mathcal{F}_t}. \tag{27}$$

Furthermore the Laplace functional of $\mathbb{P}^*$ satisfies (see [28], Theorem 3):

$$\mathbb{E}^*(\exp -\langle X_t, f\rangle \mid X_0 = m) = \frac{\langle m, V_t f\rangle}{\langle m, 1\rangle} e^{-\mu t} e^{-\langle m, U_t f\rangle} = \frac{\langle m, \tilde{V}_t f\rangle}{\langle m, 1\rangle} e^{-\langle m, U_t f\rangle} \tag{28}$$

where

$$\frac{\partial \tilde{V}_t f}{\partial t} = \Delta \tilde{V}_t f - c\, U_t f \tilde{V}_t f, \quad \tilde{V}_0 f = 1. \tag{29}$$

The total mass process $x_t = \langle X_t, 1\rangle$ is a (sub)critical Feller branching diffusion under $\mathbb{P}$. By (28) its Laplace transform under $\mathbb{P}^*$ is

$$\mathbb{E}^*(\exp -\lambda x_t \mid x_0 = x) = \tilde{v}_t^\lambda\, e^{-x u_t^\lambda}, \qquad \lambda \in \mathbb{R}_+, \tag{30}$$

with

$$\frac{d\tilde{v}_t^\lambda}{dt} = -c u_t^\lambda \tilde{v}_t^\lambda, \quad \tilde{v}_0^\lambda = 1.$$

Recall that the cumulant u_t^λ satisfies

$$\frac{du_t^\lambda}{dt} = \mu\, u_t^\lambda - \frac{c}{2}\, (u_t^\lambda)^2, \quad u_0^\lambda = \lambda. \tag{31}$$

This yields in the subcritical case the explicit formulas

$$u_t^\lambda = \frac{\lambda\, e^{\mu t}}{1 + \frac{c}{2|\mu|}\lambda(1 - e^{\mu t})}, \quad \lambda \geq 0, \tag{32}$$

$$\text{and} \quad \tilde{v}_t^\lambda = \exp\left(-c \int_0^t u_s^\lambda ds\right) = \frac{1}{\left(1 + \frac{c}{2|\mu|}\lambda(1 - e^{\mu t})\right)^2}. \tag{33}$$

In the critical case ($\mu = 0$) one obtains (see [20] Equation (2.14))

$$u_t^\lambda = \frac{\lambda}{1 + \frac{c}{2}\lambda t} \qquad \text{and} \qquad v_t^\lambda = \tilde{v}_t^\lambda = \frac{1}{\left(1 + \frac{c}{2}\lambda t\right)^2}. \tag{34}$$

We are now ready to state the following asymptotic result.

Proposition 3.1

(a) *In the critical case ($\mu = 0$), the process x_t explodes in $\mathbb{P}^*$-probability when $t \to \infty$, i.e. for any $M > 0$,*

$$\lim_{t\to+\infty} \mathbb{P}^*(x_t \leq M) = 0.$$

(b) *In the subcritical case ($\mu < 0$),*

$$\lim_{t\to+\infty} \mathbb{P}^*(x_t \in \cdot) \overset{(d)}{=} \Gamma(2, \frac{2|\mu|}{c})$$

where this notation means that x_t converges in $\mathbb{P}^*$-distribution to a Gamma distribution with parameters 2 and $2|\mu|/c$.

One can find in [19] Theorem 4.2 a proof of this theorem for a more general model, based on a pathwise approach. We propose here a different proof, based on the behavior of the cumulant semigroup and moment properties, which will be useful in the sequel.

Proof For $\mu = 0$, by (34), $u_t^\lambda \to 0$ and $v_t^\lambda \to 0$ when $t \to \infty$ for any $\lambda \neq 0$. This implies by (30) the asymptotic explosion of x_t in $\mathbb{P}^*$-probability. Actually, the rate of explosion is also known: in [10] Lemma 2.1, the authors have proved that $\frac{x_t}{t}$ converges in distribution as $t \to \infty$ to a Gamma-distribution. This can also be deduced from (34), since $u_t^{\lambda/t}$ and $v_t^{\lambda/t}$ converge to 0 and $1/(1 + c\lambda/2)^2$ respectively, as $t \to \infty$.

For $\mu < 0$, by (30), (32) and (33), the process x_t has the same law as the sum of two independent random variables, the first one with distribution $\Gamma(2, \frac{2|\mu|}{c(1-e^{\mu t})})$ and the second one vanishing for $t \to \infty$. The conclusion is now clear. □

Remark 3.2 The presence of a Gamma-distribution in the above Proposition is not surprising.

- As we already mentioned it appears in the critical case as the limit law of x_t/t [10].
- It also goes along with the fact that these distributions are the equilibrium distributions for subcritical Feller branching diffusions with constant immigration. (See [1], and Lemma 6.2.2 in [4]). *We are grateful to A. Wakolbinger for proposing this interpretation.*
- Another interpretation is given in [19]. The Yaglom distribution of the process x_t, defined as the limit law as $t \to \infty$ of x_t conditioned on $x_t > 0$, is the exponential distribution with parameter $2|\mu|/c$ (see Proposition 3.3 below, with $\theta = 0$). The Gamma distribution appears as the size-biased distribution of the Yaglom limit ($\mathbb{P}^*(x_\infty \in dr) = r\mathbb{P}(Y \in dr)/\mathbb{E}(Y)$, where $Y \sim \mathcal{E}xp(\frac{2|\mu|}{c})$), which is actually a general fact ([19, Th.4.2(ii)(b)]). ♢

We have just proved that, for $\mu < 0$, the law of x_t conditioned on $x_{t+\theta} > 0$ converges to a Gamma distribution when taking first the limit $\theta \to \infty$ and next the limit $t \to \infty$. It is then natural to ask whether the order of the two limits can be exchanged: what happens if one first fix θ and let t tend to infinity, and then let θ increase? We obtain the following answer.

Proposition 3.3 *When $\mu < 0$, conditionally on $x_{t+\theta} > 0$, x_t converges in distribution when $t \to \infty$ to the sum of two independent exponential r.v. with respective parameters $\frac{2|\mu|}{c}$ and $\frac{2|\mu|}{c}(1 - e^{\mu\theta})$.*

Therefore, one can interchange both limits in time:

$$\lim_{\theta\to\infty} \lim_{t\to\infty} \mathbb{P}(x_t \in \cdot \mid x_{t+\theta} > 0) \overset{(d)}{=} \lim_{t\to\infty} \lim_{\theta\to\infty} \mathbb{P}(x_t \in \cdot \mid x_{t+\theta} > 0) \overset{(d)}{=} \Gamma(2, \frac{2|\mu|}{c}).$$

Proof First, observe that, by (32),

$$\lim_{\bar{\lambda}\to\infty} u_t^{\bar{\lambda}} = \frac{2|\mu|}{c} \frac{e^{\mu t}}{1 - e^{\mu t}} \text{ and } \lim_{t\to\infty} \frac{u_t^{\lambda}}{e^{\mu t}} = \frac{\lambda}{1 - \frac{c}{2\mu}\lambda}.$$

As in (7), it holds

$$\begin{aligned}
\mathbb{E}(e^{-\lambda x_t} \mid x_{t+\theta} > 0) &= \frac{\mathbb{E}\big(e^{-\lambda x_t}(1 - \mathbb{P}(x_{t+\theta} = 0 \mid \mathcal{F}_t))\big)}{1 - \mathbb{P}(x_{t+\theta} = 0)} \\
&= \frac{\mathbb{E}\big(e^{-\lambda x_t}(1 - e^{-x_t \lim_{\bar{\lambda}\to\infty} u_\theta^{\bar{\lambda}}})\big)}{1 - e^{-x \lim_{\bar{\lambda}\to\infty} u_{t+\theta}^{\bar{\lambda}}}} \\
&= \frac{e^{-xu_t^{\lambda}} - e^{-xu_t^{\lambda + \lim_{\bar{\lambda}\to\infty} u_\theta^{\bar{\lambda}}}}}{1 - e^{-x \lim_{\bar{\lambda}\to\infty} u_{t+\theta}^{\bar{\lambda}}}} \\
&= \frac{e^{-xu_t^{\lambda}} - e^{-xu_t^{\lambda + \lim_{\bar{\lambda}\to\infty} u_\theta^{\bar{\lambda}}}}}{1 - \exp(-x \frac{2|\mu|}{c} \frac{e^{\mu(t+\theta)}}{1 - e^{\mu(t+\theta)}})}
\end{aligned}$$

Thus

$$\begin{aligned}
\lim_{t\to\infty} \mathbb{E}(e^{-\lambda x_t} \mid x_{t+\theta} > 0) &= \frac{c}{2|\mu|} e^{-\mu\theta} \lim_{t\to\infty} e^{|\mu|t} \big(u_t^{\lambda + \lim_{\bar{\lambda}\to\infty} u_\theta^{\bar{\lambda}}} - u_t^{\lambda}\big) \\
&= \frac{c}{2|\mu|} e^{-\mu\theta} \Big(\frac{\lambda + \lim_{\bar{\lambda}\to\infty} u_\theta^{\bar{\lambda}}}{1 - \frac{c}{2\mu}(\lambda + \lim_{\bar{\lambda}\to\infty} u_\theta^{\bar{\lambda}})} - \frac{\lambda}{1 - \frac{c}{2\mu}\lambda} \Big) \\
&= \frac{1}{1 + \frac{c}{2|\mu|}\lambda} \cdot \frac{1}{1 + \frac{c}{2|\mu|}(1 - e^{\mu\theta})\lambda}
\end{aligned}$$

where the first (resp. the second) factor is equal to the Laplace transform of an exponential r.v. with parameter $2|\mu|/c$ (resp. with parameter $\frac{2|\mu|}{c}(1-e^{\mu\theta})$). This means that

$$\lim_{t\to\infty} \mathbb{P}(x_t \in \cdot \mid x_{t+\theta} > 0) \overset{(d)}{=} \mathcal{E}xp(\frac{2|\mu|}{c}) \otimes \mathcal{E}xp(\frac{2|\mu|}{c}(1-e^{\mu\theta})).$$

It is now clear that

$$\lim_{\theta\to\infty}\lim_{t\to\infty} \mathbb{P}(x_t \in \cdot \mid x_{t+\theta} > 0) \overset{(d)}{=} \mathcal{E}xp(\frac{2|\mu|}{c}) \otimes \mathcal{E}xp(\frac{2|\mu|}{c}) = \Gamma(2, \frac{2|\mu|}{c}).$$

Thus, the limits in time interchange. □

Remark 3.4 The previous computation is also possible in the critical case and gives a similar interchangeability result. More precisely, for any $\theta > 0$, x_t explodes conditionally on $x_{t+\theta} > 0$ in $\mathbb{P}$-probability when $t \to +\infty$. In particular, for any $M > 0$,

$$\lim_{\theta\to\infty}\lim_{t\to\infty} \mathbb{P}(x_t \le M \mid x_{t+\theta} > 0) = \lim_{t\to\infty}\lim_{\theta\to\infty} \mathbb{P}(x_t \le M \mid x_{t+\theta} > 0) = 0.$$

◇

Remark 3.5 One can develop the same ideas as before when the branching mechanism with finite variance c is replaced by a β-stable branching mechanism, $0 < \beta < 1$, with infinite variance (see [2] Section 5 for a precise definition). In this case, equation (31) has to be replaced by $\frac{du_t^\lambda}{dt} = \mu\, u_t^\lambda - c(u_t^\lambda)^{1+\beta}$ which implies that

$$u_t^\lambda = \frac{\lambda\, e^{\mu t}}{\Big(1 + \frac{c\lambda^\beta}{|\mu|}(1-e^{\beta\mu t})\Big)^{1/\beta}}.$$

Therefore, with a similar calculation as above, one can easily compute the Laplace transform of the limit conditional law of x_t when $t \to \infty$ and prove the exchangeability of limits:

$$\lim_{\theta\to\infty}\lim_{t\to\infty} \mathbb{E}(e^{-\lambda x_t} \mid x_{t+\theta} > 0) = \lim_{t\to\infty}\lim_{\theta\to\infty} \mathbb{E}(e^{-\lambda x_t} \mid x_{t+\theta} > 0) = \frac{1}{\Big(1 + \frac{c}{|\mu|}\lambda^\beta\Big)^{1+1/\beta}}.$$

As before, this distribution can be interpreted as the size-biased Yaglom distribution corresponding to the stable branching mechanism. This conditional limit law for the subcritical branching process has been obtained in [21] Theorem 4.2. We also refer to [19] Theorem 5.2 for a study of the critical stable branching process. ◇

3.2 Multitype irreducible case

We now present the multitype generalization of Proposition 3.1 on the asymptotic behavior of the conditioned multitype Feller diffusion with irreducible mutation matrix D.

Theorem 3.6 (a) *In the critical case ($\mu = 0$), when the mutation matrix D is irreducible, x_t explodes in $\mathbb{P}^*$-probability when $t \to \infty$, i.e.*

$$\forall i \in \{1, \dots, k\},\ \forall M > 0, \quad \lim_{t\to+\infty} \mathbb{P}^*(x_{t,i} \leq M) = 0.$$

(b) *In the subcritical case ($\mu < 0$) when D is irreducible, the law of x_t converges in distribution under $\mathbb{P}^*$ when $t \to \infty$ to a non-trivial limit which does not depend on the initial condition x.*

Proof One obtains from (24) that

$$Dv_t^\lambda - c\sup_i(u_{t,i}^\lambda)\, v_t^\lambda \leq \frac{dv_t^\lambda}{dt} \leq Dv_t^\lambda - c\inf_i(u_{t,i}^\lambda)\, v_t^\lambda.$$

Then, by Lemma 2.4,

$$\exp\Big(\mu t - c\int_0^t \sup_i u_{s,i}^\lambda\, ds\Big)\xi \leq v_t^\lambda \leq \exp\Big(\mu t - c\int_0^t \inf_i u_{s,i}^\lambda\, ds\Big)\xi. \qquad (35)$$

Therefore, in the critical case, v_t^λ vanishes for t large if $\lambda > 0$, due to the divergence of $\int_0^\infty \inf_i u_{s,i}^\lambda ds$, which is itself a consequence of Lemma 2.3 (iii). If $\lambda_i = 0$ for some type i, by Lemma 2.3 (i) and the semigroup property of $t \mapsto u_t$, we can use once again Lemma 2.3 (iii) starting from a positive time, to prove that $\lim_{t\to\infty} v_t^\lambda = 0$. Then, the explosion of x_t in $\mathbb{P}^*$-probability follows directly from (23) and from the fact that $\lim_{t\to\infty} u_t^\lambda = 0$.

To prove (b), we study the convergence of $\tilde{v}_t^\lambda := e^{-\mu t} v_t^\lambda$ when $t \to \infty$. By (35) and Lemma 2.3 (ii) we know that $t \mapsto \tilde{v}_t^\lambda$ is bounded and bounded away from 0. Fix $\varepsilon \in (0, 1)$ and t_0 such that $\int_{t_0}^\infty \sup_i u_{t,i}^\lambda\, dt < \varepsilon$. Then, for any $t \geq 0$,

$$e^{-c\varepsilon} e^{(D-\mu I)t} \tilde{v}_{t_0}^\lambda \leq \tilde{v}_{t_0+t}^\lambda \leq e^{(D-\mu I)t} \tilde{v}_{t_0}^\lambda \qquad (36)$$

and so, for any $s, t \geq 0$,

$$|\tilde{v}_{t_0+t+s}^\lambda - \tilde{v}_{t_0+t}^\lambda| \leq \sup_{\delta\in\{-1,1\}} \left|\left(e^{c\delta\varepsilon} I - e^{(D-\mu I)s}\right) e^{(D-\mu I)t} \tilde{v}_{t_0}^\lambda\right|.$$

By Perron-Frobenius' theorem, $\lim_{t\to\infty} e^{(D-\mu I)t} = P := (\xi_i \eta_j)_{i,j}$ and thus, when $t \to \infty$,

$$|\tilde{v}_{t_0+t+s}^\lambda - \tilde{v}_{t_0+t}^\lambda| \leq (e^{c\varepsilon} - 1)(\tilde{v}_{t_0}^\lambda, \eta)\, \xi + |\tilde{v}_{t_0}^\lambda| o(1)$$

where the negligible term $o(1)$ does not depend on t_0, s, ε and λ, since $\varepsilon < 1$ and $\exp((D-\mu I)s)$ is a bounded function of s. Therefore, $(\tilde{v}_t^\lambda)_{t\geq 0}$ satisfies the Cauchy criterion and converges to a finite positive limit $\tilde{v}_\infty^\lambda$ when $t\to\infty$.

We just proved the convergence of the Laplace functional (23) of x_t under $\mathbb{P}^*$ when $t\to\infty$. In order to obtain the convergence in law of x_t, we have to check the continuity of the limit for $\lambda = 0$, but this is an immediate consequence of $\lim_{\lambda\to 0}\lim_{t\to\infty}\tilde{v}_t^\lambda = \xi$.

Finally, letting t go to infinity in (36), we get

$$|\tilde{v}_\infty^\lambda - P\tilde{v}_{t_0}^\lambda| \leq (1-e^{-c\varepsilon})P\tilde{v}_{t_0}^\lambda,$$

where $P\tilde{v}_{t_0}^\lambda = (\tilde{v}_{t_0}^\lambda,\eta)\xi$. It follows that $\tilde{v}_\infty^\lambda$ is proportional to ξ, as limit of quantities proportional to ξ. Therefore $(x,\tilde{v}_\infty^\lambda)/(x,\xi) = (\tilde{v}_\infty^\lambda,\mathbf{1})$ is independent of x and the limit law of x_t too. □

We can also generalize the exchange of limits of Proposition 3.3 to the multitype irreducible case.

Theorem 3.7 *In the subcritical case, conditionally on $(x_{t+\theta},\mathbf{1})>0$, x_t converges in distribution when $t\to+\infty$ to a non-trivial limit which depends only on θ. Furthermore, one can interchange both limits in t and θ :*

$$\lim_{\theta\to\infty}\lim_{t\to\infty}\mathbb{P}(x_t\in\cdot \mid (x_{t+\theta},\mathbf{1})>0) \overset{(d)}{=} \lim_{t\to\infty}\lim_{\theta\to\infty}\mathbb{P}(x_t\in\cdot \mid (x_{t+\theta},\mathbf{1})>0).$$

Proof Following a similar computation as in the proof of Proposition 3.3,

$$\begin{aligned}\lim_{t\to\infty}\mathbb{E}(e^{-(x_t,\lambda)} \mid (x_{t+\theta},\mathbf{1})>0) &= \lim_{t\to\infty}\frac{\exp(-(x,u_t^\lambda)) - \exp(-(x,u_t^{\lambda+\lim_{\bar\lambda\hookrightarrow\infty}u_\theta^{\bar\lambda}}))}{1-\exp(-(x,\lim_{\bar\lambda\hookrightarrow\infty}u_{t+\theta}^{\bar\lambda}))}\\ &= \lim_{t\to\infty}\frac{(x,u_t^{\lambda+\lim_{\bar\lambda\hookrightarrow\infty}u_\theta^{\bar\lambda}} - u_t^\lambda)}{(x,\lim_{\bar\lambda\hookrightarrow\infty}u_{t+\theta}^{\bar\lambda})}.\end{aligned}$$

Since

$$Du_t^\lambda - \frac{c}{2}(\sup_i u_{t,i}^\lambda)\,u_t^\lambda \leq \frac{du_t^\lambda}{dt} \leq Du_t^\lambda - \frac{c}{2}(\inf_i u_{t,i}^\lambda)\,u_t^\lambda,$$

one gets :

$$\exp\Big(-\frac{c}{2}\int_0^t \sup_i u_{s,i}^\lambda\,ds\Big)e^{Dt}\lambda \leq u_t^\lambda \leq \exp\Big(-\frac{c}{2}\int_0^t \inf_i u_{s,i}^\lambda\,ds\Big)e^{Dt}\lambda.$$

This inequality, similar to (35), can be used exactly as in the proof of Theorem 3.6 (b) to prove that, for any ε, there exists t_0 large enough such that

$$e^{-c\varepsilon/2}e^{(D-\mu I)t}e^{-\mu t_0}u_{t_0}^\lambda \leq e^{-\mu(t+t_0)}u_{t_0+t}^\lambda \leq e^{(D-\mu I)t}e^{-\mu t_0}u_{t_0}^\lambda. \tag{37}$$

and to deduce from (37) that $\tilde{u}_t^\lambda := e^{-\mu t} u_t^\lambda$ converges as $t \to \infty$ to a non-zero limit $\tilde{u}_\infty^\lambda$ proportional to the vector ξ.

Moreover, because of (5), u_t^λ is increasing with respect to each coordinate of λ. Therefore, it is elementary to check that $t \mapsto \lim_{\bar{\lambda} \hookrightarrow \infty} u_t^{\bar{\lambda}} = \sup_n u_t^{n\mathbf{1}}$ is also solution of the non-linear differential system (6), but only defined on $(0, \infty)$ (recall that, by Lemma 2.3 (ii), $\lim_{\bar{\lambda} \hookrightarrow \infty} u_t^{\bar{\lambda}} < \infty$ for any $t > 0$). Indeed, assume that $b_t = \sup_n a_t^n$ where $\dot{a}_t^n = F(a_t^n)$ for a locally Lipschitz function F. Fix t such that $b_t < +\infty$ and a small $\eta > 0$, and let $\underline{F} := \inf_{|x-b_t| \leq \eta} F(x)$ and $\bar{F} := \sup_{|x-b_t| \leq \eta} F(x)$. There exists n_0 such that, for $n \geq n_0$, $|a_t^n - b_t| \leq \eta/2$. Moreover, for any s in a neighborhood of t, $|a_s^n - b_t| \leq \eta$, where the neighborhood depends on $\bar{F}$ and $\underline{F}$, but is uniform in $n \geq n_0$. Therefore, for sufficiently small s and for n sufficiently large, $\underline{F} \leq (a_{t+s}^n - a_t^n)/s \leq \bar{F}$. Letting $n \to \infty$, $s \to 0$ and finally $\eta \to 0$, since $\bar{F} - \underline{F} \to 0$ when $\eta \to 0$, b_t is differentiable at time t and $\dot{b}_t = F(b_t)$.

Therefore, the semigroup property of the flow of (6) implies that, for any $t \geq 0$,

$$\lim_{\bar{\lambda} \hookrightarrow \infty} u_{t+\theta}^{\bar{\lambda}} = u_t^{\lim_{\bar{\lambda} \hookrightarrow \infty} u_\theta^{\bar{\lambda}}},$$

so that $e^{-\mu t} \lim_{\bar{\lambda} \hookrightarrow \infty} u_t^{\bar{\lambda}}$ also converges as $t \to \infty$ to a positive limit $\tilde{u}_\infty^\infty$, proportional to ξ too.

Hence,

$$\begin{aligned} \lim_{t \to \infty} \mathbb{E}(e^{-(x_t, \lambda)} \mid (x_{t+\theta}, 1) > 0) &= e^{-\mu\theta} \frac{(x, \tilde{u}_\infty^{\lambda + \lim_{\bar{\lambda} \hookrightarrow \infty} u_\theta^{\bar{\lambda}}} - \tilde{u}_\infty^\lambda)}{(x, \tilde{u}_\infty^\infty)} \\ &= e^{-\mu\theta} \frac{(\tilde{u}_\infty^{\lambda + \lim_{\bar{\lambda} \hookrightarrow \infty} u_\theta^{\bar{\lambda}}} - \tilde{u}_\infty^\lambda, \mathbf{1})}{(\tilde{u}_\infty^\infty, \mathbf{1})}, \end{aligned} \quad (38)$$

which is independent of the initial condition x.

In order to prove the convergence in distribution as $t \to \infty$ of x_t conditionally on $(x_{t+\theta}, \mathbf{1}) > 0$ to a random variable with Laplace transform (38), it remains to prove the continuity of this expression as a function of λ for $\lambda \to 0$. To this aim and also to prove the exchangeability of limits, we use the following Lemma, the proof of which is postponed at the end of the subsection. This lemma gives the main reason why the limits can be exchanged: v_t^λ is solution of the *linearized* equation of (6), and therefore, the gradient of u_t^λ with respect to λ is solution of the *same* system of ODEs as v_t^λ. The function v_t^λ was involved in the computation of $\lim_t \lim_\theta \mathbb{P}(x_t \in \cdot \mid x_{t+\theta} > 0)$, whereas the gradient of u_t^λ will be involved in the computation of $\lim_\theta \lim_t \mathbb{P}(x_t \in \cdot \mid x_{t+\theta} > 0)$.

Lemma 3.8 *The function $\lambda \mapsto u_t^\lambda$ is differentiable, and its derivative in the direction η, denoted by $\nabla_\eta u_t^\lambda$, is solution of the same differential system*

(24) *as* v_t^λ *except for the initial condition given by* $\nabla_\eta u_0^\lambda = \eta$*. Furthermore,* $\lambda \mapsto \tilde{u}_\infty^\lambda$ *is differentiable too and its derivative in the direction* η*, denoted by* $\nabla_\eta \tilde{u}_\infty^\lambda$ *satisfies*

$$\nabla_\eta \tilde{u}_\infty^\lambda = \lim_{t\to\infty} e^{-\mu t} \nabla_\eta u_t^\lambda.$$

Since $\nabla_\eta u_t^\lambda$ satisfies the same differential equation as v_t^λ, in particular, $\nabla_\xi u_t^\lambda = v_t^\lambda$ and, with the notations of the proof of Theorem 3.6, $\nabla_\xi \tilde{u}_\infty^\lambda = \tilde{v}_\infty^\lambda$.

It also follows from the above lemma that $\tilde{u}_\infty^\lambda$ is continuous as a function of λ. As a result,

$$\begin{aligned}\lim_{\lambda\to 0}\lim_{t\to\infty} \mathbb{E}(e^{-(x_t,\lambda)} \mid (x_{t+\theta}, \mathbf{1}) > 0) &= \lim_{\lambda\to 0} e^{-\mu\theta} \frac{(\tilde{u}_\infty^{\lambda+\lim_{\bar\lambda\hookrightarrow\infty} u_\theta^{\bar\lambda}} - \tilde{u}_\infty^\lambda, \mathbf{1})}{(\tilde{u}_\infty^\infty, \mathbf{1})} \\ &= e^{-\mu\theta} \frac{(\tilde{u}_\infty^{\lim_{\bar\lambda\hookrightarrow\infty} u_\theta^{\bar\lambda}}, \mathbf{1})}{(\tilde{u}_\infty^\infty, \mathbf{1})} = 1\end{aligned}$$

since

$$\tilde{u}_\infty^{\lim_{\bar\lambda\hookrightarrow\infty} u_\theta^{\bar\lambda}} = \lim_{t\to\infty} e^{-\mu t} u_t^{\lim_{\bar\lambda\hookrightarrow\infty} u_\theta^{\bar\lambda}} = \lim_{t\to\infty} e^{-\mu t} \lim_{\bar\lambda\hookrightarrow\infty} u_{t+\theta}^{\bar\lambda} = e^{\mu\theta} \tilde{u}_\infty^\infty.$$

Finally, let us check that the limits in t and θ can be exchanged. Since $\lim_{\bar\lambda\hookrightarrow\infty} u_\theta^{\bar\lambda} \sim e^{\mu\theta}\tilde{u}_\infty^\infty = e^{\mu\theta}(\tilde{u}_\infty^\infty, \mathbf{1})\xi$ when $\theta \to \infty$, it follows from Lemma 3.8 that

$$\lim_{\theta\to\infty} e^{-\mu\theta} \frac{(\tilde{u}_\infty^{\lambda+\lim_{\bar\lambda\hookrightarrow\infty} u_\theta^{\bar\lambda}} - \tilde{u}_\infty^\lambda, \mathbf{1})}{(\tilde{u}_\infty^\infty, \mathbf{1})} = (\nabla_\xi \tilde{u}_\infty^\lambda, \mathbf{1}) = (\tilde{v}_\infty^\lambda, \mathbf{1}) = \lim_{t\to\infty} \mathbb{E}^*(e^{-(x_t,\lambda)}),$$

which completes the proof of Theorem 3.7. □

Proof of Lemma 3.8 The differentiability of u_t^λ with respect to λ and the ODE satisfied by its derivatives are classical results on the regularity of the flow of ODEs (see e.g. Perko [26]).

Moreover, since $\nabla_\eta u_t^\lambda$ and v_t^λ are both solution of the ODE (24) (with different initial conditions), it is trivial to transport the properties of v_t^λ proved in the proof of Theorem 3.6 to $\nabla_\eta u_t^\lambda$. In particular, $e^{-\mu t}\nabla_\eta u_t^\lambda$ converges as $t \to +\infty$ to a non-zero vector w_η^λ which is proportional to ξ. We only have to check that $\nabla_\eta \tilde{u}_\infty^\lambda$ exists and that $w_\eta^\lambda = \nabla_\eta \tilde{u}_\infty^\lambda$. Moreover, as for (35),

$$\exp\Big(-\frac{c}{2}\int_0^t \sup_i u_{s,i}^\lambda\, ds\Big) e^{Dt}\eta \le \nabla_\eta u_t^\lambda \le \exp\Big(-\frac{c}{2}\int_0^t \inf_i u_{s,i}^\lambda\, ds\Big) e^{Dt}\eta.$$

Therefore, since $\exp(Dt) \geq 0$,

$$|\nabla_\eta u_t^\lambda| \leq e^{Dt}|\eta|,$$

which implies that $e^{-\mu t}\nabla_\eta u_t^\lambda$ is uniformly bounded for $t \geq 0$, $\lambda \geq 0$ and η in a compact subset of $\mathbb{R}^k$.

Now, letting $t \to +\infty$ in (37) one gets for any $h \geq 0$,

$$\begin{aligned}|\tilde{u}_\infty^{\lambda+h\eta} - \tilde{u}_\infty^\lambda - P\int_0^h e^{-\mu t_0}\nabla_\eta u_{t_0}^{\lambda+r\eta}dr| &= |\tilde{u}_\infty^{\lambda+h\eta} - \tilde{u}_\infty^\lambda - Pe^{-\mu t_0}(u_{t_0}^{\lambda+h\eta} - u_{t_0}^\lambda)| \\ &\leq (1-e^{-c\varepsilon/2})P(e^{-\mu t_0}u_{t_0}^\lambda + e^{-\mu t_0}u_{t_0}^{\lambda+h\eta}).\end{aligned}$$

Letting $\varepsilon \to 0$ (and thus $t_0 \to +\infty$) in the previous inequality, Lebesgue's convergence theorem yields

$$\tilde{u}_\infty^{\lambda+h\eta} - \tilde{u}_\infty^\lambda = \int_0^h w_\eta^{\lambda+r\eta}dr.$$

Therefore, $\tilde{u}_\infty^\lambda$ is differentiable with respect to λ and $\nabla_\eta \tilde{u}_\infty^\lambda = w_\eta^\lambda$. The proof of Lemma 3.8 is completed. □

4 Some decomposable cases conditioned by different remote survivals

In this section, we study some models for which the mutation matrix D is not irreducible: it is called 'reducible' or 'decomposable'. In this case, the general theory developed above does not apply. In contrast with the irreducible case, the asymptotic behavior of the MDW process and the MF diffusion depends on the type.

Decomposable critical multitype pure branching processes (without motion and renormalization) were the subject of several works since the seventies. See e.g. [23, 11, 12, 30, 33, 31].

4.1 A first critical model

Our first example is a 2-types DW process with a reducible mutation matrix of the form

$$D = \begin{pmatrix} -\alpha & \alpha \\ 0 & 0 \end{pmatrix}, \quad \alpha > 0. \tag{39}$$

For this model type 1 (resp. type 2) is subcritical (resp. critical). Moreover mutations can occur from type 1 to type 2 but no mutations from type 2 to

type 1 are allowed.
In this section we analyze not only the law $\mathbb{P}^*$ of MDW process conditioned on the non-extinction of the whole population, but also the MDW process conditioned on the survival of each type separately.

Theorem 4.1 *Let $\mathbb{P}$ be the distribution of the MDW process X with mutation matrix* (39) *and non-zero initial condition m. Let us define $\mathbb{P}^*, \hat{\mathbb{P}}^*$ and $\check{\mathbb{P}}^*$ for any $t > 0$ and $B \in \mathcal{F}_t$ by*

$$\begin{aligned}
\mathbb{P}^*(B) &= \lim_{\theta\to\infty} \mathbb{P}(B \mid \langle X_{t+\theta}, \mathbf{1}\rangle > 0) \\
\hat{\mathbb{P}}^*(B) &= \lim_{\theta\to\infty} \mathbb{P}(B \mid \langle X_{t+\theta,1}, 1\rangle > 0) \quad (\textit{if } m_1 \neq 0) \\
\check{\mathbb{P}}^*(B) &= \lim_{\theta\to\infty} \mathbb{P}(B \mid \langle X_{t+\theta,2}, 1\rangle > 0).
\end{aligned}$$

Then, all these limits exist and, for any $t > 0$,

$$\check{\mathbb{P}}^*\big|_{\mathcal{F}_t} = \mathbb{P}^*\big|_{\mathcal{F}_t} = \frac{\langle X_t, \mathbf{1}\rangle}{\langle m, \mathbf{1}\rangle}\, \mathbb{P}\big|_{\mathcal{F}_t} \tag{40}$$

$$\textit{and} \quad \hat{\mathbb{P}}^*\big|_{\mathcal{F}_t} = \frac{\langle X_{t,1}, 1\rangle}{\langle m_1, 1\rangle}\, e^{\alpha t}\, \mathbb{P}\big|_{\mathcal{F}_t}. \tag{41}$$

Proof Let us first prove (41). Using the method leading to (12), we get that if $m_1 \neq 0$

$$\hat{\mathbb{P}}^*(B) = \lim_{\theta\to\infty} \mathbb{E}\Big(\mathbb{1}_B \frac{(x_t, \lim_{\lambda_1\to\infty} u_\theta^{(\lambda_1,0)})}{(x, \lim_{\lambda_1\to\infty} u_{t+\theta}^{(\lambda_1,0)})}\Big).$$

The cumulant u_t^λ of the mass process satisfies for any $\lambda = (\lambda_1, \lambda_2)$

$$\begin{cases}
\dfrac{du_{t,1}^\lambda}{dt} = -\alpha u_{t,1}^\lambda + \alpha u_{t,2}^\lambda - \dfrac{c}{2}(u_{t,1}^\lambda)^2 & u_{0,1}^\lambda = \lambda_1 \\
\dfrac{du_{t,2}^\lambda}{dt} = -\dfrac{c}{2}(u_{t,2}^\lambda)^2 & u_{0,2}^\lambda = \lambda_2.
\end{cases}$$

The second equation admits as solution

$$u_{t,2}^\lambda = \frac{\lambda_2}{1 + \frac{c}{2}\lambda_2 t} \tag{42}$$

and $u_{t,1}^\lambda$ can be computed explicitly if $\lambda_2 = 0$:

$$u_{t,1}^{(\lambda_1,0)} = \frac{\lambda_1 e^{-\alpha t}}{1 + \frac{c\lambda_1}{2\alpha}(1 - e^{-\alpha t})} \quad \text{and} \quad u_{t,2}^{(\lambda_1,0)} = 0. \tag{43}$$

We then get

$$\hat{\mathbb{P}}^*(B) = e^{\alpha t}\,\mathbb{E}\big(\mathbb{1}_B \frac{x_{t,1}}{x_1}\big),$$

which yields (41).

Concerning $\check{\mathbb{P}}^*$, remark first that it is well defined even if $m_2 = 0$ (but $m_1 \neq 0$) since particles of type 2 can be created by particles of type 1. We are going to prove (40) by a similar method as Theorem 2.2. Let us first compute ξ. The matrix D has two eigenvalues, 0 and $-\alpha$, each of them with one-dimensional eigenspace. The (normalized) right and left eigenvectors of the greatest eigenvalue $\mu = 0$ are respectively $\xi = (\frac{1}{2}; \frac{1}{2})$ and $\eta = (0; 2)$. Since $\xi > 0$, the proof of Lemma 2.3 (and therefore Lemma 2.3 itself) is still valid for this specific matrix, except for the assertion (i), which has to be reduced to the following: if $\lambda_2 > 0$, then, for any $t > 0$, $u^\lambda_t > 0$ (if $\lambda_2 = 0$, then $u^\lambda_{t,2} \equiv 0$).

Therefore, as in the proof of Theorem 2.2, we can prove that

$$\forall \lambda \geq 0,\ \forall t \geq 0, \quad |u^\lambda_t - e^{Dt}\lambda| \leq K\|\lambda\| t e^{Dt}\lambda$$

and thus that, for θ and t_0 such that $\|u^\lambda_{t_0}\| \leq 1/K(\theta + t)$,

$$\left|\frac{(a, u^\lambda_{t_0+\theta})}{(b, u^\lambda_{t_0+\theta+t})} - \frac{(a, e^{D\theta}u^\lambda_{t_0})}{(b, e^{D(\theta+t)}u^\lambda_{t_0})}\right| \leq \frac{2K\|a\|\|u^\lambda_{t_0}\|\theta\|e^{D\theta}u^\lambda_{t_0}\|}{(b, e^{D(\theta+t)}u^\lambda_{t_0})} + \frac{2K\|a\|\|b\|\|e^{D\theta}u^\lambda_{t_0}\|\|u^\lambda_{t_0}\|(t+\theta)\|e^{D(\theta+t)}u^\lambda_{t_0}\|}{(b, e^{D(\theta+t)}u^\lambda_{t_0})^2}. \quad (44)$$

Let us compute explicitly the exponential of the matrix Dt. Since $D^n = (-\alpha)^n N$ where $N = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$, then

$$e^{Dt} = P + e^{-\alpha t}N, \text{ with } P = (\xi_i \eta_j)_{1 \leq i,j \leq 2} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}.$$

This has the same form as (15) in the irreducible case, except that $P \not> 0$. Because of this, we cannot obtain a bound for (44) uniform in $\lambda \in \mathbb{R}^2$ as in the proof of Theorem 2.2. However, we can restrict to a subset of $\mathbb{R}^2$ for which the convergence is uniform and which covers the two limits involved in the computation of $\check{\mathbb{P}}^*$ ($\lambda_1 = 0$ and $\lambda_2 \to +\infty$) and $\mathbb{P}^*$ ($\lambda_1 \to +\infty$ and $\lambda_2 \to +\infty$).

This can be done as follows: if $\lambda_1 = \lambda_2$, then $u^\lambda_{t,1} = u^\lambda_{t,2}$ for any $t \geq 0$. Therefore, for any $\lambda \neq \{0\}$ such that $\lambda_2 \geq \lambda_1$, $u^\lambda_{t,2} \geq u^\lambda_{t,1} > 0$ for any $t > 0$

(if at some time these quantities are equal, they remain equal for any larger time). Then, since

$$e^{D\theta}u^{\lambda}_{t_0} = (u^{\lambda}_{t_0,2} + e^{-\alpha\theta}(u^{\lambda}_{t_0,1} - u^{\lambda}_{t_0,2}), u^{\lambda}_{t_0,2}),$$

this quantity converges when $\theta \to \infty$ to $(u^{\lambda}_{t_0,2}, u^{\lambda}_{t_0,2})$, *uniformly* in λ such that $\lambda_2 > \lambda_1 \geq 0$.

From this follows as in the proof of Theorem 2.2 that

$$\lim_{\theta\to\infty} \frac{u^{\lambda}_{\theta}}{(x, u^{\lambda}_{t+\theta})} = \frac{\xi}{(x, \xi)}$$

uniformly for λ in the set of $(\lambda_1; \lambda_2) \neq 0$ such that $\lambda_2 \geq \lambda_1 \geq 0$. This ends the proof of (40). □

From the local density of $\mathbb{P}^*$ (resp. $\hat{\mathbb{P}}^*$) with respect to $\mathbb{P}$, we easily obtain as in Section 2.2 the following expressions for the Laplace functionals of the different conditioned processes.

Theorem 4.2 *The probability measure $\mathbb{P}^*(= \check{\mathbb{P}}^*)$ is characterized by*

$$\forall f \in C_b(\mathbb{R}, \mathbb{R}^2)_+, \quad \mathbb{E}^*(\exp -\langle X_t, f\rangle \mid X_0 = m) = \frac{\langle m, V_t f\rangle}{\langle m, \xi\rangle} e^{-\langle m, U_t f\rangle}$$

where $V_t f$ is the unique semigroup solution of the PDE (22). *The probability measure $\hat{\mathbb{P}}^*$ is characterized by*

$$\forall f \in C_b(\mathbb{R}, \mathbb{R}^2)_+, \quad \hat{\mathbb{E}}^*(\exp -\langle X_t, f\rangle \mid X_0 = m) = \frac{\langle m, \hat{V}_t f\rangle}{\langle m_1, 1\rangle} e^{\alpha t} e^{-\langle m, U_t f\rangle}$$

where $\hat{V}_t f$ satisfies the same PDE as $V_t f$ except for the initial condition $\hat{V}_0 f = (1; 0)$.

4.2 Long time behaviors of the Feller diffusions

Let us now analyze the long time behavior of the various conditioned MF diffusions.

Proposition 4.3 (a) *The first type vanishes in $\mathbb{P}^*$-probability when $t \to \infty$, that is*

$$\forall \varepsilon > 0, \quad \lim_{t\to\infty} \mathbb{P}^*(x_{t,1} > \varepsilon) = 0.$$

(b) *Under $\hat{\mathbb{P}}^*$, the first type converges in distribution to the probability measure $\Gamma(2, 2\alpha/c)$*

$$\lim_{t\to\infty} \hat{\mathbb{P}}^*(x_{t,1} \in \cdot) \stackrel{(d)}{=} \Gamma(2, 2\alpha/c).$$

(c) *The second type $x_{t,2}$ explodes in $\mathbb{P}^*$- and in $\hat{\mathbb{P}}^*$-probability when $t \to \infty$.*

Proof We first compute the vector $v_t^\lambda := V_t\lambda, \lambda \in \mathbb{R}^2$.

$$\mathbb{E}^*(\exp -(x_t,\lambda) \mid x_0 = x) = \frac{(x, v_t^\lambda)}{(x,\xi)} e^{-(x,u_t^\lambda)}$$

with

$$\begin{cases} \dfrac{dv_{t,1}^\lambda}{dt} = -\alpha v_{t,1}^\lambda + \alpha v_{t,2}^\lambda - c u_{t,1}^\lambda v_{t,1}^\lambda, & v_{0,1}^\lambda = 1/2, \\ \dfrac{dv_{t,2}^\lambda}{dt} = -c u_{t,2}^\lambda v_{t,2}^\lambda, & v_{0,2}^\lambda = 1/2. \end{cases}$$

Therefore, replacing $u_{t,2}^\lambda$ by its value obtained in (42),

$$v_{t,2}^\lambda = \frac{1}{2(1+\frac{c}{2}\lambda_2 t)^2}$$

and

$$v_{t,1}^\lambda = \frac{1}{2} e^{-\alpha t - c\int_0^t u_{s,1}^\lambda ds}\Big(1 + \alpha \int_0^t \frac{e^{\alpha s + c\int_0^s u_{\tau,1}^\lambda d\tau}}{(1+\frac{c}{2}\lambda_2 s)^2}\, ds\Big). \tag{45}$$

In particular, if $\lambda_2 = 0$, $v_{t,2}^{(\lambda_1;0)} = 1/2$ and one gets from the explicit expression (43) of $u_{t,1}^{(\lambda_1;0)}$

$$v_{t,1}^{(\lambda_1,0)} = \frac{1}{2} - \frac{c\lambda_1(1+\frac{c}{2\alpha}\lambda_1)te^{-\alpha t} + \frac{c^2}{\alpha^2}\lambda_1^2 e^{-2\alpha t}}{2\left(1+\frac{c}{2\alpha}\lambda_1(1-e^{-\alpha t})\right)^2}.$$

Similarly, for $\hat{v}_t^\lambda := \hat{V}_t\lambda, \lambda \in \mathbb{R}^2$, we get

$$\hat{\mathbb{E}}^*(\exp -(x_t,\lambda) \mid x_0 = x) = \frac{(x, \hat{v}_t^\lambda)}{(x,\xi)} e^{\alpha t} e^{-(x,u_t^\lambda)}$$

with

$$\begin{cases} \dfrac{d\hat{v}_{t,1}^\lambda}{dt} = -\alpha \hat{v}_{t,1}^\lambda + \alpha \hat{v}_{t,2}^\lambda - c u_{t,1}^\lambda \hat{v}_{t,1}^\lambda, & \hat{v}_{0,1}^\lambda = 1, \\ \dfrac{d\hat{v}_{t,2}^\lambda}{dt} = -c u_{t,2}^\lambda \hat{v}_{t,2}^\lambda, & \hat{v}_{0,2}^\lambda = 0. \end{cases}$$

Therefore, $\hat{v}_{t,2}^\lambda = 0$ and

$$\hat{v}_{t,1}^\lambda = \exp\Big(-\alpha t - c\int_0^t u_{s,1}^\lambda\, ds\Big). \tag{46}$$

In particular, if $\lambda_2 = 0$, using (43) again,

$$\hat{v}_{t,1}^{(\lambda_1;0)} = \frac{e^{-\alpha t}}{\left(1+\frac{c}{2\alpha}\lambda_1(1-e^{-\alpha t})\right)^2}. \tag{47}$$

Now (a) and (b) can be deduced from the facts that $\lim_{t\to\infty} v_t^{(\lambda_1;0)} = \xi$ and $\lim_{t\to\infty} e^{\alpha t}\hat{v}_t^{(\lambda_1;0)} = (1/(1+c\lambda_1/2\alpha)^2; 0)$.
The explosion of $x_{t,2}$ in $\mathbb{P}^*$-probability in (c) is a consequence of the fact that $\lim_{t\to\infty} e^{\alpha t}\hat{v}_t^{(\lambda_1;\lambda_2)} = \big(\exp(-c\int_0^\infty u^\lambda_{s,1}\,ds); 0\big) = (0;0)$, by Lemma 2.3 (iii).
Finally, it follows from (45) that

$$\begin{aligned} v^\lambda_{t,1} &\leq \frac{e^{-\alpha t}}{2} + \frac{\alpha}{2}\int_0^t \frac{e^{-\alpha(t-s)-c\int_s^t u^\lambda_{\tau,1}d\tau}}{(1+\frac{c}{2}\lambda_2 s)^2}ds \\ &\leq \frac{e^{-\alpha t}}{2} + \frac{e^{-\alpha t/2}}{2} + \frac{\alpha}{c\lambda_2(1+\frac{c}{4}\lambda_2 t)} \end{aligned}$$

where the last inequality is obtained by splitting the integral over the time interval $[0,t]$ into the sum of the integrals over $[0,\frac{t}{2}]$ and $[\frac{t}{2},t]$. This implies the first part of (c). □

We interpret this proposition as follows. Conditionally on the survival of the whole population, the weakest type gets extinct and the strongest type has the same behavior as in the critical monotype case. Conversely, conditionally on the long time survival of the weakest type, the weakest type behaves at large time as in the monotype subcritical case and the strongest type explodes.

4.3 A more general subcritical decomposable model

We consider a generalization of the previous model. The mutation matrix is now given by

$$D = \begin{pmatrix} -\alpha & \alpha \\ 0 & -\beta \end{pmatrix} \tag{48}$$

where $\alpha > 0$ (as before) and $\beta > 0$ with $\beta \neq \alpha$.

In this case, the whole population is subcritical. Here again, mutations are only possible from type 1 to type 2. If $\beta < \alpha$, type 2 is "less subcritical" than type 1 (as in the previous case) but if $\alpha < \beta$, type 1 is "less subcritical" than type 2. We will see below that the behavior of the various conditioned processes is strongly related to the so-called dominating type, which is the first one if $\alpha < \beta$ and the second type if $\beta < \alpha$.
Before treating separately both cases with different techniques, we define the common ingredients we need.

We can easily compute the normalized right eigenvector ξ for the greatest eigenvalue μ. If $\beta < \alpha$, $\mu = -\beta$ and $\xi = \frac{1}{2\alpha-\beta}(\alpha; \alpha-\beta)$ and if $\alpha < \beta$,

$\mu = -\alpha$ and $\xi = (1; 0)$. We can also explicitly compute the exponential of the mutation matrix:

$$e^{Dt} = e^{-\beta t} \begin{pmatrix} 0 & \frac{\alpha}{\alpha-\beta} \\ 0 & 1 \end{pmatrix} + e^{-\alpha t} \begin{pmatrix} 1 & -\frac{\alpha}{\alpha-\beta} \\ 0 & 0 \end{pmatrix}.$$

The cumulant u_t^λ of the mass process satisfies

$$\begin{cases} \dfrac{du_{t,1}^\lambda}{dt} = -\alpha u_{t,1}^\lambda + \alpha u_{t,2}^\lambda - \dfrac{c}{2}(u_{t,1}^\lambda)^2 & u_{0,1}^\lambda = \lambda_1 \\ \dfrac{du_{t,2}^\lambda}{dt} = -\beta u_{t,2}^\lambda - \dfrac{c}{2}(u_{t,2}^\lambda)^2 & u_{0,2}^\lambda = \lambda_2 \end{cases} \tag{49}$$

Thus $u_{t,2}^\lambda$ is given by

$$u_{t,2}^\lambda = \frac{\lambda_2 e^{-\beta t}}{1 + \frac{c}{2\beta}\lambda_2(1 - e^{-\beta t})}. \tag{50}$$

One can compute $u_{t,1}^\lambda$ explicitly only when $\lambda_2 = 0$, and in this case, as in (43),

$$u_{t,1}^{(\lambda_1;0)} = \frac{\lambda_1 e^{-\alpha t}}{1 + \frac{c}{2\alpha}\lambda_1(1 - e^{-\alpha t})} \quad \text{and} \quad u_{t,2}^{(\lambda_1;0)} = 0.$$

We now consider the system of equations

$$\begin{cases} \dfrac{dh_{t,1}}{dt} = -\alpha h_{t,1} + \alpha h_{t,2} - c u_{t,1}^\lambda h_{t,1} \\ \dfrac{dh_{t,2}}{dt} = -\beta h_{t,2} - c u_{t,2}^\lambda h_{t,2} \end{cases} \tag{51}$$

which solutions are given by

$$h_{t,2} = \frac{h_{0,2}\, e^{-\beta t}}{\left(1 + \frac{c\lambda_2}{2\beta}(1 - e^{-\beta t})\right)^2}. \tag{52}$$

and

$$h_{t,1} = e^{-\alpha t - c\int_0^t u_{s,1}^\lambda ds}\Big(h_{0,1} + \alpha \int_0^t e^{\alpha s + c\int_0^s u_{\tau,1}^\lambda d\tau} h_{s,2} ds\Big). \tag{53}$$

We denote as before by v_t^λ, $\hat{v}_t^\lambda$ or $\check{v}_t^\lambda$ the respective solutions of (51) with initial conditions $v_0^\lambda = \xi$, $\hat{v}_0^\lambda = (1; 0)$ and $\check{v}_0^\lambda = (0; 1)$.

4.3.1 Case $\beta < \alpha$

We now identify the laws obtained by conditioning with respect to the various remote survivals.

Theorem 4.4 *Let $\mathbb{P}^*$ (resp. $\hat{\mathbb{P}}^*, \check{\mathbb{P}}^*$) be the conditioned laws defined in Theorem 4.1 where $\mathbb{P}$ is the law of the MDW process with mutation matrix given by (48) with $\beta < \alpha$ and non-zero initial condition m. It holds*

$$\check{\mathbb{P}}^*\big|_{\mathcal{F}_t} = \mathbb{P}^*\big|_{\mathcal{F}_t} = \frac{\langle X_t, \xi \rangle}{\langle m, \xi \rangle} e^{\beta t}\, \mathbb{P}\big|_{\mathcal{F}_t}$$
$$\textit{and} \quad \hat{\mathbb{P}}^*\big|_{\mathcal{F}_t} = \frac{\langle X_{t,1}, 1 \rangle}{\langle m_1, 1 \rangle} e^{\alpha t}\, \mathbb{P}\big|_{\mathcal{F}_t} \quad (\textit{if } m_1 \neq 0).$$

Sketch of the proof The greatest eigenvalue of D is $\mu = -\beta$, the normalized right eigenvector for μ is $\xi = \frac{1}{2\alpha-\beta}(\alpha; \alpha - \beta)$ and the normalized left eigenvector is $\eta = (0; \frac{2\alpha-\beta}{\alpha-\beta})$.
As in the proof of Theorem 4.1, $\xi > 0$, so that Lemma 2.3 holds (except assertion (i)) and we can use a similar method. The only difficulty is to find a domain $E \subset \mathbb{R}^2_+$ such that, for each initial condition $\lambda \in E$, the cumulant semigroup u^λ_t takes its values in E and $\{\lambda_1/\lambda_2, \lambda \in E\}$ is bounded. To this aim, one can check that, if $0 \leq u^\lambda_{t,1} = \frac{\alpha}{\alpha-\beta} u^\lambda_{t,2}$ at some time $t \geq 0$, then $\frac{du^\lambda_{t,1}}{dt} \leq \frac{\alpha}{\alpha-\beta} \frac{du^\lambda_{t,2}}{dt}$. Therefore, if $0 \leq \lambda_1 \leq \frac{\alpha}{\alpha-\beta}\lambda_2$ with $\lambda_2 > 0$, one has $0 < u^\lambda_{t,1} \leq \frac{\alpha}{\alpha-\beta} u^\lambda_{t,2}$ for any positive t. □

Let us now analyze the behavior for large t of the mass process under the three measures $\mathbb{P}^*$, $\hat{\mathbb{P}}^*$ and $\check{\mathbb{P}}^*$. Since $\hat{v}_{0,2} = 0$, $\hat{v}_{t,2} \equiv 0$ as in Section 4.2 and (46) holds. Therefore the behavior of x_t under $\hat{\mathbb{P}}^*$ is exactly the same as for $\beta = 0$, treated in Proposition 4.3 (b) and (c).
The long time behavior of x_t under $\mathbb{P}^*$ is different from Section 4.2 and is given in the following proposition.

Proposition 4.5 (a) *The first type vanishes in $\mathbb{P}^*$-probability when $t \to \infty$.*

(b) *Under $\mathbb{P}^*$, the second type converges in distribution to the probability measure* $\Gamma(2, 2\beta/c)$

$$\lim_{t\to\infty} \mathbb{P}^*(x_{t,2} \in \cdot) \overset{(d)}{=} \Gamma(2, 2\beta/c).$$

Proof Since

$$\mathbb{E}^*(e^{-(x_t,\lambda)} \mid x_0 = x) = \frac{(x, v_t^\lambda)}{(x,\xi)} e^{\beta t} e^{-(x,u_t^\lambda)}$$

we have to compute $\lim_{t\to\infty} v_t^\lambda e^{\beta t}$.
For the proof of (a) we remark that, from (52) and (53), $v_{t,2}^{(\lambda_1;0)} = \xi_2\, e^{-\beta t}$ and

$$v_{t,1}^{(\lambda_1;0)} = e^{-\alpha t} \exp(-c \int_0^t u_{s,1}^{(\lambda_1;0)} ds) \Big(\xi_1 + \alpha\, \xi_2 \int_0^t e^{(\alpha-\beta)s} \exp(c \int_0^s u_{\tau,1}^{(\lambda_1;0)} d\tau) ds\Big).$$

Since

$$\exp(-c \int_0^t u_{s,1}^{(\lambda_1;0)} ds) = \exp\Big(-c \int_0^t \frac{\lambda_1 e^{-\alpha s}}{1+\frac{c}{2\alpha}\lambda_1(1-e^{-\alpha s})} ds\Big) = \frac{1}{\big(1+\frac{c}{2\alpha}\lambda_1(1-e^{-\alpha t})\big)^2}$$

one obtains

$$\begin{aligned} v_{t,1}^{(\lambda_1;0)} e^{\beta t} &= \exp\Big(-\frac{(\alpha-\beta)t}{(1+\frac{c}{2\alpha}\lambda_1(1-e^{-\alpha t}))^2}\Big)\xi_1 \\ &\quad + \frac{\alpha}{\big(1+\frac{c}{2\alpha}\lambda_1(1-e^{-\alpha t})\big)^2} \int_0^t e^{-(\alpha-\beta)(t-s)} \big(1+\frac{c}{2\alpha}\lambda_1(1-e^{-\alpha s})\big)^2 ds\, \xi_2. \end{aligned}$$

The integral can be computed explicitly and is equal, for t large, to

$$\frac{1}{\alpha-\beta}(1+\frac{c}{2\alpha}\lambda_1)^2 + O(e^{-(\alpha-\beta)t}) \,.$$

Thus,

$$\lim_{t\to\infty} v_{t,1}^{(\lambda_1;0)} e^{\beta t} = \frac{\alpha}{\alpha-\beta}\, \xi_2 = \xi_1.$$

For the proof of (b), it suffices to show that

$$\lim_{t\to\infty} v_t^{(0;\lambda_2)} e^{\beta t} = \frac{1}{(1+\frac{c}{2\beta}\lambda_2)^2}\, \xi.$$

From (52), it is clear that $\lim_{t\to\infty} v_{t,2}^{(0;\lambda_2)}\, e^{\beta t} = \frac{1}{(1+\frac{c}{2\beta}\lambda_2)^2}\xi_2$. It then remains to compute the limit, for $\lambda_2 > 0$, of $v_{t,1}^{(0;\lambda_2)}$ as $t \to \infty$. Using (53), we get

$$v_{t,1}^{(\lambda_1;\lambda_2)} e^{\beta t} = e^{-(\alpha-\beta)t - c\int_0^t u_{s,1}^\lambda ds}\, \xi_1 + \alpha \xi_2 \int_0^t \frac{e^{-(\alpha-\beta)(t-s) - c\int_s^t u_{\tau,1}^\lambda d\tau}}{\big(1+\frac{c\lambda_2}{2\beta}(1-e^{-\beta s})\big)^2} ds.$$

The first term is $O(e^{-(\alpha-\beta)t})$ and goes to 0 as $t \to \infty$. The limit of the integral can be computed as follows:

$$\begin{aligned}
&\Big|\int_0^t \frac{e^{-(\alpha-\beta)(t-s)-c\int_s^t u^\lambda_{\tau,1}d\tau}}{\big(1+\frac{c}{2\beta}\lambda_2(1-e^{-\beta s})\big)^2}ds - \frac{1}{(1+\frac{c}{2\beta}\lambda_2)^2}\int_0^t e^{-(\alpha-\beta)(t-s)}ds\Big| \\
&\quad \le \bar{K}\int_0^t e^{-(\alpha-\beta)(t-s)}\Big|1-e^{-c\int_s^t u^\lambda_{\tau,1}d\tau}\Big(1-\frac{\frac{c\lambda_2}{2\beta}e^{-\beta s}}{1+\frac{c\lambda_2}{2\beta}}\Big)^{-2}\Big|ds \\
&\quad \le \bar{K}\Bigg(e^{-(\alpha-\beta)t/2}+\frac{t}{2}\Big(1-e^{-c\int_{t/2}^{+\infty}u^\lambda_{s,1}ds}\Big)\vee\Big(\Big(1-\frac{\frac{c\lambda_2}{2\beta}e^{-\beta t/2}}{1+\frac{c}{2\beta}\lambda_2}\Big)^{-2}-1\Big)\Bigg)
\end{aligned}$$

where the positive constant $\bar{K}$ may vary from line to line and where the last inequality is obtained by splitting the integration over the time intervals $[0, \frac{t}{2}]$ and $[\frac{t}{2}, t]$.
Now, by Lemma 2.3 (ii), $\lim_{t\to\infty}\int_{t/2}^{+\infty} u^\lambda_{s,1}ds = 0$. Therefore,

$$\lim_{t\to\infty} v^\lambda_{t,1}e^{\beta t} = \frac{\alpha\xi_2}{\big(1+\frac{c}{2\beta}\lambda_2\big)^2}\int_0^\infty e^{-(\alpha-\beta)s}ds = \frac{\xi_1}{\big(1+\frac{c}{2\beta}\lambda_2\big)^2}$$

as required. □

Here again, this result can be interpreted as follows: for $i = 1, 2$, conditionally on the survival of the type i, this type i behaves as if it was alone, and the other type j explodes or goes extinct according to whether it is stronger or weaker.

4.3.2 Case $\alpha < \beta$

When $\alpha < \beta$, the greatest eigenvalue of D is $\mu = -\alpha$ and the normalized right eigenvector to μ is $\xi = (1; 0)$. In particular, $\xi \not> 0$, so that Lemma 2.3 does not hold and we cannot use the previous method anymore. However, in our specific example, $u^\lambda_{t,2}$ can be explicitly computed, and, by (49), $u^\lambda_{t,1}$ is solution of the one-dimensional differential equation

$$\frac{dy_t}{dt} = -\alpha y_t - \frac{c}{2}y_t^2 + \frac{\alpha\lambda_2 e^{-\beta t}}{1+\frac{c}{2\beta}\lambda_2(1-e^{-\beta t})}. \tag{54}$$

This equation can be (formally) extended to the case $\lambda_2 = \infty$ as

$$\frac{dy_t}{dt} = -\alpha y_t - \frac{c}{2}y_t^2 + \frac{2\alpha\beta e^{-\beta t}}{c(1-e^{-\beta t})}. \tag{55}$$

The following technical lemma gives (non-explicit) long-time estimates of the solutions of (54) that are sufficient to compute the various conditioned laws of the MDW process. We postpone its proof at the end of the subsection.

Lemma 4.6 *For any $\lambda_2 \in [0,\infty]$, let $\mathcal{Y}(\lambda_2)$ denote the set of solutions y_t of* (54) *(or of* (55) *if $\lambda_2 = \infty$) defined (at least) on $(0,\infty)$. For any $y \in \mathcal{Y}(\lambda_2)$, the limit $C(\lambda_2, y) := \lim_{t\to\infty} y_t e^{\alpha t}$ exists and satisfies*

$$0 < \inf_{\lambda_2\geq 1,\, y\in\mathcal{Y}(\lambda_2)} C(\lambda_2, y) \leq \sup_{\lambda_2\geq 1,\, y\in\mathcal{Y}(\lambda_2)} C(\lambda_2, y) < +\infty$$

We now identify the laws obtained by conditioning $\mathbb{P}$ with respect to the various remote survivals.

Theorem 4.7 *Let $\mathbb{P}^*$ (resp. $\hat{\mathbb{P}}^*, \check{\mathbb{P}}^*$) be the conditioned laws defined in Theorem* 4.1*, where $\mathbb{P}$ is the MDW process with mutation matrix given by* (48) *with $\alpha < \beta$ and initial condition $m = (m_1; m_2)$ with $m_1 \neq 0$. It holds*

$$\hat{\mathbb{P}}^*\big|_{\mathcal{F}_t} = \check{\mathbb{P}}^*\big|_{\mathcal{F}_t} = \mathbb{P}^*\big|_{\mathcal{F}_t} = \frac{\langle X_t, \xi\rangle}{\langle m, \xi\rangle}\, e^{\alpha t}\, \mathbb{P}\big|_{\mathcal{F}_t}.$$

When $m_1 = 0$ and $m_2 \neq 0$,

$$\check{\mathbb{P}}^*\big|_{\mathcal{F}_t} = \mathbb{P}^*\big|_{\mathcal{F}_t} = \frac{\langle X_{t,2}, 1\rangle}{\langle m_2, 1\rangle}\, e^{\beta t}\, \mathbb{P}\big|_{\mathcal{F}_t}.$$

and $\hat{\mathbb{P}}^*$ is not defined.

Proof Our usual method consists in computing the following limits as θ goes to $+\infty$:

$$\frac{\lim_{\lambda_1\to\infty} u_\theta^{(\lambda_1;0)}}{(x, \lim_{\lambda_1\to\infty} u_{t+\theta}^{(\lambda_1;0)})}, \quad \frac{\lim_{\lambda_2\to\infty} u_\theta^{(0;\lambda_2)}}{(x, \lim_{\lambda_2\to\infty} u_{t+\theta}^{(0;\lambda_2)})}, \quad \frac{\lim_{\lambda_1,\lambda_2\to\infty} u_\theta^{(\lambda_1;\lambda_2)}}{(x, \lim_{\lambda_1,\lambda_2\to\infty} u_{t+\theta}^{(\lambda_1;\lambda_2)})}. \tag{56}$$

It is elementary to prove that, as monotone limits of solutions of (54), the function $\theta \mapsto \lim_{\lambda_1\to\infty} u_{\theta,1}^{(\lambda_1;0)}$ is still solution of (54) with $\lambda_2 = 0$, and the functions $\theta \mapsto \lim_{\lambda_2\to\infty} u_{\theta,1}^{(0;\lambda_2)}$ and $\theta \mapsto \lim_{\lambda_1,\lambda_2\to\infty} u_{\theta,1}^{(\lambda_1;\lambda_2)}$ are solutions of (55) (a priori only defined for $t > 0$).
Therefore, we can use Lemma 4.6 and the explicit formula (50) for $u_{t,2}^\lambda$ to compute the three limits of (56). In each case, the dominant term is the one including $u_{t,1}^\lambda$, except when $m_1 = 0$, where the only remaining term is the one including $u_{t,2}^\lambda$. □

Finally, we give the long time behavior of the mass process under $\mathbb{P}^*$ (which is equal to $\check{\mathbb{P}}^*$ and $\hat{\mathbb{P}}^*$ when this last measure exists).

Proposition 4.8 (a) *If $m_1 \neq 0$, the laws under $\mathbb{P}^*$ of the mass of both types $x_{t,1}$ and $x_{t,2}$ converge when $t \to \infty$. More precisely*

$$\lim_{t\to\infty} \mathbb{P}^*(x_{t,1} \in \cdot) \stackrel{(d)}{=} \Gamma(2, 2\alpha/c).$$

$x_{t,2}$ converges in $\mathbb{P}^*$-distribution to a non-trivial (and non-explicit) distribution on $\mathbb{R}_+$.

(b) *If $m_1 = 0$ ($m_2 \neq 0$), $x_{t,1} \equiv 0 \quad \mathbb{P}^* - a.s.$ and*

$$\lim_{t\to\infty} \mathbb{P}^*(x_{t,2} \in \cdot) \stackrel{(d)}{=} \Gamma(2, 2\beta/c).$$

Proof With the previous notation, when $m_1 \neq 0$,

$$\mathbb{E}^*(\exp -(x_t, \lambda)) = \frac{(x, \hat{v}_t^\lambda)}{(x, \hat{v}_0^\lambda)} e^{\alpha t} e^{-(x, u_t^\lambda)}.$$

Since $\hat{v}_{t,2} = 0$ and $\hat{v}_{t,1} = \exp(-\alpha t - c\int_0^t u_{s,1}^\lambda ds)$,

$$\lim_{t\to\infty} e^{\alpha t}\hat{v}_t = \Big(\exp - \int_0^\infty u_{s,1}^\lambda ds; 0\Big)$$

where $\exp - \int_0^\infty u_{s,1}^\lambda ds \in (0, 1)$ by Lemma 4.6. In order to prove the convergence in distribution of $x_{t,2}$, it remains to prove that

$$\lim_{\lambda_2\to 0} \int_0^\infty u_{s,1}^{(0,\lambda_2)} ds = 0.$$

Because of (54), $\dot{u}_{t,1}^\lambda \leq -\alpha u_{t,1}^\lambda + \alpha\lambda_2 e^{-\beta t}$. Therefore, it is easy to check that $u_{t,1}^\lambda \leq (\beta + 2)\lambda_2 e^{-\beta t}$ for all $t \geq 0$ if $\lambda_1 \leq (\beta + 2)\lambda_2$ (simply differentiate the difference). This implies the required result.

When $\lambda_2 = 0$, the computations can be made explicitly as in the proof of Proposition 4.3 (b) and give the usual Gamma limit distribution for $x_{t,1}$ under $\mathbb{P}^*$ when $t \to +\infty$.

If $m_1 = 0$ (i.e. $x_1 = \langle m_1, 1\rangle = 0$),

$$\mathbb{E}^*(\exp -(x_t, \lambda)) = \frac{(x, \check{v}_t^\lambda)}{(x, \check{v}_0^\lambda)} e^{\beta t} e^{-(x, u_t^\lambda)} = \check{v}_{t,2}^\lambda e^{\beta t} e^{-(x, u_t^\lambda)}$$

and the computations are the same as in the monotype case. □

We interpret this last result as follows: when the first type is present, it dominates the asymptotic behavior of both types, since its subcriticality is weaker than the one of the second type, although mutations from type 2 to type 1 do not occur.

Remark 4.9 Theorem 4.7 and Proposition 4.8 are still valid in the remaining case $\alpha = \beta$. In this case, it is actually possible to be more precise than in Lemma 4.6 by proving, using a similar method, that a solution y_t to (54) or (55) (with $\alpha = \beta$) satisfies $y_t \sim C(\lambda_2) t e^{-\alpha t}$ where $C(\lambda_2) = \alpha\lambda_2/(1 + c\lambda_2/2\alpha)$ if $\lambda_2 < \infty$ and $C(\infty) = 2\alpha^2/c$.

Proof of lemma 4.6 Let $z_t := e^{\alpha t} y_t$. It solves the equation

$$\frac{dz_t}{dt} = -\frac{c}{2} e^{-\alpha t} z_t^2 + \frac{\alpha\lambda_2 e^{-(\beta-\alpha)t}}{1 + \frac{c}{2\beta}\lambda_2(1 - e^{-\beta t})}. \tag{57}$$

Let us first prove that z_t is bounded for $t \in [1, \infty)$, uniformly in λ_2 and independently of the choice of the solution y_t of (54) or (55) defined on $(0, \infty)$. For any $t \geq 1$ and $\lambda_2 \in [0, \infty]$,

$$\frac{dz_t}{dt} \leq \frac{2\alpha\beta}{c(1 - e^{-\beta})} e^{-(\beta-\alpha)t}.$$

Since the integral of the above r.h.s. over $[1, \infty)$ is finite, we only have to prove that z_1 is bounded uniformly in $\lambda_2 \geq 0$ and independently of the choice of y_t. Now, for any $t \in [\frac{1}{2}, 1]$ and $\lambda_2 \in [0, \infty]$,

$$\frac{dz_t}{dt} \leq \frac{2\alpha\beta}{c} \frac{e^{-(\beta-\alpha)t}}{1 - e^{-\beta/2}} - \frac{ce^{-\alpha}}{2} z_t^2.$$

In particular, for any $t \in [\frac{1}{2}, 1]$, the first term in the r.h.s. above is bounded and bounded away from 0. Thus, there exists a constant K such that, if $z_t \geq K$ and $\frac{1}{2} \leq t \leq 1$, $\frac{dz_t}{dt} \leq -\frac{ce^{-\alpha}}{4} z_t^2$. Therefore, distinguishing between $z_{1/2} \leq K$ and $z_{1/2} > K$, one obtains

$$z_1 \leq \frac{z_{1/2}}{1 + \frac{c}{4} e^{-\alpha} z_{1/2}} \vee K \leq \frac{4e^\alpha}{c} \vee K =: K' < \infty.$$

Second, it follows from (57) and from the boundedness of z_t that $|dz_t/dt| \leq K''(e^{-(\beta-\alpha)t} + e^{-\alpha t})$ for some constant K'' for any $t \geq 1$. Therefore, z_t converges as $t \to \infty$. Moreover, this function is uniformly bounded from above for $t \geq 1$ by some constant K''' independent of the particular function z_t considered. Therefore, since $dz_t/dt \geq -cK''' e^{-\alpha t} z_t/2$ for $t \geq 1$, the limit of z_t when $t \to +\infty$ is also greater than $z_1 \exp(-cK''' e^{-\alpha}/2)$.

Then, it only remains to prove that z_1 is bounded away from 0, uniformly in $\lambda_2 \in [1, \infty]$. This follows from the fact that, for any $\lambda_2 \geq 1$, there exists a constant $M > 0$ independent of λ_2 and t such that, for $t \in [\frac{1}{2}, 1]$,

$$\frac{dz_t}{dt} \geq M - \frac{c}{2} z_t^2.$$

This implies that there exists M' such that, if $z_t \leq M'$ for $t \in [\frac{1}{2}, 1]$, $dz_t/dt \geq M/2$, and thus

$$z_1 \geq \left(z_{1/2} + \frac{M}{4}\right) \wedge M' \geq \frac{M}{4} \wedge M' > 0,$$

which completes the proof of Lemma 4.6. □

4.4 Exchange of long time limits

As in the irreducible case, one can study the interchangeability of the long time limit ($t \to +\infty$) of the conditioned Feller diffusion and the limit of long time survival ($\theta \to +\infty$). The same method as in Proposition 3.7 yields, for $i, j \in \{1, 2\}$ and $\lambda \in \mathbb{R}_+$,

$$\lim_{t\to\infty} \mathbb{E}(e^{-\lambda x_{t,i}} \mid x_{t+\theta,j} > 0) = \lim_{t\to\infty} \frac{(x, u_t^{\lambda^i + \lim_{\bar\lambda\to\infty} u_\theta^{\bar\lambda^j}} - u_t^{\lambda^i})}{(x, \lim_{\bar\lambda\to\infty} u_{t+\theta}^{\bar\lambda^j})}$$

where $\lambda^1 = (\lambda; 0)$ and $\lambda^2 = (0; \lambda)$.

However, the computation of these quantities requires precise information about the behavior of u_t^λ as a function of its initial condition λ for t large. The cases we could handle are the one with non degenerate limits. In the model introduced in section 4.1, it corresponds to $i = j = 1$ and the computation reduces to the monotype case studied in Proposition 3.3. In the model introduced in section 4.3, with $\beta < \alpha$, it corresponds to $i = j = 1$ and $i = j = 2$, and with $\alpha < \beta$, to all cases. For $i = j = 1$ the proof is based on explicit expressions like in the monotype case, and for the other cases, the arguments are similar to those used in the proof of Theorem 3.7 (except in the case $\alpha < \beta$ and $m_1 = 0$, where the computation can also be done explicitly). To summarize:

Proposition 4.10 *In the cases described above, one can interchange both limits in time:*

$$\begin{aligned}
\lim_{\theta\to\infty}\lim_{t\to\infty} \mathbb{P}(x_{t,1} \in \cdot \mid x_{t+\theta,1} > 0) &\overset{(d)}{=} \lim_{t\to\infty}\lim_{\theta\to\infty} \mathbb{P}(x_{t,1} \in \cdot \mid x_{t+\theta,1} > 0) \\
&\overset{(d)}{=} \Gamma(2, 2\alpha/c) \quad \text{if } 0 \leq \beta < \alpha, \\
\lim_{\theta\to\infty}\lim_{t\to\infty} \mathbb{P}(x_{t,2} \in \cdot \mid x_{t+\theta,2} > 0) &\overset{(d)}{=} \lim_{t\to\infty}\lim_{\theta\to\infty} \mathbb{P}(x_{t,2} \in \cdot \mid x_{t+\theta,2} > 0) \\
&\overset{(d)}{=} \Gamma(2, 2\beta/c) \quad \text{if } 0 < \beta < \alpha \\
&\qquad \text{or } 0 < \alpha < \beta \text{ and } m_1 = 0,
\end{aligned}$$

$$
\begin{aligned}
&\lim_{\theta\to\infty}\lim_{t\to\infty}\mathbb{P}(x_{t,1}\in\cdot\mid x_{t+\theta,1}>0)\overset{(d)}{=}\lim_{\theta\to\infty}\lim_{t\to\infty}\mathbb{P}(x_{t,1}\in\cdot\mid x_{t+\theta,2}>0)\\
\overset{(d)}{=}&\lim_{t\to\infty}\lim_{\theta\to\infty}\mathbb{P}(x_{t,1}\in\cdot\mid x_{t+\theta,1}>0)\overset{(d)}{=}\lim_{t\to\infty}\lim_{\theta\to\infty}\mathbb{P}(x_{t,1}\in\cdot\mid x_{t+\theta,2}>0)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\overset{(d)}{=}\Gamma(2,2\alpha/c)\quad \textit{if } 0<\alpha<\beta
\end{aligned}
$$

and

$$
\begin{aligned}
&\lim_{\theta\to\infty}\lim_{t\to\infty}\mathbb{P}(x_{t,2}\in\cdot\mid x_{t+\theta,1}>0)\overset{(d)}{=}\lim_{\theta\to\infty}\lim_{t\to\infty}\mathbb{P}(x_{t,2}\in\cdot\mid x_{t+\theta,2}>0)\\
\overset{(d)}{=}&\lim_{t\to\infty}\lim_{\theta\to\infty}\mathbb{P}(x_{t,2}\in\cdot\mid x_{t+\theta,1}>0)\overset{(d)}{=}\lim_{t\to\infty}\lim_{\theta\to\infty}\mathbb{P}(x_{t,2}\in\cdot\mid x_{t+\theta,2}>0)
\end{aligned}
$$

if $0<\alpha<\beta$ *and* $m_1\neq 0$ *(in this last case the limit is not known explicitly).*

Acknowledgments The first author is grateful to the DFG, which supported his Post-Doc in the Dutch-German Bilateral Research Group "Mathematics of Random Spatial Models from Physics and Biology", at the Weierstrass Institute for Applied Analysis and Stochastics in Berlin, where part of this research was made.

References

[1] Athreya, K.B. and Ney, P. Branching processes. Springer (1972).

[2] Dawson, D.A. Limit theorems for interaction free geostochastic systems. Col. Math. Soc. Bolyai, 22–47 (1978).

[3] Dawson, D.A. Measure-valued Markov processes. In Saint Flour Summer School in Probability Theory, XXI, 1991, pp. 1–260. Lecture Notes in Mathematics, 1541. Springer-Verlag, Berlin (1993).

[4] Dawson, D.A., Gorostiza, L. and Wakolbinger,A. Hierarchical Equilibria of Branching Populations. Elec. Jour. Probab. 9, 316–381 (2004).

[5] Dynkin, E.B. An introduction to Branching Measure-valued Processes, CRM Monographs 6, Amer. Math. Soc. Providence (1994).

[6] El Karoui, N. and Rœlly, S. Propriétés de martingales, explosion et représentation de Lévy-Khintchine d'une classe de processus de branchement à valeurs mesures. Stoch. Proc. Appl., 38, 239–266 (1991).

[7] Etheridge, A. An introduction to superprocesses, Univ. Lecture Series vol.20, Amer. Math. Soc. Providence (2000).

[8] Etheridge, A. and Williams, D.R.E. A decomposition of the (1+beta)-superprocess conditioned on survival. Proceed. of the Royal Soc. of Edinburgh (2004)

[9] Evans, S.N. Two representations of a conditioned superprocess. Proceed. of the Royal Soc. of Edinburgh 123A, 959–971 (1993)

[10] Evans, S.N. and Perkins, E. Measure-valued Markov Branching processes conditioned on non-extinction. Israel J. Math., 71, 329–337 (1990).

[11] Foster, J. and Ney, P. Decomposable critical multi-type branching processes. Sankhyã, the Indian Journal of Statistics, 38 Series A, 28–37 (1976).

[12] Foster, J. and Ney, P. Limit laws for decomposable critical branching processes. Z. Wahrsch. Verw. Gebiete 46, 13–43 (1978/1979).

[13] Frobenius, G. Über Matrizen aus positiven Elementen Sitz.-Ber. der Preußischen Akademie der Wissenschaft 456–477 (1912).

[14] Gantmacher, F.R. Matrizentheorie. VEB, Berlin (1986).

[15] Gorostiza, L.G. and Lopez-Mimbela, J.A. The multitype measure branching process. Adv. Appl. Probab. 22, 49–67 (1990).

[16] Gorostiza, L. and Rœlly, S. Some properties of the multitype measure branching process. Stoch. Proc. Appl. 37, 259–274 (1991).

[17] Jirina, M. Branching processes with measure-valued states. Trans. Third Prague Conf. Information Theory, Statist. Decision Functions, Random Processes (Liblice, 1962), pp. 333–357, Publ. House Czech. Acad. Sci., Prague (1964).

[18] Kawazu, K. and Watanabe, S. Branching processes with immigration and related limit theorems. Theor. Probab. Appl. 16, 36–49 (1971).

[19] Lambert, A. Quasi-stationary distributions and the continuous-state branching process conditioned to be never extinct. Elec. J. Prob. 12 420–446 (2007).

[20] Lamperti, J. and Ney, P. Conditioned branching processes and their limiting diffusions. Theor. Probab. Appl. 13, 128–139 (1968).

[21] Li, Z.-H. Asymptotic behaviour of continuous time and state branching processes. J. Austr. Math. Soc. (Series A) 68, 68–84 (2000).

[22] Meyer, P.-A. Fonctionnelles multiplicatives et additives de Markov. Annales de l'Institut Fourier 12, 125–230 (1962).

[23] Ogura, Y. Asymptotic behavior of multitype Galton-Watson processes. J. Math. Kyoto Univ. 15, 251–302 (1975).

[24] Overbeck, L. Conditioned super-Brownian motion. Probab. Theory Relat. Fields 96, 545–570 (1993).

[25] Perkins, E. Dawson-Watanabe Superprocesses and Measure-valued Diffusions. In Saint Flour Summer School in Probability Theory, XXIX, 1999, pp. 125–329. Lecture Notes in Mathematics, 1781. Springer-Verlag, Berlin (2002).

[26] Perko, L. Differential equations and dynamical systems. Springer-Verlag (1991).

[27] Perron, O. Über Matrizen. Math. Annalen 64, 248–263 (1907)

[28] Roelly, S. and Rouault, A. Processus de Dawson-Watanabe conditionné par le futur lointain. C. R. Acad. Sci. Paris Sér. I, 309, 867–872 (1989).

[29] Seneta, E. Non-negative matrices and Markov chains. Springer Verlag 2nd rev. ed. (2006).

[30] Sugitani, S. On the limit distributions of decomposable Galton-Watson processes with the Peron-Frobenius root 1. Osaka Jour. Math. 18, 175–224 (1981).

[31] Vatutin, V.A. and Sagitov, S.M. A decomposable critical branching processes with two types of particles. Proc. Steklov Inst. of Math. 4, 1–19 (1988)

[32] Watanabe, S. A limit theorem of branching processes and continuous state branching processes. Jour. Math. Kyoto Univ. 8, 141–167 (1968).

[33] Zubkov, A.M. Limit behavior of decomposable critical branching processes with two types of particles. Theor. Probab. Appl. 27, 235–247 (1982).