Yet another introduction to rough paths

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Abstract. This article provides another point of view on the theory of rough paths, which starts with simple considerations on ordinary integrals, and endows the importance of the Green-Riemann formula, as in the work of D. Feyel and A. de La Pradelle. This point of view allows us to introduce gently the required algebraic structures and provides alternative ways to understand why the construction of T. Lyons *et al.* is a natural generalization of the notion of integral of differential forms, in the sense it shares the same properties as integrals along smooth paths, when we use the "right notion" of path.

1. Introduction

The theory of rough paths [Lyo98, LQ02, Lej03, LCL07] is now an active field of research, especially among the probabilistic community. Although this theory is motivated by stochastic analysis, it takes its roots in analysis and control theory, and is also connected to differential geometry and algebra.

Given a path x of finite p-variation with $p \geq 2$ on $[0, T]$ with values in $\mathbb{R}^d$ or an α-Hölder continuous path with $\alpha \leq 1/2$, this theory allows us to define the integral $\int_x f$ of a differential form f along x, which is $\int_x f = \int_0^T f(x_s)\,\mathrm{d}x_s$. Using a fixed point theorem, it is then possible to solve differential equations driven by x of type

$$y_t = y_0 + \int_0^t g(y_s)\,\mathrm{d}x_s.$$

The case $1 \leq p < 2$ (or $\alpha > 1/2$) is covered by the Young integrals introduced by L.C. Young in [You36]. Some of the most common stochastic processes, including the Brownian motion, have trajectories that are of finite p-variation with $p > 2$. So, being able to define almost surely an integral along such irregular paths is of great practical interest, both

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theoretically and numerically. Yet we know this is not possible in general, and integrals of type Itô and Stratonovich are defined only as limits in probability of Riemann sums.

Introduced in the 50's by K.-T. Chen (see for example [Che58]), the notion of iterated integrals provides an algebraic tool to deal with a geometrical object which is a smooth path, and allows us to manipulate controlled differential equations using formal computations (see for example [Fli81, Isi95]).

The main feature of the rough paths theory is then to assert that, if it is possible to consider not only a path x but a path $\mathbf{x}$ which encodes the iterated integrals (that cannot be canonically defined if x is of finite p-variation with $p \geq 2$), then one may properly define the integral $z_t = \int_0^t f(x_s)\,\mathrm{d}\mathbf{x}_s$ and solve the differential equation $y_t = y_0 + \int_0^t g(y_s)\,\mathrm{d}\mathbf{x}_s$ provided that f and g are smooth enough. In addition, the maps $\mathbf{x} \mapsto z$ and $\mathbf{x} \mapsto y$ are continuous, with respect to the topology induced by the p-variation distance. The dimension of the path $\mathbf{x}$, or equivalently the number of "iterated integrals" to consider, depends on the regularity of x. For $p \in [2,3)$ (or $\alpha \in (1/3,1/2]$), then one has to consider only the iterated integrals of x along itself. This can be justified by the first order Taylor development of $\int_s^t f(x_r)\,\mathrm{d}x_r$:

$$\sum_{i=1}^d \int_s^t f_i(x_r)\,\mathrm{d}x_r^i \approx \sum_{i=1}^d f_i(x_s)(x_t^i - x_s^i) + \sum_{i,j=1}^d \frac{\partial f_i}{\partial x_j}(x_s) \int_s^t (x_r^j - x_s^j)\,\mathrm{d}x_r^i.$$

If x is α-Hölder continuous with $\alpha \in (1/3,1/2]$ and one has succeed in constructing $K_{s,t}^{i,j}(x) = \int_s^t (x_r^j - x_s^j)\,\mathrm{d}x_r^j$, then one can expect that $|K_{s,t}^{i,j}(x)| \leq C|t-s|^{2\alpha}$. Hence, we will use to approximate $\int_0^T f(x_r)\,\mathrm{d}x_r$ the sums

$$\sum_{k=0}^{n-1}\sum_{i=1}^d f_i(x_{kT/n})(x_{(k+1)T/n}^i - x_{kT/n}^i) + \sum_{k=0}^{n-1}\sum_{i,j=1}^d \frac{\partial f_i}{\partial x_j}(x_{kT/n}) K_{kT/n,(k+1)T/n}^{i,j}(x)$$

and show it converges as $n \to \infty$. Hence, the integral will be defined not along a path x, but along $\mathbf{x}_{s,t}$ given by

$$\mathbf{x}_{s,t} = (1, x_t^i - x_s^i, \dots, x_t^d - x_s^d, K_{s,t}^{1,1}(x), \dots, K_{s,t}^{d,d}(x)),$$

where the first component 1 is here for algebraic reasons. The element $\mathbf{x}$ can be seen as an element of the truncated tensor space $\mathrm{T}(\mathbb{R}) = \mathbb{R} \oplus \mathbb{R}^d \oplus (\mathbb{R}^d \otimes \mathbb{R}^d)$. By similarity with what happens for the power series constructed from the iterated integrals — sometimes called the *signature* of the path —, one has that for all $0 \leq s \leq r \leq t \leq T$,

$$\mathbf{x}_{s,t} = \mathbf{x}_{s,r} \otimes \mathbf{x}_{r,t},$$

where $\otimes$ is the tensor product on $\mathrm{T}(\mathbb{R})$ (where we keep only the tensor products of no more than 2 terms). In addition, it is possible to consider the formal logarithm of $\mathbf{x}$, and following also the properties of the Chen series, we look for paths $\mathbf{x}$ such that $\log(\mathbf{x}_{s,t})$ belongs to $\mathrm{A}(\mathbb{R}^d) = \mathbb{R}^d \oplus [\mathbb{R}^d, \mathbb{R}^d]$, where $[\mathbb{R}^d, \mathbb{R}^d]$ is the space generated by all the Lie brackets between two elements of $\mathbb{R}^d$. This algebraic property allows us to give proper definitions of rough paths and geometric rough paths from an algebraic point of view. The articles [Lyo98, Lej03] and the books [LQ02, LCL07] use this point of view.

As noted first by N. Victoir, since $(\mathrm{T}_1(\mathbb{R}^d), \otimes)$ — the subset of $\mathrm{T}(\mathbb{R}^d)$ whose element have a first term equal to 1 — is a Lie group, one may describe $\mathbf{x}_{s,t}$ by $\mathbf{x}_{s,t} = (-\mathbf{x}_{0,s})^{-1} \otimes \mathbf{x}_{0,t}$, and then, instead of considering the family $(\mathbf{x}_{s,t})_{0 \leq s < t \leq T}$, one may work with the path $\mathbf{x}_t = \mathbf{x}_{0,t}$, which lives in the non-commutative space $(\mathrm{T}_1(\mathbb{R}^d), \otimes)$. This provides some simplifications on the statement of some theorems, but also opens the door to look for more connections with differential geometry.

Shortly after the publication of the article [Lyo98], other authors provided alternative constructions of the differential equations and integrals, still by using some of the ideas provided by the theory of rough paths. One of this work — from D. Feyel and A. de la Pradelle [FdLP06] — uses a point of view from the differential geometry and endows the importance of the Gauss/Green-Riemann/Stokes formula to understand the need to "enhanced" the path with more information to get a rigorous definition. Another approach, by M. Gubinelli, rather relies on algebraic considerations [Gub04].

The idea of this article is then to justify the construction of the algebraic structures (tensor space, Lie groups) needed in the theory of rough paths from basic considerations on integrals of differential forms. To simplify, we consider that the dimension d of the state space is $d = 2$ (for $d = 1$, there is no real problem since (i) Any differential form is the differential of a function; (ii) The controlled differential equation $y_t = g(y_t)\,\mathrm{d}x_t$ is solved under reasonable assumptions on g by $y_t = \Phi(x_t)$ with $\Phi'(z) = g(\Phi(z))$ if x is smooth so that a density argument may be used. On that topic, see for example the work of Doss and Sussmann [Dos77]). By considering all the pairs of components, it is easy to pass from $d = 2$ to $d > 2$. In addition, we restrict ourselves to α-Hölder continuous paths, which is not a stringent assumption at all, since a time change allows us to transform any path with p-finite variation into a path which is $1/p$-Hölder continuous.

Given a differential form, we wish to construct a map $x \mapsto \int_x f$ which is continuous on the space C^α of α-Hölder continuous paths. If $\alpha > 1/2$, the existence of $\int_x f$ is provided by the theory of Young integrals. We also get that $x \mapsto \int_x f$ is continuous on C^α equipped with the α-Hölder norm. Yet we construct some sequence $(x^n)_{n\in\mathbb{N}}$ of functions in C^α that converges to x in C^β with $\beta < 1/2$, and such that $\int_0^T f(x^n_s)\,\mathrm{d}x^n_s$ does not converges to $\int_x f$, but to $\int_0^T f(x_s)\,\mathrm{d}x_s + \int_0^T [f,f](x_s)\,\mathrm{d}\varphi_s$ where $[f,f]$ is the Lie bracket of f and φ is an arbitrary function. This counter-examples makes use of the Green-Riemann functions, and see that, if one consider not a path x, but a path (x,φ) with values in $\mathbb{R}^3$, then one can extend the notion of the integral to C^α with $\alpha \in (1/3, 1/2]$. In some sense, the third component records the area enclosed between that path and its chord between times s and t. We can then provided an algebraic setting for describing such paths, still with a non-commutative operation. Then, we construct paths with values in $\mathrm{A}(\mathbb{R}^2)$, a space of dimension 3, where the first two coordinates corresponds to an "ordinary" path in the Euclidean vector space $\mathbb{R}^2$. The non-commutativity comes from the fact that the area enclosed between $x \cdot y$ — the concatenation of two paths x and y — and its chord is different from the area enclosed between $y \cdot x$ and its chord. The degree of freedom we gain comes from the fact that small loops allows us to move in the third direction while staying roughly at the same position in $\mathbb{R}^2$. Any α-Hölder continuous path with values in $\mathrm{A}(\mathbb{R}^2)$ (with the right distance) with $\alpha > 1/3$ may be approximated by smooth paths lifted in $\mathrm{A}(\mathbb{R}^2)$ using their area. In addition, the convergence of paths with values in $\mathrm{A}(\mathbb{R}^2)$ in the α-Hölder topology implies that the corresponding integrals form a Cauchy sequence in C^β for any $\beta < \alpha$. It is then possible to extend the notion of Young integrals to α-Hölder continuous functions with values in $\mathrm{A}(\mathbb{R}^2)$, and also to get the continuity result we need.

The basic idea to approximate some α-Hölder continuous path $\mathbf{x}$ taking its values in $\mathrm{A}(\mathbb{R}^2)$ with $\alpha > 1/3$ consists in lifting paths x^n that take the same values as $\mathbf{x}$ on the points of a partition of $[0,T]$ and that links two successive times by a loop and a straight line. The loop is a way to "encode the area". It may then be tempting to look for real geodesics. For this, we will interpret the space $\mathrm{A}(\mathbb{R}^2)$ as the subspace of the tangent space at any point of the tensor space $\mathrm{T}(\mathbb{R}^2)$, and we will look for simple curves linking two points in $\mathrm{T}(\mathbb{R}^2)$. There are several possibilities. One consists in using the tools from the sub-Riemannian geometry [FV06b, FV08]. Another one consists in studying paths with values in a sub-manifold $\mathrm{G}(\mathbb{R}^2)$ of $\mathrm{T}(\mathbb{R}^2)$, which is also a subgroup of $(\mathrm{T}(\mathbb{R}^2), \otimes)$, and which is the Lie group whose Lie algebra may be identified with $\mathrm{A}(\mathbb{R}^2)$. We give another way to define the integral by extending the differential form f to a differential form on $\mathrm{G}(\mathbb{R}^2)$ and construct curves that connect two points of $\mathrm{G}(\mathbb{R}^2)$. Hence, instead of considering paths with values in $\mathrm{A}(\mathbb{R}^2)$, we will consider paths with values in $\mathrm{G}(\mathbb{R}^2)$, and the difference between two points in $\mathrm{A}(\mathbb{R}^2)$ corresponds then to a direction.

With this, we may redefine the integral as the limit of some Riemann sums — which is the original definition given by T. Lyons —, but where the addition has been replaced by some tensor product. Moreover, it becomes then possible to extend the notion of integrals to paths living in the bigger space $\mathrm{T}_1(\mathbb{R}^2)$.

Consequently, using the concept of path living in a non-commutative space, the rough path theory provides a way to define an integral $\int f(x_s)\,\mathrm{d}x_s$ that shares the same properties as ordinary integrals:

(a) It is a limit of expressions similar to Riemann sums.
(b) It is a limit of integrals along approximations of the path construct from sampling the path at a finite set of points and connecting successive sample points by "simple" curves.

In addition, this map $x \mapsto \int f(x_s)\,\mathrm{d}x_s$ is continuous from $\mathrm{C}^\alpha([0,T];\mathrm{T}_1(\mathbb{R}^2))$ to $\mathrm{C}^\alpha([0,T];\mathrm{T}_1(\mathbb{R}^2))$ and may be used to solve differential equations driven by x, still with a continuity property.

The theory of rough paths turns out to be the natural extension of integrals on the space of α-Hölder continuous paths with $\alpha \in (1/3, 1/2]$, in the same way Young integrals is the natural notion of integral against α-Hölder continuous paths with $\alpha \in (1/2, 1]$.

Outline. In Section 2, we introduce our notations and recall some elementary facts about integrals of differential forms along smooth paths as well as about Hölder continuous paths. In section 3, we quickly present results about Young integrals, and thus show the properties of integrals along α-Hölder continuous paths with $\alpha > 1/2$. In Section 4, we assume that one can integrate differential forms along α-Hölder continuous path with $\alpha \in (1/3, 1/2]$, and we show how to transform this integral into a continuous one with respect to the path. In Section 5, we consider paths taking their values in $\mathrm{A}(\mathbb{R}^2)$, and show how to define the integral $\int_x f$ as limits of ordinary integrals. In Section 6, we continue our analysis of the space $\mathrm{A}(\mathbb{R}^2)$ and introduce the tensor space $\mathrm{T}(\mathbb{R}^2)$. In Section 7, we give another way to define the integral of f along x, using an expression of Riemann sum type. This construction corresponds to the original one of T. Lyons [Lyo98, LQ02, LCL07]. In Section 8, we give some related results: case of the d-dimensional space, Chen series, other constructions for paths with quadratic variation, link with stochastic integrals. In Section 9, we solve differential equations. We end this article with appendix on the Heisenberg group and we recall a technical result about almost rough paths, on which the original construction of $\int_x f$ is based.

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2. Notations

2.1. **Differential forms.** Let $f_1, \dots, f_d$ be some functions from $\mathbb{R}^d$ to $\mathbb{R}^m$. We consider the differential form

$$f(x) = f_1(x^1)\,\mathrm{d}x^1 + \cdots + f_d(x^d)\,\mathrm{d}x^d$$

on $\mathbb{R}^d$.

Definition 1. For $\gamma > 0$, f is said to be γ-*Lipschitz* if the f_i's for $i = 1, \dots, d$ are of class $\mathrm{C}^{\lfloor\gamma\rfloor}(\mathbb{R}^d;\mathbb{R}^m)$ with bounded derivative up to order $\lfloor\gamma\rfloor$, and the $f_i^{\lfloor\gamma\rfloor}$'s are $(\gamma - \lfloor\gamma\rfloor)$-Hölder continuous with a $(\gamma - \lfloor\gamma\rfloor)$-Hölder constant $H_\gamma^i(f)$. The class of such γ-Lipschitz differential forms is denoted by $\mathrm{Lip}(\gamma;\mathbb{R}^d \to \mathbb{R}^m)$.

For $f \in \mathrm{Lip}(\gamma;\mathbb{R}^d \to \mathbb{R}^m)$, define

$$\|f\|_{\mathrm{Lip}} = \max_{i=1,\dots,d} \max\{\|f_i^{(0)}\|_\infty, \dots, \|f_i^{(\lfloor\gamma\rfloor)}\|_\infty, H_\gamma^i(f)\},$$

which is a norm on $\mathrm{Lip}(\gamma;\mathbb{R}^d \to \mathbb{R}^m)$.

Remark 1. If $\gamma = 1$, this definition is slightly different from the notion of Lipschitz functions, since this definition implies that f is of class $\mathrm{C}^1(\mathbb{R}^d;\mathbb{R}^m)$, while with the definition that $|f(x) - f(y)|/|x-y|$ is bounded as $x \to y$ for all $y \in \mathbb{R}^d$, this means only that f is almost everywhere differentiable. Anyway, in our context, the case $\gamma \in \mathbb{N}$ is never considered.

Given a path $x \in \mathrm{C}^1([0,T];\mathbb{R}^d)$ and a continuous differential form f, we define the *integral of f along x* by

$$\int_x f = \int_0^T f(x_s) \left.\frac{\mathrm{d}x}{\mathrm{d}t}\right|_{t=s} \mathrm{d}s = \sum_{i=1}^d \int_0^T f_i(x_s) \left.\frac{\mathrm{d}x^i}{\mathrm{d}t}\right|_{t=s} \mathrm{d}s.$$

Let us recall a few facts on such integrals, that will be heavily used:

(i) If $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is strictly increasing and continuous, then $\int_{x\circ\varphi} f = \int_x f$. In other words, the integral of f along x does not depend on the parametrization of x.

(ii) If $\varphi : [0,T] \to [0,T]$ is $\varphi(t) = T - t$, then $\int_{x\circ\varphi} f = -\int_x f$. In other words, reversing the time changes the sign of $\int_x f$.

(iii) If $x, y \in \mathrm{C}^1_{\mathrm{p}}([0,T];\mathbb{R}^d)$ (the class of functions from $[0,T]$ to $\mathbb{R}^d$ which are piecewise in C^1) and $x \cdot y$ is the concatenation of x and y, then $\int_{x\cdot y} f = \int_x f + \int_y f$. This is the *Chasles relation.*

(iv) If $x \in \mathrm{C}^1_{\mathrm{p}}([0,T];\mathbb{R}^2)$ is a closed loop in $\mathbb{R}^2$, that is $x_T = x_0$, then

$$\int_x f = \iint_{\mathrm{Surface}(x)} [f,f](x^1,x^2)\,\mathrm{d}x^1\,\mathrm{d}x^2, \tag{1}$$

where $\mathrm{Surface}(x)$ is the oriented surface surrounded by x and

$$[f,f] = \frac{\partial f_1}{\partial x_2} - \frac{\partial f_2}{\partial x_1}.$$

This is the *Green-Riemann/Stokes/Gauss formula.*

2.2. **Paths of finite p-variation.** Fix $T > 0$. Let x be a continuous path from $[0, T]$ to $\mathbb{R}^d$ and $\Pi = \{t_i\}_{i=0,\dots,k}$ be a partition of $[0, T]$ with k elements. For $p \geq 1$, define

$$\mathfrak{P}(x; \Pi, p) = \sum_{i=0}^{k-1} |x_{t_{i+1}} - x_{t_i}|^p.$$

The p-variation of x on $[s, t] \subset [0, T]$ is defined by

$$\mathrm{Var}_{p,[s,t]}(x) = \sup_{\Pi \text{ partition of } [0,T]} \mathfrak{P}(x|_{[s,t]}; \Pi \cap [s, t], p)^{1/p}.$$

Definition 2. A function $x : [0, T] \to \mathbb{R}^d$ is said to be of *finite p-variation* if $\mathrm{Var}_{p,[0,T]}(x)$ is finite.

If x is of finite p-variation, then we get easily that

$$\mathrm{Var}_{q,[0,T]}(x) \leq 2^{(q-p)/q} \|x\|_\infty^{(q-p)/q} (\mathrm{Var}_{p,[0,T]}(x))^{p/q} \tag{2}$$

and then x is of finite q-variation for all $q > p$. Note that $\mathrm{Var}_{p,[0,T]}(x)$ defines a semi-norm on the space of functions of finite p-variation, but not a norm, since $\mathrm{Var}_{p,[0,T]}(x) = 0$ implies only that x is constant. In addition, on the space of functions x with $x_0 = 0$ and $\mathrm{Var}_{p,[0,T]}(x) < +\infty$, $\mathrm{Var}_{p,[0,T]}$ defines a norm which is however *not* equivalent to the uniform norm $\|\cdot\|_\infty$, and counter-examples are easily constructed.

Following a recent remark due to P. Friz [Fri05], we may work with a more precise norm than the norm constructed from p-variation. Indeed, to simplify our approach, we work only with Hölder continuous paths and the Hölder norm.

If x is a path of finite p-variation and

$$\varphi(t) = \inf \left\{ s > 0 \,\middle|\, \mathrm{Var}_{p,[0,s]}(x)^p > t \right\},$$

then φ is increasing and $x \circ \varphi$ is $1/p$-Hölder continuous. As the integral of a differential form keeps the same value under a continuous, increasing time change, there is no difficulty in considering the $1/p$-Hölder norm, which is simpler to use than the p-variation norm (for some results on the relationship between p-variation and $1/p$-Hölder continuity, see for example [CG98]). Yet for convergence problems, this is not the most general framework, and dealing with the p-variation norm allows us to treat with more complete results (for example, in [Lej06, Lej08], we prove the convergence only in p-variation although the path is α-Hölder continuous, and this is due to a singularity at 0 of some term).

Let us denote by $H_\alpha(x)$ the Hölder continuity modulus of a path $x : [0, T] \to \mathbb{R}^d$ which is α-Hölder continuous, that is

$$H_\alpha(x) = \sup_{0 \leq s < t \leq T} \frac{|x_t - x_s|}{|t - s|^\alpha}.$$

Of course, any α-Hölder continuous path is also β-Hölder continuous for any $\beta \leq \alpha$. In addition, the equivalent of (2) is

$$\text{for } \beta \leq \alpha,\ H_\beta(x) \leq 2^{1-\beta/\alpha} \|x\|_\infty^{1-\beta/\alpha} H_\alpha(x)^{\beta/\alpha}. \tag{3}$$

If $H_\alpha(x) = 0$ then x is constant, and H_α defines only a semi-norm.

Notation 1. If $x : [0,T] \to \mathbb{R}^d$ α-Hölder continuous, then we set

$$\|x\|_\alpha = |x_0| + H_\alpha(x)$$

and by $\mathrm{C}^\alpha([0,T];\mathbb{R}^d)$ the subset of functions x in $\mathrm{C}([0,T];\mathbb{R}^d)$ such that $\|x\|_\alpha$ is finite.

Equipped with $\|\cdot\|_\alpha$, this space $\mathrm{C}^\alpha([0,T],\mathbb{R}^d)$ is a Banach space. In addition, we get the following Lemma which is a consequence of the Ascoli Theorem and (3).

Lemma 1. *Let $(x^n)_{n\in\mathbb{N}}$ such that $x^n \in \mathrm{C}^\alpha([0,T];\mathbb{R}^d)$ and $(\|x^n\|_\alpha)_{n\in\mathbb{N}}$ is bounded. Then there exists x in $\mathrm{C}^\alpha([0,T];\mathbb{R}^d)$ and a subsequence of $(x^n)_{n\in\mathbb{N}}$ that converges to x with respect to $\|\cdot\|_\beta$ for each $\beta < \alpha$.*

Remark 2. It is important to note that here, we used the $\|\cdot\|_\beta$ norm for the space $\mathrm{C}^\alpha([0,T];\mathbb{R}^d)$ with $\beta < \alpha$. When equipped with this norm, $(\mathrm{C}^\alpha([0,T];\mathbb{R}^d), \|\cdot\|_\beta)$ becomes a separable space, while the space $(\mathrm{C}^\alpha([0,T];\mathbb{R}^d), \|\cdot\|_\alpha)$ is not separable: See [MS61] for example.

The next corollary follows easily.

Corollary 1. *Let Π be a partition of $[0,T]$ and x^Π be the linear approximation of $x \in \mathrm{C}^\alpha([0,T];\mathbb{R}^d)$ along Π. Then $\|x^\Pi\|_\alpha \leq 3^{1-\alpha}\|x\|_\alpha$.*

If $(\Pi^n)_{n\in\mathbb{N}}$ is a sequence of partitions of $[0,T]$ whose meshes converge to 0, then $(x^{\Pi^n})_{n\in\mathbb{N}}$ converges to x in $(\mathrm{C}^\alpha([0,T];\mathbb{R}^d), \|\cdot\|_\beta)$ for all $\beta < \alpha$.

Proof. Let $\Pi = \{t_i\}_{i=1,\dots,J}$. For $0 \leq s < t \leq T$, let $s' = \min \Pi \cap [s,T]$ and $t' = \max \Pi \cap [0,t]$. If $s \notin \Pi$ (resp. $t \notin \Pi$), denote by $s'' = \max \Pi \cap [0,s]$ (resp. $t'' = \min \Pi \cap [t,T]$). As $s', t' \in \Pi$, if $s, t \notin \Pi$,

$$\begin{aligned}
|x_t^\Pi - x_s^\Pi| &\leq |x_t^\Pi - x_{t'}^\Pi| + |x_{t'}^\Pi - x_{s'}^\Pi| + |x_{s'}^\Pi - x_s^\Pi| \\
&\leq \frac{t-t'}{t''-t'}|x_{t''} - x_{t'}| + |x_{t'} - x_{s'}| + \frac{s-s'}{s''-s'}|x_{s'} - x_s| \\
&\leq \|x\|_\alpha (t-t')^\alpha + \|x\|_\alpha (s'-s)^\alpha + \|x\|_\alpha (t'-s')^\alpha \\
&\leq 3^{1-\alpha}\|x\|_\alpha (t-s)^\alpha,
\end{aligned}$$

the last inequality coming from the Jensen inequality applied by $x \mapsto x^{1/\alpha}$. The case where s or t belongs to Π is treated similarly. This proves that $\|x^\Pi\|_\alpha \leq 3^{1-\alpha}\|x\|_\alpha$.

The second part of this corollary is an immediate consequence of Lemma 1. □

Remark 3. One may wonder if it is possible to approximate a function $x \in \mathrm{C}^\alpha([0,T];\mathbb{R}^2)$ by piecewise linear functions that converge with respect to $\|\cdot\|_\alpha$, and not with respect to $\|\cdot\|_\beta$ for $\beta < \alpha$. As shown in [MS61] (see also [DN98, § 4.3]), this is only possible if x belongs to the class of functions such that

$$\lim_{\delta\to 0} \sup_{\substack{0\leq s<t\leq T\\ |t-s|\leq\delta}} \frac{|x_t - x_s|}{(t-s)^\alpha} = 0,$$

or, in other words, if $|x(t+h) - x(t)| = \mathrm{o}(h^\alpha)$. Of course, this class of functions is strictly included in $\mathrm{C}^\alpha([0,T];\mathbb{R}^d)$: the function $f(x) = \sum_{k=0}^{+\infty} c^{-k\alpha}\sin(c^k x)$ for c large enough provides us with a counter-example, as it is easily proved using the results from [Cie60].

3. Integrals along α-Hölder continuous paths, $\alpha \in (1/2, 1]$

For the sake of simplicity, consider $d = 2$. The construction of $\mathfrak{I}$ on $\mathrm{C}^\alpha([0,T];\mathbb{R}^2)$ for $\alpha > 1/2$ is first deduced from the Young integral.

3.1. **Defining the integrals.** We recall here the construction of the integral of a β-Hölder continuous path driven by a α-Hölder continuous path, provided that $\alpha+\beta > 1$. This theorem is due to L.C. Young [You36] (see also [DN98] for example).

Theorem 1. *Let $\alpha, \beta \in (0, 1]$ with $\alpha + \beta > 1$. Then*

$$(x, y) \mapsto \left(t \mapsto \int_0^t y_s \,\mathrm{d}x_s\right)$$

is bilinear and continuous from $\mathrm{C}^\alpha([0,T];\mathbb{R})\times\mathrm{C}^\beta([0,T];\mathbb{R})$ to $\mathrm{C}^\alpha([0,T];\mathbb{R})$.

Sketch of the proof. Fix $n \in \mathbb{N}^*$, and let us set, for $t_k^n = Tk/2^n$,

$$J^n = \sum_{k=0}^{2^n-1} y_{t_k^n}(x_{t_{k+1}^n} - x_{t_k^n}).$$

Then

$$\begin{aligned} |J^{n+1} - J^n| &= \left|\sum_{k=0}^{2^n-1} (y_{t_{2k+1}^{n+1}} - y_{t_{2k}^{n+1}})(x_{t_{2k+2}^{n+1}} - x_{t_{2k+1}^{n+1}})\right| \\ &\leq \sum_{k=0}^{2^n-1} H_\beta(y)H_\alpha(x)T^{\alpha+\beta}2^{-(n+1)(\alpha+\beta)} \\ &\leq 2^{-n(\alpha+\beta-1)}H_\beta(y)H_\alpha(x). \end{aligned}$$

As $\alpha+\beta-1 > 0$, we deduce that the series $\sum_{n\geq 0}(J_{n+1} - J_n)$ converges and thus that, if $J \stackrel{\text{def}}{=} J_0 + \sum_{n\geq 0}(J_{n+1} - J_n)$, then

$$|J - y_0(x_T - x_0)| \leq \zeta(\alpha+\beta-1)T^{\alpha+\beta}H_\beta(y)H_\alpha(x), \tag{4}$$

where $\zeta(\theta) = \sum_{n\geq 0} 1/n^\theta$. Of course, we define $\int_0^T y_s\,\mathrm{d}x_s$ as J. From the last inequality in which t is substituted to T and s to 0, this also proves that $t \mapsto \int_0^t y_s\,\mathrm{d}x_s$ is α-Hölder continuous.

The other properties of the integral are easily, although tedious, deduced from this construction. □

Remark 4. Indeed, using the argument of Lemma 2.2.1, p. 244 [Lyo98], there is no need to consider dyadic partitions, but we keep them for simplicity. Note that however, especially when dealing with stochastic processes, some results in the rough paths theory are dependent from the choice of a dyadic partition (see for example [CL05]).

One may then define for $0 \leq s \leq t \leq T$,

$$\mathfrak{I}(x;s,t) = \int_{x_{|[s,t]}} f = \int_s^t f_1(x_r)\,\mathrm{d}x_r^1 + \int_s^t f_2(x_r)\,\mathrm{d}x_r^2 \tag{5}$$

as Young integrals with $y_t = f(x_t)$. Yet a global regularity condition is imposed on (x,y) with implies in particular that $\alpha > 1/2$ and the minimal assumptions on the regularity of f also depends on α.

Notation 2. For a path x defined on the time interval $[S,T]$, We will use $\mathfrak{I}(x;s,t)$ to denote the integral $\int_{x_{|[s,t]}} f$ when $S \leq s < t \leq T$, and $\mathfrak{I}(x)$ to denote the function $t \in [S,T] \mapsto \mathfrak{I}(x;S,t)$.

The following corollaries follow from the construction of the Young integrals and (4): see in particular [LZ94, Lej03].

Corollary 2. *Fix* $\alpha \in (1/2,1]$ *and* $f \in \mathrm{Lip}(\gamma;\mathbb{R}^2 \to \mathbb{R}^m)$ *with* $\gamma > 1/\alpha - 1$. *Then* $\mathfrak{I}$ *defined in* (5) *is well defined as a Young integral on* $\mathrm{C}^\alpha([0,T];\mathbb{R}^2)$ *and is a locally Lipschitz map from* $(\mathrm{C}^\alpha([0,T];\mathbb{R}^2), \|\cdot\|_\alpha)$ *to* $(\mathrm{C}^\alpha([0,T];\mathbb{R}^m), \|\cdot\|_\alpha)$.

Corollary 3. *Fix* $\alpha \in (1/3,1/2]$ *and let* $f \in \mathrm{Lip}(\gamma;\mathbb{R}^2 \to \mathbb{R}^m)$ *with* $\gamma > 1/\alpha - 1$. *Then*

$$\begin{aligned} \mathrm{C}^{2\alpha}([0,T];\mathbb{R}) \times \mathrm{C}^\alpha([0,T];\mathbb{R}^2) &\to \mathrm{C}^{2\alpha}([0,T];\mathbb{R}^m) \\ (\varphi, x) &\mapsto \left(t \mapsto \int_0^t [f,f](x_s)\,\mathrm{d}\varphi_s \right) \end{aligned}$$

is well defined as a Young integral and is a locally Lipschitz map from $(\mathrm{C}^{2\alpha}([0,T];\mathbb{R}^2), \|\cdot\|_{2\alpha}) \times (\mathrm{C}^\alpha([0,T];\mathbb{R}^2), \|\cdot\|_\alpha)$ *to* $(\mathrm{C}^{2\alpha}([0,T];\mathbb{R}^2), \|\cdot\|_{2\alpha})$.

3.2. **A problem of continuity.** We have to take great care of the meaning of the continuity result in Corollary 2: the norm $\|\cdot\|_\alpha$ is *not* equivalent to the uniform norm. Convergence in C^α implies uniform convergence but the converse is not true.

The following counter-example is the cornerstone to understand how $\mathfrak{I}$ will be defined for dealing with irregular paths.

Let $(x^n)_{n\in\mathbb{N}}$ and x be continuous paths such that x^n converges to x in $\mathrm{C}^\alpha([0,T];\mathbb{R}^2)$ with $\alpha \in (1/2,1]$.

Let φ be a function in $\mathrm{C}^\beta([0,T];\mathbb{R})$ with $\beta \in (2/3,1]$. Let us also assume that f belongs to $\mathrm{Lip}(\gamma;\mathbb{R}^d \to \mathbb{R})$

$$(\gamma + 1)\,\beta > 2$$

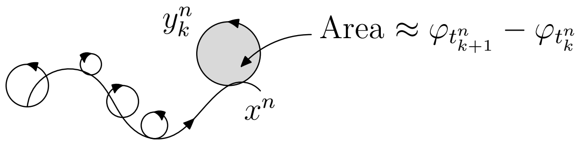


FIGURE 1. The path $x^n \bowtie \Phi^n$.

which implies that $2 > \gamma > 1$. Let $\Pi^n = \{t_k^n\}_{k=0,\dots,2^n-1}$ be the dyadic partition of $[0,T]$ at level n, that is $t_k^n = Tk/2^n$. For each $n = 1, 2, \dots$, we denote by $\Phi^n = \{y_k^n\}_{k=0,\dots,2^n-1}$ a set of functions piecewise of class C^1 such that for a fixed $\kappa > 1$,

$$y_k^n : [t_k^n, t_{k+1}^n] \to \mathbb{R}^2 \text{ with } y_k^n(t_k^n) = y_k^n(t_{k+1}^n) = x^n(t_k^n), \tag{6a}$$

$$\sup_{n=1,2,\dots,\ k=0,\dots,2^n} \|y_k^n\|_{\beta/2} < +\infty, \tag{6b}$$

$$\text{uniformly in } n, k,\ |\operatorname{Area}(y_k^n) - (\varphi(t_{k+1}^n) - \varphi(t_k^n))| \leq CT^\kappa 2^{-n\kappa}, \tag{6c}$$

where $\operatorname{Area}(y_k^n)$ is the algebraic area of the loop y_k^n defined by

$$\begin{aligned}\operatorname{Area}(y_k^n) = \frac{1}{2}\int_{t_k^n}^{t_{k+1}^n} (y_k^{1,n}(s) - y_k^{1,n}(t_k^n))\,\mathrm{d}y_k^{2,n}(s) \\ - \frac{1}{2}\int_{t_k^n}^{t_{k+1}^n} (y_k^{2,n}(s) - y_k^{2,n}(t_k^n))\,\mathrm{d}y_k^{1,n}(s).\end{aligned}$$

For such a sequence, we say that φ *encodes asymptotically the areas of* $(\Phi^n)_{n\in\mathbb{N}}$.

We denote by $x^n \bowtie \Phi^n$ the path from $[0, 2T]$ to $\mathbb{R}^2$ defined by

$$x^n \bowtie \Phi^n = y_0^n \cdot x^n_{|[t_0^n,t_1^n]} \cdot y_1^n \cdot x^n_{|[t_1^n,t_2^n]} \cdots y_{2^n-1}^n \cdot x^n_{|[t_{2^n-1}^n,t_{2^n}^n]},$$

where $x \cdot y$ is the concatenation between two path x and y (see Figure 1). This path $x^n \bowtie \Phi^n$ is defined on the time interval $[0, 2T]$.

Then, by the Chasles property of the integral,

$$\mathfrak{I}(x^n \bowtie \Phi^n; 0, 2T) = \mathfrak{I}(x^n; 0, T) + \sum_{k=0}^{2^n-1} \int_{t_k^n}^{t_{k+1}^n} f(y_k^n(s))\,\mathrm{d}y_k^n(s).$$

By the Green-Riemann formula (1),

$$\int_{t_k^n}^{t_{k+1}^n} f(y_k^n(s))\,\mathrm{d}y_k^n(s) = \iint_{\operatorname{Surface}(y_k^n)} [f, f](x^1, x^2)\,\mathrm{d}x^1\,\mathrm{d}x^2.$$

The idea is now the following,

$$\begin{aligned}\iint_{\operatorname{Surface}(y_k^n)} [f, f](x^1, x^2)\,\mathrm{d}x^1\,\mathrm{d}x^2 &\approx [f, f](x_{t_k^n}) \operatorname{Area}(y_k^n) \\ &\approx [f, f](x_{t_k^n})(\varphi(t_{k+1}^n) - \varphi(t_k^n)).\end{aligned}$$

To be more precise, using our hypotheses on f and Φ^n, with $\Delta_n t = T2^{-n}$,

$$
(7) \quad \left|\iint_{\mathrm{Surface}(y_k^n)} [f,f](x^1,x^2)\,\mathrm{d}x^1\,\mathrm{d}x^2 - [f,f](x_{t_k^n})(\varphi(t_{k+1}^n)-\varphi(t_k^n))\right|
$$

$$
\leq 2\|\nabla f\|_{\gamma-1}\|y_k^n\|_{\beta/2}^{\gamma-1}(C\Delta_n t^\kappa + \|\varphi\|_\beta \Delta_n t^\beta)\Delta_n t^{(\gamma-1)\beta/2} + 2C\|\nabla f\|_\infty \Delta_n t^\kappa
$$

$$
\leq 2\|\nabla f\|_{\gamma-1}\|y_k^n\|_{\beta/2}^{\gamma-1}(C\Delta_n t^{\kappa-\beta} + \|\varphi\|_\beta)\Delta_n t^{(\gamma+1)\beta/2} + 2C\|\nabla f\|_\infty \Delta_n t^\kappa.
$$

There are now 2^n of such terms to sum. By hypothesis, $(\gamma+1)\beta/2 > 1$ and $\kappa > 1$ so that, the sum of the right-hand side of (7) vanishes as $n \to \infty$. In addition, necessarily $\beta + \gamma\alpha > 1$ so that one can consider $\int [f,f](x_s)\,\mathrm{d}\varphi_s$ as a Young integral. Thus, we easily get that

$$
\sum_{k=0}^{2^n-1} \int_{t_k^n}^{t_{k+1}^n} f(y_k^n(s))\,\mathrm{d}y_k^n(s) \xrightarrow[n\to\infty]{} \int_0^T [f,f](x_s)\,\mathrm{d}\varphi_s.
$$

In other words,

$$
\mathfrak{I}(x^n \bowtie \Phi^n; 0, 2T) \xrightarrow[n\to\infty]{} \mathfrak{I}(x;s,t) + \int_0^T [f,f](x_r)\,\mathrm{d}\varphi_r.
$$

It is important to note that here, $(x^n \bowtie \Phi^n)_{n\in\mathbb{N}}$ is in general not bounded in $\mathrm{C}^\alpha([0,2T];\mathbb{R}^2)$, but it is bounded in $\mathrm{C}^{\beta/2}([0,2T];\mathbb{R}^2)$. Let us remark that for $t \in [0,2T]$, if $t/2 \in [t_k^{n+1}, t_{k+1}^{n+1}]$ and k is odd, then $x^n \bowtie \Phi^n(t) = x^n(t/2)$. If k is even, then $x^n \bowtie \Phi^n(t) = y_k^n(t/2)$. Thus,

$$
|x^n \bowtie \Phi^n(t) - x^n \bowtie \Phi^n(s)|
$$

$$
\leq \begin{cases}
|x^n(t/2) - x^n(s/2)| \\
\quad \text{if } s/2 \in [t_{2k+1}^{n+1}, t_{2k+2}^{n+1}],\ t/2 \in [t_{2\ell+1}^{n+1}, t_{2\ell+2}^{n+1}], \\
|y_\ell^n(t/2) - y_\ell^n(t_\ell^n)| + |x^n(t_k^n) - x^n(s/2)| \\
\quad \text{if } s/2 \in [t_{2k+1}^{n+1}, t_{2k+2}^{n+1}],\ t/2 \in [t_{2\ell}^{n+1}, t_{2\ell+1}^{n+1}], \\
|y_\ell^n(t/2) - y_\ell^n(t_\ell^n)| + |y_\ell^n(t_\ell^n) - y_k^n(t_k^n)| + |y_k^n(t_k^n) - y_k^n(s/2)| \\
\quad \text{if } s/2 \in [t_{2k}^{n+1}, t_{2k+1}^{n+1}],\ t/2 \in [t_{2\ell}^{n+1}, t_{2\ell+1}^{n+1}],\ k \neq \ell, \\
|y_\ell^n(t/2) - y_\ell^n(s/2)| \\
\quad \text{if } s/2 \in [t_{2\ell}^{n+1}, t_{2\ell+1}^{n+1}],\ t/2 \in [t_{2\ell}^{n+1}, t_{2\ell+1}^{n+1}], \\
|x^n(t/2) - x^n(t_k^n)| + |y_k^n(t_k^n) - y_k^n(s/2)| \\
\quad \text{if } s/2 \in [t_{2k}^{n+1}, t_{2k+1}^{n+1}],\ t/2 \in [t_{2\ell+1}^{n+1}, t_{2\ell+2}^{n+1}].
\end{cases}
$$

Using the convexity inequality, one gets that for some constant C that depends only on α and β,

$$
|x^n \bowtie \Phi^n(t) - x^n \bowtie \Phi^n(s)|
\leq C \max\{\|x\|_\alpha, \sup_{k=0,\dots,2^n-1} \|y_k^n\|_{\beta/2}\} \max\{(t-s)^{\beta/2}, (t-s)^\alpha\}.
$$

Since $\beta/2 \leq \alpha$, it follows that $(x^n \bowtie \Phi^n)_{n\in\mathbb{N}}$ is bounded in $\mathrm{C}^{\beta/2}([0,2T];\mathbb{R}^2)$ assuming that of course, the y_k^n have a $\beta/2$-Hölder norm different from zero. As we required that φ is β-Hölder continuous, and if we we choose for y_n^k some circles with area $\varphi(t_{k+1}^n)-\varphi(t_k^n)$, then their radius are $\sqrt{|\varphi(t_{k+1}^n)-\varphi(t_k^n)|/\pi}$ and this is why we look for y_k^n's that are $\beta/2$-Hölder continuous.

This also means that when one considers a sequence $(x^n)_{n\in\mathbb{N}}$ of elements in $\mathrm{C}^\alpha([0,T];\mathbb{R}^2)$ and a path x of $\mathrm{C}^\alpha([0,T];\mathbb{R}^2)$ with $\alpha > 1/2$, one has to consider the fact that $(x^n)_{n\in\mathbb{N}}$ may converge to x with respect to some β-Hölder norm with $\beta \leq 1/2$. In addition, this counter-example ruins all hope to extend $\mathfrak{I}$ naturally to $\mathrm{C}^\alpha([0,T];\mathbb{R}^2)$ for $\alpha < 1/2$, since may construct at least two bounded sequences $(x^n)_{n\in\mathbb{N}}$ and $(z^n)_{n\in\mathbb{N}}$ in $\mathrm{C}^\alpha([0,T];\mathbb{R}^2)$ with $\alpha < 1/2$ converging uniformly to x — hence that converge to x in $\mathrm{C}^\beta([0,T];\mathbb{R}^2)$ for any $\beta < \alpha$ — such that $\mathfrak{I}(x^n;0,T) \xrightarrow[n\to\infty]{} \mathfrak{I}(x;0,T)$ and $\mathfrak{I}(z^n;0,T) \xrightarrow[n\to\infty]{} \mathfrak{I}(x;0,T) + \int_0^T [f,f](x_s)\,\mathrm{d}\varphi_s$, which is different from $\mathfrak{I}(x;0,T)$ unless $[f,f]=0$ or φ is constant.

3.3. **A practical counter-example in the stochastic setting.** In [Lej02, LL06a], we give a stochastic example of such a phenomenon coming from the homogenization theory. Let us consider some coefficients σ from $\mathbb{R}^d$ to the space of $d\times d$-matrices and $b:\mathbb{R}^d\to\mathbb{R}^d$ smooth enough which are 1-periodic. We consider the SDE

$$X_t^\varepsilon = \int_0^t \sigma(X_s^\varepsilon/\varepsilon)\,\mathrm{d}B_s + \frac{1}{\varepsilon}\int_0^t b(X_s^\varepsilon/\varepsilon)\,\mathrm{d}s$$

for some Brownian motion B. It is well known from the homogenization theory (see [BLP78] for example) that X^ε converges as $\varepsilon\to 0$ to $\overline{\sigma}W$ for some Brownian motion W and a $d\times d$-matrix $\overline{\sigma}$ which is constant, provided that the drift b satisfies some averaging property. One of the application of this theory is to provide a tool to replace (for modelling or numerical computations) a PDE of type $\partial_t u^\varepsilon(t,x) + L^\varepsilon u^\varepsilon(t,x) = 0$, $u^\varepsilon(T,x) = g(x)$ with $L^\varepsilon = \sum_{i,j=1}^d \frac{1}{2}a_{i,j}(\cdot/\varepsilon)\partial^2_{x_ix_j} + \sum_{i=1}^d \frac{1}{\varepsilon}b_i(\cdot/\varepsilon)\partial_{x_i}$ and $a=\sigma\sigma^{\mathrm{t}}$ by the simpler PDE $\partial_t u(t,x) + \overline{L}u(t,x) = 0$ with $\overline{L} = \sum_{i,j=1}^d \frac{1}{2}\overline{a}_{i,j}\partial^2_{x_ix_j}$ and $\overline{a} = \overline{\sigma}\overline{\sigma}^{\mathrm{t}}$. From the probabilistic point of view, this means that X^ε behaves — thanks to a functional Central Limit Theorem and the ergodic behavior of its projection on the torus $\mathbb{R}^d/[0,1]^d$ — like a non-standard Brownian motion. However, one has to take care when using X^ε as the driver of some SDE, since

$$i,j=1,\dots,d,\ \mathfrak{A}^{i,j}(X^\varepsilon;0,t) \xrightarrow[\varepsilon\to 0]{} \mathfrak{A}^{i,j}(\overline{\sigma}W;0,t) + t\overline{c}_{i,j}$$

uniformly and in p-variation for $p>2$, where $(\overline{c}_{i,j})_{i,j=1,\dots,d}$ is a $d\times d$-antisymmetric matrix that can be computed from a and b, and $\mathfrak{A}^{i,j}$ is the Lévy area of (Y^i, Y^j), *i.e.*,

$$\mathfrak{A}^{i,j}(Y;0,t) = \frac{1}{2}\int_0^t (Y_s^i - Y_0^i)\circ \mathrm{d}Y_s^j - \frac{1}{2}\int_0^t (Y_s^j - Y_0^j)\circ \mathrm{d}Y_s^i$$

for a d-dimensional semi-martingale Y. If $b=0$, then $\overline{c}=0$, so that this effect comes from the presence of the drift.

From the Wong-Zakai theorem (see for example [IW89]), the Stratonovich integral appears as the natural extension of $\mathfrak{I}$ on the subset $\mathrm{SM}([0,T];\mathbb{R}^2)$ of $\mathrm{C}^\alpha([0,T];\mathbb{R}^2)$ with $\alpha < 1/2$ that contains trajectories of semi-martingales. Let us note however that for $Y \in \mathrm{SM}([0,T];\mathbb{R}^2)$ and $(f_1, f_2) = \frac{1}{2}(-x_j, x_i)$,

$$\mathfrak{I}(Y;0,t) = \mathfrak{A}^{1,2}(Y;0,t)$$

for $t \in [0,T]$, if $\mathfrak{I}$ is defined on $\mathrm{SM}([0,T];\mathbb{R}^2)$ as the Stratonovich integral $\mathfrak{I}(Y;0,t) = \int_0^t f(Y_s)\circ \mathrm{d}Y_s$. Since both X^ε and $\overline{\sigma}W$ belong to $\mathrm{SM}([0,T];\mathbb{R}^2)$, the previous example shows that $\mathfrak{I}(X^\varepsilon;0,t)$ does not converge in general to $\mathfrak{I}(B;0,t)$. This proves that $\mathfrak{I}$ cannot be continuous on $\mathrm{SM}([0,T];\mathbb{R}^2) \subset \mathrm{C}^\alpha([0,T];\mathbb{R}^2)$.

Counter-examples to the Wong-Zakai theorem (see [McS72, IW89]) also rely on the construction of approximations of the trajectories of the Brownian motion by using a "perturbation" of the piecewise linear approximation that gives rise, in the limit, to a non-vanishing supplementary area and then, for the SDE, to a drift term. The theory of rough paths gives a better understanding of this phenomena [LL06a].

This problem of convergence may arise in natural setting and has then a practical interest.

4. Integrals along α-Hölder continuous paths, $\alpha \in (1/3, 1/2]$: Heuristic considerations

We present in this section a construction of the integral which is not the best possible one, but which allows us to understand the main ideas and problems.

The counter-example of Section 3.2 has endowed a few ideas: (1) We may use the Green-Riemann formula to deal with close loops. (2) For some $\alpha > 1/2$, we may add to our paths small loops whose radius are of order $2^{-n\alpha/2}$ and thus whose area is of order $2^{-n\alpha}$. (3) As many loops are added, the sum of the areas does not vanish and gives rise to a supplementary term.

Our construction will now take these facts into account.

4.1. Construction of the integral along a subset of $\mathrm{C}^\alpha([0,T];\mathbb{R}^2)$.

As we wish our definition of the integral to be continuous, a naive construction is the following: Fix $K > 0$, $\alpha \in (1/3, 1/2]$ and $f \in \mathrm{Lip}(\gamma;\mathbb{R}^2 \to \mathbb{R})$ with $\gamma > 1/\alpha - 1$ (and then $\gamma > 1$). Denote by Π^n the dyadic partition of $[0,T]$ at level n, and by $\mathrm{L}^\alpha([0,T];\mathbb{R}^2)$ the set of functions $x \in \mathrm{C}^\alpha([0,T];\mathbb{R}^2)$ for which the linear approximations $(x^{\Pi^n})_{n\in\mathbb{N}}$ satisfy

$$\mathfrak{I}(x) \stackrel{\text{def}}{=} \lim_{n\in\mathbb{N}} \mathfrak{I}(x^{\Pi^n}) \text{ exists in } \mathrm{C}^\alpha([0,T];\mathbb{R})$$
$$\text{and } |\mathfrak{I}(x_{|[s,t]}) - \mathfrak{I}(x^{\Pi^n}_{|[s,t]})| \leq K\|x - x^{\Pi^n}\|_\alpha |t-s|^\alpha,\ 0 \leq s < t \leq T.$$

If K is large enough, it follows from Corollary 2 that $\mathrm{L}^\alpha([0,T];\mathbb{R}^2)$ contains subsets of $\mathrm{C}^\beta([0,T];\mathbb{R}^2)$ for all $\beta > 1/2$ (this depends on f and the choice of K, since from Corollary 2, $x \mapsto \mathfrak{I}(x)$ is locally Lipschitz) and it is also known (but for this, we need a more complete theory) that it contains paths that are not β-Hölder continuous for $\beta > 1/2$, such as Brownian motion's trajectories (see for example [Sip93, CL05]). Any element x of $\mathrm{L}^\alpha([0,T];\mathbb{R}^2)$ may be identified with the sequence $(x^{\Pi^n})_{n\in\mathbb{N}}$.

Now, consider $\varphi \in \mathrm{C}^{2\alpha}([0,T];\mathbb{R}^2)$ and $(\Phi^n)_{n\in\mathbb{N}}$ a sequence of loops at each level n whose areas are asymptotically encoded by φ. Then, as previously,

$$\mathfrak{I}(x^{\Pi^n} \bowtie \Phi^n) \xrightarrow[n\to\infty]{\mathrm{C}^\alpha} \mathfrak{I}(x,\varphi) \stackrel{\text{def}}{=} \mathfrak{I}(x) + \int [f,f](x_s)\,\mathrm{d}\varphi_s.$$

For $(x,\varphi) \in \mathrm{L}^{\bowtie,\alpha}([0,T];\mathbb{R}^3) \stackrel{\text{def}}{=} \mathrm{L}^\alpha([0,T];\mathbb{R}^2) \times \mathrm{C}^{2\alpha}([0,T];\mathbb{R})$, we may then define

$$\mathfrak{I}(x,\varphi) = \lim_{n\to\infty} \mathfrak{I}(x^{\Pi^n} \bowtie \Phi^n)$$

where φ encodes asymptotically the areas of $(\Phi^n)_{n\in\mathbb{N}}$. The space $\mathrm{L}^{\bowtie,\alpha}([0,T];\mathbb{R}^3)$ is naturally a Banach space when equipped with the norm $\|(x,\varphi)\|_{\bowtie,\alpha} = \|x\|_\alpha + \|\varphi\|_{2\alpha}$.

The interesting point with this definition of the map $(x,\varphi) \mapsto \mathfrak{I}(x,\varphi)$ is that its continuity follows naturally from its very construction.

Proposition 1. *For all $\beta < \alpha$ with $\alpha \in (1/3, 1/2]$, the map $\mathfrak{I}$ is continuous from $(\mathrm{L}^{\bowtie,\alpha}([0,T];\mathbb{R}^3), \|\cdot\|_{\bowtie,\alpha})$ to $(\mathrm{C}^\alpha([0,T];\mathbb{R}), \|\cdot\|_\beta)$*

Proof. Let $(x^n,\varphi^n)_{n\in\mathbb{N}}$ be a sequence of paths converging to (x,φ) in $\mathrm{L}^{\bowtie,\alpha}([0,T];\mathbb{R}^3)$.

By definition, $\mathfrak{I}(x^n,\varphi^n;s,t) = \mathfrak{I}(x^n;s,t) + \int_s^t [f,f](x^n_r)\,\mathrm{d}\varphi^n_r$. From Corollary 3, we know that $\int_0^\cdot [f,f](x^n)\,\mathrm{d}\varphi^n$ converges to $\int_0^\cdot [f,f](x)\,\mathrm{d}\varphi$ in $\mathrm{C}^{2\alpha}([0,T];\mathbb{R})$.

From the very definition of $\mathrm{L}^\alpha([0,T];\mathbb{R}^2)$,

$$\|\mathfrak{I}(x^{n,\Pi^m}) - \mathfrak{I}(x^n)\|_\alpha \leq K\|x^{n,\Pi^m} - x^n\|_\alpha.$$

But it is easily shown with Corollary 1 that for all $\beta < \alpha$ and some constant K_2, $\|x^{n,\Pi^m} - x^n\|_\beta \leq K_2\|x^n\|_\alpha / 2^{m(\beta-\alpha)}$ and thus $(\mathfrak{I}(x^{n,\Pi^m}))_{m\in\mathbb{N}}$ converges to $\mathfrak{I}(x^n)$ in $\mathrm{C}^\beta([0,T];\mathbb{R})$ at a rate which is uniform in n since $(\|x^n\|_\alpha)_{n\in\mathbb{N}}$ is bounded.

It follows that for all $\beta < \alpha$, $\mathfrak{I}(x^{n,\Pi^m})$ converges uniformly in n to $\mathfrak{I}(x^n)$ in $\mathrm{C}^\beta([0,T];\mathbb{R})$ as $m \to \infty$.

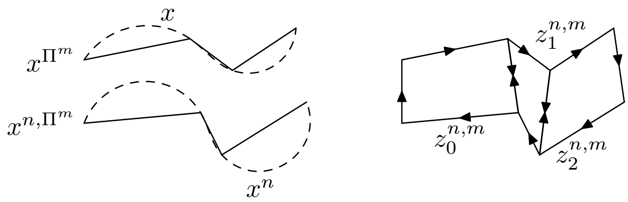


FIGURE 2. The paths $z_k^{n,m}$.

For $s < t$ fixed, there exist some integers i_m and j_m such that $t^m_{i_m-1} \leq s < t^m_{i_m}$ and $t^m_{j_m} < t \leq t^m_{j_m+1}$. To simplify the notations, we set $t^m_{i_m-1} = s$ and $t^m_{j_m+1} = t$. For $k = i_m - 1, \dots, j_m + 1$, denote by $z_k^{n,m}$ the following path (see Figure 2)

$$z_k^{n,m} = x^{\Pi^m}_{|[t^m_k, t^m_{k+1}]} \cdot \overline{x^{\Pi^m}_{t^m_{k+1}} x^{n,\Pi^m}_{t^m_{k+1}}} \cdot x^{n,\Pi^m}_{|[t^m_{k+1}, t^m_k]} \cdot \overline{x^{n,\Pi^m}_{t^m_k} x^{\Pi^m}_{t^m_k}}.$$

Hence, with the previous convention on $t^m_{i_m-1}$ and $t^m_{j^m}$,

$$\mathfrak{I}(x^{\Pi^m}; s,t) - \mathfrak{I}(x^{n,\Pi^m}; s,t) = \sum_{k=i_m-1}^{j_m} \int_{z_k^{n,m}} f + \int_{\overline{x^{\Pi^m}_s x^{n,\Pi^m}_s}} f + \int_{\overline{x^{n,\Pi^m}_t x^{\Pi^m}_t}} f. \tag{8}$$

Let us note that

$$\begin{aligned}\left| \int_{z_k^{n,m}} f \right| &= \left| \iint_{\mathrm{Surface}(z_k^{n,m})} [f,f](x^1,x^2)\, \mathrm{d}x^1\, \mathrm{d}x^2 \right| \\ &\leq \frac{1}{2} \|f\|_{\mathrm{Lip}} |x_{t^m_{k+1}} - x_{t^m_k}| \times |x^n_{t^m_k} - x_{t^m_k}| \\ &\leq \frac{(t^m_{k+1} - t^m_k)^\alpha}{2} \|x\|_\alpha \|f\|_{\mathrm{Lip}} \|x^n - x\|_\infty.\end{aligned}$$

Using the convexity inequality with $x \mapsto x^{1/\alpha}$, since there are at most 2^m terms in the series in the right-hand-side of (8), we get

$$\begin{aligned}\sum_{k=i_m-1}^{j_m} \left| \int_{z_k^{n,m}} f \right| &\leq 2^{m(1-\alpha)} \left(\sum_{k=i_m-1}^{j_m} \left(\left| \int_{z_k^{n,m}} f \right| \right)^{1/\alpha} \right)^\alpha \\ &\leq \frac{2^m}{2} \|f\|_{\mathrm{Lip}} \|x\|_\alpha \|x^n - x\|_\infty (t-s)^\alpha.\end{aligned}$$

On the other hand, setting $\Delta_r^n = x_r^{n,\Pi^m} - x_r^n$ for $r \in \{s,t\}$,

$$\begin{aligned}\left|\int_{\overline{x_s^{\Pi^m} x_s^{n,\Pi^m}}} f + \int_{\overline{x_t^{n,\Pi^m} x_t^{\Pi^m}}} f\right| &= \left|\int_{\overline{x_s^{\Pi^m} x_s^{n,\Pi^m}}} f - \int_{\overline{x_t^{\Pi^m} x_t^{n,\Pi^m}}} f\right| \\ &\leq \left|\int_0^1 (f(x_s^{\Pi^m} + r\Delta_s^n) - f(x_t^{\Pi^m} + r\Delta_t^n))\Delta_s^n \,\mathrm{d}r\right| \\ &\quad + \left|\int_0^1 f(x_t^{\Pi^m} + r\Delta_t^n)(\Delta_t^n - \Delta_s^n)\,\mathrm{d}r\right| \\ &\leq \|f\|_{\mathrm{Lip}}|\Delta_s^n|(\|x^n\|_\alpha + \|x^{n,\Pi^m}\|_\alpha)(t-s)^\alpha + \|f\|_{\mathrm{Lip}}|\Delta_t^n - \Delta_s^n|.\end{aligned}$$

But, for any $\delta \in [0,1)$,

$$\begin{aligned}|\Delta_t^n - \Delta_s^n| &\leq |x_t^{\Pi^m} - x_s^{\Pi^m} - x^{n,\Pi^m} + x_s^{n,\Pi^m}| \\ &\leq (|x_t^{\Pi^m} - x_s^{\Pi^m}|^\delta + |x_t^{n,\Pi^m} - x_s^{n,\Pi^m}|^\delta) 2\|x^{\Pi^m} - x^{n,\Pi^m}\|_\infty^{1-\delta} \\ &\leq (t-s)^{\alpha\delta} 2\max\{\|x^{\Pi^m}\|_\alpha^\delta, \|x^{n,\Pi^m}\|_\alpha^\delta\} 2\|x^{\Pi^m} - x^{n,\Pi^m}\|_\infty^{1-\delta}.\end{aligned}$$

This proves the convergence of $\mathfrak{I}(x^{n,\Pi^m})$ to $\mathfrak{I}(x^{\Pi^m})$ in $\mathrm{C}^\beta([0,T];\mathbb{R})$ as $n \to \infty$ for any m for any $\beta < \alpha$.

It is now possible to complete the following diagram

$$\begin{array}{ccc} \mathfrak{I}(x^{n,\Pi^m}) & \xrightarrow[n\to\infty]{\|\cdot\|_\beta} & \mathfrak{I}(x^{\Pi^m}) \\ \|\cdot\|_\beta \downarrow \begin{smallmatrix} m\to\infty \\ \text{unif. in } n \end{smallmatrix} & & \|\cdot\|_\beta \downarrow m\to\infty \\ \mathfrak{I}(x^n) & & \mathfrak{I}(x) \end{array}$$

to obtain that $\mathfrak{I}(x^n, \varphi^n)$ converges in $\mathrm{C}^\beta([0,T];\mathbb{R})$ to $\mathfrak{I}(x,\varphi)$. □

Moreover, the following stability result is easily proved.

Lemma 2. *If ψ (resp. φ) is given in $\mathrm{C}^{2\alpha}([0,T];\mathbb{R})$ and that encodes asymptotically the areas of $(\Psi^n)_{n\in\mathbb{N}}$ (resp. $(\Phi^n)_{n\in\mathbb{N}}$), then*

$$\lim_{n\to+\infty} \mathfrak{I}(x^{\Pi^n} \bowtie \Phi^n \bowtie \Psi^n) = \mathfrak{I}(x, \varphi + \psi).$$

The function φ can be chosen arbitrary, so that we have gained a degree of freedom. In other words, to get a proper definition of $\mathfrak{I}$ that respect the continuity, we have to consider not a path with values in $\mathbb{R}^2$ but a path with values in $\mathbb{R}^3$. Indeed, this construction is far to be optimal, *i.e.,* the set $\mathrm{L}^{\bowtie,\alpha}([0,T];\mathbb{R}^3)$ is not the biggest one that can be considered. Yet it gives a proper understanding of the problem.

4.2. **Is this construction natural?** Of course, the real question is to consider whether or not is it natural to extend $\mathfrak{I}$ on (at least) a subset of $\mathrm{C}^\alpha([0,T];\mathbb{R}^2)$ with $\alpha \in (1/3, 1/2]$ by considering paths not with values in $\mathbb{R}^2$ but with values in $\mathbb{R}^3$?

Let us consider a path $x \in \mathrm{C}^\alpha([0,T];\mathbb{R}^2)$. The piecewise linear path x^{Π^n} is an approximation of x, and for each $m \geq n$, we may define

$$\widehat{x}^{\Pi^m} \stackrel{\mathrm{def}}{=} (x^{\Pi^m}_{|[t_0^n,t_1^n]} \cdot x^{\Pi^n}_{|[t_1^n,t_0^n]}) \cdot x^{\Pi^n}_{|[t_0^n,t_1^n]} \cdots (x^{\Pi^m}_{|[t_{2^n-1}^n,t_{2^n}^n]} \cdot x^{\Pi^n}_{|[t_{2^n}^n,t_{2^n-1}^n]}) \cdot x^{\Pi^n}_{|[t_{2^n-1}^n,t_{2^n}^n]}$$

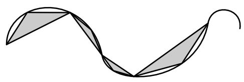


FIGURE 3. The paths x, x^{Π^n}, $x^{\Pi^{n+1}}$ and the areas defined by $\Phi^{n,n+1}$ (in gray).

on the time interval $[0,3T]$. As we go back on forth on the segments composing x^{Π^n}, we get that $\mathfrak{I}(\widehat{x}^{\Pi^m};0,3T) = \mathfrak{I}(x^{\Pi^m};0,T)$. We then define $y_k^{n,m} = x^{\Pi^m}_{|[t_k^n,t_{k+1}^n]} \cdot x^{\Pi^n}_{|[t_{k+1}^n,t_k^n]}$, that satisfies (6a)–(6b) and $\Phi^{n,m} = \{y_k^{n,m}\}_{k=0,\dots,2^n-1}$. Since $\widehat{x}^{\Pi^m} = x^n \bowtie \Phi^{n,m}$,

$$\mathfrak{I}(x^{\Pi^m};0,T) = \mathfrak{I}(\widehat{x}^{\Pi^m};0,3T) = \mathfrak{I}(x^{\Pi^n} \bowtie \Phi^{n,m};0,3T).$$

If we now set for example $m = n^2$, then *a priori* nothing ensures, unless $x \in \mathrm{L}^\alpha([0,T];\mathbb{R}^2)$, that the areas of $(\Phi^{n,n^2})_{n\in\mathbb{N}}$ are asymptotically encoded by the function $\varphi \equiv 0$, nor that there exists a function $\varphi \in \mathrm{C}^{2\alpha}([0,T];\mathbb{R})$ that encodes the areas of $(\Phi^{n,n^2})_{n\in\mathbb{N}}$. In the last two cases, how then to consider the limit of $\mathfrak{I}(x^{\Pi^{n^2}})$, since it may differs from the limit of $\mathfrak{I}(x^{\Pi^n} \bowtie \Phi^{n,n^2})$? Indeed,

$$\mathfrak{I}(x^{\Pi^n} \bowtie \Phi^{n,n^2};0,T) = \mathfrak{I}(x^{\Pi^n};0,T) + \sum_{k=0}^{2^n-1} \mathfrak{I}(y_k^{n,n^2};t_k^n,t_{k+1}^n).$$

Yet with the Green-Riemann formula,

$$\mathfrak{I}(y_k^{n,n^2};t_k^n,t_{k+1}^n) \approx [f,f](x_{t_k^n})\,\mathrm{Area}(y_k^{n,n^2}).$$

As we have seen it, the function $\mathfrak{A}$ on $\mathrm{C}^\beta([0,T];\mathbb{R}^2)$, $\beta > 1/2$, defined by

$$\mathfrak{A}(x;s,t) = \frac{1}{2}\int_s^t (x_r^1 - x_s^1)\,\mathrm{d}x_r^2 - \frac{1}{2}\int_s^t (x_r^2 - x_s^2)\,\mathrm{d}x_r^1 \tag{9}$$

is *not* continuous with respect to the uniform norm: One has only to take $f(x) = \frac{1}{2}x^1\,\mathrm{d}x^2 - \frac{1}{2}x^2\,\mathrm{d}x^1$ and to use the previous counter-examples. As $\mathrm{Area}(y_k^{n,n^2}) = \mathfrak{A}(x^{n^2};t_k^n,t_{k+1}^n)$ and although y_k^{n,n^2} converges uniformly to 0, it may happens that $\mathrm{Area}(y_k^{n,n^2};t_k^n,t_{k+1}^n)$ is of order $2^{-2\alpha n}$ (this is possible since the distance between $x_{t_k^n}$ and $x_{t_{k+1}^n}$ is roughly of order $2^{-\alpha n}$ if x is α-Hölder continuous, see Figure 4). In this case, $\sum_{k=0}^{2^n-1}\mathfrak{I}(y_k^{n,n^2})$ may have a limit different from 0 or have no limit at all.

In other words, the area contained between a path x and its chord for all couple of times (s,t) is "hidden" in x and has to be determined in an arbitrary manner[1].

[1]Consider the case of Brownian trajectories, where the Lévy area is a natural, but not the only one, choice and was the first example of stochastic integral [Lév65]. In addition, it is then defined as a limit in probability.

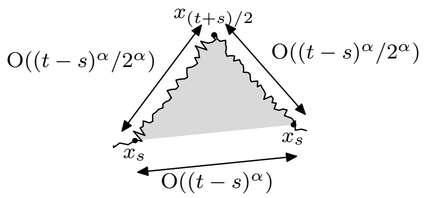


FIGURE 4. The area of some α-Hölder continuous path between times s and t is of order $(t-s)^{2\alpha}$.

For some $(x,\varphi) \in \mathrm{L}^{\bowtie,\alpha}([0,T];\mathbb{R}^3)$, which is identified with a sequence converging uniformly to x, the element φ means in some sense that some area has been chosen and then that our integral is properly determined. Once this choice of φ has been done, Lemma 2 shows us how to deduce from it different integrals by choosing other areas.

4.3. **Justifications for a new setting.** The previous construction does not answer our main question: "How to construct an integral for paths in $\mathrm{C}^\alpha([0,T];\mathbb{R}^2)$ for $\alpha \in (1/3,1]$?". Yet it endows the fact one cannot define a map $x \mapsto \mathfrak{I}$ which extend the map $x \mapsto \int_x f$ on $\mathrm{C}^\alpha([0,T];\mathbb{R}^2)$ with $\alpha > 1/2$ unless one adds some extra information. Here, this information corresponds to the choice of a function φ, so that we consider indeed a subset of $\mathrm{C}^\alpha([0,T];\mathbb{R}^2) \times \mathrm{C}^{2\alpha}([0,T];\mathbb{R})$ (for $\alpha \leq 1/2$) such that, when equipped with the norm $\|(x,\varphi)\| = \|x\|_\alpha + \|\varphi\|_{2\alpha}$, the map $\mathfrak{I}$ is continuous.

We have also seen that in Section 4.2 above that for considering integral along path in $\mathrm{C}^\alpha([0,T];\mathbb{R}^2)$ with $\alpha \in (1/3,1/2]$, it is natural to consider the area contained between the path and its chord in view of defining some integral, although there is no way to define it canonically in general.

The drawback of our construction is that we assumed the convergence of the integrals along the piecewise linear approximations of x.

The idea is now to construct directly a path in $\mathbb{R}^3$ in a way that such a path may be identified with a limit of converging sequence of piecewise smooth paths in $\mathbb{R}^2$ whose integrals also converge. This allows us to to get rid of the loops themselves, since the only information we need is the asymptotic limit of the area, while keeping enough information to construct the integral. Besides, this proves that the choice of a converging subsequence does not depend on the choice of the differential form which is integrated.

5. INTEGRALS ALONG α-HÖLDER CONTINUOUS PATHS, $\alpha \in (1/3,1/2]$: CONSTRUCTION BY APPROXIMATIONS

It is time to turn to the whole picture, now that the importance of knowing the area has been shown.

5.1. **Motivations.** The main idea in the previous approach was to replace an irregular path $(x,\varphi)\in \mathrm{L}^{\bowtie,\alpha}([0,T];\mathbb{R}^3)$ with a simpler path $x^n \in \mathrm{C}^1_{\mathrm{p}}([0,T];\mathbb{R}^2)$ which "approximates" x in the following sense: $x^n_{t^n_k} = x_{t^n_k}$ for the dyadics point $\{t^n_k\}_{k=0,\dots,2^n}$ of $[0,T]$, and on $[t^n_k, t^n_{k+1}]$ is composed of a loop $y^n_k : [t^n_k, t^n_k + T2^{-n-1}] \to \mathbb{R}^2$ and then a segment joining $x^n_{t^n_k}$ and $x^n_{t^n_{k+1}}$.

Once this family $(x^n)_{n\in\mathbb{N}}$ has been constructed, one may study the convergence of the ordinary integrals $\mathfrak{I}(x^n)$, where the integrals of f on the loops have been transformed with the Green-Riemann formula into double integrals whose values are given approximatively by the areas of the loops times the Lie brackets of f at the starting points of the loop.

A simple approximation of $\mathfrak{I}(x)$ is then given by, if x^n is then defined on $[0,T]$ with loops on $[t^n_k, t^n_{k+1} + T2^{-n-1}]$ and straight lines on $[t^n_k + T2^{-n-1}, t^n_{k+1}]$, is then given by

(10)
$$J^n = \sum_{k=0}^{2^n-1} \left(\int_{t^n_k + T2^{-n-1}}^{t^n_{k+1}} f(x^n_s)\,\mathrm{d}x^n_s + [f,f](x_{t^n_k})\mathfrak{A}(x^n; t^n_k, t^n_k + T2^{-n-1}) \right),$$

where $\mathfrak{A}(x;s,t)$ has been defined by (9). Now, following the heuristic reasoning of Section 4.2, we replace the assumption

(H1) The path (x,φ) belongs to $\mathrm{L}^{\bowtie,\alpha}([0,T];\mathbb{R}^3)$.

by the assumption

(H2) There exists some function $\mathfrak{A}(x;s,t)$ which is the limit of $\mathfrak{A}(x^n;s,t)$ for all $0 \le s \le t \le T$.

Let us note that the assumption (H1) implies (H2) if f is the differential form $f(x) = \frac{1}{2}(x^1\,\mathrm{d}x^2 - x^2\,\mathrm{d}x^1)$. In (H2), there is no more reference to f, while the set $\mathrm{L}^{\bowtie,\alpha}([0,T];\mathbb{R}^3)$ depends *a priori* on f.

The assumption (H2) means that $\mathfrak{A}(x^{n^2}; t^n_k, t^n_{k+1})$ (which is equal to $\mathfrak{A}(x^{n^2}; t^n_k, t^n_{k+1} + T2^{-n-1})$) is equivalent to $\mathfrak{A}(x; t^n_k, t^n_{k+1})$ as $n \to \infty$. Hence, one may replace (10) by

$$(11)\quad J^n = \sum_{k=0}^{2^n-1} \left(\int_{t^n_k + T2^{-n-1}}^{t^n_{k+1}} f(x^{\Pi^n}_s)\,\mathrm{d}x^{\Pi^n}_s + [f,f](x_{t^n_k})\mathfrak{A}(x; t^n_k, t^n_{k+1}) \right).$$

This form has the following advantage over the previous one: Under (H2), one can study, in the same way as for the proof of the Young integrals, the convergence of J^n by studying $J^{n+1} - J^n$ in order to prove that $\sum_{n\ge 0}(J^{n+1} - J^n)$ converges and to define the integral of x as the limit of this series plus J^0. This method is one of the core of the theory of rough paths.

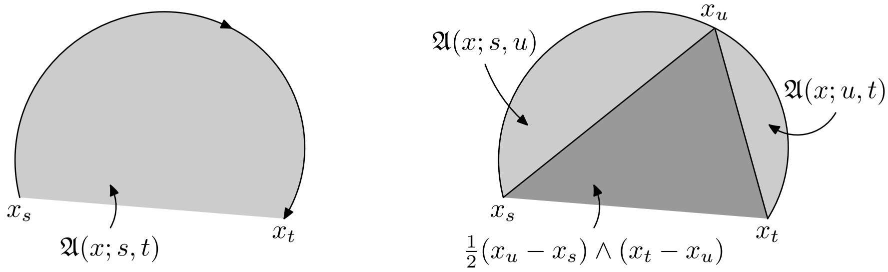


FIGURE 5. A geometrical illustration of (12).

Still using some approximation, we change (11) into

$$J^n=\sum_{k=0}^{2^n-1}\int_{t_k^n+T2^{-n-1}}^{t_{k+1}^n} f(x_s^{\Pi^n})(x_{t_{k+1}^n}-x_{t_k^n})\frac{\mathrm{d}s}{\Delta_n t}\\ +\sum_{k=0}^{2^n-1}\int_{t_k^n}^{t_{k+1}^n}[f,f](x_s^{\Pi^N})\mathfrak{A}(x;t_k^n,t_{k+1}^n)\frac{\mathrm{d}s}{\Delta_n t},$$

with $\Delta_n t=T2^{-n}$. We use this expression to motivate our introduction of some algebraic structures. Our wish is then to interpret $\mathfrak{A}(x;s,t)$ as some "vector", in the same way as we can seen, from the geometrical point of view, x_t-x_s as the vector that link the two points x_s and x_t and $\mathbb{R}^2$ seen as some affine space. As we will see it below, $\mathfrak{A}(x;s,t)$ is different in general from $\mathfrak{A}(x;0,t)-\mathfrak{A}(x;0,s)$. Hence, the Euclidean structure is not adapted.

We will now construct some space $\mathrm{A}(\mathbb{R}^2)$ of dimension 3, that will play the role both of an affine and a vector space, and the kind of vector we will consider will be $(x_t^1-x_s^1,x_t^2-x_s^2,\mathfrak{A}(x;s,t))$. Nevertheless, there will be constructed from the paths $(x_t^1,x_t^2,\mathfrak{A}(x;0,t))_{t\geq 0}$ living in $\mathrm{A}(\mathbb{R}^2)$ seen as some affine space.

In a first time, we define this space $\mathrm{A}(\mathbb{R}^2)$, then we study the approximation of paths living in this space and finally, we define an integral as limit of ordinary integrals using the previously constructed approximations.

5.2. **What happens to the area?** For a continuous path $x\in \mathrm{C}^\alpha([0,T];\mathbb{R}^2)$ with $\alpha>1/2$, let $y_t=\mathfrak{A}(x;0,t)$ be the area enclosed between the curve $x_{|[0,t]}$ and its chord $\overline{x_0x_t}$, where $\mathfrak{A}$ have been defined by (9). This path y is well defined by (9) and belongs to $\mathrm{C}^\alpha([0,T];\mathbb{R})$.

As we have seen that $x\mapsto\mathfrak{A}(x;0,\cdot)$ is not continuous in general on $\mathrm{C}^\alpha([0,T];\mathbb{R}^2)$ for $\alpha\leq 1/2$, we are nonetheless willing to define the equivalent of a process y for an irregular path. This can be achieved using an algebraic setting. Remark first that if $x\in\mathrm{C}^\alpha([0,T];\mathbb{R}^2)$ with

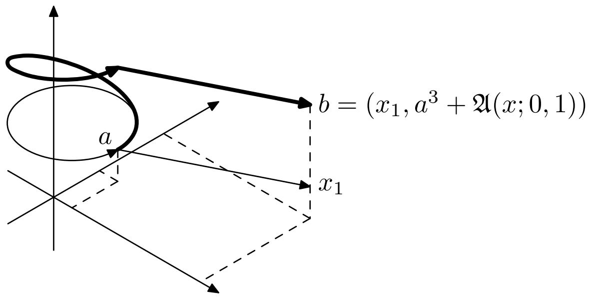


FIGURE 6. A simple path (x, y) from a to b controlled by a path x in $\mathbb{R}^2$.

$\alpha \in (1/2, 1]$,

$$\mathfrak{A}(x; s, t) = \mathfrak{A}(x; s, u) + \mathfrak{A}(x; u, t) + \frac{1}{2}(x_u - x_s) \wedge (x_t - x_u) \tag{12}$$

for all $0 \leq s < u < t \leq T$ (See Figure 5). Here, $\wedge$ is the vector product between two vectors: $a \wedge b = a^1 b^2 - a^2 b^1$.

5.3. **Linking points.** In a first time, we consider, for a piecewise smooth paths x, the path $(x^1, x^2, \mathfrak{A}(x))$ living in a three dimensional space. If u belongs to $\mathbb{R}$, then we set

$$\mathfrak{C}(x, u; t) = (x_t^1, x_t^2, u + \mathfrak{A}(x; 0, t)) \tag{13}$$

for $t \in [0, T]$. In the following, we may think that x represents a 2-dimensional control trajectory of the position of a particle moving in $\mathbb{R}^3$.

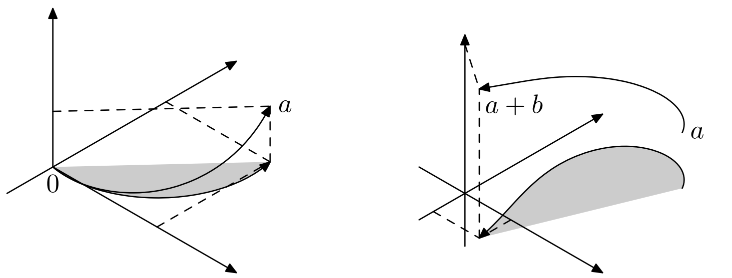
Given two points $a = (a^1, a^2, a^3)$ and $b = (b^1, b^2, b^3)$, we wish to construct a piecewise smooth path x from $[0, 1]$ to $\mathbb{R}^2$ such that the continuous path $(x_t, a^3 + \mathfrak{A}(x; 0, t))$ from $[0, 1]$ to $\mathbb{R}^3$ goes from a to b.

Such a path is easily constructed. We give here a simple example, that serves as a prototype for our approach. Our choice, drawn in Figure 6, is

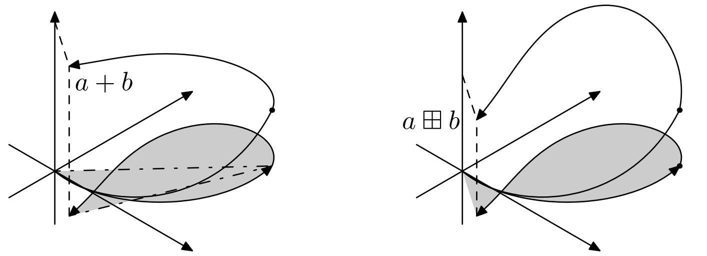
$$x_t = \begin{bmatrix} a^1 \\ a^2 \end{bmatrix} + \frac{\sqrt{|b^3 - a^3|}}{\sqrt{\pi}} \begin{bmatrix} \cos(4\pi t) - 1, \\ \operatorname{sgn}(b^3 - a^3)\sin(4\pi t) \end{bmatrix} \text{ if } t \in \left[0, \frac{1}{2}\right],$$
$$\text{and } x_t = \begin{bmatrix} a^1 \\ a^2 \end{bmatrix} + (2t - 1) \begin{bmatrix} b^1 - a^1 \\ b^2 - a^2 \end{bmatrix} \text{ if } t \in \left[\frac{1}{2}, 1\right].$$

Given two points a and b in $\mathbb{R}^3$, let us consider two paths x and y in $\mathrm{C}^1_{\mathrm{p}}([0, T]; \mathbb{R}^2)$ such that $x_0 = y_0 = 0$ and $\mathfrak{C}(x, 0; T) = a$, $\mathfrak{C}(y, 0; T) = b$.

The concatenation $x \cdot y$ of x and y gives rise to a path that goes from 0 to $\pi(a + b)$ by passing through $\pi(a)$, where π is the projection $\pi(a^1, a^2, a^3) = (a^1, a^2)$. What can then be said on $\mathfrak{C}(x \cdot y; 0, 2T)$? Due to (12), we get that $\mathfrak{C}(x \cdot y)$ is a path that goes from 0 to the point we



(a) The path $\mathfrak{C}(x, 0)$. (b) The path $\mathfrak{C}(\pi(a) + y, a^3)$.



(c) The path $\mathfrak{C}(x, 0) \cdot \mathfrak{C}(\pi(a) + y, a^3)$. (d) The path $\mathfrak{C}(x \cdot y, 0)$.

FIGURE 7. Reaching a point with the constraint of passing through another point.

denote by $a \boxplus b$ defined by

$$a \boxplus b = \left(a^1 + b^1, a^2 + b^2, a^3 + b^3 + \frac{1}{2} \begin{bmatrix} a^1 \\ a^2 \end{bmatrix} \wedge \begin{bmatrix} b^1 \\ b^2 \end{bmatrix} \right).$$

With this notation, $\boxplus$ clearly defines an operation on $\mathbb{R}^3$, which is different from the usual addition (geometrically equivalent to some translation) in this space $\mathbb{R}^3$. In addition, $\mathfrak{C}(x \cdot y, 0)$ passes through the point a.

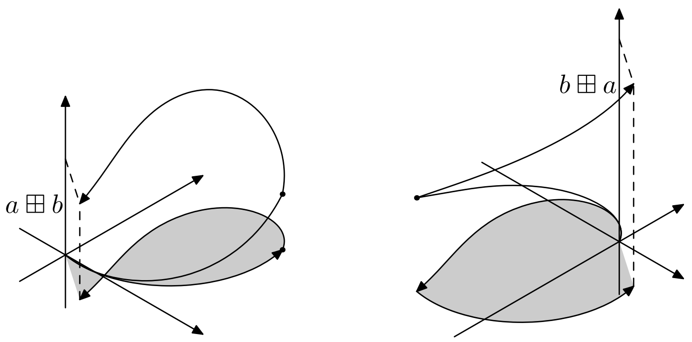
As illustrated in Figure 7, this gives rise to a different path as the one obtained by the concatenation of $\mathfrak{C}(x, 0)$ and $\mathfrak{C}(\pi(a) + y; a^3)$, which ends at $a + b$.

5.4. **The space $\mathbb{R}^3$ as a non-commutative group.** We have now equipped $\mathbb{R}^3$ with an operation $\boxplus$, which is easily proved to be associative. When equipped with this operation $\boxplus$, we denote $\mathbb{R}^3$ by $\mathrm{A}(\mathbb{R}^2)$. We also set

$$[a, b] = a \boxplus b - b \boxplus a = \left(0, 0, \frac{1}{2} \begin{bmatrix} a^1 \\ a^2 \end{bmatrix} \wedge \begin{bmatrix} b^1 \\ b^2 \end{bmatrix} \right).$$

This bracket $[\cdot, \cdot]$ is our course linked to the fact that $(\mathrm{A}(\mathbb{R}^2), \boxplus)$ is a non-commutative group, called the *Heisenberg group* (see Section 6.3).

Lemma 3. *The space* $(\mathrm{A}(\mathbb{R}^2), \boxplus)$ *is a* non-commutative group *with* 0 *as the neutral element. The inverse of any element* $a = (a^1, a^2, a^3)$ *is* $-a = (-a^1, -a^2, -a^3)$.



(a) The path $\mathfrak{C}(x \cdot y, 0)$. (b) The path $\mathfrak{C}(y \cdot x, 0)$.

FIGURE 8. Illustration of the non-commutativity of $\mathrm{A}(\mathbb{R}^2)$.

Proof. That the inverse of a is $-a$ is easily verified since

$$(-a^1, -a^2, -a^3) \boxplus (a^1, a^2, a^3) = -\frac{1}{2}[a, a] = 0.$$

The non-commutativity of $\boxplus$ in general follows from $b \boxplus a = a \boxplus b \boxplus [b, a]$. □

The non-commutativity of $\boxplus$ is illustrated in Figure 8. Of course, if $a, b \in \mathbb{R}^3$ are of type $a = (a^1, a^2, 0)$ and $b = (b^1, b^2, 0)$, then $a \boxplus b = b \boxplus a$: the non-commutativity concerns only the third component. If $x : [0, 1] \to \mathbb{R}^2$ goes from a to b and $y : [0, 1] \to \mathbb{R}^2$ goes from b to c, then $x \cdot y$ goes from a to c and $(y - b + a) \cdot (b - a + x)$ goes from also from a to c. Yet the area enclosed between these two paths and its chord is not the same.

It is now easy to remark that $\mathrm{A}(\mathbb{R}^2)$ is both a Lie algebra and a Lie group. For some introduction on these notion, see among many other books [War83, Var84, SW93, DK00, Hal03].

Lemma 4. *The space* $(A(\mathbb{R}^2), [\cdot, \cdot])$ *is a* Lie algebra.

Proof. Clearly, $(a, b) \mapsto [a, b]$ is bilinear, $[a, b] = -[b, a]$ and the Jacobi identity is easily satisfied:

$$[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0, \ \forall a, b, c \in \mathrm{A}(\mathbb{R}^2).$$

This proves the Lemma. □

As for $\mathbb{R}^3$, $\mathrm{A}(\mathbb{R}^2)$ may be equipped with the multiplication by a scalar, which is $(\lambda, x) = \lambda \cdot x \stackrel{\text{def}}{=} (\lambda x^1, \lambda x^2, \lambda x^3)$ if $x = (x^1, x^2, x^3) \in \mathrm{A}(\mathbb{R}^2)$ and $\lambda \in \mathbb{R}$. But unlike $\mathbb{R}^3$, this operation is not distributive, since

$$\lambda \cdot (x \boxplus y) = (\lambda x) \boxplus (\lambda y) + \lambda(1 - \lambda)[x, y].$$

Thus, $(\mathrm{A}(\mathbb{R}^2), \boxplus, \cdot)$, where $\cdot$ denotes the multiplication by a scalar, is not a module.

Another natural external law equip naturally $\mathrm{A}(\mathbb{R}^2)$, which is the *dilatation*. Given $\lambda \in \mathbb{R}$, we set

$$\delta_\lambda x = (\lambda x^1, \lambda x^2, \lambda^2 x^3) \text{ for } x = (x^1, x^2, x^3) \in \mathrm{A}(\mathbb{R}^2). \tag{14}$$

Let us note that

$$\delta_\lambda(x \boxplus y) = (\delta_\lambda x) \boxplus \delta_\lambda y \text{ and } \delta_\lambda \delta_\mu x = \delta_{\lambda\mu} x$$

for $\lambda, \mu \in \mathbb{R}$ and $x \in \mathrm{A}(\mathbb{R}^2)$. However, we do not have that $\delta_{\lambda+\mu} x = \delta_\lambda x \boxplus \delta_\mu x$. Hence, $(\mathrm{A}(\mathbb{R}^2), \boxplus, \delta)$ is not a module.

This space $\mathrm{A}(\mathbb{R}^2)$ is equipped with a norm defined by

$$|a|_\star = \max\{|a^1|, |a^2|, |a^3|\} \tag{15}$$

and a *homogeneous norm* defined by

$$|a| = \max\left\{|a^1|, |a^2|, \sqrt{\frac{1}{2}|a^3|}\right\}, \tag{16}$$

which means that $|a| = 0$ if and only if $a = 0$,

$$|\delta_\lambda x| = |\lambda| \cdot |x| \text{ for } \lambda \in \mathbb{R} \text{ and } x \in \mathrm{A}(\mathbb{R}^2),$$

and $|-x| = |x|$ for all $x \in \mathrm{A}(\mathbb{R}^2)$ (see also Section A).

Remark that this choice ensures that $|a \boxplus b| \leq 3/2(|a| + |b|)$. We will see below in Sections 5.9 and A that this homogeneous norm is equivalent to another homogeneous norm $\|\cdot\|_{\mathrm{CC}}$ which allows us to define a distance between two points a and b in $\mathrm{A}(\mathbb{R}^2)$ by $\|(-a) \boxplus b\|_{\mathrm{CC}}$ (with the $\|\cdot\|_{\mathrm{CC}}$, the triangular inequality is satisfied, which is not the case with $|\cdot|$). Because of the square root in the definition of $|\cdot|$, this distance is not equivalent to the one generated by $|\cdot|_\star$. Yet it generates the same topology.

Remark 5. Because $|\cdot|$ does not satisfy the triangle inequality, $d : (a, b) \mapsto |(-a) \boxplus b|$ does not define a distance. However, this may be called a *near-metric* because $d(a, b) \leq C(d(a, c) + d(c, b))$ for some constant $C > 0$ and all $a, c, b \in \mathrm{A}(\mathbb{R}^2)$.

From this, we easily deduce that $\mathrm{A}(\mathbb{R}^2)$ is also a Lie group. We recall that a *Lie group* $(G, \times)$ is a group with a differentiable manifold structure (and in particular a norm) such that $(x, y) \mapsto x \times y$ and $x \mapsto x^{-1}$ are continuous (see for example [SW93, War83, Var84, Hal03] and many other books).

Lemma 5. *The space* $(\mathrm{A}(\mathbb{R}^2), \boxplus)$ *is a* Lie group.

Proof. The continuity of $(x, y) \mapsto x \boxplus y$ and $x \mapsto -x$ is easily proved. □

5.5. **Enhanced paths and their approximations.** Of course, we have constructed the space $\mathrm{A}(\mathbb{R}^2)$ with the idea of considering paths living in $\mathrm{A}(\mathbb{R}^2)$, the third component giving all the information we need.

Basically, a continuous path with values in $\mathrm{A}(\mathbb{R}^2)$ is a continuous path with values in the Euclidean space $\mathbb{R}^3$ (recall that the norm $|\cdot|_\star$ we put on $\mathrm{A}(\mathbb{R}^2)$ is equivalent to the Euclidean norm). However, we will use the group operation $\boxplus$ of $\mathrm{A}(\mathbb{R}^2)$ in replacement as the translation by a vector in $\mathbb{R}^3$, and thus the paths we consider will be seen differently from the usual paths.

Let us recall that $(\mathbb{R}^2,+)$ is in some sense contained in $(\mathrm{A}(\mathbb{R}^2),\boxplus)$, and plays then a special role.

Definition 3. Given a continuous path x with values in $\mathbb{R}^2$, a continuous path $\mathbf{x}$ with values in $\mathrm{A}(\mathbb{R}^2)$ with $x=(\mathbf{x}^1,\mathbf{x}^2)$ may then be called an *enhanced path*, or a *path lying above* x. Given a path $x:[0,T]\to\mathbb{R}^2$, a path $\mathbf{x}:[0,T]\to\mathrm{A}(\mathbb{R}^2)$ with lies above x is called a *lift of* x.

Let x and y be two smooth paths lifted as $\mathbf{x}=\mathfrak{C}(x,0)$ and $\mathbf{y}=\mathfrak{C}(y,0)$, where $\mathfrak{C}$ has been defined by (13). We have seen that the usual concatenation $\mathbf{x}\cdot\mathbf{y}$ of $\mathbf{x}$ and $\mathbf{y}$ seen as paths with values in $\mathbb{R}^3$ is different from the path $\mathfrak{C}(x\cdot y,0)$. We introduce then a new kind of concatenation of two paths $\mathbf{x}:[0,T]\to\mathrm{A}(\mathbb{R}^2)$ and $\mathbf{y}:[0,S]\to\mathrm{A}(\mathbb{R}^2)$. This concatenation is defined by

$$(\mathbf{x}\boxdot\mathbf{y})_t=\begin{cases}\mathbf{x}_t & \text{if } t\in[0,T],\\ \mathbf{x}_T\boxplus((-\mathbf{y}_0)\boxplus\mathbf{y}_{t-T}) & \text{if } t\in[T,S+T]\end{cases}$$

and gives rise to a continuous path from $[0,T+S]$ to $\mathrm{A}(\mathbb{R}^2)$ when $\mathbf{x}$ and $\mathbf{y}$ are continuous. In addition, $\mathbf{x}\boxdot\mathbf{y}$ lies above $x\cdot y$ if $\mathbf{x}$ (resp. $\mathbf{y}$) lies above x (resp. y). Yet we have to be warned of an important that this concatenation is different from the usual concatenation in $\mathbb{R}^3$.

If $x:[0,T]\to\mathbb{R}^2$ and $y:[0,S]\to\mathbb{R}^2$ are two piecewise smooth paths, then this concatenation satisfies

$$\mathfrak{C}(x\cdot y,0)=\mathfrak{C}(x,0)\boxdot\mathfrak{C}(y,0).$$

For two points a and b in $\mathrm{A}(\mathbb{R}^2)$, let $\psi_{a,b}\in\mathrm{C}^1_{\mathrm{p}}([0,1];\mathbb{R}^3)$ be a smooth path joining a and b lying above $\zeta_{a,b}:[0,1]\to\mathbb{R}^2$ (for example, we can use the one of Section 5.3). By definition of $\zeta_{a,b}$ and $\psi_{a,b}$, $\psi_{a,b}(t)=\mathfrak{C}(\zeta_{a,b},a^3;t)$. Moreover, for a,b,c in $\mathrm{A}(\mathbb{R}^2)$,

$$\psi_{a,b}\boxdot\psi_{b,c}=\mathfrak{C}(\zeta_{a,b}\cdot\zeta_{b,c},a^3).$$

Thus, $\psi_{a,b}\boxdot\psi_{b,c}$ is a path that goes from a to c by passing through b.

Let $\mathbf{x}$ be a continuous path from $[0,T]$ living in $\mathrm{A}(\mathbb{R}^2)$. It is then natural to look for an approximation of $\mathbf{x}$ given by the sequence a paths

$$\mathbf{x}^n=\psi_{\mathbf{x}_{t_0^n},\mathbf{x}_{t_1^n}}\boxdot\psi_{\mathbf{x}_{t_1^n},\mathbf{x}_{t_2^n}}\boxdot\cdots\boxdot\psi_{\mathbf{x}_{t_{n-1}^n},\mathbf{x}_{t_n^n}}.$$

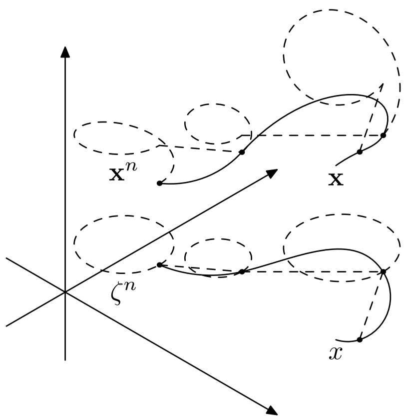


FIGURE 9. Approximation of a path $\mathbf{x}$ in $\mathrm{A}(\mathbb{R}^2)$.

The path $\mathbf{x}^n$ satisfies $\mathbf{x}^n(t) = \mathbf{x}(t)$ for the dyadics times t at level n. In addition,

$$\mathbf{x}^n = \mathfrak{C}(\zeta^n, \mathbf{x}^3_0) \text{ with } \zeta^n = \zeta_{\mathbf{x}_{t_0^n}, \mathbf{x}_{t_1^n}} \cdot \zeta_{\mathbf{x}_{t_1^n}, \mathbf{x}_{t_2^n}} \cdot \dots \cdot \zeta_{\mathbf{x}_{t_{n-1}^n}, \mathbf{x}_{t_n^n}},$$

and it is easily proved that ζ^n converges uniformly to x, the path above which $\mathbf{x}$ lives (See Figure 9).

Now, there are two natural questions: (1) Provided that $\mathbf{x}$ is regular enough, does $\mathbf{x}^n$ converge to $\mathbf{x}$, in which sense? (2) Is it possible to construct $\mathfrak{I}(\mathbf{x})$ as the limit of the $\mathfrak{I}(\zeta^n)$'s, which are then ordinary integrals?

5.6. **Hölder continuous enhanced paths.** We have defined the space $\mathrm{A}(\mathbb{R}^2)$ as the space $\mathbb{R}^3$ with a special non-commutative group structures, which is different from the translation.

Let $x \in \mathrm{C}^\alpha([0,T];\mathbb{R}^2)$ with $\alpha > 1/2$ and $x_0 = 0$. Set $\mathbf{x} = (x^1, x^2, \mathfrak{A}(x))$. With (12),

$$(-\mathbf{x}_s) \boxplus \mathbf{x}_t = (x^1_t - x^1_s, x^2_t - x^2_s, \mathfrak{A}(x; s, t)),$$

which means that $(-\mathbf{x}_s) \boxplus \mathbf{x}_t$ can be constructed from the path x restricted to $[s,t]$. The same is true even if $x_0 \neq 0$.

For a path $\mathbf{x}$ from $[0,T]$ to $\mathrm{A}(\mathbb{R}^2)$, $\mathbf{x}_{s,t} \stackrel{\text{def}}{=} (-\mathbf{x}_s) \boxplus \mathbf{x}_t$ may then be interpreted as an "increment" of $\mathbf{x}$, and we indeed get the following trivial identity $\mathbf{x}_t = \mathbf{x}_s \boxplus \mathbf{x}_{s,t}$ for all $0 \leq s \leq t \leq T$, which is the equivalent to $x_t = x_s + (x_t - x_s)$ in $\mathbb{R}^2$. Let us note that in general $\mathbf{x}^3_{s,t}$ is different from $\mathbf{x}^3_t - \mathbf{x}^3_t$, although $\mathbf{x}^i_{s,t} = \mathbf{x}^i_t - \mathbf{x}^i_s$ for $i = 1, 2$.

Similarly, we may write the value of $\mathbf{x}_t$ at time t in function the values of $\mathbf{x}$ at times $s \leq r \leq t$:

$$\mathbf{x}_t = \mathbf{x}_s \boxplus \mathbf{x}_{s,r} \boxplus \mathbf{x}_{r,t} \qquad (17)$$

for all $0 \leq s \leq r \leq t \leq T$. When one sees $\mathbf{x}$ as a geometric object, (17) yields

$$\mathbf{x}_{|[s,t]} = \mathbf{x}_{|[s,r]} \boxdot \mathbf{x}_{|[r,t]}, \tag{18}$$

for all $0 \leq s \leq r \leq t \leq T$.

From now, to endow that fact that we work in $\mathrm{A}(\mathbb{R}^2)$, we have to think to paths from $[0, T]$ to $\mathrm{A}(\mathbb{R}^2)$ as continuous paths $\mathbf{x}$ satisfying (18), although this relation is verified by any continuous path from $[0, T]$ to $\mathbb{R}^3$ (which means also that there are an infinite number of paths lying above a continuous path from $[0, T]$ to $\mathbb{R}^2$). But we will see below that if $\mathbf{x}$ lies above a smooth path x and is also quite regular (in a sense to be defined), then (18) and the regularity condition will impose some "constraint" on the path $\mathbf{x}$.

Lemma 6 ([Lyo98, Lemma 2.2.3, p. 250])**.** *Let $\mathbf{x}$ and $\mathbf{y}$ be two continuous paths from $[0, T]$ to $\mathrm{A}(\mathbb{R}^2)$ such that $(\mathbf{x}^1, \mathbf{x}^2) = (\mathbf{y}^1, \mathbf{y}^2)$. Then there exists a continuous path $\varphi : [0, T] \to \mathbb{R}$ such that $\mathbf{y} = (\mathbf{x}^1, \mathbf{x}^2, \mathbf{x}^3 + \varphi)$, which means that*

$$((-\mathbf{y}_s) \boxplus \mathbf{y}_t)^3 = ((-\mathbf{x}_s) \boxplus \mathbf{x}_t)^3 + \varphi_t - \varphi_s \tag{19}$$

for all $0 \leq s \leq t \leq T$.

Proof. It is sufficient to set $\varphi_t = ((-\mathbf{y}_0) \boxplus \mathbf{y}_t)^3 - ((-\mathbf{x}_0) \boxplus \mathbf{x}_t)^3$, which clearly satisfies (19). □

Notation 3. We denote by $\mathrm{C}^\alpha([0, T]; \mathrm{A}(\mathbb{R}^2))$ the set of continuous paths $\mathbf{x} : [0, T] \to \mathrm{A}(\mathbb{R}^2)$ and such that

$$\|\mathbf{x}\|_\alpha = |\mathbf{x}_0| + \sup_{0 \leq s < t \leq T} \frac{|(-\mathbf{x}_s) \boxplus \mathbf{x}_t|}{|t - s|^\alpha}$$

is finite. If $x = (x^1, x^2)$ is a path in $\mathrm{C}^\alpha([0, T]; \mathbb{R}^2)$ and $\mathbf{x} = (x^1, x^2, y)$ a path in $\mathrm{C}^\alpha([0, T]; \mathrm{A}(\mathbb{R}^2))$, then we say that $\mathbf{x}$ *lies above* x.

Lemma 7. *Let $x \in \mathrm{C}^\alpha([0, T]; \mathbb{R}^2)$ with $\alpha > 1/2$. Then $\mathbf{x} = (x, \mathfrak{A}(x; 0, \cdot))$ belongs to $\mathrm{C}^\alpha([0, T]; \mathrm{A}(\mathbb{R}^2))$. In addition the map $x \mapsto \mathbf{x}$ is Lipschitz continuous from $(\mathrm{C}^\alpha([0, T]; \mathbb{R}^2), \| \cdot \|_\alpha)$ to $(\mathrm{C}^\alpha([0, T]; \mathrm{A}(\mathbb{R}^2)), \| \cdot \|_\alpha)$.*

Proof. By construction, $\mathbf{x}$ is a path with value in $\mathrm{A}(\mathbb{R}^2)$. Let us note that $(-\mathbf{x}_s) \boxplus \mathbf{x}_t = (x^1_t - x^1_s, x^2_t - x^2_s, \mathfrak{A}(x; s, t))$. From the construction of the Young integral (more specifically, from a variation of (4)),

$$|\mathfrak{A}(x; s, t)| \leq \zeta(2\alpha - 1)(t - s)^{2\alpha} \|x\|^2_\alpha \tag{20}$$

and then the result is proved. □

Let us note that in the previous proof, (20) does not means that $t \mapsto \mathfrak{A}(x; 0, t)$ is 2α-Hölder continuous (in which case $2\alpha > 1$!). Indeed, $t \mapsto \mathfrak{A}(x; 0, t)$ is only α-Hölder continuous, since x is α-Hölder continuous.

On the other hand, any path in $\mathrm{C}^\alpha([0, T]; \mathrm{A}(\mathbb{R}^2))$ with $\alpha > 1/2$ can be expressed as a path $x \in \mathrm{C}^\alpha([0, T]; \mathbb{R}^2)$ lifted using its area $\mathfrak{A}(x)$.

Lemma 8. *Let* $\mathbf{x} \in \mathrm{C}^\alpha([0,T];\mathrm{A}(\mathbb{R}^2))$ *with* $\alpha > 1/2$. *Then* $\mathbf{x} = \mathfrak{C}(x, \mathbf{x}_0^3) = (x, \mathbf{x}_0^3 + \mathfrak{A}(x))$ *with* $x = (\mathbf{x}^1, \mathbf{x}^2)$.

Remark 6. If for some $\alpha > 1/2$, $(\mathbf{x}^n)_{n\in\mathbb{N}}$ belongs to $\mathrm{C}^\alpha([0,T];\mathrm{A}(\mathbb{R}^2))$ is composed of paths of type $\mathbf{x}^n = (x^n, \mathfrak{A}(x^n))$ with $x^n \in \mathrm{C}^\alpha([0,T];\mathbb{R}^2)$ and $\mathbf{x}^n$ converges in $\mathrm{C}^\alpha([0,T];\mathrm{A}(\mathbb{R}^2))$ to some $\mathbf{x}$, then $\mathbf{x} \in \mathrm{C}^\alpha([0,T];\mathrm{A}(\mathbb{R}))$ is necessarily of type $\mathbf{x} = (x, \mathfrak{A}(x))$ with $x \in \mathrm{C}^\alpha([0,T];\mathbb{R}^2)$. In Proposition 2 below, we will see how to construct a family of paths x^n in $\mathrm{C}^1([0,T];\mathbb{R}^2)$ for which $\mathbf{x}^n = (x, \mathfrak{A}(x))$ converges to $\mathbf{x} \in \mathrm{C}^\alpha([0,T];\mathrm{A}(\mathbb{R}^2))$ with $\alpha > 1/3$. Thus, if one considers a path with values in $\mathrm{A}(\mathbb{R}^2)$ which is not of type $(x, \mathfrak{A}(x))$ but which is piecewise smooth, one has to interpret it as a path in $\mathrm{C}^{1/2}([0,T];\mathrm{A}(\mathbb{R}^2))$ in order to identify it with a family of converging paths.

Proof. From Lemma 7, $\mathbf{y} = \mathfrak{C}(x, \mathbf{x}_0^3)$ belongs to $\mathrm{C}^\alpha([0,T];\mathrm{A}(\mathbb{R}^2))$, and from Lemma 6, there exists a function $\varphi : [0,T] \to \mathbb{R}$ such that $((-\mathbf{x}_s) \boxplus \mathbf{x}_t)^3 = ((-\mathbf{y}_s) \boxplus \mathbf{y}_t)^3 + \varphi_t - \varphi_s$ for all $0 \leq s \leq t \leq T$. Hence, $\sqrt{|\varphi_t - \varphi_s|} \leq \|\mathbf{x}\|_\alpha |t-s|^\alpha$ and then $|\varphi_t - \varphi_s| \leq \|\mathbf{x}\|_\alpha^2 |t-s|^{2\alpha}$. As $\alpha > 1/2$, necessarily φ is constant. □

As we saw it, one can add a path with values in $\mathbb{R}$ to the third component of a path with values in $\mathrm{A}(\mathbb{R}^2)$ to get a new path with values in $\mathrm{A}(\mathbb{R}^2)$. Although a path with values in $\mathbb{R}^2$ which is regular enough can be naturally lifted as a path with values in $\mathbb{R}^3$, we gain one degree of freedom: there are an infinite number of paths that lie above a path in $\mathbb{R}^2$. The next lemma, whose proof is immediate, precises the kind of paths we have to use to stay in $\mathrm{C}^\alpha([0,T];\mathrm{A}(\mathbb{R}^2))$.

Lemma 9. *For* $\alpha \leq 1/2$, *let* $\mathbf{x} \in \mathrm{C}^\alpha([0,T];\mathrm{A}(\mathbb{R}^2))$ *and* $\varphi \in \mathrm{C}^{2\alpha}([0,T];\mathbb{R})$. *Then* $\mathbf{y} = (\mathbf{x}^1, \mathbf{x}^2, \mathbf{x}^3 + \varphi)$ *belongs to* $\mathrm{C}^\alpha([0,T];\mathrm{A}(\mathbb{R}^2))$.

Any path in $\mathrm{C}^\alpha([0,T];\mathrm{A}(\mathbb{R}^2))$ can be seen as the limit of paths naturally constructed above path of finite variation. Before proving this, we state a lemma on relative compactness, which is just an adaptation of Lemma 2.

Lemma 10. *Let* $(\mathbf{x}^n)_{n\in\mathbb{N}}$ *such that* $\mathbf{x}^n \in \mathrm{C}^\alpha([0,T];\mathrm{A}(\mathbb{R}^2))$ *and is bounded. Then there exists* $\mathbf{x}$ *in* $\mathrm{C}^\alpha([0,T];\mathrm{A}(\mathbb{R}^2))$ *and a subsequence of* $(\mathbf{x}^n)_{n\in\mathbb{N}}$ *that converges to* $\mathbf{x}$ *in* $(\mathrm{C}^\alpha([0,T];\mathrm{A}(\mathbb{R}^2)), \|\cdot\|_\beta)$ *for each* $\beta < \alpha$.

We shall now prove the main result of this section: any path $\mathbf{x}$ in $\mathrm{C}^\alpha([0,T];\mathrm{A}(\mathbb{R}^2))$ with $\alpha \in (1/3, 1/2)$ may be identified as the limit of $\mathfrak{C}(x^n, \mathbf{x}_0^3)$, where x^n are paths in $\mathrm{C}_\mathrm{p}^\infty([0,T];\mathbb{R}^2)$. Paths taking their values in $\mathrm{A}(\mathbb{R}^2)$ are then objects that are easier to deal with than sequences of paths with loops as we did previously.

Let $\mathbf{x} \in \mathrm{C}^\alpha([0,T];\mathrm{A}(\mathbb{R}^2))$ with $\alpha \in (1/3, 1/2)$ lying above x. Denote by x^{Π^n} the linear interpolation of x along the dyadic partition $\Pi^n = \{t_k^n\}_{k=0,\dots,2^n}$ at level n, with $t_k^n = Tk/2^n$. Also define

$$\theta_k^n = ((-\mathbf{x}_{t_{k+1}^n}) \boxplus \mathbf{x}_{t_k^n})^3. \tag{21a}$$

Set $\Phi^n = \{y^n_k\}_{k=0,\dots,2^n-1}$ with $y^n_k : [t^n_k, t^n_{k+1}] \to \mathbb{R}^2$ and

(21b) $$y^n_k(t) = \sqrt{\frac{|\theta^n_k|}{\pi}} \begin{bmatrix} \cos\left(2\pi \frac{t-t^n_k}{t^n_{k+1}-t^n_k}\right) - 1 \\ \operatorname{sgn}(\theta^n_k)\sin\left(2\pi\frac{t-t^n_k}{t^n_{k+1}-t^n_k}\right)\end{bmatrix}.$$

Finally, set

(21c)
$$x^n_t = x^{\Pi^n} \bowtie \Phi^n(t/2) \text{ for } t \in [0,T] \text{ and } \mathbf{x}^n = (x^n, \mathbf{x}^3_0 + \mathfrak{A}(x^n; 0, \cdot)).$$

This corresponds to join the points of $\{\mathbf{x}_{t^n_k}\}_{k=0,\dots,2^n}$ by the simple paths constructed in Section 5.3 (see Figure 6).

Proposition 2. *With the previous notations* (21a)-(21c)*,* $(\mathbf{x}^n)_{n\in\mathbb{N}}$ *is uniformly bounded in* $\mathrm{C}^\alpha([0,T];\mathrm{A}(\mathbb{R}^2))$ *and converges to* $\mathbf{x}$ *with respect to* $\|\cdot\|_\beta$ *for all* $\beta < \alpha$*.*

Remark 7. We have considered a path $\mathbf{x}$ in $\mathrm{C}^\alpha([0,T];\mathrm{A}(\mathbb{R}^2))$ above a path $x \in \mathrm{C}^\alpha([0,T];\mathbb{R}^2)$, but we have not shown how to construct such a path, except of $\alpha > 1/2$. For that, we may either use the results in [LV07], that assert it is always possible to do so, or to deal with particular cases. For example, many trajectories of stochastic processes have been dealt with (Brownian motion [Sip93], semi-martingales [CL05], fractional Brownian motion [CQ02, MSS05, Cou07], Wiener process [LLQ02], Gaussian processes [FV07b, FV07a], free Brownian motion [Vic04], ... The book [FV06a] contains a large number of these constructions). All these results are done in general in connection with an approximation results of Wong-Zakai type.

Choosing a path $\mathbf{x}$ above x corresponds to a determination of the limit of $\mathfrak{A}(x^n; s,t)$ where x^n converges to x, and is then a slightly weaker hypothesis than (H2).

Proof of Proposition 2. Let us note first that for $t = t^n_k$, $\mathbf{x}^n_{t^n_k} = \mathbf{x}_{t^n_k}$. For $t \in [0,T)$, let $\underline{M}(t,n)$ be the biggest integer such that $t^n_{\underline{M}(t,n)} \leq t$. Then, for $0 \leq t < T$,

$$\begin{aligned}|\mathbf{x}^n_t - \mathbf{x}_t| &\leq |\mathbf{x}^n_t - \mathbf{x}^n_{t^n_{\underline{M}(t,n)}}| + |\mathbf{x}_t - \mathbf{x}_{t^n_{\underline{M}(t,n)}}| \\ &\leq \max\{\sqrt{|\theta^n_k|/\pi}, |\mathbf{x}_{t^n_{\underline{M}(t,n)+1}} - \mathbf{x}_{t^n_{\underline{M}(t,n)}}|\} + \|\mathbf{x}\|_\alpha (t - t^n_{\underline{M}(t,n)})^\alpha \\ &\leq 2\|\mathbf{x}\|_\alpha T^\alpha 2^{-\alpha n}.\end{aligned}$$

This prove that $\mathbf{x}^n$ converges uniformly to $\mathbf{x}$.

The convergence in $\mathrm{C}^\beta([0,T];\mathrm{A}(\mathbb{R}^2))$ follows from the uniform boundedness of the α-Hölder norm of $\mathbf{x}^n$ and Lemma 10.

So, it remains to estimate the α-Hölder norm of $\mathbf{x}^n$ in $\mathrm{A}(\mathbb{R}^2)$. For $0 \leq s < t \leq T$, let $\overline{M}(s,n)$ be the smallest integer such that $s \leq t^n_{\overline{M}(s,n)}$. Then, unless s,t belongs to the same dyadic interval $[t^n_k, t^n_{k+1}]$ for some

$k = 0, \dots, 2^n - 1$,

$$\mathbf{x}^n_{s,t} = \mathbf{x}^n_{s,t^n_{\overline{M}(s,n)}} \boxplus \mathbf{x}^n_{t^n_{\overline{M}(s,n)},t^n_{\underline{M}(t,n)}} \boxplus \mathbf{x}^n_{t^n_{\underline{M}(t,n)},t}.$$

for all $0 \leq s < t \leq T$. In addition, $\mathbf{x}^n_{t^n_{\overline{M}(s,n)},t^n_{\underline{M}(t,n)}} = \mathbf{x}_{t^n_{\overline{M}(s,n)},t^n_{\underline{M}(t,n)}}$ for any integer n. Since $|\cdot|$ is a homogeneous norm on $\mathrm{A}(\mathbb{R}^2)$, it follows that for some universal constant C_0,

$$\begin{aligned}|\mathbf{x}^n_{s,t}| &\leq C_0|\mathbf{x}^n_{s,t^n_{\overline{M}(s,n)}}| + C_0|\mathbf{x}_{t^n_{\overline{M}(s,n)},t^n_{\underline{M}(t,n)}}| + C_0|\mathbf{x}^n_{t^n_{\underline{M}(t,n)},t}| \\ &\leq C_0|\mathbf{x}^n_{s,t^n_{\overline{M}(s,n)}}| + C_0\|\mathbf{x}\|_\alpha(t^n_{\underline{M}(t,n)} - t^n_{\overline{M}(s,n)})^\alpha + C_0|\mathbf{x}^n_{t^n_{\underline{M}(t,n)},t}|.\end{aligned}$$

Let us assume that we have proved that for some constant K,

$$(22) \quad |\mathbf{x}^n_{s,t}| \leq K(t-s)^\alpha \text{ for all } t^n_k \leq s \leq t \leq t^n_{k+1},\ k = 0, \dots, 2^n - 1,$$

then the boundedness of $(\|\mathbf{x}^n\|_\alpha)_{n\in\mathbb{N}}$ follows easily as in the proof of Corollary 1 by applying (22) to s, t in the same dyadic interval, and to $|\mathbf{x}^n_{s,t^n_{\overline{M}(s,n)}}|$ as well as to $|\mathbf{x}^n_{t^n_{\underline{M}(t,n)},t}|$.

We now turn to the proof of (22). First, let us consider that for some $k \in \{0, \dots, 2^n - 1\}$, either $s, t \in [t^n_k, t^n_k - T2^{-n-1}]$ or $s, t \in [t^n_k + T2^{-n-1}, t^n_k]$. In the latter case,

$$\mathbf{x}^n_{s,t} \stackrel{\text{def}}{=} (-\mathbf{x}^n_s) \boxplus \mathbf{x}^n_t = \begin{bmatrix} T^{-1}2^{n+1}(t-s)(x^1_{t^n_{k+1}} - x^2_{t^n_k}) \\ T^{-1}2^{n+1}(t-s)(x^1_{t^n_{k+1}} - x^2_{t^n_k}) \\ 0 \end{bmatrix}$$

and then $|\mathbf{x}^n_{s,t}| \leq \|\mathbf{x}\|_\alpha|t-s|^\alpha$. In the former case, setting $\Delta_n t = T2^{-n}$,

$$\mathbf{x}^n_{s,t} = \begin{bmatrix} \sqrt{\frac{|\theta^n_k|}{\pi}}\left(\cos\left(\frac{\pi}{\Delta_{n+1}t}(t-t^n_k)\right) - \cos\left(\frac{\pi}{\Delta_{n+1}t}(s-t^n_k)\right)\right) \\ \operatorname{sgn}(\theta^n_k)\sqrt{\frac{|\theta^n_k|}{\pi}}\left(\sin\left(\frac{\pi}{\Delta_{n+1}t}(t-t^n_k)\right) - \sin\left(\frac{\pi}{\Delta_{n+1}t}(s-t^n_k)\right)\right) \\ \theta^n_k\frac{t-s}{\Delta_{n+1}t} \end{bmatrix}.$$

Thus, for some universal constant C_1,

$$|\mathbf{x}^n_{s,t}| \leq C_1 2^{n+1}\sqrt{|\theta^n_k|}\frac{t-s}{T} \leq 2C_1 2^{n(1-\alpha)}\|\mathbf{x}\|_\alpha\frac{t-s}{T} \leq C_2\|\mathbf{x}\|_\alpha(t-s)^\alpha,$$

where C_2 depends only on C_1 and T.

Now, if $t^n_k \leq s \leq t^n_k + T2^{-n-1} \leq t \leq t^n_{k+1}$, we get by combining the previous estimates that

$$\begin{aligned}|\mathbf{x}^n_{s,t}| \leq C_0C_2\|\mathbf{x}\|_\alpha((t - T2^{-n-1})^\alpha + (T2^{-n-1} - s)^\alpha) \\ \leq 2^{\alpha-1}C_0C_2\|\mathbf{x}\|_\alpha(t-s)^\alpha.\end{aligned}$$

We have then proved (22) with a constant which is in addition proportional to $\|\mathbf{x}\|_\alpha$. □

Let us come back to the Remark 6 following Lemma 8. For $\alpha \in (1/3, 1/2]$, let us consider $\mathbf{x}_t = (0, 0, \varphi_t)$ where $\varphi \in \mathrm{C}^{2\alpha}([0,T];\mathbb{R})$, then one can find $x^n \in \mathrm{C}^1_{\mathrm{p}}([0,T];\mathbb{R})$ such that x^n converges uniformly to 0,

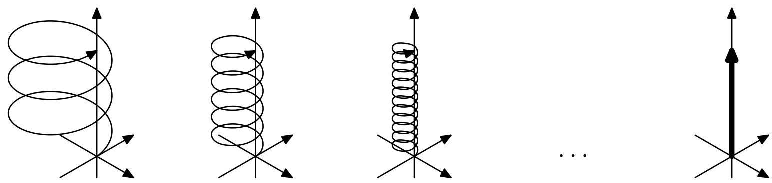


FIGURE 10. Moving freely in the third direction.

$\mathbf{x}^n = (x^n, \mathfrak{A}(x^n; 0, \cdot))$ is uniformly bounded in $\mathrm{C}^\alpha([0,T]; \mathrm{A}(\mathbb{R}^2))$ and converges in $\mathrm{C}^\beta([0,T]; \mathrm{A}(\mathbb{R}^2))$ to $\mathbf{x}$ for any $\beta < \alpha$. For this, one may simply consider (see Figure 10)

$$z_t^n = \frac{1}{n\sqrt{\pi}}(\cos(2\pi t n^2) - 1, \sin(2\pi t n^2)),$$

and then set $x_t^n = z_{\varphi_t}^n$.

Thus, moving freely in the "third direction" is equivalent to accumulate the areas of small loops. Using the language of differential geometry, which we develop below, this new degree of freedom comes from the lack of commutativity of $(\mathrm{A}(\mathbb{R}^2), \boxplus)$: a small loop of radius $\sqrt{\varepsilon}$ around the origin in the plane $\mathbb{R}^2$ is equivalent in some sense to a small displacement of length ε in the third direction. To rephrase Remark 6, even if $\varphi \in \mathrm{C}^1([0,T]; \mathbb{R})$, then one has to see $\mathbf{x}$ as a path in $\mathrm{C}^{1/2}([0,T]; \mathrm{A}(\mathbb{R}^2))$ that may be approximated by paths in $\mathrm{C}^1_{\mathrm{p}}([0,T]; \mathrm{A}(\mathbb{R}^2))$ (here, Lipschitz continuous paths with values in $\mathrm{A}(\mathbb{R}^2)$) which converge to $\mathbf{x}$ only in $\|\cdot\|_\beta$ for any $\beta < 1/2$. Hence, we recover the problem underlined in Section 3.2.

5.7. **Construction of the integral.** If $\mathbf{x} \in \mathrm{C}^\alpha([0,T]; \mathrm{A}(\mathbb{R}^2))$ with $\alpha > 1/2$, then from Lemma 8, $\mathbf{x} = (x, \mathbf{x}_0^3 + \mathfrak{A}(x))$ with $x = (\mathbf{x}^1, \mathbf{x}^2)$. For a differential form $f \in \mathrm{Lip}(\gamma; \mathbb{R}^2 \to \mathbb{R})$ with $\gamma > 1/\alpha - 1$, we set $\mathfrak{I}(\mathbf{x}) \stackrel{\text{def}}{=} \mathfrak{I}(x) = \int_{x_{|[0,\cdot]}} f$ which is well defined as a Young integral.

The next proposition will be refined later.

Proposition 3. *Let* $\mathbf{x} \in \mathrm{C}^\alpha([0,T]; \mathrm{A}(\mathbb{R}^2))$ *with* $\alpha \in (1/3, 1/2]$ *and* f *be a differential form in* $\mathrm{Lip}(\gamma; \mathbb{R}^2 \to \mathbb{R})$ *with* $\gamma > 1/\alpha - 1$. *Let* $(\mathbf{x}^n)_{n\in\mathbb{N}}$ *be constructed by* (21a)–(21c). *Then* $(\mathfrak{I}(x^n))_{n\in\mathbb{N}}$ *has a unique limit in* $(\mathrm{C}^\alpha([0,T]; \mathbb{R}), \|\cdot\|_\beta)$ *for all* $\beta < \alpha$, *which we denote by* $\mathfrak{I}(\mathbf{x})$ *(of course, the limit does not depend on* β*). Both the* α*-Hölder continuity modulus of* $\mathfrak{I}(\mathbf{x})$ *and the rate of convergence with respect to* $\|\cdot\|_\beta$ *depends only on* T, α, γ, β, $\|\mathbf{x}\|_\alpha$ *and* $\|f\|_{\mathrm{Lip}}$.

Other properties of this map $x \mapsto \mathfrak{I}(\mathbf{x})$ will be proved below. Indeed, this map is obviously an extension of the one we have constructed beforehand on $\mathrm{L}^{\bowtie,\alpha}([0,T]; \mathbb{R}^3)$, with a more convenient way to encode the loops.

Proof. Let us fix a dyadic level n. We remark first that for $k \in \{0,\dots,2^n-1\}$, $t_k^n \le s < t \le t_{k+1}^n$,

$$\mathfrak{I}(x^n;s,t) = \begin{cases} \iint_{\mathrm{Part}^n(s,t)} [f,f](z^1,z^2)\,\mathrm{d}z^1\,\mathrm{d}z^2 + \int_{\overline{x_s^n x_t^n}} f \\ \quad \text{if } t_k^n \le s \le t \le t_k^n + T2^{-n-1}, \\ \iint_{\mathrm{Part}^n(s,t_k^n+T2^{-n-1})} [f,f](z^1,z^2)\,\mathrm{d}z^1\,\mathrm{d}z^2 + \int_{\overline{x_s^n x_{t_k^n+T2^{-n-1}}^n}} f \\ \quad + \int_{t_k^n}^{t_k^n+2(t-t_k^n-2^{n+1}T)} f(x_r^{\Pi^n})\,\mathrm{d}x_r^{\Pi^n} \text{ if } t_k^n \le s \le t_k^n + T2^{-n-1} \le t \le t_{k+1}^n, \\ \int_{t_k^n+2(s-t_k^n-2^{n+1}T)}^{t_k^n+2(t-t_k^n-2^{n+1}T)} f(x_r^{\Pi^n})\,\mathrm{d}x_r^{\Pi^n} \text{ if } t_k^n + T2^{-n-1} \le s < t \le t_{k+1}^n, \end{cases}$$

where $\mathrm{Part}^n(s,t)$ is the portion of the disk enclosed between the loop $x^n_{|[t_k^n,t_k^n+T2^{-n-1}]}$ and the segment $\overline{x_s^n x_t^n}$. Of course, the integral of f over $\mathrm{Part}^n(t_k^n, t_k^n+T2^{-n-1})$ is the integral of $[f,f]$ over the surface of the loop $x^n_{|[t_k^n,t_k^n+T2^{-n-1}]}$.

If $t_k^n \le s < t \le t_k^n + T2^{-n-1}$, then the algebraic area of $\mathrm{Part}^n(s,t)$ is $\theta_k^n(t-s)2^{n+1}/T$. In addition, the maximal distance between two points in $\mathrm{Part}^n(s,t)$ smaller than $\sqrt{|\theta_k^n|}\sqrt{2}(t-s)2^{n+1}/T$. As $[f,f]$ is $(\gamma-1)$-Hölder continuous, we deduce that for $r \in [s,t]$, there exists a constant C that depends only on T such that

$$\left| \iint_{\mathrm{Part}^n(s,t)} [f,f](z^1,z^2)\,\mathrm{d}z^1\,\mathrm{d}z^2 - [f,f](x_s)\theta_k^n \frac{t-s}{T2^{-n-1}} \right| \le C\|f\|_{\mathrm{Lip}}\|\mathbf{x}\|_\alpha^{1+\gamma}(t-s)^{\alpha(1+\gamma)} \tag{23}$$

since $|\theta_k^n| \le \|\mathbf{x}\|_\alpha^2 2^{-2n\alpha}$. We also deduce that for some constant C' that depends only on T, $\|\mathbf{x}\|_\alpha$ and $\|f\|_{\mathrm{Lip}}$,

$$\left| \iint_{\mathrm{Part}^n(s,t)} [f,f](z^1,z^2)\,\mathrm{d}z^1\,\mathrm{d}z^2 \right| \le C'(t-s)^{2\alpha}. \tag{24}$$

In addition, since from Proposition 2, x^n is α-Hölder continuous with some constant that depends only on $\|\mathbf{x}\|_\alpha$, there exists some constant C'' such that

$$\left| \int_{\overline{x_s^n x_t^n}} f \right| \le \|f\|_\infty C''(t-s)^\alpha. \tag{25}$$

If $t^n_k + T2^{-n-1} \leq s < t \leq t^n_{k+1}$, then

$$\left|\int_{t^n_k+2(s-t^n_k-2^{n+1}T)}^{t^n_k+2(t-t^n_k-2^{n+1}T)} f(x^{\Pi^n}_r)\,\mathrm{d}x^{\Pi^n}_r\right| \leq \|f\|_{\mathrm{Lip}}\|\mathbf{x}\|_\alpha(T2^n)^{1-\alpha}(t-s) \leq \|f\|_{\mathrm{Lip}}\|\mathbf{x}\|_\alpha(t-s)^\alpha. \tag{26}$$

It follows from (24), (25) and (26) that for some constant C_1 that depends only on $\|f\|_{\mathrm{Lip}}$ and $\|\mathbf{x}\|_\alpha$,

$$|\mathfrak{I}(x^n;s,t)| \leq C_1(t-s)^\alpha \tag{27}$$

for all $t^n_k \leq s \leq t \leq t^n_{k+1}$, $k=0,\dots,2^n-1$.

Yet this is not sufficient to bound $|\mathfrak{I}(x^n;s,t)|$ by $C(t-s)^\alpha$ for all $0 \leq s < t \leq T$. We then use another computation.

Let us remark first that $t^{n+1}_{2k} = t^n_k$, $t^{n+1}_{2k+2} = t^n_{k+1}$ and that

$$\mathfrak{I}(x^{\Pi^{n+1}};t^{n+1}_{2k},t^{n+1}_{2k+1}) + \mathfrak{I}(x^{\Pi^{n+1}};t^{n+1}_{2k+1},t^{n+1}_{2k+2}) - \mathfrak{I}(x^{\Pi^n};t^{n+1}_{2k},t^{n+1}_{2k+2}) = \iint_{T^n_k}[f,f](z)\,\mathrm{d}z,$$

where T^n_k is Triangle$(x_{t^n_{2k}}, x_{t^n_{2k+1}}, x_{t^n_{2k+2}})$ with area

$$\mathrm{Area}(T^n_k) = -\frac{1}{2}(x_{t^{n+1}_{2k+1}} - x_{t^{n+1}_{2k}}) \wedge (x_{t^{n+1}_{2k+2}} - x_{t^{n+1}_{2k+1}}).$$

In addition,

$$\mathfrak{I}(x^n;t^n_k,t^n_k+T2^{-n-1}) = \iint_{\mathrm{Part}^n(t^n_k,t^n_k+T2^{-n-1})}[f,f](z^1,z^2)\,\mathrm{d}z^1\,\mathrm{d}z^2 = [f,f](x_{t^n_k})\theta^n_k + \zeta^n_k,$$

where, from (23), $|\zeta^n_k| \leq C_2 2^{-n\alpha(1+\gamma)}$ for some constant C_2 that depends only on $\|\mathbf{x}\|_\alpha$, $\|f\|_{\mathrm{Lip}}$ and T.

Let us recall that from (12),

$$\theta^{n+1}_{2k} + \theta^{n+1}_{2k+1} + \frac{1}{2}(x_{t^{n+1}_{2k+1}} - x_{t^{n+1}_{2k}}) \wedge (x_{t^{n+1}_{2k+2}} - x_{t^{n+1}_{2k+1}}) = \theta^n_k.$$

Hence, we get easily that

$$\begin{aligned}&\mathfrak{I}(x^{n+1};t^{n+1}_{2k},t^{n+1}_{2k+1}) + \mathfrak{I}(x^{n+1};t^{n+1}_{2k+1},t^{n+1}_{2k+2}) - \mathfrak{I}(x^n;t^{n+1}_{2k},t^{n+1}_{2k+2})\\ &\quad = \zeta^{n+1}_{2k} + \zeta^{n+1}_{2k+1} - \zeta^n_k + ([f,f](x_{t^{n+1}_{2k+1}}) - [f,f](x_{t^{n+1}_{2k}}))\theta^{n+1}_{2k+1} + \xi^n_k,\end{aligned}$$

where

$$\xi^n_k = \iint_{T^n_k}[f,f](z^1,z^2)\,\mathrm{d}z^1\,\mathrm{d}z^2 - [f,f](x_{t^{n+1}_{2k}})\,\mathrm{Area}(T^n_k).$$

As in (23),

$$|\xi^n_k| \leq \|f\|_{\mathrm{Lip}}\|\mathbf{x}\|^{1+\gamma}_\alpha \Delta_n t^{\alpha(\gamma+1)},$$

where $\Delta_n t = T2^{-n}$. Thus, for some constant C_3 that depends only on $\|f\|_{\mathrm{Lip}}$, $\|\mathbf{x}\|_\alpha$,

$$|\mathfrak{I}(x^{n+1}; t^{n+1}_{2k}, t^{n+1}_{2k+1}) + \mathfrak{I}(x^{n+1}; t^{n+1}_{2k+1}, t^{n+1}_{2k+2}) - \mathfrak{I}(x^n; t^{n+1}_{2k}, t^{n+1}_{2k+2})| \leq C_3 2^{-n\alpha(\gamma+1)}. \tag{28}$$

For $m \leq n$ and $k \in \{0, \dots, 2^m - 1\}$,

$$\mathfrak{I}(x^n; t^m_k, t^m_{k+1}) - \mathfrak{I}(x^m; t^m_k, t^m_{k+1}) = \sum_{\ell=m}^{n-1} (\mathfrak{I}(x^{\ell+1}; t^m_k, t^m_{k+1}) - \mathfrak{I}(x^\ell; t^m_k, t^m_{k+1})).$$

As there are exactly $2^{\ell-m}$ dyadics intervals of the form $[t^\ell_i, t^\ell_{i+1}]$ contained in $[t^m_k, t^m_{k+1}]$ for all $\ell \geq m$, we deduce from the Chasles relation and (28) that

$$|\mathfrak{I}(x^n; t^m_k, t^m_{k+1}) - \mathfrak{I}(x^m; t^m_k, t^m_{k+1})| \leq C_3 \sum_{\ell=m}^{n-1} \frac{2^{\ell-m}}{2^{\ell\alpha(\gamma+1)}} \leq \frac{C_4}{2^{m\alpha(\gamma+1)}}, \tag{29}$$

where C_4 depends on C_3 and the choice of α and γ (note that our choice of α and γ ensures that the involved series converges as $n \to \infty$).

We now choose for $m(0)$ the smallest integer such that there exists some $k \in \{0, \dots, 2^{m(0)} - 1\}$ for which $[t^{m(0)}_k, t^{m(0)}_{k+1}] \subset [t^n_{\overline{M}(s,n)}, t^n_{\underline{M}(t,n)}]$, where $\overline{M}(s,n)$ (resp. $\underline{M}(t,n)$) is the smallest (resp. the largest) integer such that $s \leq t^n_{\overline{M}(s,n)}$ (resp. $t \geq t_{\underline{M}(t,n)}$).

From the Chasles relation,

$$\mathfrak{I}(x^n; t^n_{\overline{M}(s,n)}, t^n_{\underline{M}(t,n)}) = \mathfrak{I}(x^n; t^n_{\overline{M}(s,n)}, t^{m(0)}_k) + \mathfrak{I}(x^n; t^{m(0)}_k, t^{m(0)}_{k+1}) + \mathfrak{I}(x^n; t^{m(0)}_{k+1}, t^n_{\underline{M}(t,n)}).$$

By combining (27) and (29), we get that $|\mathfrak{I}(x^n; t^{m(0)}_k, t^{m(0)}_{k+1})| \leq C_5 2^{-m(0)\alpha}$ for some constant C_5 that depends only on T, α, γ, $\|f\|_{\mathrm{Lip}}$ and $\|\mathbf{x}\|_\alpha$.

We may now find some integers $m(1)$ and $k(1)$ such that $[t^{m(1)}_{k(1)}, t^{m(1)}_{k(1)+1}]$ is the biggest interval of this type contained in $[t^n_{\overline{M}(s,n)}, t^{m(0)}_k]$, in order to estimate $\mathfrak{I}(x^n; t^n_{\overline{M}(s,n)}, t^{m(0)}_k)$. Similarly, we can find some integers $m'(1)$ and $k'(1)$ such that $[t^{m'(1)}_{k'(1)}, t^{m'(1)}_{k'(1)+1}]$ is the biggest interval of this type contained in $[t^{m(0)}_{k+1}, t^n_{\underline{M}(t,n)}]$, in order to estimate $\mathfrak{I}(x^n; t^{m(0)}_{k+1}, t^n_{\underline{M}(t,n)})$. Note that necessarily, $m(1)$ and $m'(1)$ are strictly greater than $m(0)$.

Hence, proceeding recursively, we obtain with (23) and (29) that

$$|\mathfrak{I}(x^n; t^n_{\overline{M}(s,n)}, t^n_{\underline{M}(t,n)})| \leq \frac{C_5}{2^{m(0)\alpha}} + \sum_{j\in J} \frac{C_5}{2^{m(j)\alpha}} + \sum_{j\in J'} \frac{C_5}{2^{m'(j)\alpha}},$$

where $(m(j))_{j\in J}$ and $(m'(j))_{j\in J'}$ are two finite increasing families of integers, that are bounded by n and greater than $m(0)$. This kind of computation is the core of the proof of the Kolmogorov Lemma (see for example Corollary of Theorem 4.5 in [IW89]) and is also an important tool in the theory of rough paths. It also is close to the one used in [FdLP06].

For some constant C_6, we then obtain that

$$|\mathfrak{I}(x^n; t^n_{\overline{M}(s,n)}, t^n_{\underline{M}(t,n)})| \leq \frac{C_6}{2^{m(0)\alpha}}.$$

Let us note that $T2^{-m(0)} \leq t^n_{\underline{M}(t,n)} - t^n_{\overline{M}(s,n)} < T2^{-m(0)+1}$. With (27) and the Chasles relation, we then obtain that

$$\begin{aligned}(30)\quad |\mathfrak{I}(x^n; s,t)| &\leq C_1(t^n_{\overline{M}(s,n)} - s)^\alpha + \frac{C_6}{2^{m\alpha}} + C_1(t - t^n_{\underline{M}(t,n)})^\alpha \\ &\leq \max\{C_1, C_6/T^\alpha\}(t-s)^\alpha.\end{aligned}$$

Since $\mathfrak{I}(x^n; 0) = 0$, this proves that $\mathfrak{I}(x^n; s,t)$ is uniformly bounded in $(\mathrm{C}^\alpha([0,T];\mathbb{R}), \|\cdot\|_\alpha)$. It follows that there exists a convergent subsequence in $(\mathrm{C}^\alpha([0,T];\mathbb{R}), \|\cdot\|_\beta)$, whose limit is denoted by $\mathfrak{I}(x)$, which is also a α-Hölder continuous function.

We may however give more information on the limit. With (29) and (30), for some constant C_7 and any integers $0 \leq m \leq n$ and any $0 \leq s \leq t \leq T$ with $t - s > T2^{-m}$,

$$\begin{aligned}|\mathfrak{I}(x^n; s,t) - \mathfrak{I}(x^m, s,t)| \leq C_7(t^m_{\overline{M}(s,m)} - s)^\alpha + C_7(t - t^m_{\underline{M}(t,m)})^\alpha \\ + \frac{C_4(\underline{M}(t,m) - \overline{M}(s,m))}{2^{m\alpha(\gamma+1)}}.\end{aligned}$$

As $\underline{M}(t,m) - \overline{M}(s,m) \leq 2^m$ and $\varepsilon = \alpha(\gamma+1) - 1 > 0$, it follows that

$$\begin{aligned}(31)\quad &|\mathfrak{I}(x^n; s,t) - \mathfrak{I}(x^m, s,t)| \\ &\qquad\leq C_7(t^m_{\overline{M}(s,m)} - s)^\alpha + C_7(t - t^m_{\underline{M}(t,m)})^\alpha + \frac{C_4}{2^{m\varepsilon}}.\end{aligned}$$

Set

$$R_m(s,t;\alpha,\varepsilon) = \max\left\{C_7(t^m_{\overline{M}(s,m)} - s)^\alpha, C_7(t - t^m_{\underline{M}(t,m)})^\alpha, \frac{C_4}{2^{m\varepsilon}}\right\}.$$

As $R_m(s,t;\alpha,\varepsilon)$ converges to 0 when $m \to \infty$, the sequence $(\mathfrak{I}(x^n; s,t))_{n\in\mathbb{N}}$ is a Cauchy sequence for any $0 \leq s \leq t \leq T$, which has a unique limit. Necessarily, this limit is $\mathfrak{I}(\mathbf{x}; s,t)$. Besides, we get from (31) that for some constant C_8 and any $\beta < \min\{\alpha, \varepsilon\}$,

$$|\mathfrak{I}(x; s,t) - \mathfrak{I}(x^m, s,t)| \leq C_8(t-s)^\beta R_m(s;t,\alpha-\beta,\varepsilon-\beta),$$

when m is large enough so that $T2^{-m} < t - s$. If $T2^{-m} > t - s$, then there is at most one point t^m_k such that $s \leq t^m_k \leq t$ and then for some constant C_9,

$$|\mathfrak{I}(x;s,t)-\mathfrak{I}(x^m,s,t)|\leq|\mathfrak{I}(x;s,t)|+|\mathfrak{I}(x^m,s,t)|\\ \leq C_9(t-s)^\alpha\leq\frac{C_9T^{\alpha\wedge\varepsilon-\beta}}{2^{-m(\alpha\wedge\varepsilon-\beta)}}(t-s)^\beta.$$

We get that the whole sequence $(\mathfrak{I}(x^n))_{n\in\mathbb{N}}$ converges to $\mathfrak{I}(x)$ in the space $(\mathrm{C}^\alpha([0,T];\mathbb{R}),\|\cdot\|_\beta)$ for any $\beta<\alpha\wedge\varepsilon$. Since $(\mathfrak{I}(x^n))_{n\in\mathbb{N}}$ is bounded in $\mathrm{C}^\alpha([0,T];\mathbb{R})$ and $\mathrm{C}^\varepsilon([0,T];\mathbb{R})$ is contained in $\mathrm{C}^\alpha([0,T];\mathbb{R})$ for $\varepsilon<\alpha$, $(\mathfrak{I}(x^n))_{n\in\mathbb{N}}$ converges to $\mathfrak{I}(x)$ in the space $(\mathrm{C}^\alpha([0,T];\mathbb{R}),\|\cdot\|_\beta)$ for any $\beta<\alpha$.

The proposition is then proved. □

Corollary 4. *Let $(\mathbf{x}^n)_{n\in\mathbb{N}}$ be a sequence of paths converging to $\mathbf{x}$ in the space $(\mathrm{C}^\alpha([0,T];\mathrm{A}(\mathbb{R}^2)),\|\cdot\|_\alpha)$. Then for all $\beta<\alpha$, $\mathfrak{I}(\mathbf{x}^n;0,\cdot)$ converges to $\mathfrak{I}(\mathbf{x};0,\cdot)$ in $(\mathrm{C}^\alpha([0,T];\mathbb{R}),\|\cdot\|_\beta)$.*

Proof. The proof follows the same line as the proof of Proposition 1.

To simplify the notation, we denote $\mathbf{x}$ by $\mathbf{x}^\infty$.

Since $\mathbf{x}^n$ is convergent in $\mathrm{C}^\alpha([0,T];\mathrm{A}(\mathbb{R}^2))$, the sequence $(\|\mathbf{x}^n\|_\alpha)_{n\in\mathbb{N}}$ is bounded and then, from Proposition 3, $(\mathfrak{I}(\mathbf{x}^n))_{n\in\mathbb{N}}$ is bounded in the space $(\mathrm{C}^\alpha([0,T];\mathbb{R}),\|\cdot\|_\alpha)$.

For $n\in\mathbb{N}\cup\{\infty\}$, let $(\mathbf{x}^{n,m})_{m\in\mathbb{N}}$ be the sequence of paths converging to $\mathbf{x}^n$ given by Proposition 2. We have seen in Proposition 3 for for any $\beta<\alpha$, there exists some constant K^n that depends on $\|\mathbf{x}^n\|_\alpha$ such that $\|\mathfrak{I}(\mathbf{x}^{n,m})-\mathfrak{I}(\mathbf{x}^n)\|_\beta\leq K^n2^{m(\beta-\alpha)}$. In addition, the sequence $(K^n)_{n\in\mathbb{N}}$ is bounded if $(\|\mathbf{x}^n\|_\alpha)_{n\in\mathbb{N}}$ is bounded. As $\mathfrak{I}(\mathbf{x}^{n,m})$ is a Young integral, it follows from Corollary 2 that $\mathfrak{I}(\mathbf{x}^{n,m})$ converges to $\mathfrak{I}(\mathbf{x}^{\infty,m})$ in $(\mathrm{C}^\alpha([0,T];\mathbb{R}),\|\cdot\|_\beta)$. Hence, this is sufficient to prove that $\mathfrak{I}(\mathbf{x}^n)$ converges to $\mathfrak{I}(\mathbf{x})$ in $(\mathrm{C}^\alpha([0,T];\mathbb{R}),\|\cdot\|_\beta)$, as in the proof of Proposition 1. □

Remark 8. Let us consider the following equivalence relation $\sim$ between two sequences $(x^n)_{n\in\mathbb{N}}$ and $(y^n)_{n\in\mathbb{N}}$ of sequences of path converging in $(\mathrm{C}^\alpha([0,T];\mathbb{R}^2),\|\cdot\|_\beta)$ with $\alpha>1/2$ and $\beta\in(1/3,1]$: $(x^n)_{n\in\mathbb{N}}\sim(y^n)_{n\in\mathbb{N}}$ if $\mathbf{x}\stackrel{\text{def}}{=}\lim_{n\in\mathbb{N}}\mathfrak{C}(x^n,0)=\lim_{n\in\mathbb{N}}\mathfrak{C}(y^n,0)$ in $(\mathrm{C}^\gamma([0,T];\mathrm{A}(\mathbb{R}^2),\|\cdot\|_\beta)$ for some $\gamma>\beta$. This implies that $\mathfrak{I}(x^n;s,t)$ and $\mathfrak{I}(y^n;s,t)$ converge to the same limit $\mathfrak{I}(\mathbf{x};s,t)$. Hence, one may identify $\mathrm{C}^\gamma([0,T];\mathrm{A}(\mathbb{R}^2),\|\cdot\|_\gamma)$ with the quotient space $(\mathrm{C}^\alpha([0,T];\mathbb{R}^2),\|\cdot\|_\beta)^{\mathbb{N}}/\sim$, and two elements in the same class of equivalence give rise to the same integral.

Here, we have used the dyadics partitions[2], so that one may ask whether $\mathfrak{I}(\mathbf{x};s,t)$ is equal to $\mathfrak{I}(\mathbf{x}_{|[s,t]})$? As this is true for ordinary integrals, we easily get the following result.

[2]We will give below another construction of $\mathfrak{I}$ for which there family of partitions different from the dyadics ones can be used.

Lemma 11. *Let* $\mathbf{x}$ *in* $\mathrm{C}^\alpha([0,T];\mathrm{A}(\mathbb{R}^2))$. *Then, for all* $0\leq s\leq t\leq T$, $\mathfrak{I}(\mathbf{x};s,t)=\mathfrak{I}(\mathbf{x}_{|[s,t]})$.

From this lemma, we deduce that if $\mathbf{x}\in\mathrm{C}^\alpha([0,T];\mathrm{A}(\mathbb{R}^2))$ and $\mathbf{y}\in\mathrm{C}^\alpha([0,S];\mathrm{A}(\mathbb{R}^2))$, then

$$\mathfrak{I}(\mathbf{x}\boxdot\mathbf{y};0,t)=\begin{cases}\mathfrak{I}(\mathbf{x};0,t) & \text{if } t\in[0,T],\\ \mathfrak{I}(\mathbf{x};0,T)+\mathfrak{I}(\mathbf{y};0,t-T) & \text{if } t\in[S,T].\end{cases}$$

Proof. This lemma means that the integral constructed using the dyadics on $[0,T]$ but restricted to $[s,t]$ corresponds to the integral constructed using the dyadics on $[s,t]$. One knows that such a relation holds for ordinary integrals, since the integral does not depend on the choice of the family of partitions on which approximations of the integrals are defined.

Let $(\mathbf{x}^n)_{n\in\mathbb{N}}$ be the approximation of $\mathbf{x}$ given by Proposition 2. Then $\mathfrak{I}(\mathbf{x}^n)$ is an ordinary integral. Hence $\mathfrak{I}(\mathbf{x}^n;s,t)=\mathfrak{I}(\mathbf{x}^n_{|[s,t]};0,t-s)$ (the last integral means that T is replaced by $t-s$ and thus that we consider the dyadic partitions of $[0,t-s]$. The result follows from passing to the limit. □

Let us end this section with an important remark. Consider $\mathbf{x}\in\mathrm{C}^\alpha([0,T];\mathrm{A}(\mathbb{R}^2))$ with $\alpha\in(1/3,1/2)$ and φ in $\mathrm{C}^{2\alpha}([0,T];\mathbb{R})$. We saw in Lemma 9 that $\mathbf{y}=(\mathbf{x}^1,\mathbf{x}^2,\mathbf{x}^3+\varphi)$ also belongs to $\mathrm{C}^\alpha([0,T];\mathrm{A}(\mathbb{R}^2))$.

Hence, we set $y^n_t=x^{\Pi^n}\bowtie\Phi^n\bowtie\Psi^n(t/3)$ for $t\in[0,3T]$ where $\Psi^n=\{z^n_k\}_{k=0,\dots,2^n-1}$ with $z^n_k:[t^n_k,t^n_{k+1}]\to\mathbb{R}^2$ defined by

$$z^n_k(t)=\frac{\varphi_{t^n_{k+1}}-\varphi_{t^n_k}}{\sqrt{\pi}}\begin{bmatrix}\cos\left(2\pi\frac{t-t^n_k}{t^n_{k+1}-t^n_k}\right)-1\\ \sin\left(2\pi\frac{t-t^n_k}{t^n_{k+1}-t^n_k}\right)\end{bmatrix},$$

so that φ asymptotically encodes the area of $(\Psi^n)_{n\in\mathbb{N}}$.

Similarly as in Section 3.2, it is then easily shown that

$$\mathfrak{I}(y^n;0,t)\xrightarrow[n\to\infty]{}\mathfrak{I}(\mathbf{y};0,t)=\mathfrak{I}(\mathbf{x};0,t)+\int_0^t[f,f](x_s)\,\mathrm{d}\varphi_s.$$

Hence, adding a path φ to the third component of $\mathbf{x}$ consists in in adding a term $\int_0^\cdot[f,f](x_s)\,\mathrm{d}\varphi_s$ to $\mathfrak{I}(\mathbf{x})$.

5.8. **A sub-Riemannian point of view.** Our definition of $\mathfrak{I}$ consists in approximating a path $\mathbf{x}\in\mathrm{C}^\alpha([0,T];\mathrm{A}(\mathbb{R}^2))$ by a family of paths $(\mathbf{x}^n)_{n\in\mathbb{N}}$ in $\mathrm{C}^1([0,T];\mathrm{A}(\mathbb{R}^2))$ such that $\mathfrak{I}(\mathbf{x}^n)$ converges with respect to the β-Hölder norm in $\mathrm{C}^\alpha([0,T];\mathbb{R})$ as $n\to\infty$ for all $\beta<\alpha$. The integral $\mathfrak{I}(\mathbf{x})$ is then defined as the limit if $\mathfrak{I}(\mathbf{x}^n)$. In addition, necessarily, it follows from Lemma 8 that $\mathbf{x}^n=(x^{1,n},x^{2,n},\mathbf{x}^3_0+\mathfrak{A}(x^n))$, where x^n is a family of functions in $\mathrm{C}^1_{\mathrm{p}}([0,T];\mathbb{R}^2)$.

The paths $\mathbf{x}^n$ were constructed by replacing $\mathbf{x}_{|[t^n_k,t^n_{k+1}]}$ by some paths obtained by combining loops and segments. Of course, other choices are possible, and a natural one consists in using geodesics.

Let a be a point in $\mathrm{A}(\mathbb{R}^2)$. How to find a path $\mathbf{x} : [0,1] \to \mathrm{A}(\mathbb{R}^2)$ with $\mathbf{x}_0 = 0$, $\mathbf{x}_1 = a$ and whose length (or whose energy) is minimal? Of course, one can use the segment $\mathbf{y} = (ta^1, ta^2, ta^3)_{t\in[0,1]}$ that goes from 0 to a, which is the natural geodesic in $\mathbb{R}^3$. But $\mathfrak{A}(\mathbf{y}^1, \mathbf{y}^2; t) = 0$ and thus $\mathbf{y}$ is not of type $(y, \mathfrak{A}(y))$ and does not belong to $\mathrm{C}^1([0,T]; \mathrm{A}(\mathbb{R}^2))$. We will use this point of view in Section 7.2, and this will help us to bridge our construction with another one of Riemann sum type. So, we may reformulate our question by imposing the condition that $\mathbf{y}$ is of type $\mathbf{y} = (y, \mathfrak{A}(y))$, which means that $\mathbf{y}^3_t = \mathfrak{A}(\mathbf{y}^1, \mathbf{y}^2; 0, t)$ for $t \in [0,1]$. This kind of problem is related to sub-Riemannian geometry: see [Gro96, BBI01, Mon02, Bau04] for example.

The notion of length we use is then the length of the path $(\mathbf{y}^1, \mathbf{y}^2)$:

$$\mathrm{Length}(\mathbf{y}) = \int_0^1 \sqrt{(\dot{\mathbf{y}}^1_s)^2 + (\dot{\mathbf{y}}^2_s)^2}\, \mathrm{d}s.$$

Such a path — which will be characterized from the differentiable point of view in the next section —, is called *horizontal*. It is then possible to introduce a distance between two points of $\mathrm{A}(\mathbb{R}^2)$ by

$$d(a,b) = \inf_{\substack{\mathbf{y}:[0,1]\to\mathrm{A}(\mathbb{R}^2)\text{ horizontal}\\ \mathbf{y}_0=a,\ \mathbf{y}_1=b}} \mathrm{Length}(\mathbf{y}),$$

which is called the *Carnot-Carathéodory* distance. We may then define $\|x\|_{\mathrm{CC}} = d(0,x)$, which becomes a homogeneous sub-additive norm on $\mathrm{A}(\mathbb{R}^2)$ (see Section A) *i.e.,* $\|x\|_{\mathrm{CC}} = 0$ if and only $x = 0$ and for all $x, y \in \mathrm{A}(\mathbb{R}^2)$ and $\lambda \in \mathbb{R}$, $\|\delta_\lambda x\|_{\mathrm{CC}} = |\lambda| \cdot \|x\|_{\mathrm{CC}}$, $\|x^{-1}\|_{\mathrm{CC}} = \|x\|_{\mathrm{CC}}$ and $\|x \boxplus y\|_{\mathrm{CC}} \leq \|x\|_{\mathrm{CC}} + \|y\|_{\mathrm{CC}}$, which is the *sub-additive property.*

For any $a \in \mathrm{A}(\mathbb{R}^2)$, we succeed in constructing in Section 5.3 a path that goes from 0 to a, so that $\|a\|_{\mathrm{CC}}$ is finite. Of course, $d(a,b) = \|a^{-1} \boxplus b\|_{\mathrm{CC}}$ for all $a, b \in \mathrm{A}(\mathbb{R}^2)$. If $a^3 = 0$, then the shortest horizontal path from 0 to a is the segment that goes from 0 to a. If $a = (0,0,a^3)$ with $a^3 \neq 0$, then this problem is equivalent to the isoperimetric problem, whose solution is known to be circle.

In the general case, this problem is called the *Dido problem*, and the solutions are known to be arcs of circle (see for example [Str87, Mon02]), but they are less practical to use than our construction with circles and loops (see below in the proof of Proposition 4).

These solutions are not real geodesics in $\mathrm{A}(\mathbb{R}^2)$, but they are called *sub-Riemannian geodesics*. The sub-Riemannian geodesic that links a to b is then denoted by $\psi_{a,b}$ and belongs to $\mathrm{C}^1([0,T]; \mathrm{A}(\mathbb{R}^2))$.

If we define the energy of a path by $\mathrm{Energy}(\mathbf{y}) = \frac{1}{2}\int_0^1 ((\dot{\mathbf{y}}^1_s)^2 + (\dot{\mathbf{y}}^2_s)^2)\,\mathrm{d}s$, then $\psi_{a,b}$ is also a minimizer for the energy among all the paths with constant speed $\mathrm{Length}(\psi_{a,b})$.

To a path $\mathbf{x}$ in $\mathrm{C}^\alpha([0,T]; \mathrm{A}(\mathbb{R}^2))$, we associate

$$\mathbf{x}^n_t = \psi_{\mathbf{x}_{t^n_k}, \mathbf{x}_{t^n_{k+1}}}\left(\frac{t - t^n_k}{t^n_{k+1} - t^n_k}\right) \text{ for } t \in [t^n_k, t^n_{k+1}], \tag{32}$$

for $n = 0, 1, 2, \ldots$.

Proposition 4. *The sequence of paths* $(\mathbf{x}^n)_{n\in\mathbb{N}}$ *constructed by* (32) *is a family of paths in* $\mathrm{C}^1([0,T];\mathrm{A}(\mathbb{R}^2))$ *which converges to* $\mathbf{x}$ *in* $\mathrm{C}^\alpha([0,T];\mathrm{A}(\mathbb{R}^2))$ *with respect to* $\|\cdot\|_\beta$ *for any* $\beta < \alpha$.

Proof. The proof is similar to the one of Corollary 1 or of Proposition 2.

Obviously, $(\mathbf{x}^n)_{n\in\mathbb{N}}$ converges uniformly to $\mathbf{x}$. Let us remark that $\mathbf{x}^n_{s,t} = \mathbf{x}^n_{s,t^n_k} \boxplus \mathbf{x}^n_{t^n_k,t^n_{k+1}} \boxplus \mathbf{x}^n_{t^n_{k+1},t}$ and that $\mathbf{x}^n_{t^n_k,t^n_{k+1}} = \mathbf{x}_{t^n_k,t^n_{k+1}}$. Using the same argument as in Corollary 1, the α-Hölder norm of $\mathbf{x}^n$ is then deduced from estimates on $\mathbf{x}^n_{s,t_k n}$ and $\mathbf{x}^n_{t^n_{k+1},t}$ for $t \in [t^n_k, t^n_{k+1}]$ for $k = 0, \ldots, 2^n - 1$.

After a translation, we are looking for establishing an estimate of type $|\psi_{0,x}(t)| \leq Ct|x|$ for $t \in [0,1]$ for some constant C. If this holds, then for $t \in [t^n_k, t^n_{k+1}]$,

$$|\psi_{\mathbf{x}_{t^n_k},\mathbf{x}_{t^n_{k+1}}}(t/\Delta_n t)| \leq C\frac{t}{\Delta_n t}|\mathbf{x}_{t^n_k,t^n_{k+1}}| \leq C\frac{t\Delta_n t^\alpha}{\Delta_n t}\|\mathbf{x}\|_\alpha \leq Ct^\alpha\|\mathbf{x}\|_\alpha.$$

We now give two proofs: one is done "by hand", and the second one uses the properties of the Carnot-Carathéodory distance.

∘ If $x^3 = 0$, then $\psi_{0,x}(t)$ is a segment and for $t \in [0,1]$,

$$|\psi_{0,x}(t)| \leq |x|t.$$

which gives the desired result.

Now, if $x^3 \neq 0$, let us note first that for some constants $a \neq 0$ and $r, \varphi \in [0, 2\pi)$,

$$\begin{cases} \psi^1_{0,x}(t) & = a(\cos(rt+\varphi) - \cos(\varphi)), \\ \psi^2_{0,x}(t) & = a(\sin(rt+\varphi) - \sin(\varphi)), \\ \psi^3_{0,x}(t) & = a^2 rt \end{cases}$$

since the minimizers lies above arcs of circles. Hence, $a^2 r = x^3$ and

$$(x^1)^2 + (x^2)^2 = \psi^1_{0,x}(1)^2 + \psi^2_{0,x}(1)^2 = 2a^2(1 - \cos(r)).$$

It is easily seen that one may find a and r in order to satisfy $\psi_{0,x}(1) = x$.

If $r \in [\pi/2, 3\pi/2]$, then $1 \leq 1 - \cos(r) \leq 2$, $a^2 \leq \max\{|x^1|^2, |x^2|^2\}$ and

$$\max\{|\psi^1_{0,x}(t)|, |\psi^2_{0,x}(t)|\} \leq \sqrt{2}\pi t \max\{|x^1|, |x^2|\},$$

and $|\psi^3_{0,t}(t)| \leq 4\pi^{-1} t \max\{|x^1|, |x^2|\}^2$. This is sufficient to conclude.

In the other case, since cos and sin are Lipschitz continuous and $|a^2 r| \leq |x^2|$, we get that

$$\psi^1_{0,x}(t)^2 + \psi^2_{0,x}(t)^2 = 2a^2(1 - \cos(rt)) \leq 2|x^3|t \leq 2|x|^2 t.$$

Hence, $|\psi_{0,x}(t)| \leq \sqrt{2}|x|t$.

It follows that $(\mathbf{x}^n)_{n\in\mathbb{N}}$ is bounded in $\mathrm{C}^\alpha([0,T];\mathrm{A}(\mathbb{R}^2))$ and this is sufficient to conclude.

∘ *(Alternative proof)*. As the Carnot-Carathéodory norm is equivalent to any homogeneous norm (see Proposition 10 in Section A), it follows that for some universal constants C and C',

$$\forall t\in[0,1],\ |\psi_{0,x}(t)|\leq C\|\psi_{0,x}(t)\|_{\mathrm{CC}}=Ct\|x\|_{\mathrm{CC}}\leq CC't|x|, \tag{33}$$

since $\psi_{0,x}(t)$ is a sub-Riemannian geodesic and then $\|\psi_{0,x}(t)\|_{\mathrm{CC}} = td(0,x)$. The inequalities (33) yields the result. □

The point of view of the sub-Riemannian geometry, which is natural in the context of Heisenberg groups, have been used by P. Friz and N. Victoir in [FV06b] and [FV08].

5.9. **A sub-Riemannian point of view: differentiable paths in** $\mathrm{A}(\mathbb{R}^2)$**.** We have introduced the set of paths $\mathrm{C}^\alpha([0,T];\mathrm{A}(\mathbb{R}^2))$ for $\alpha\in[1/2,1/3)$, but we have that the value of α does not really refer to the regularity of the path $\mathbf{x}$ in such a set, but to the norms to be used to approximate $\mathbf{x}$ by a family of paths x^n that are naturally lifted as $\mathbf{x}^n=(x^n,\mathfrak{A}(x^n))$. It is then possible to consider some path $\mathbf{x}\in\mathrm{C}^\alpha([0,T];\mathrm{A}(\mathbb{R}^2))$ with $\alpha<1/2$ that are differentiable: for example, if $\mathbf{x}$ in $\mathrm{C}^1([0,T];\mathrm{A}(\mathbb{R}^2))$ and φ in $\mathrm{C}^1([0,T];\mathbb{R})$, then $\mathbf{y}_t=(\mathbf{x}^1_t,\mathbf{x}^2_t,\mathbf{x}^3_t+\varphi_t)$ is almost everywhere differentiable, in the sense that

$$i=1,2,3,\ \lim_{\varepsilon\to0}\frac{\mathbf{y}^i_{t+\varepsilon}-\mathbf{y}^i_t}{\varepsilon}=\alpha^i(t) \tag{34}$$

exists for almost every t. Another natural way of thinking the derivative of $\mathbf{y}$ consists in setting

$$i=1,2,3,\ \lim_{\varepsilon\to0}\frac{1}{\varepsilon}(-\mathbf{y}_t)\boxplus\mathbf{y}^i_{t+\varepsilon}=\beta^i(t) \tag{35}$$

when this limit exists. If $t\in[0,T]$ is such that (34) holds, then $\beta^i(t)$ exists and

$$\beta(t)=\alpha(t)-\frac{1}{2}[\mathbf{y}_t,\alpha(t)].$$

Reciprocally, if (35) holds, then (34) holds and

$$\alpha(t)=\beta(t)+\frac{1}{2}[\mathbf{y}_t,\beta(t)].$$

Of course, $(\alpha^1(t),\alpha^2(t))=(\beta^1(t),\beta^2(t))$ for all t at which $\mathbf{y}_t$ is differentiable.

If the path $\mathbf{y}$ is of type $(y,\mathfrak{A}(y))$, then

$$\alpha^1(t)=\frac{\mathrm{d}y^1_t}{\mathrm{d}t},\ \alpha^2(t)=\frac{\mathrm{d}y^2_t}{\mathrm{d}t}\text{ and }\alpha^3(t)=\frac{1}{2}y_t\wedge\frac{\mathrm{d}y_t}{\mathrm{d}t}=\frac{1}{2}y_t\wedge\begin{bmatrix}\alpha^1(t)\\ \alpha^2(t)\end{bmatrix}.$$

At each point a of $\mathrm{A}(\mathbb{R}^2)$, we associate the 2-dimensional vector space

$$\Theta(a)=\left\{(v^1,v^2,v^3)\in\mathbb{R}^3\,\middle|\,v^3=\frac{1}{2}\begin{bmatrix}a^1\\ a^2\end{bmatrix}\wedge\begin{bmatrix}v^1\\ v^2\end{bmatrix}\right\}$$

as well as the space $\Xi(a)$ orthogonal to $\Theta(a)$ with respect to the usual scalar product in $\mathbb{R}^3$. The one-dimensional space $\Xi(a)$ is generated by the vector $(-a^2/2, a^1/2, 1)^{\mathrm{T}}$. It is easily seen that $a \mapsto (a, \Xi(a))$ and $a \mapsto (a, \Theta(a))$ forms two sub-bundles of the tangent bundle of $\mathrm{A}(\mathbb{R}^2)$.

We then obtain the next result.

Lemma 12. *A differentiable curve* $\mathbf{y}$ *is the natural lift* $(y, \mathfrak{A}(y))$ *of a differentiable curve* y *if and only if* $\dot{\mathbf{y}}_t$ *belongs to* $\Theta(\mathbf{y}_t)$ *for any* $t \in [0, T]$.

For a differentiable path $\mathbf{y} : [0, T] \to \mathrm{A}(\mathbb{R}^2)$, let $\beta(t)$ be given by (35). The condition that $\dot{\mathbf{y}}_t$ belongs to $\Theta(\mathbf{y}_t)$ is equivalent to $\beta(t) = (\dot{\mathbf{y}}_t^1, \dot{\mathbf{y}}_t^2, 0)$. More generally, if $\pi_{\Xi(a)}$ is the projection from $\mathbb{R}^3$ identified with the tangent plane of $\mathrm{A}(\mathbb{R}^2)$ at a onto $\Xi(a)$, then for $t \in [0, T]$,

$$\beta(t) = (\dot{\mathbf{y}}_t^1, \dot{\mathbf{y}}_t^2, \pi_{\Xi(\mathbf{y}_t)}(\dot{\mathbf{y}}_t)).$$

Thus, a differentiable path $\mathbf{y}$ from $[0, T]$ to $(\mathrm{A}(\mathbb{R}^2), \boxplus)$ is necessarily of type $(y, \mathfrak{A}(y) + \varphi)$ where $y = (\mathbf{y}^1, \mathbf{y}^2)$ and φ is differentiable, and $\beta(t) = (\dot{y}_t^1, \dot{y}_t^2, \dot{\varphi}_t)$ for $t \in [0, T]$.

We will see in Section 6.12 how to interpret this condition.

6. Geometric and algebraic structures

6.1. **Motivations.** Up to now, we have introduced a space $\mathrm{A}(\mathbb{R}^2)$ and considered paths in $\mathrm{C}^\alpha([0, T]; \mathrm{A}(\mathbb{R}^2))$. For a path $\mathbf{x} \in \mathrm{C}^\alpha([0, T]; \mathrm{A}(\mathbb{R}^2))$, we have seen how to construct a sequence $(\mathbf{x}^n)_{n\in\mathbb{N}}$ of paths converging to $\mathrm{C}^\beta([0, T]; \mathrm{A}(\mathbb{R}^2))$ with $\beta < \alpha$ such that $x^n = (\mathbf{x}^{1,n}, \mathbf{x}^{2,n})$ is piecewise smooth and $\mathbf{x}^{3,n} = \mathbf{x}_0^3 + \mathfrak{A}(x^n)$. As $\mathbf{x}^n$ lies above a piecewise smooth path x^n, $\mathfrak{I}(\mathbf{x}^n)$ is well defined as a Young integral, and we have shown in Proposition 3 that the sequence $(\mathfrak{I}(\mathbf{x}^n))_{n\in\mathbb{N}}$ converges and its limit defines $\mathfrak{I}(\mathbf{x})$.

On the other hand, we may rewrite

$$\mathfrak{I}(\mathbf{x}^n; 0, T) = \sum_{k=0}^{2^n-1} \mathfrak{I}(\mathbf{x}^n_{|[t_k^n, t_{k+1}^n]}) \text{ and } \mathfrak{I}(\mathbf{x}; 0, T) = \sum_{k=0}^{2^n-1} \mathfrak{I}(\mathbf{x}_{|[t_k^n, t_{k+1}^n]}).$$

The path $\mathbf{x}^n_{|[t_k^n, t_{k+1}^n]}$ was constructed in Section 5.3 from the values of $\mathbf{x}_{t_{k+1}^n}$ and $\mathbf{x}_{t_k^n}$. Hence, $\int_{t_k^n}^{t_{k+1}^n} f(x_s^n)\,\mathrm{d}x_s^n$ is an approximation of $\mathfrak{I}(\mathbf{x}_{|[t_k^n, t_{k+1}^n]})$, and $\mathfrak{I}(\mathbf{x}^n)$ is constructed only from the values of $\{\mathbf{x}_{t_k^n}\}_{k=0,\dots,2^n-1}$.

We have proposed two constructions of integrals that rely on approximation of the path. We are now looking for a Riemann sum like expression, which consists in finding approximations of $\mathfrak{I}(\mathbf{x}; 0, T)$ and to sum them over the dyadic partitions of $[0, T]$.

Let us note first that if $\mathbf{x}$ belongs to $\mathrm{C}^\alpha([0, T]; \mathrm{A}(\mathbb{R}^2))$ with $\alpha > 1/2$ and x^{Π^n} is the piecewise linear approximation of x along the dyadic partition Π^n, then

$$|\mathfrak{I}(x^{\Pi^n}; t_k^n, t_{k+1}^n) - \mathfrak{I}(\mathbf{x}; t_k^n, t_{k+1}^n)| \leq \|f\|_{\mathrm{Lip}} |\mathfrak{A}(x; t_k^n, t_{k+1}^n)| \leq \frac{T^{2\alpha}\|f\|_{\mathrm{Lip}}\|x\|_\alpha^2}{2^{2n\alpha}}$$

and thus, since $\alpha > 1/2$,

$$\begin{aligned}\mathfrak{I}(\mathbf{x};0,T) &= \lim_{n\to\infty}\sum_{k=0}^{2^n-1}\int_{t_k^n}^{t_{k+1}^n} f(x_s^{\Pi^n})\frac{x_{t_{k+1}^n}-x_{t_k^n}}{t_{k+1}^n-t_k^n}\,\mathrm{d}s\\
&= \lim_{n\to\infty}\sum_{k=0}^{2^n-1}\int_{t_k^n}^{t_{k+1}^n} f(x_s^{\Pi^n})\frac{\mathrm{d}x_s^{\Pi^n}}{\mathrm{d}s}\,\mathrm{d}s \qquad (36)\end{aligned}$$

where x is the path above which $\mathbf{x}$ lies.

The first idea is then to find a formulation similar to (36), by looking for another way to draw a piecewise differentiable path $\mathbf{y}^n$ lying above a path $y^n : [0,T] \to \mathbb{R}^2$ with $y^n(t_k^n) = x_{t_k^n}$ for $k = 0,\dots,2^n$ and for which the expression $\xi_k^n = \int_{t_k^n}^{t_{k+1}^n} f(y_s^n)\frac{\mathrm{d}\mathbf{y}_s^n}{\mathrm{d}s}\,\mathrm{d}s$ provides a good approximation of $\mathfrak{I}(\mathbf{x};t_k^n,t_{k+1}^n)$, in the sense that for some $\theta > 1$ and $C > 0$,

$$|\xi_k^n - \mathfrak{I}(\mathbf{x};t_k^n,t_{k+1}^n)| \leq \frac{C}{2^{n\theta}}.$$

The space in which $\mathbf{y}$ lives has to be precised, but it is natural to assume that $\frac{\mathrm{d}\mathbf{y}_k^n(s)}{\mathrm{d}s}$ belongs to $\mathrm{A}(\mathbb{R}^2)$, and then one has to extend the definition of f into a differential form on $\mathrm{A}(\mathbb{R}^2)$ accordingly.

The second idea would then to get an expression of type $\sum_{k=0}^{2^n-1} f(x_{t_k^n})\Delta_k^n\mathbf{x}$ where $\Delta_k^n\mathbf{x}$ depends only on $\mathbf{x}_{t_{k+1}^n}$ and $\mathbf{x}_{t_k^n}$. As we deal with second-order calculus, the things are not that simple: think to the difference between the Stratonovich and the Itô integrals for the Brownian motion.

6.2. **Another formulation for the integral.** We rewrite $\mathfrak{I}(\mathbf{x}^n;t_k^n,t_{k+1}^n)$ as

$$\mathfrak{I}(\mathbf{x}^n;t_k^n,t_{k+1}^n) = \int_{t_k^n}^{t_{k+1}^n} f(x_s^{\Pi^n})\,\mathrm{d}x_s^{\Pi^n} + \iint_{\mathrm{Surface}(y_k^n)} [f,f](z)\,\mathrm{d}z$$

where y_k^n has been defined by (21b). Setting $\mathbf{x}_{s,t} = (-\mathbf{x}_s)\boxplus\mathbf{x}_t$ and $\Delta_n t = T2^{-n}$, we have already seen that

$$\left|\iint_{\mathrm{Surface}(y_k^n)} [f,f](z)\,\mathrm{d}z - \mathbf{x}^3_{t_k^n,t_{k+1}^n}[f,f](x_{t_k^n})\right| \leq \Delta_n t^{\alpha(1+\gamma)}\|f\|_{\mathrm{Lip}}\|\mathbf{x}\|_\alpha^{1+\gamma}.$$

On the other hand,

$$\left|\mathbf{x}^3_{t_k^n,t_{k+1}^n}[f,f](x_{t_k^n}) - \mathbf{x}^3_{t_k^n,t_{k+1}^n}\int_{t_k^n}^{t_{k+1}^n}[f,f](x_s^{\Pi^n})\frac{\mathrm{d}s}{\Delta_n t}\right| \leq \Delta_n t^{\alpha(1+\gamma)}\|f\|_{\mathrm{Lip}}\|\mathbf{x}\|_\alpha^{1+\gamma}.$$

Hence, this means that one can replace $\mathfrak{I}(\mathbf{x}^n; t^n_k, t^n_{k+1})$ by

$$\xi^n_k = \mathbf{x}^1_{t^n_k, t^n_{k+1}} \int_{t^n_k}^{t^n_{k+1}} f_1(x^{\Pi^n}_s) \frac{\mathrm{d}s}{\Delta_n t} + \mathbf{x}^2_{t^n_k, t^n_{k+1}} \int_{t^n_k}^{t^n_{k+1}} f_2(x^{\Pi^n}_s) \frac{\mathrm{d}s}{\Delta_n t} + \mathbf{x}^3_{t^n_k, t^n_{k+1}} \int_{t^n_k}^{t^n_{k+1}} [f,f](x^{\Pi^n}_s) \frac{\mathrm{d}s}{\Delta_n t},$$

in the sense that $\mathfrak{I}(\mathbf{x}; 0, T) = \lim_{n\to\infty} \sum_{k=0}^{2^n-1} \xi^n_k$.

Let us denote by $\{e_1, e_2, [e_1, e_2]\}$ the canonical basis of $\mathrm{A}(\mathbb{R}^2)$, and by $\{e^1, e^2, [e^1, e^2]\}$ its dual basis. For $z = (z^1, z^2, z^3) \in \mathrm{A}(\mathbb{R}^2)$, let us define the differential form

$$\mathfrak{E}_{\mathrm{A}(\mathbb{R}^2)}(f)(z) = f_1(z^1, z^2)e^1 + f_2(z^1, z^2)e^2 + [f,f](z^1, z^2)[e^1, e^2]. \tag{37}$$

With $\mathbf{x}^{\Pi^n} = (x^{\Pi^n}, 0)$, the term ξ^n_k may be put in a more synthetic form

$$\xi^n_k = \int_{t^n_k}^{t^n_{k+1}} \mathfrak{E}_{\mathrm{A}(\mathbb{R}^2)}(f)(\mathbf{x}^{\Pi^n}_s) \mathbf{x}_{t^n_k, t^n_{k+1}} \frac{\mathrm{d}s}{\Delta_n t}.$$

Remark 9. We have to note the following point: using the same technique as in Corollary 1, one can show that for $\mathbf{x} \in \mathrm{C}^\alpha([0,T]; \mathrm{A}(\mathbb{R}^2))$, the path $\mathbf{x}^n$ defined by

$$\mathbf{x}^n_t = \mathbf{x}_{t^n_k} \boxplus \delta_{(t-t^n_k)/(t^n_{k+1}-t^n_k)}((-\mathbf{x}_{t^n_k}) \boxplus \mathbf{x}_{t^n_{k+1}}) \text{ for } t \in [t^n_k, t^n_{k+1}]$$

converges to $\mathbf{x}$ in $(\mathrm{C}^\alpha([0,T]; \mathrm{A}(\mathbb{R}^2)), \|\cdot\|_\beta)$ for any $\beta < \alpha$ when the mesh of of partition $\{t^n_k\}_{k=0,\dots,n}$ converges to 0. Here, $\delta.$ is the dilatation operator introduced in (14). We have then that $\mathfrak{I}(\mathbf{x}^n)$ converges to $\mathfrak{I}(\mathbf{x})$ in $(\mathrm{C}^\alpha([0,T]; \mathrm{A}(\mathbb{R}^2)), \|\cdot\|_\beta)$ for any $\beta < \alpha$ if $\alpha \in (1/3, 1]$.

Here, we consider the piecewise linear approximation

$$\widehat{\mathbf{x}}^n_t = \mathbf{x}_{t^n_k} \boxplus \frac{t - t^n_k}{t^n_{k+1} - t^n_k}((-\mathbf{x}_{t^n_k}) \boxplus \mathbf{x}_{t^n_{k+1}}) \text{ for } t \in [t^n_k, t^n_{k+1}]$$

which a piecewise smooth path with values in $\mathrm{A}(\mathbb{R}^2)$. If $\alpha > 1/2$, we may show that $(\widehat{\mathbf{x}}^n)_{n\in\mathbb{N}}$ is bounded in $\mathrm{C}^\beta([0,T]; \mathrm{A}(\mathbb{R}^2))$ with $\beta = 2\alpha - 1$. We do not know whether or not $\widehat{\mathbf{x}}^n$ is bounded in $\mathrm{C}^\beta([0,T]; \mathrm{A}(\mathbb{R}^2))$ when $\alpha < 1/2$ for $\beta < \alpha$. However, we may define $\mathfrak{I}(\mathbf{x})$ using $(\widehat{\mathbf{x}}^n)_{n\in\mathbb{N}}$ by changing the definition of the integral.

The important point is the following: as we primarily want to focus on the increments of the paths, we leave the world of sub-Riemannian geometry, in which paths in $\mathrm{A}(\mathbb{R}^2)$ are seen basically as 2-dimensional paths with a constraint on their areas. We are now willing to deal with paths that are seen directly as paths with values in $\mathrm{A}(\mathbb{R}^2)$ (or other spaces that will be introduced later).

We are now looking for a curve $\mathbf{y}^n(t)$ on $[0,T]$ which is piecewise differentiable and such that

$$\frac{\mathrm{d}\mathbf{y}^n(t)}{\mathrm{d}t} = \frac{1}{\Delta_n t} \mathbf{x}_{t^n_k, t^n_{k+1}},\ t \in (t^n_k, t^n_{k+1}). \tag{38}$$

Of course, from (38), such a path lies above x^{Π^n}. The problem is now to find the space in which $\mathbf{y}^n$ lives.

Let us recall the results from Section 5.4: The space $(\mathrm{A}(\mathbb{R}^2))$ is a non-commutative group when equipped with $\boxplus$, and it is also a *Lie algebra* when equipped with the brackets $[\cdot,\cdot]$.

We have already denoted the basis of $\mathrm{A}(\mathbb{R}^2)$ by $\{e_1, e_2, [e_1, e_2]\}$. The choice of $[e_1, e_2]$ to denote the third component follows naturally from the bilinearity of $[\cdot,\cdot]$.

The Lie algebra structure is particularly important here, since one knows that $\mathrm{A}(\mathbb{R}^2)$ may be identified with the tangent space at any point of a Lie group. We will now construct such a Lie group.

6.3. **Matrix groups.** We give here a very brief presentation of matrix groups. This part can also serve as a presentation of Lie groups, for which matrix groups are prototype with the advantage that of having an explicit coordinate system. For a more detailed insight, there are many books (see specifically [Bak02, Tap05] or some books on Lie groups as [DK00]).

Let us consider a matrix group M that is a subset of $d \times d$-matrices such that for $p, q \in \mathrm{M}$, $p \times q$ also belongs to M and p^{-1} belongs to M, and which is closed. This matrix group can be equipped with the induced topology of the set $\mathrm{M}_d(\mathbb{R})$ of $d \times d$-matrices.

A general result is that a matrix group forms a *smooth manifold* [Tap05, Theorem 7.17, p. 106], which means that around each point p of M, there exists an open set $U(p)$ in $\mathbb{R}^m$ (for some fixed m) and an open neighbourhood V_p of p in $\mathrm{M}_d(\mathbb{R})$ (see as $\mathbb{R}^{d^2}$) such that there exists a map Φ_p which is a homeomorphism from U_p to $V_p \cap \mathrm{M}$. In addition, we require that for two points p and q of M, $V_p \cap V_q \neq \emptyset$, $\Phi_p \circ \Phi_q^{-1}$ and $\Phi_q \circ \Phi_p^{-1}$ are smooth on their domain of definition. In other word, one can describe locally M using a smooth one-to-one map from an open set of $\mathbb{R}^m$ (indeed, the dimension m does not depends on the points around which the neighbourhood is considered) to M.

Example 1. Basic examples of Lie group are given by the sets of invertible matrices, of orthogonal matrices, ...

Example 2. A particular example for us is the *Heisenberg group* H, which is the set of matrices

$$\mathrm{H} = \left\{ \begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \middle| \, a, b, c \in \mathbb{R} \right\}.$$

which is easily seen to be stable under the matrix multiplication.

The Heisenberg group has been widely studied, as appears in sub-Riemannian geometry, quantum physics, ... (see for example [Fol89, Mon02, Bau04]).

For a given point p in M, we can consider a smooth path γ from $(-\varepsilon, \varepsilon)$ to $\mathrm{M} \subset \mathrm{M}_d(\mathbb{R})$ for some $\varepsilon > 0$ and with $\gamma(0) = p$. As $\gamma(t) = [\gamma_{i,j}(t)]_{i,j=1,\dots,d}$, we may consider its derivative $\gamma'(t) = [\gamma'_{i,j}(t)]_{i,j=1,\dots,d}$.

As γ moves only on M, $\gamma'(t)$ can only belongs to a subspace of $\mathrm{M}_d(\mathbb{R})$ at each time. We denote by $T_p\mathrm{M}$ the subset of $\mathrm{M}_d(\mathbb{R})$ given by all the derivatives of the possible curves γ as above. This is the *tangent plane*, which is obviously a vector space.

Example 3. For the Heisenberg group, it is easily computed that the tangent plane $T_p\mathrm{M}$ at each point $p \in \mathrm{H}$ is

$$T_p\mathrm{H} = \left\{ \begin{bmatrix} 0 & a & c \\ 0 & 0 & b \\ 0 & 0 & 0 \end{bmatrix} \middle| \, a, b, c \in \mathbb{R} \right\}$$

Let us now consider a map φ from a matrix group M to a matrix group M′. Let p a point of M and set $p' = \varphi(p')$. Given two neighbourhood V_p and $V'_{p'}$ of p and p' in M′ and the associated maps Φ_p and $\Phi'_{p'}$ defined on open subset of $\mathbb{R}^m$ and $\mathbb{R}^{m'}$, we assume that $(\Phi'_{p'})^{-1} \circ \varphi \circ \Phi_p$ is smooth. We may the define the *differential* $\mathrm{d}_p\varphi$ of φ at p as the linear map from $T_p\mathrm{M}$ to $T_{\varphi(p)}\mathrm{M}'$ defined by

$$d_p\varphi(v) = \left. \frac{\mathrm{d}\varphi \circ \gamma'}{\mathrm{d}t} \right|_{t=0}$$

where $\gamma : (-\varepsilon, \varepsilon) \to \mathrm{M}$ is any smooth path such that $\gamma(0) = p$ and $\gamma'(0) = v$ for $v \in T_p\mathrm{M}$.

Remark 10. The advantage with matrix groups is that $\mathrm{M}_d(\mathbb{R})$ gives a global systems of coordinates for M and any tangent planes. However, as usual in differential geometry, even if we may identify $T_p\mathrm{M}$ with $T_q\mathrm{M}$, they are really different spaces.

Two particular smooth maps are the following: for a given p in M, let us set

$$R_p(q) = q \times p \text{ and } L_p(q) = p \times q$$

for all $q \in \mathrm{M}$.

The differentials of $R_p : T_q\mathrm{M} \to T_{q\times p}\mathrm{M}$ and $L_p : T_q\mathrm{M} \to T_{p\times q}\mathrm{M}$ are easily computed:

$$\mathrm{d}_q R_p(v) = v \times p \text{ and } \mathrm{d}_q R_p(v) = p \times v \text{ for any } q \in \mathrm{M},\ v \in T_q\mathrm{M}.$$

In particular, this implies that the left or right multiplication of an element of T_qM by an element of M gives an element in some tangent space of M.

Using for p the inverse q^{-1} of $q \in \mathrm{M}$, we deduce that any element of the tangent plane $T_q\mathrm{M}$ at any is in bijection with an tangent plane $T_{\mathrm{Id}}\mathrm{M}$ at the identity matrix Id (which necessarily belongs to M). Hence, the dimension of $T_q\mathrm{M}$ does not depend on q, and the dimension of $T_{\mathrm{Id}}\mathrm{M}$ is then called the *dimension of the matrix group* M.

Let us denote by $T\mathrm{M}$ the set $\cup_{p\in\mathrm{M}} T_p\mathrm{M}$, which is called the *tangent bundle of* M. This set has itself a manifold structure. A smooth *vector field* is an application that associates at any point p of M a tangent vector X_p in $T_p\mathrm{M}$ and such that the dependence is smooth (the precise definition uses local coordinates, as above). An *integral curve* along X is a smooth path $\gamma : [0,T] \to \mathrm{M}$ such that $\gamma'(t) = X_{\gamma(t)}$.

Given two matrix groups M and M′ with a smooth map φ between them and two vectors fields X and X' on M and M′, we say that X and X' are *related* if $X'_\varphi(p)$ is equal to $\mathrm{d}_p\varphi(X_p)$ at any point p of M. In particular, this means that if γ is an integral curve of X, then $\varphi\circ\gamma$ is an integral curve of X'.

A *left-invariant vector field* is a vector field X such that $\mathrm{d}_q L_p(X_q) = X_{L_p q}$. For a matrix group, this means that $p \times X_q = X_{p\times q}$. Using $q = \mathrm{Id}$, the value of a left-invariant vector field X may be deduced from the value of X at Id, that is from a vector in $T_{\mathrm{Id}}\mathrm{M}$.

Let γ be the integral curve of a left-invariant vector field X, with $\gamma(0) = p$ (and then $\gamma'(0) = X_p = p \times X_{\mathrm{Id}}$). We then obtain that

$$\gamma'(t) = X_{\gamma(t)} = \gamma(t) \times X_{\mathrm{Id}} = \gamma(t) \times p^{-1} \times X_p.$$

When $p = \mathrm{Id}$ and $X_{\mathrm{Id}} = v$, we deduce that $\gamma'(t) = \gamma(t) \times v$ which we know how to solve:

$$\gamma(t) = \exp(tv) \text{ for } t \geq 0,$$

where exp is the matrix exponential:

$$\exp(v) = \mathrm{Id} + \sum_{k\geq 1} \frac{1}{k!} v^k.$$

As $\exp(-v)$ is the inverse of $\exp(v)$, one can extend γ to $\mathbb{R}$. In addition, we also easily obtain that $\gamma(t+s) = \gamma(t) \times \gamma(s)$, so that $\gamma : \mathbb{R} \to \mathrm{M}$ is a group homomorphism.

Proposition 5 (See for example [DK00, Proposition 1.3.4, p. 19])**.** *There exists some open neighbourhood U of* 0 *in* $T_{\mathrm{Id}}\mathrm{M}$ *and some neighbourhood V of* Id *in* M *such that the application* exp *is a* C^1 *diffeomorphism between U and V.*

Example 4. For the Heisenberg group H, we have that $P^3 = 0$ for $P \in T_{\mathrm{Id}}\mathrm{H}$ (which means that H is a step 2 nilpotent group) and then

$$\exp(P) = \mathrm{Id} + P + \frac{1}{2}P^2.$$

In addition, for $Q \in \mathrm{H}$, $P = \mathrm{Id} - Q \in T_{\mathrm{Id}}\mathrm{H}$ and one can define

$$\log(\mathrm{Id} + P) = P - \frac{1}{2}P^2.$$

Here, both $\exp : T_{\mathrm{Id}}\mathrm{H} \to \mathrm{H}$ and $\log : \mathrm{H} \to T_{\mathrm{Id}}\mathrm{H}$ are one-to-one map that are reciprocal, and exp is a global C^1 diffeomorphism.

More generally, the inverse of the exponential is also denoted by log, and as it maps a neighbourhood of V_{Id} of M containing V_{Id} to the vector space $T_{\mathrm{Id}}\mathrm{M}$, this gives a local system of coordinates $\Psi_{\mathrm{Id}} : V_{\mathrm{Id}} \to \mathbb{R}^m$ (where m is the dimension of the matrix group) by $\Psi_{\mathrm{Id}} = i \circ \log$, where $i : T_{\mathrm{Id}}\mathrm{M} \mapsto \mathbb{R}^m$ is the map which naturally identifies $T_{\mathrm{Id}}\mathrm{M}$ with $\mathbb{R}^m$. This function $\Phi : V \to \mathbb{R}^m$ is called the *normal chart* or the *logarithmic chart*.

We then deduce a local system of coordinates in a neighbourhood V of a point p of M by $\Phi_p : V_p \to \mathbb{R}^m$ with $\Phi_p(x) = i(\log(p^{-1} \otimes x))$ for $x \in V_p$.

Another map from M to M of interest is the *adjoint* defined by

$$\mathrm{Ad}(p)(q) = p \times q \times p^{-1} \text{ for } p, q \in \mathrm{M}.$$

Of course, the interest of this map comes from the fact that in general, M is not an Abelian group and then that $p \times q \neq q \times p$. It can be turned into a map from $\mathrm{T}_{\mathrm{Id}}M$ to $\mathrm{T}_{\mathrm{Id}}M$, still denoted by $\mathrm{Ad}(p)$, by setting $\mathrm{Ad}(p) = p \times q \times p^{-1}$ for $q \in T_{\mathrm{Id}}\mathrm{M}$. This new map $\mathrm{Ad}(p)$ is simply the differential at Id of $\mathrm{Ad}(p)$.

Given some smooth path $\gamma : (-\varepsilon, \varepsilon) \to \mathrm{M}$ with $\gamma(0) = \mathrm{Id}$ and $\gamma'(0) = p \in T_{\mathrm{Id}}\mathrm{M}$,

$$\mathrm{ad}(p)(q) \stackrel{\mathrm{def}}{=} \left.\frac{\mathrm{d}\,\mathrm{Ad}(\gamma(t))(q)}{\mathrm{d}t}\right|_{t=0} = p \times q - q \times p.$$

For two matrices $p, q \in \mathrm{M}_d(\mathbb{R})$, we denote by $[p, q]$ their brackets — called their *Lie brackets* — $[p, q] = p \times q - q \times p$. Hence, $\mathrm{ad}(p)(q) = [p, v]$, and we see that from the definition of ad, $[p, q]$ belongs to $T_{\mathrm{Id}}\mathrm{M}$ when $p, q \in T_{\mathrm{Id}}\mathrm{M}$.

The space $(T_{\mathrm{Id}}\mathrm{M}, [\cdot, \cdot])$ has then a Lie algebra structure.

The Lie brackets are useful for the following property: let p and q in $T_{\mathrm{Id}}\mathrm{M}$, and let t be small enough. Then

$$\begin{aligned} (39) \quad & \exp(tp) \times \exp(tq) \\ & \quad = \exp\left(tp + tq + \frac{t^2}{2}[p, q] + \frac{t^3}{12}[p, [p, q]] + \frac{t^3}{12}[q, [q, p]] + \cdots\right). \end{aligned}$$

This is the Dynkin formula (or also called the Baker-Campbell-Hausdorff formula), for which the complete (infinite) expansion may be given with the help of the Lie brackets (See for example [DK00, § 1.7, p. 29]).

If we identify an element p of the tangent space $T_{\mathrm{Id}}\mathrm{M}$ with the flow $t \mapsto \exp(tp)$ is generates, a geometric interpretation of the Lie bracket follows from (39), as for ε small enough,

$$\exp(\varepsilon p) \times \exp(\varepsilon q) \times \exp(-\varepsilon p) \times \exp(-\varepsilon q) = \exp(\varepsilon^2[p, q] + \mathrm{o}(\varepsilon^2))$$

which means that if we follow the flow $t \mapsto \exp(tp)$ in direction of p up to a time ε, then the flow $t \mapsto \exp(q)$ in the direction of q before coming back in the direction of $-p$ and then of $-q$, always up to a time ε, we arrive close to a point given by the value of the flow $t \mapsto \exp(t[p,q])$ at time ε^2.

Example 5. For the Heisenberg group H, we easily obtain that the product of two matrices P and Q in $T_{\mathrm{Id}}\mathrm{H}$ is of type

$$PQ = \begin{bmatrix} 0 & 0 & c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ for some } c \in \mathbb{R}$$

and then that the product of the matrices P, Q and R in $T_{\mathrm{Id}}\mathrm{H}$ is equal to 0. Then Formula (39) becomes an exact formula

$$\exp(P) \times \exp(Q) = \exp\left(P + Q + \frac{1}{2}[P,Q]\right)$$

and is true whatever the norms of P and Q.

We now consider an element $x = (a,b,c) \in \mathrm{A}(\mathbb{R}^2)$, and

$$\Phi(x) = \begin{bmatrix} 0 & a & c \\ 0 & 0 & b \\ 0 & 0 & 0 \end{bmatrix}. \tag{40}$$

Clearly, Φ is a one-to-one map between $\mathrm{A}(\mathbb{R}^d)$ and $T_{\mathrm{Id}}\mathrm{H}$. In addition, it is easily obtained that

$$\Phi([x,y]) = [\Phi(x), \Phi(y)] \text{ for all } x, y \in \mathrm{A}(\mathbb{R}^2),$$

or in other words, that Φ is a Lie algebra isomorphism between $(\mathrm{A}(\mathbb{R}^d), [\cdot,\cdot])$ and $(T_{\mathrm{Id}}\mathrm{H}, [\cdot,\cdot])$. With the exponential application exp, we may then identify an path $\mathbf{x}$ in $\mathrm{A}(\mathbb{R}^2)$ with a path $\mathbf{y} = \exp(\mathbf{x})$ living in the Heisenberg group. The path $\mathbf{x}$ takes its values in the vector space $\mathrm{A}(\mathbb{R}^d)$ and $\mathbf{x}_t$ gives the "direction" to follow to reach $\mathbf{y}_t$ by the use of the integral curves of the integral curves of left-invariant vector fields.

6.4. **Lie groups.** We have already seen that $(\mathrm{A}(\mathbb{R}^2), \boxplus)$ is a Lie group, that is a group $(G, \times)$ such that $(x,y) \mapsto x \times y$ and $x \mapsto x^{-1}$ are continuous. We denote by 1 the neutral element of G.

Here, we consider groups $(G, \times)$ that are finite-dimensional manifold of class C^2 and such that $(x,y) \mapsto x \times y$ and $x \mapsto x^{-1}$ are also of class C^2. Any matrix group is a Lie group.

We recall here some general results about G, which are merely a copy of the previous statements on matrix groups. For $x \in G$, let us denote by $T_x(G)$ the tangent plane at x. A *vector field* X is a differentiable application $X : x \in G \mapsto X_x \in T_xG$.

A *left-invariant* vector field X is a vector field such that $X_{L_x(y)} = \mathrm{d}_y L_x X_y$ for all $x, y \in G$, where $L_x(y) = x \times y$. It is easily shown that for such a vector field,

$$X_x = \mathrm{d}_1 L_x X_1, \ \forall x \in G,$$

where 1 is the neutral element of the Lie group G. In other words, a left-invariant vector field is fully characterized by the tangent vector X_1 in the tangent plane $T_1(G)$ at the identity of G.

An *integral curve of* X is a differentiable curve $\gamma : \mathbb{R}_+ \to G$ such that

$$\frac{\mathrm{d}\gamma(t)}{\mathrm{d}t} = X_{\gamma(t)}.$$

A *one-parameter subgroup of* G is a differentiable curve $\gamma : \mathbb{R} \to G$ such that $\gamma(t+s) = \gamma(t) \times \gamma(s)$ for all $s, t \in \mathbb{R}$ (note that $\gamma(-t) = \gamma(t)^{-1}$ for all $t \in \mathbb{R}$). This implies in particular that $\gamma(0) = 1$. If γ is an integral curve of a left-invariant vector field X, then γ is deduced from the tangent vector $X_1 \in T_1 G$ at the identity 1 of G. This vector X_1 is then called the *generator* of γ. Given a vector v in $T_1 G$, it is usual to denote by $(\exp(tv))_{t\in\mathbb{R}}$ the one-parameter subgroup of G generated by v.

One may define a map Ad on G such that $\mathrm{Ad}(x) : y \mapsto x \times y \times x^{-1}$. Its differential $\mathrm{Ad}'(x) \stackrel{\text{def}}{=} \mathrm{d}_1 \mathrm{Ad}(x)$ at 1 maps $T_1 G$ to $T_1 G$, which is linear. Hence, $x \mapsto \mathrm{Ad}'(x)$ can be seen as a map from G to $\mathrm{L}(T_1 G, T_1 G)$, the vector space of linear maps from $T_1 G$ to $T_1 G$, and its differential $\mathrm{ad}(x) \stackrel{\text{def}}{=} \mathrm{d}_1 \mathrm{Ad}'$ is a linear map from $T_1 G$ to $\mathrm{L}(T_1 G, T_1 G)$. Thus, for $(x, y) \in T_1 G^2 \mapsto \mathrm{ad}(x)(y)$ is a bilinear map with values in $T_1 G$, which is anti-symmetric: $\mathrm{ad}(y)(x) = -\mathrm{ad}(x)(y)$. We then define by $[x, y] \stackrel{\text{def}}{=} \mathrm{ad}(x)(y)$ the Lie bracket of x and y, and $(T_1 G, [\cdot, \cdot])$ is a Lie algebra. This space is called the *Lie algebra* of G.

For a matrix group, this Lie bracket correspond to the Lie bracket of matrices.

6.5. **Tensor algebra.** We have introduced matrix groups, and we have seen that $(\mathrm{A}(\mathbb{R}^d), [\cdot, \cdot])$ is isomorphic to the Lie algebra $T_{\mathrm{Id}}\mathrm{H}$ of the Heisenberg group. We will now construct a bigger space, that will contain also the Heisenberg group.

We consider now the following *tensor algebra* $\mathrm{T}(\mathbb{R}^2) = \mathbb{R} \oplus \mathbb{R}^2 \oplus (\mathbb{R}^2 \otimes \mathbb{R}^2)$ where $\mathbb{R}^2 \otimes \mathbb{R}^2$ is the tensor product of $\mathbb{R}^2$ (on this notion see for example [DP91]). If $\{e_1, e_2\}$ is the canonical basis of $\mathbb{R}^2$, then $\mathbb{R}^2 \otimes \mathbb{R}^2$ is the vector space of dimension 4 with basis $\{e_1 \otimes e_1, e_1 \otimes e_2, e_2 \otimes e_1, e_2 \otimes e_2\}$. For $x, y \in \mathbb{R}^2$,

$$x \otimes y = (x^1 e_1 + x^2 e_2) \otimes (y^1 e_1 + y^2 e_2) = \sum_{i,j=1,2} x^i y^j e_i \otimes e_j,$$
$$\lambda(x \otimes y) = (\lambda x) \otimes y = x \otimes (\lambda y), \ \forall \lambda \in \mathbb{R}.$$

Any element $x \in \mathrm{T}(\mathbb{R}^2)$ may be decomposed as $x = (x^0, x^1, x^2)$ where $x^0 \in \mathbb{R}$, $x^1 \in \mathbb{R}^2$ and $x^2 \in \mathbb{R}^2 \otimes \mathbb{R}^2$. This space $\mathrm{T}(\mathbb{R}^2)$ is equipped with the addition term-wise addition $+$, and the multiplication $\otimes$ defined by the tensor product between two elements of $\mathbb{R}^2$ and

$$x \otimes y = xy \text{ if } x \in \mathbb{R},\ y \in \mathrm{T}(\mathbb{R}^2),$$
$$x \otimes y \otimes z = 0 \text{ if } x, y, z \in \mathbb{R}^2.$$

The element $e_0 = 1 = (1, 0, 0)$ is the neutral element of $\mathrm{T}(\mathbb{R}^2)$ for $\otimes$, while $0 = (0, 0, 0)$ is the neutral element of $+$. The space $(\mathrm{T}(\mathbb{R}^2), +, \otimes)$ is an associative algebra, which is obtained by quotienting the tensor algebra $\mathbb{R} \oplus \mathbb{R}^2 \oplus \mathbb{R}^2 \otimes \mathbb{R}^2 \oplus \cdots$ by the ideal formed by all the elements which belongs to $(\mathbb{R}^2)^{\otimes 3} \oplus (\mathbb{R}^2)^{\otimes 4} \oplus \cdots$.

Remark 11. Let us consider the space $\mathbb{R}\langle X_1, X_2\rangle$ of polynomials with two non-commutative variables X_1 and X_2, as well as the equivalence relation $\sim$ on $\mathbb{R}\langle X_1, X_2\rangle$ defined by $P \sim Q$ if $P - Q$ is the sum of terms of total degree at least 3. Then there exists an associative algebra isomorphism Φ from $(\mathrm{T}(\mathbb{R}^2), +, \otimes)$ to $(\mathbb{R}\langle X_1, X_2\rangle/\sim, +, \times)$ such that $\Phi(e_i) = X_i$ for $i = 1, 2$. In other words, the elements of $\mathrm{T}(\mathbb{R}^2)$ are manipulated as polynomials where the terms of total degree at most 2 are kept.

For $\xi \in \{0, 1\}$, we denote by $\mathrm{T}_\xi(\mathbb{R}^2)$ the subset of $\mathrm{T}(\mathbb{R}^2)$ defined by

$$\mathrm{T}_\xi(\mathbb{R}^2) = \left\{(\xi, x^1, x^2) \middle| x^1 \in \mathbb{R}^2, x^2 \in \mathbb{R}^2 \otimes \mathbb{R}^2\right\}.$$

Lemma 13. *The space* $(\mathrm{T}_1(\mathbb{R}^2), \otimes)$ *is a non-commutative group.*

Proof. Clearly, if $x, y \in \mathrm{T}_1(\mathbb{R}^2)$, then $x \otimes y \in \mathrm{T}_1(\mathbb{R}^2)$. That $(\mathrm{T}_1(\mathbb{R}^2), \otimes)$ is non-commutative follows from the very definition of $\otimes$. To show it is a group, it remains to compute the inverse of each element. If $x = (1, x^1, x^2)$, then $x^{-1} = (1, -x^1, -x^2 + x^1 \otimes x^1)$ is the inverse of x. □

For $x, y \in \mathrm{T}(\mathbb{R}^2)$, we define the bracket of x and y by

$$[x, y] = x \otimes y - y \otimes x.$$

If $x = (x^0, x^1, x^2)$ and $y = (y^0, y^1, y^2)$ belong to $\mathrm{T}(\mathbb{R}^2)$, then

$$[x, y] = [x^1, y^1] = (x^1 \wedge y^1)[e_1, e_2].$$

Let us also note that $[x, y] = -[y, x]$.

A natural sub-vector space of $(\mathrm{T}_0(\mathbb{R}^2), +) \subset (\mathrm{T}(\mathbb{R}^2), +)$ is then

$$\mathrm{g}(\mathbb{R}^2) = \left\{x \in \mathrm{T}_0(\mathbb{R}^2) \middle| x = x^1 + x^a[e_1, e_2],\ x^1 \in \mathbb{R}^2,\ x^a \in \mathbb{R}\right\}.$$

Although not stable under $\otimes$, $\mathrm{g}(\mathbb{R}^2)$ is stable under $[\cdot, \cdot]$: if $x = (x^1, x^a)$, $y = (y^1, y^a) \in \mathrm{g}(\mathbb{R}^2)$, then

$$[x, y] = x^1 \wedge y^1[e_1, e_2] \in \mathrm{g}(\mathbb{R}^2).$$

This space $\mathrm{g}(\mathbb{R}^2)$ is of dimension 3. For $x = x^1 + x^a[e_1, e_2]$ and $y = y^1 + x^a[e_1, e_2]$, we set

$$x \boxplus y = x^1 + y^1 + (x^a + y^a)[e_1, e_2] + \frac{1}{2}[x^1, y^1]$$
$$= x^1 + y^1 + (x^a + y^a + \frac{1}{2}x^1 \wedge y^1)[e_1, e_2].$$

Finally, we define $i_{\mathrm{g}(\mathbb{R}^2),\mathrm{A}(\mathbb{R}^2)}$ by

$$i_{\mathrm{g}(\mathbb{R}^2),\mathrm{A}(\mathbb{R}^2)}(x) = (x^{1,1}, x^{1,2}, x^a) \text{ if } x = x^{1,1}e_1 + x^{1,2}e_2 + x^a[e_1, e_2].$$

It is clear that $i_{\mathrm{g}(\mathbb{R}^2),\mathrm{A}(\mathbb{R}^2)}$ is one-to-one from $\mathrm{g}(\mathbb{R}^2)$ to $\mathrm{A}(\mathbb{R}^2)$, and an additive group homomorphism from $(\mathrm{g}(\mathbb{R}^2), \boxplus)$ to $(\mathrm{A}(\mathbb{R}^2), \boxplus)$. In addition, $[i_{\mathrm{g}(\mathbb{R}^2),\mathrm{A}(\mathbb{R}^2)}(x), i_{\mathrm{g}(\mathbb{R}^2),\mathrm{A}(\mathbb{R}^2)}(y)] = i_{\mathrm{g}(\mathbb{R}^2),\mathrm{A}(\mathbb{R}^2)}[x, y]$ for all $x, y \in \mathrm{g}(\mathbb{R}^2)$, which means that $i_{\mathrm{g}(\mathbb{R}^2),\mathrm{A}(\mathbb{R}^2)}$ is also a Lie homomorphism. Hence, we identify the spaces $\mathrm{g}(\mathbb{R}^2)$ and $\mathrm{A}(\mathbb{R}^2)$. Lemmas 4 and 5 are then rewritten in the following way.

Lemma 14. *The space* $(\mathrm{g}(\mathbb{R}^2), [\cdot, \cdot])$ *is a Lie algebra, and* $(\mathrm{g}(\mathbb{R}^2), \boxplus)$ *is a Lie group with the neutral element* 0.

On $\mathrm{T}_0(\mathbb{R}^2)$, let us define

$$\exp(x) = 1 + x^1 + x^2 + \frac{1}{2}x^1 \otimes x^1 \text{ for } x = (0, x^1, x^2). \tag{41}$$

This map exp is given by the first terms of the formal expansion of the exponential, since we are working in a truncated tensor algebra.

Similarly, let us define on $\mathrm{T}_1(\mathbb{R}^2)$,

$$\log(x) = x^1 + x^2 - \frac{1}{2}x^1 \otimes x^1 \text{ for } x = (1, x^1, x^2) \in \mathrm{T}_1(\mathbb{R}^2).$$

It is easily seen that for $\exp \circ \log$ and $\log \circ \exp$ are equal to the identity respectively on $\mathrm{T}_1(\mathbb{R}^2)$ and on $\mathrm{T}_0(\mathbb{R}^2)$.

If $x, y \in \mathrm{T}_0(\mathbb{R}^2)$, then

$$\exp(x) \otimes \exp(y) = 1 + x^1 + y^1 + x^2 + y^2 + \frac{1}{2}x^1 \otimes x^1 + \frac{1}{2}y^1 \otimes y^1 + x^1 \otimes y^1$$

and then

$$\log(\exp(x) \otimes \exp(y)) = x \boxplus y \tag{42}$$

with

$$x \boxplus y = x^1 + y^1 + x^2 + y^2 + \frac{1}{2}x^1 \otimes y^1 - \frac{1}{2}y^1 \otimes x^1 = x + y + \frac{1}{2}[x, y].$$

This is the truncated version of the *Baker-Campbell-Hausdorff-Dynkin formula* (see for example [Hal03, Reu93]).

Lemma 15. *If* $\mathrm{G}(\mathbb{R}^2) = \exp(\mathrm{g}(\mathbb{R}^2))$, *then* $\mathrm{G}(\mathbb{R}^2)$ *is a subgroup of* $(\mathrm{T}_1(\mathbb{R}^2), \otimes)$ *and* exp *is a group isomorphism from* $(\mathrm{g}(\mathbb{R}^2), \boxplus)$ *to* $(\mathrm{G}(\mathbb{R}^2), \otimes)$.

Let us note that $\exp(-x)$ is the inverse of $\exp(x)$ in $\mathrm{G}(\mathbb{R}^2)$, for all $x \in \mathrm{g}(\mathbb{R}^2)$.

For a sub-vector space V of $\mathrm{T}(\mathbb{R}^2)$, π_{V} denotes the projection onto V. If $\mathrm{V} = \mathrm{Vect}(e)$ for some $e \in \mathrm{T}(\mathbb{R}^2)$, then we denote $\pi_{\mathrm{Vect}(e)}$ simply by π_e. For $x \in \mathrm{T}(\mathbb{R}^2)$, set

$$\mathfrak{s}(x) = \sum_{i,j=1,2} \frac{1}{2}(\pi_{e_i \otimes e_j}(x) + \pi_{e_j \otimes e_i}(x)) e_i \otimes e_j,$$

$$\mathfrak{a}(x) = \frac{1}{2}(\pi_{e_1 \otimes e_2}(x) - \pi_{e_2 \otimes e_1}(x))[e_1, e_2].$$

If x belongs to $\mathbb{R}^2 \otimes \mathbb{R}^2$, then

$$x = \mathfrak{s}(x) + \mathfrak{a}(x), \tag{43}$$

and $\mathfrak{s}(x)$ (resp. $\mathfrak{a}(x)$) corresponds to the symmetric (resp. anti-symmetric) part of x. Finally, let us note that for $x \in \mathrm{T}(\mathbb{R}^2)$,

$$\mathfrak{s}(x \otimes x) = \pi_{\mathbb{R}^2 \otimes \mathbb{R}^2}(x \otimes x), \tag{44}$$

For $z = \exp(x) \in \mathrm{G}(\mathbb{R}^2)$, we have

$$\mathfrak{s}(z) = \frac{1}{2}\mathfrak{s}(x \otimes x) = \frac{1}{2} x \otimes x \tag{45}$$

and

$$\mathfrak{a}(z) = \pi_{[e_1, e_2]}(x)[e_1, e_2].$$

Hence, for $x \in \mathrm{g}(\mathbb{R}^2)$, one may rewrite

$$\exp(x) = 1 + \pi_{\mathbb{R}^2}(x) + \frac{1}{2} x \otimes x + \mathfrak{a}(x) \text{ and } x = \pi_{\mathbb{R}^2}(x) + \mathfrak{a}(x). \tag{46}$$

In particular, for $z \in \mathrm{G}(\mathbb{R}^2)$, $\mathfrak{a}(\log(z)) = \mathfrak{a}(z)$.

6.6. **The tensor space as a Lie group.** It is possible to find a norm $|\cdot|$ on $\mathbb{R}^2 \otimes \mathbb{R}^2$ such that $|x \otimes y| \leq |x| \cdot |y|$ for all $x, y \in \mathbb{R}^2$ (there are indeed several possibilities [Rya02]).

For $x = (1, x^1, x^2) \in \mathrm{T}_1(\mathbb{R}^2)$ or for $x = (0, x^1, x^2) \in \mathrm{T}_0(\mathbb{R}^2)$, we set

$$\|x\|_\star = \max\{|x^1|, |x^2|\}$$

and

$$\|x\| = \max\left\{|x^1|, \sqrt{\frac{1}{2}|x^2|}\right\}.$$

Then $\|\cdot\|$ is an *homogeneous gauge* for the dilatation operator δ_t defined by $\delta_t x = (1, tx^1, t^2 x^2)$, $t \in \mathbb{R}$, since $\|\delta_t x\| = |t| \cdot \|x\|$ (see Section A). Besides, $\|x \otimes y\| \leq 3/2(\|x\| + \|y\|)$ for all $x, y \in \mathrm{T}_1(\mathbb{R}^2)$. We have introduced in Section 5.4 a dilatation operator, also denoted by δ, in a similar way. Note that for $x \in \mathrm{A}(\mathbb{R}^2)$ and $t \in \mathbb{R}$, $\exp(\delta_t x) = \delta_t \exp(x)$.

The the next lemma is easily proved.

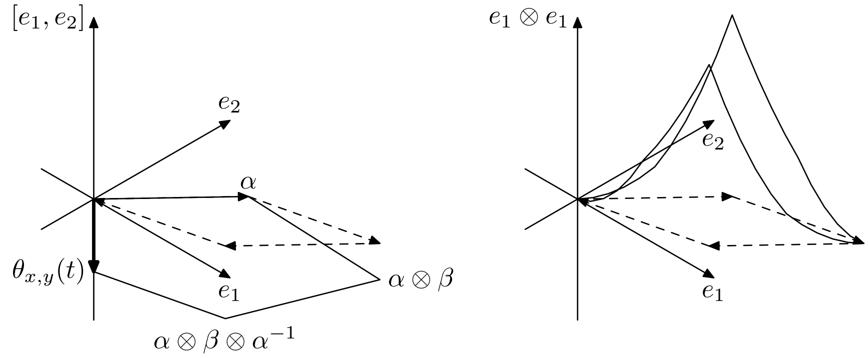


FIGURE 11. Illustration of the non-commutativity with $\alpha = \gamma_x(\sqrt{t})$ and $\beta = \gamma_y(\sqrt{t})$.

Lemma 16. *With the norm* $\|\cdot\|_\star$*, the spaces* $(\mathrm{T}_1(\mathbb{R}^2), \otimes)$ *and* $(\mathrm{G}(\mathbb{R}^2), \otimes)$ *are Lie groups, and* $\mathrm{G}(\mathbb{R}^2)$ *is a closed subgroup of* $\mathrm{T}_1(\mathbb{R}^2)$*.*

For $x \in \mathrm{g}(\mathbb{R}^2)$, $t \in \mathbb{R} \mapsto \gamma_x(t) \stackrel{\text{def}}{=} \exp(tx) \in \mathrm{G}(\mathbb{R}^2)$ is a one-parameter subgroup of $(\mathrm{G}(\mathbb{R}^2), \otimes)$. The point x is the tangent vector to $\gamma_x(t)$ for $t = 0$:

$$\left.\frac{\mathrm{d}\gamma_x}{\mathrm{d}t}\right|_{t=0} = x.$$

Hence, $\mathrm{g}(\mathbb{R}^2)$ may be identified with the tangent plane of $\mathrm{G}(\mathbb{R}^2)$ at the point 1, and indeed at any point $y \in \mathrm{G}(\mathbb{R}^2)$.

The bracket allows us to characterize the lack of commutativity of $\mathrm{G}(\mathbb{R}^2)$, as it follows from the next result, which is classical in the theory of Lie group (see Figure 11): For $x, y \in \mathrm{g}(\mathbb{R}^2)$ and for $t \geq 0$, set

$$\theta_{x,y}(t) = \gamma_x(\sqrt{t}) \otimes \gamma_y(\sqrt{t}) \otimes (\gamma_x(-\sqrt{t})) \otimes \gamma_y(-\sqrt{t}).$$

Then $\theta_{x,y}(0) = 1$ and

$$\left.\frac{\mathrm{d}\theta_{x,y}}{\mathrm{d}t}\right|_{t=0} = [x, y].$$

In our case, it follows from the truncated version the Baker-Campbell-Hausdorff-Dynkin formula (42) that $\theta_{x,y}(t) = \exp(t[x, y])$ for all $t \geq 0$.

To any Lie group corresponds a Lie algebra, which is identified to the tangent plane at the neutral elements, and then at any point. Of course, $\mathrm{g}(\mathbb{R}^2) \cong \mathrm{A}(\mathbb{R}^2)$ has been constructed to be the tangent plane of $\mathrm{G}(\mathbb{R}^2)$ at any point.

Lemma 17. *The tangent plane of* $\mathrm{G}(\mathbb{R}^2)$ *at any point may be identified with* $\mathrm{A}(\mathbb{R}^2)$*, and the tangent plane of* $\mathrm{T}_1(\mathbb{R}^2)$ *at any point may be identified with* $\mathrm{T}_0(\mathbb{R}^2)$*.*

Remark 12. We have seen that $(\mathrm{A}(\mathbb{R}^d), [\cdot, \cdot])$ is isomorphic to the Lie algebra $(T_{\mathrm{Id}}\mathrm{H}, [\cdot, \cdot])$ of the Heisenberg group.

Let us consider the map $\Psi : \mathrm{T}(\mathbb{R}^2)$ to H defined by

$$\Psi(x) = \begin{bmatrix} 1 & x_1 & x_{1,2} \\ 0 & 1 & x_2 \\ 0 & 0 & 1 \end{bmatrix} \text{ for } x = x_0 e_0 + \sum_{i=1}^{2} x_i e_i + \sum_{i,j=1}^{2} x_{i,j} e_i \otimes e_j.$$

Then we note that

$$\Psi(x \otimes y) = \Psi(x) \times \Psi(y) \text{ for } x, y \in \mathrm{T}(\mathbb{R}^2),$$

so that Ψ is a group homomorphism from $(\mathrm{T}_1(\mathbb{R}^2), \otimes)$ or $(\mathrm{G}(\mathbb{R}^2), \otimes)$ to $(\mathrm{H}, \times)$. As Ψ is linear, we easily get that $\Psi(\exp(x)) = \exp(\Phi(x))$, where Φ is the Lie algebra isomorphism given by (40). We then deduce that Ψ is indeed an isomorphism between $(\mathrm{G}(\mathbb{R}^2), \otimes)$ and the Heisenberg group $(\mathrm{H}, \times)$. The Heisenberg group is then a representation of the group $(\mathrm{G}(\mathbb{R}^2), \otimes)$.

We end with a very useful lemma, whose proof is straightforward. The notion of Lipschitz functions on spaces with homogeneous gauges is similar to the notion of Lipschitz functions (See Definition 9 in Section A).

Lemma 18. *The applications* exp *is Lipschitz continuous from* $(\mathrm{A}(\mathbb{R}^2), |\cdot|)$ *to* $(\mathrm{G}(\mathbb{R}^2), \|\cdot\|)$*, and* log *is Lipschitz continuous from* $(\mathrm{G}(\mathbb{R}^2), \|\cdot\|)$ *to* $(\mathrm{A}(\mathbb{R}^2), |\cdot|)$*.*

The applications exp *is locally Lipschitz continuous from* $(\mathrm{A}(\mathbb{R}^2), |\cdot|_\star)$ *to* $(\mathrm{G}(\mathbb{R}^2), \|\cdot\|_\star)$*, and* log *is locally Lipschitz continuous from* $(\mathrm{G}(\mathbb{R}^2), \|\cdot\|_\star)$ *to* $(\mathrm{A}(\mathbb{R}^2), |\cdot|_\star)$*.*

6.7. **The Riemannian structure on $\mathrm{T}_1(\mathbb{R}^2)$ induced by Euclidean coordinates.** A natural system of coordinates — which we call the Euclidean chart — follows from the identification of $\mathrm{T}_1(\mathbb{R}^2)$ with the vector space $\mathbb{R}^2 \oplus (\mathbb{R}^2 \otimes \mathbb{R}^2)$. If $\gamma(t) = 1 + \sum_{i=1,2} \gamma_i(t) e_i + \sum_{i,j=1,2} \gamma_{i,j}(t) e_i \otimes e_j$ is a smooth path with from $(-\varepsilon, \varepsilon)$ to $\mathrm{T}_1(\mathbb{R}^2)$ with $\gamma(0) = x \in \mathrm{T}_1(\mathbb{R}^2)$, then the derivative $\gamma'(0)$ of γ at time 0 may be expressed simply as

$$\gamma'(0) = \sum_{i=1,2} \gamma_i'(0) e_i(x) + \sum_{i,j=1,2} \gamma_{i,j}'(0) e_{i,j}(x),$$

where $e_i(x) \in T_x \mathrm{T}_1(\mathbb{R}^2)$ is the tangent vector at 0 of the path $\varphi_i(t) = x + t e_i$ and $e_{i,j}(x) \in T_x \mathrm{T}_1(\mathbb{R}^2)$ is the tangent vector at 0 of the path $\varphi_{i,j}(t) = x + t e_i \otimes e_j$.

Let us introduce the natural map A_x (*attach*) from $\mathrm{T}_0(\mathbb{R}^2)$ to $T_x \mathrm{T}_1(\mathbb{R}^2)$ which is linear and satisfies $A_x(e_i) = e_i(x)$ and $A_x(e_i \otimes e_j) = e_{i,j}(x)$ for $i, j = 1, 2$.

With this map, the derivative of γ at $t = 0$ is easily computed by

$$\gamma'(0) = A_x \left(\lim_{t \to 0} \frac{1}{t} (\gamma(t) - \gamma(0)) \right). \tag{47}$$

Hence, it is possible to endow $\mathrm{T}_1(\mathbb{R}^2)$ with a Riemannian structure $\langle\cdot,\cdot\rangle$ by setting for $x\in\mathrm{T}_1(\mathbb{R}^2)$,

$$\langle e_i(x),e_j(x)\rangle_x=\delta_{i,j},\ \langle e_i(x),e_{j,k}(x)\rangle_x=0,\langle e_{i,j}(x),e_{k,\ell}(x)\rangle_x=\delta_{i,k}\delta_{j,\ell}$$

for $i,j,k,\ell=1,2$, where $\delta_{i,j}=1$ if $i=j$ and $\delta_{i,j}=0$ otherwise. We then define $\langle\cdot,\cdot\rangle_x$ as a bilinear form on $T_x\mathrm{T}_1(\mathbb{R}^2)$.

6.8. **The left-invariant Riemannian structure on $\mathrm{T}_1(\mathbb{R}^2)$.** We have defined the logarithm map log as a map from $\mathrm{T}_1(\mathbb{R}^2)$ to the vector space $\mathrm{T}_0(\mathbb{R}^2)\cong\mathbb{R}^2\oplus(\mathbb{R}^2\otimes\mathbb{R}^2)$. Given a point $x\in\mathrm{T}_1(\mathbb{R}^2)$, another system of coordinates Φ_x from $\mathrm{T}_1(\mathbb{R}^2)$ to $\mathbb{R}^2\oplus(\mathbb{R}^2\otimes\mathbb{R}^2)$ around x is given by

$$\Phi_x(y)=i_{\mathrm{T}_0(\mathbb{R}^2)\to\mathbb{R}^2\oplus(\mathbb{R}^2\otimes\mathbb{R}^2)}\left(\log(x^{-1}\otimes y)\right),$$

where $i_{\mathrm{T}_0(\mathbb{R}^2)\to\mathbb{R}^2\oplus(\mathbb{R}^2\otimes\mathbb{R}^2)}$ is the natural identification of $\mathrm{T}_0(\mathbb{R}^2)$ with $\mathbb{R}^2\oplus(\mathbb{R}^2\otimes\mathbb{R}^2)$ for which we use the basis $\{e_i,e_j\otimes e_k\}_{i,j,k=1,2}$. For $y\in\mathrm{T}_1(\mathbb{R}^2)$, we then set

$$\Phi_x(y)=\sum_{i=1,2}\Phi_x^i(y)e_i+\sum_{i,j=1,2}\Phi_x^{i,j}(y)e_i\otimes e_j.$$

This system of coordinates is called the *normal chart* or the *logarithmic chart*.

Let $\gamma:(-\varepsilon,\varepsilon)\to\mathrm{T}_1(\mathbb{R}^2)$ be a smooth map with $\gamma(0)=x$. The derivative $\gamma'(0)$ of γ at 0 in this system of coordinate is then given by

$$\gamma'(0)=\sum_{i=1,2}(\Phi_x^i\circ\gamma)'(0)\widetilde{e}_i(x)+\sum_{i,j=1,2}(\Phi_x^{i,j}\circ\gamma)'(0)\widetilde{e}_{i,j}(x),$$

where $\widetilde{e}_i(x)$ (resp. $\widetilde{e}_{i,j}(x)$) is the tangent vector in $T_x\mathrm{T}_1(\mathbb{R}^2)$ which is the derivative at 0 of the path ψ_x^i (resp. $\psi_x^{i,j}$) such that $(\Phi_x\circ\psi_x^i)'(0)=e_i$ (resp. $(\Phi_x\circ\psi_x^{i,j})'(0)=e_i\otimes e_j$). These paths are easily computed: $\psi_x^i(t)=x\otimes\exp(te_i)$ for $i=1,2$ and $\psi_x^{i,j}(t)=x\otimes\exp(te_i\otimes e_j)$ for $i,j=1,2$.

If we write $\gamma(t)=x\otimes\exp(\lambda(t))$ for $\lambda:(-\varepsilon,\varepsilon)\to\mathrm{T}_0(\mathbb{R}^2)$ with $\lambda(0)=0$ and

$$\lambda(t)=\sum_{i=1,2}\lambda_i(t)e_i+\sum_{i,j=1,2}\lambda_{i,j}(t)e_i\otimes e_j,$$

then

$$\gamma'(0)=\sum_{i=1,2}\lambda_i'(0)\widetilde{e}_i(x)+\sum_{i,j=1,2}\lambda_{i,j}'(0)\widetilde{e}_{i,j}(x).$$

In the Euclidean structure, it follows from (47) that if $x=1+\sum_{i=1,2}x_ie_i+\sum_{i,j=1,2}x_{i,j}e_i\otimes e_j$, then

$$\begin{aligned}\widetilde{e}_i(x)&=A_x(x\otimes e_i)=e_i(x)+\sum_{j=1,2}x_je_{j,i}(x)\\ \text{and }\widetilde{e}_{i,j}(x)&=A_x(x\otimes(e_i\otimes e_j))=e_{i,j}(x).\end{aligned}\tag{48}$$

Let D_x (*detach*) be the linear map from $T_x\mathrm{T}_1(\mathbb{R}^2)$ which is the inverse of A_x, that is which transform $e_i(x)$ (resp. $e_{i,j}(x)$) into e_i (resp. $e_i\otimes e_j$).

For $x \in \mathrm{T}_1(\mathbb{R}^2)$, let $L_x(y) = x \otimes y$ be the left multiplication on $\mathrm{T}_1(\mathbb{R}^2)$. Its differential at point y maps $T_y\mathrm{T}_1(\mathbb{R}^2)$ to $T_{x\otimes y}\mathrm{T}_1(\mathbb{R}^2)$ and is defined by

$$\mathrm{d}_y L_x(v) = A_{x\otimes y}(x \otimes D_y(v)).$$

A left-invariant vector field X on $\mathrm{T}_1(\mathbb{R}^2)$ satisfies $X_x = \mathrm{d}_1 L_x(X_1)$ and then $X_x = A_x(x \otimes D_1(X_1))$. From (48),

$$\widetilde{e}_i(x) = \mathrm{d}_1 L_x(e_i(1)) \text{ and } \widetilde{e}_{i,j}(x) = \mathrm{d}_1 L_x(e_{i,j}(1)).$$

In other words, the vector field $\widetilde{e}_i$ (resp. $\widetilde{e}_{i,j}$) — it is easily verified that they varies smoothly — is then the left-invariant vector field generated by $e_i(1)$ (resp. $e_{i,j}(1)$) in the Lie group $(\mathrm{T}_1(\mathbb{R}^2), \otimes)$.

We may then define another bilinear form $\langle\langle\cdot,\cdot\rangle\rangle_x$ at any point x of $\mathrm{T}_1(\mathbb{R}^2)$ by

$$\langle\langle\widetilde{e}_i(x), \widetilde{e}_j(x)\rangle\rangle_x = \delta_{i,j},\ \langle\langle\widetilde{e}_i(x), \widetilde{e}_{j,k}(x)\rangle\rangle_x = 0, \langle\langle\widetilde{e}_{i,j}(x), \widetilde{e}_{k,\ell}(x)\rangle\rangle_x = \delta_{i,k}\delta_{j,\ell}$$

for $i, j, k, \ell = 1, 2$. These bilinear forms induces another Riemannian structure $\langle\langle\cdot,\cdot\rangle\rangle$ on $\mathrm{T}_1(\mathbb{R}^2)$.

Let us note that for $v, w \in T_1\mathrm{T}_1(\mathbb{R}^2)$ and $x \in \mathrm{T}_1(\mathbb{R}^2)$,

$$\langle\langle\mathrm{d}_1 L_x(v), \mathrm{d}_1 L_x(w)\rangle\rangle_x = \langle\langle v, w\rangle\rangle_1,$$

which means that $\langle\langle\cdot,\cdot,\rangle\rangle$ is a *left-invariant metric*. For a left-invariant vector field X, the norm $\langle\langle X_x, X_x\rangle\rangle_x$ is constant.

Let us introduce the linear maps $\widetilde{A}_x : \mathrm{T}_0(\mathbb{R}^2) \to T_x\mathrm{T}_1(\mathbb{R}^2)$ and $\widetilde{D}_x : T_x\mathrm{T}_1(\mathbb{R}^2) \to \mathrm{T}_0(\mathbb{R}^2)$ such that $\widetilde{A}_x(e_i) = \widetilde{e}_i(x)$, $\widetilde{A}_x(e_i\otimes e_j) = \widetilde{e}_{i,j}(x)$ and $\widetilde{D}_x$ is the inverse of $\widetilde{A}_x$.

If $(\cdot|\cdot)$ is the natural scalar product on $\mathrm{T}_0(\mathbb{R}^2)$ for which $\{e_i, e_j \otimes e_k\}_{i,j,k=1,2}$ is orthonormal, then for $x \in \mathrm{T}_1(\mathbb{R}^2)$ and $v, w \in T_x\mathrm{T}_1(\mathbb{R}^2)$,

$$\langle v, w\rangle_x = (D_x(v)|D_x(w)) \text{ and } \langle\langle v, w\rangle\rangle_x = (\widetilde{D}_x(v)|\widetilde{D}_x(w)) \tag{49}$$

To conclude this section, let us remark that it is very easy to express a vector $v \in T_x\mathrm{T}_1(\mathbb{R}^2)$ in the basis $\{\widetilde{e}_i(x), \widetilde{e}_{j,k}(x)\}_{i,j,k=1,2}$ when we known its decomposition in $\{e_i(x), e_{j,k}(x)\}_{i,j,k=1,2}$: If $v = \sum_{i=1,2} v^i e_i(x) + \sum_{i,j=1,2} v^{i,j} e_{i,j}(x)$, then

$$v = \widetilde{A}_x(x^{-1} \otimes D_x(v)), \tag{50}$$

so that with (49),

$$\langle\langle v, w\rangle\rangle_x = (x^{-1} \otimes D_x(v)|x^{-1} \otimes D_x(w)).$$

In addition, if γ is a smooth path from $(-\varepsilon, \varepsilon)$ to $\mathrm{T}_1(\mathbb{R}^2)$, then we get from (50) a simple expression for the derivative γ' of γ at time $t \in (-\varepsilon, \varepsilon)$ in the basis $\{\widetilde{e}_i(x), \widetilde{e}_{j,k}(x)\}_{i,j,k=1,2}$ by

$$\gamma'(t) = \lim_{h\to 0} \widetilde{A}_{\gamma(t)} \left(\frac{1}{h}(\gamma(t)^{-1} \otimes \gamma(t+h) - 1)\right). \tag{51}$$

6.9. **The exponential map revisited.** Let us consider an integral curve γ along a left-invariant vector field X with $\gamma(0)=1$. If for $t\geq 0$, the path $\gamma(t)$ is written

$$\gamma(t)=1+\sum_{i=1,2}\gamma_i(t)e_i+\sum_{i,j=1,2}\gamma_{i,j}(t)e_i\otimes e_j,$$

then

$$\gamma'(t)=X_{\gamma(t)}=\mathrm{d}_1L_{\gamma(t)}(X_1)=A_{\gamma(t)}(\gamma(t)\otimes D_1(X_1))$$

and, if $X_1=\sum_{i=1,2}v_ie_i+\sum_{i,j=1,2}v_{i,j}e_i\otimes e_j$,

$$\gamma_i'(t)=v_ie_i,\ \gamma_{i,j}'v_{i,j}+\gamma_i(t)v_j$$

for $i,j=1,2$. If follows that

$$\gamma_i(t)=tv_i \text{ and } \gamma_{i,j}(t)=tv_{i,j}+\frac{t^2}{2}v_iv_j$$

which means that $\gamma(t)=\exp(tX_1)$ where exp has been defined by (41). Let us note that $\exp(tX_1)\otimes\exp(sX_1)=\exp((t+s)X_1)$, since $(tX_1)\boxplus(sX_1)=(t+s)X_1$. Hence, the one-parameter subgroup of $\mathrm{T}_1(\mathbb{R}^d)$ generated by v is given by $t\in\mathbb{R}\mapsto\exp(tX_1)$.

In the sytem of left-invariant coordinates, we get that

$$\begin{aligned}\gamma'(t)=\widetilde{A}_{\gamma(t)}(\gamma(t)^{-1}\otimes D_{\gamma(t)}(\gamma'(t)))=\widetilde{A}_{\gamma(t)}(\gamma(t)^{-1}\otimes\gamma(t)\otimes D_1(X_1))\\=\widetilde{A}_{\gamma(t)}(D_1(X_1)),\end{aligned}$$

which means that $\gamma'(t)$ is constant in the system of left-invariant coordinates.

It follows that for any $y\in\mathrm{T}_1(\mathbb{R}^2)$, it is always possible to construct an integral curve γ along a left-invariant vector field that connects x to y and which is given by $(x\otimes\exp(tv))_{t\in[0,1]}$ with $v=\log(x^{-1}\otimes y)$.

6.10. **Some particular curves for the left-invariant Riemannian metric.** For two points x and y in $\mathrm{T}_1(\mathbb{R}^2)$ and a smooth path γ from $[0,1]$ to $\mathrm{T}_1(\mathbb{R}^2)$ with $\gamma(0)=x$ and $\gamma(1)=y$, let us consider the energy $\mathrm{Energy}(\gamma)$ of the path γ as

$$\mathrm{Energy}(\gamma)\stackrel{\mathrm{def}}{=}\frac{1}{2}\int_0^1\langle\langle\gamma'(s),\gamma'(s)\rangle\rangle_{\gamma(s)}\,\mathrm{d}s.$$

For $t\in[0,1]$, set $\varphi(t)=\log(a^{-1}\otimes\gamma(t))$ so that $\gamma(t)=a\otimes\exp(\varphi(t))$ and then $\varphi(0)=0$. The path φ belongs to $\mathrm{T}_0(\mathbb{R}^2)$. With (51), we get that

$$\begin{aligned}\gamma'(t)=\widetilde{A}_{\gamma(t)}\left(\lim_{h\to 0}\frac{1}{h}(\exp((-\varphi(t))\boxplus\varphi(t+h))-1)\right)\\=\widetilde{A}_{\gamma(t)}\left(\varphi'(t)+\frac{1}{2}[\varphi'(t),\varphi(t)]\right),\end{aligned}$$

where $\varphi(t)=\sum_{i=1,2}\varphi_i(t)e_i+\sum_{i,j=1,2}\varphi_{i,j}(t)e_i\otimes e_j$ and $\varphi'(t)=\sum_{i=1,2}\varphi_i'(t)e_i+\sum_{i,j=1,2}\varphi_{i,j}'(t)e_i\otimes e_j$.

Thus, the energy of γ is given by

$$\mathrm{Energy}(\gamma) = \frac{1}{2}\int_0^1 \|\varphi'(s) + \frac{1}{2}[\varphi'(s), \varphi(s)]\|_{\mathrm{Euc}}^2 \,\mathrm{d}s.$$

where $\|\cdot\|_{\mathrm{Euc}}$ is the Euclidean norm of $\mathrm{T}_0(\mathbb{R}^2)$ identified with $\mathbb{R}^6$.

We now consider the particular path γ such that $\varphi(0) = 0$, $\varphi(1) = \log(a^{-1}\otimes b)$ and $\varphi'(t)+\frac{1}{2}[\varphi'(t), \varphi(t)]$ is constant over $[0,1]$. This means that $\varphi(t) = tv$ for some $v \in \mathrm{T}_0(\mathbb{R}^2)$. This comes from the fact that that the projection of φ on $\mathbb{R}^2$ is then constant, since $[\phi'(t), \phi(t)]$ lives in $\mathbb{R}^2\otimes\mathbb{R}^2$ and then $[\varphi'(t), \varphi(t)] = 0$ for $t \in [0,1]$. With the condition on $\varphi(1)$, $\varphi(t) = t\log(a^{-1}\otimes b)$ and $\gamma(t) = a\otimes\exp(t\log(a^{-1}\otimes b))$.

Let also $\psi : [0,1] \to \mathrm{T}_0(\mathbb{R}^d)$ be a differentiable path with $\psi(0) = \psi(1) = 0$. Set for $\varepsilon > 0$,

$$\Gamma_\varepsilon(t) = a\otimes\exp(\varphi(t)\boxplus(\varepsilon\psi(t)))$$

so that

$$\Gamma_\varepsilon'(t) = \widetilde{A}_{\Gamma_\varepsilon(t)}\Big(\varphi'(t) + \varepsilon\psi'(t) + \varepsilon[\varphi'(t), \psi(t)] \\ + \frac{1}{2}[\varphi'(t), \varphi(t)] + \frac{\varepsilon^2}{2}[\psi'(t), \psi(t)]\Big).$$

This, if $\varphi(t) = tv$ for some $v \in \mathrm{T}_0(\mathbb{R}^2)$, we then get that

$$\mathrm{Energy}(\Gamma_\varepsilon(t)) = \frac{1}{2}\|v\|_{\mathrm{Euc}}^2 + \varepsilon\int_0^1 (v|\psi'(t))\,\mathrm{d}t + \frac{\varepsilon}{2}\int_0^1 (v|[v,\psi(t)])\,\mathrm{d}t \\ + \frac{\varepsilon^2}{2}\int_0^1 \Big\|\psi'(t) + [v,\psi(t)] + \frac{\varepsilon}{2}[\psi'(t), \psi(t)]\Big\|_{\mathrm{Euc}}^2 \,\mathrm{d}t \\ + \frac{\varepsilon^2}{4}\int_0^1 (v|[\psi'(t), \psi(t)])\,\mathrm{d}t.$$

Since $\psi(0) = \psi(1) = 0$, $\int_0^1 (v|\psi'(t))\,\mathrm{d}t = 0$. But the term $\frac{\varepsilon}{2}\int_0^1 (v|[v,\psi(t)])\,\mathrm{d}t$ may be different from 0, as well as $\frac{\varepsilon^2}{4}\int_0^1 (v|[\psi'(t),\psi(t)])\,\mathrm{d}t$. Hence, we see that γ is not necessarily a path with minimal energy.

Remark 13. At first, this result seems to contradicts the result that $t \mapsto \exp(tv)$ is a path with a constant derivative in the left-invariant system of coordinates seen in Section 6.9 above. Indeed, the geodesics ξ associated to the left-invariant Riemannian structure are those for which $\nabla_{\xi'(t)}\xi'(t) = 0$ where ∇ is the Levi-Civita connection associated to $\langle\langle\cdot,\cdot\rangle\rangle$. Since there exists some elements x, y and z such that

$$\langle\langle [z,x], y\rangle\rangle \neq \langle\langle x, [z,y]\rangle\rangle$$

(consider for example $x = e_1$, $x = e_2$ and $y = e_1\otimes e_2$), this connection differs from the Cartan-Schouten (0) connection ∇^{CS} which is such that all paths of type $\gamma(t) = \exp(tv)$ are geodesics in the sense that $\nabla^{CS}_{\gamma'(t)}(\gamma'(t)) = 0$. On this topic, see for example [MM02].

However, if v belongs to $\mathrm{Vect}\{e_1, e_2\}$, then $(v|\,[v, \psi'(t)]) = 0$ and thus

$$\mathrm{Energy}(\Gamma_\varepsilon(t)) \geq \mathrm{Energy}(\gamma) = \frac{1}{2}\|\log(a^{-1}\otimes b)\|^2_{\mathrm{Euc}},\ \forall\varepsilon > 0$$

and thus γ is a *geodesic*, that is a curve with minimal energy. As usual, it can also be shown that it is a path with minimal length, and the length

$$\mathrm{Length}(\gamma) \stackrel{\mathrm{def}}{=} \int_0^1 \sqrt{\langle\langle\gamma'(s), \gamma'(s)\rangle\rangle_{\gamma(s)}}\,\mathrm{d}s$$

is then equal to $\|\log(a^{-1}\otimes b)\|_{\mathrm{Euc}}$. Another simple case is when $v \in \mathrm{Vect}\{e_i \otimes e_i\}_{i,j=1,2}$, in which case $[v, w] = 0$ for all $w \in \mathrm{T}_0(\mathbb{R}^2)$ and we also obtain that γ is a geodesic.

We also deduce that the length of the geodesic between a and b for $\langle\langle\cdot,\cdot\rangle\rangle$ is smaller than $\|\log(a^{-1}\otimes b\|$.

Let us also remark that if a and b belongs to $\mathrm{G}(\mathbb{R}^2)$, then $\gamma(t)$ belongs to $\mathrm{G}(\mathbb{R}^2)$ for $t \in [0, 1]$.

Of course, if we see $\mathrm{T}_1(\mathbb{R}^2)$ with its Euclidean structure $\langle\cdot,\cdot\rangle$ then the geodesics are simply $\varphi(t) = a + t(b - a)$. In this case, $\varphi(t)$ does not belong to $\mathrm{G}(\mathbb{R}^2)$ in general when a and b are in $\mathrm{G}(\mathbb{R}^2)$.

6.11. **A transverse decomposition of the tensor space.** We have introduced a subgroup $\mathrm{G}(\mathbb{R}^2)$ of $\mathrm{T}_1(\mathbb{R}^2)$. Is this subgroup strict or not?

The tangent space of $\mathrm{T}_1(\mathbb{R}^2)$ at any point may be identified with the vector space $(\mathrm{T}_0(\mathbb{R}^2), +)$, which is dimension 6. We have also seen that the tangent space of $\mathrm{G}(\mathbb{R}^2)$ at any point may be identified with $\mathrm{A}(\mathbb{R}^2)$, and thus is of dimension 3. Then, of course, $\mathrm{G}(\mathbb{R}^2) \neq \mathrm{T}_1(\mathbb{R}^2)$. Indeed, we may be more precise on the decomposition of $\mathrm{T}_1(\mathbb{R}^2)$.

We denote by $\mathrm{S}(\mathbb{R}^2)$ the subset of $\mathrm{T}_0(\mathbb{R}^2)$ defined by

$$\mathrm{S}(\mathbb{R}^2) = \left\{ x = (0, 0, x^2) \in \mathrm{T}_0(\mathbb{R}^2) \;\middle|\; \begin{array}{r} x^2 = \lambda e_1\otimes e_1 + \mu e_2 \otimes e_2 \\ +\nu(e_1\otimes e_2 + e_2 \otimes e_1), \\ \lambda,\mu,\nu \in \mathbb{R} \end{array} \right\}.$$

In other words, an element of $\mathrm{S}(\mathbb{R}^2)$ belongs to $\mathbb{R}^2\otimes\mathbb{R}^2$ and is symmetric. Of course, $\mathrm{S}(\mathbb{R}^2)$ is linear, stable under $\otimes$ and $+$ (indeed, if $x, y \in \mathrm{S}(\mathbb{R}^2)$, then $x \otimes y = x + y$), and is a vector space of dimension 3.

To an element e of the basis of $\mathrm{T}(\mathbb{R}^2)$, we denote by π_e the projection from $\mathrm{T}(\mathbb{R}^2)$ to $\mathrm{T}(\mathbb{R}^2)$, such that $x = \pi_1(x) + \sum_{i=1,2}\pi_{e_i}(x)e_i + \sum_{i,j=1,2}\pi_{e_i\otimes e_j}(x)e_i\otimes e_j$.

The next result follows easily from the construction of the projection operator $\widehat{\Upsilon}_{\mathrm{s}} : \mathrm{T}_0(\mathbb{R}^2) \to \mathrm{S}(\mathbb{R}^2)$ and $\widehat{\Upsilon}_{\mathrm{a}} : \mathrm{T}_0(\mathbb{R}^2) \to \mathrm{A}(\mathbb{R}^2)$ defined by

$$\widehat{\Upsilon}_{\mathrm{s}}(x) = \mathfrak{s}(x) \text{ and } \widehat{\Upsilon}_{\mathrm{a}}(x) = \pi_{\mathbb{R}^2}(x) + \mathfrak{a}(x).$$

Proposition 6. *The space* $\mathrm{T}_0(\mathbb{R}^2)$ *is the direct sum of* $\mathrm{A}(\mathbb{R}^2)$ *and* $\mathrm{S}(\mathbb{R}^2)$.

This decomposition holds at the level of the tangent spaces at any point of $\mathrm{T}_1(\mathbb{R}^2)$.

Proposition 7. *Any element x of $\mathrm{T}_1(\mathbb{R}^2)$ may be written as the sum $x = y + z$ with $y \in \mathrm{G}(\mathbb{R}^2)$ and $z \in \mathrm{S}(\mathbb{R}^2)$.*

Proof. For $x \in \mathrm{T}(\mathbb{R}^2)$, let us set

$$\begin{aligned}\Upsilon_{\mathrm{s}}(x) &= \mathfrak{s}(x) - \frac{1}{2} x \otimes x,\\ \Upsilon_{\mathrm{a}}(x) &= 1 + \pi_{\mathbb{R}^2}(x) + \mathfrak{a}(x) + \frac{1}{2} x \otimes x.\end{aligned}$$

With (43), $\Upsilon_{\mathrm{a}}(x) + \Upsilon_{\mathrm{s}}(x) = x$ for all $x \in \mathrm{T}(\mathbb{R}^2)$. Also, thanks to (44) and (46), $\Upsilon_{\mathrm{s}}(\mathrm{T}_1(\mathbb{R}^2)) \subset \mathrm{S}(\mathbb{R}^2)$ and $\Upsilon_{\mathrm{a}}(\mathrm{T}_1(\mathbb{R}^2)) \subset \mathrm{G}(\mathbb{R}^2)$. □

We have to note that with the previous decomposition, $\mathrm{G}(\mathbb{R}^2)$ is not a linear subspace of $\mathrm{T}_1(\mathbb{R}^2)$, and Υ_{a} and Υ_{s} are not linear projections, since they involve quadratic terms. This is why we do not write $\mathrm{T}_1(\mathbb{R}^2)$ as the direct sum of $\mathrm{G}(\mathbb{R}^2)$ and $\mathrm{S}(\mathbb{R}^2)$. However, as the tangent plane of $\mathrm{S}(\mathbb{R}^2)$ is $\mathrm{S}(\mathbb{R}^2)$ itself, if $\mathrm{G}(\mathbb{R}^2)$ and $\exp(\mathrm{S}(\mathbb{R}^2)) = \{1 + x | x \in \mathrm{S}(\mathbb{R}^2)\}$ are sub-manifolds of $\mathrm{T}_1(\mathbb{R}^2)$, we get that $\mathrm{G}(\mathbb{R}^2)$ and $\exp(\mathrm{S}(\mathbb{R}^2))$ provides a *transverse decomposition* of $\mathrm{T}_1(\mathbb{R}^2)$, in the sense that their tangent spaces at any point x provides an orthogonal decomposition (with respect to $\langle\langle\cdot,\cdot\rangle\rangle_x$) of the tangent space of $\mathrm{T}_1(\mathbb{R}^2)$ at x.

We define a homogeneous norm $\|\cdot\|_{\mathrm{G}(\mathbb{R}^2)\times\mathrm{S}(\mathbb{R}^2)}$ by

$$\|x\|_{\mathrm{G}(\mathbb{R}^2)\times\mathrm{S}(\mathbb{R}^2)} = \max\left\{\|\Upsilon_{\mathrm{a}}(x)\|, \sqrt{\frac{1}{2}\|\Upsilon_{\mathrm{s}}(x)\|}\right\}. \tag{52}$$

It is easily shown that this homogeneous norm is equivalent to the homogeneous gauge $\|\cdot\|$ on $\mathrm{T}^1(\mathbb{R}^2)$.

6.12. **Back to the sub-Riemannian point of view.** We now come back to the result of Section 5.9, in order to bring some precision on the sub-Riemannian geometric framework. We have already seen that $(\mathrm{A}(\mathbb{R}^2), \boxplus)$ is a Lie group (Here, we no longer consider the space $\mathrm{T}(\mathbb{R}^2)$). In addition, it is a vector space and then a smooth manifold with a natural system of coordinates given by the decomposition of $a \in \mathrm{A}(\mathbb{R}^2)$ on the basis $\{e_1, e_2, e_3\}$, where e_3 corresponds to $[e_1, e_2]$.

If $\varphi_i(t; a) = a + te_i$ for $i = 1, 2, 3$ and $t \in \mathbb{R}$ and $a \in \mathrm{A}(\mathbb{R}^2)$, we denote by $e_i(a)$ the derivative $\varphi_i'(0; a)$ at time 0 of $\varphi_i(\cdot, a)$.

As in Sections 6.7, we define for $a \in \mathrm{A}(\mathbb{R}^2)$ two linear maps A_a and D_a by $A_a(e_i) = e_i(a)$ and $D_a = A_a^{-1}$.

We now proceed as in Section 6.8. The left multiplication is $L_a(y) = a \boxplus y$, and its differential $\mathrm{d}_b L_a : T_b\mathrm{A}(\mathbb{R}^2) \to T_{a\boxplus b}\mathrm{A}(\mathbb{R}^2)$ at any point b is given by

$$\mathrm{d}_b L_a(v) = A_{a\boxplus b}\left(D_b(v) + \frac{1}{2}[a, D_b(v)]\right),$$

Here $[a,v]=(a^1v^2-a^2v^1)e_3$ for $a=a^1e_1+a^2e_2+a^3e_3$.

Thus, any left-invariant vector field $(V_a)_{a\in\mathrm{A}(\mathbb{R}^2)}$ satisfies $V_a=\mathrm{d}_0L_a(V_0)$. The left-invariant vector fields $\widetilde{e}_1$, $\widetilde{e}_2$ and $\widetilde{e}_3$ associated to e_1, e_2 and e_3 are given by

$$\widetilde{e}_1(a)=e_1(a)-\frac{1}{2}a^2e_3(a),\ \widetilde{e}_2(a)=e_2(a)+\frac{1}{2}a^1e_3(a)\text{ and }\widetilde{e}_3(a)=e_3(a)$$

for $a=a^1e_1+a^2e_2+a^3e_3$. The space $\Theta(a)$ introduced in Section 5.9 is then the vector space generated by $\widetilde{e}_1(a)$ and $\widetilde{e}_2(a)$.

Let $\widetilde{A}_a$ be the linear map from $\mathrm{A}(\mathbb{R}^2)$ to $T_a\mathrm{A}(\mathbb{R}^2)$ defined by $\widetilde{A}_a(e_i)=\widetilde{e}_i(a)$. Then a vector v in $T_a\mathrm{A}(\mathbb{R}^2)$ is easily expressed in the left-invariant basis $\{\widetilde{e}_1(a),\widetilde{e}_2(a),\widetilde{e}_3(a)\}$ by

$$v=\widetilde{A}_a((-a)\boxplus D_a(v)).$$

Similarly, if $\gamma:(-\varepsilon,\varepsilon)\to\mathrm{A}(\mathbb{R}^2)$ is a smooth path, then it is easily checked that

$$\begin{aligned}\gamma'(t)&=\widetilde{A}_a\left(\lim_{h\to 0}\frac{1}{\varepsilon}(-\gamma(t))\boxplus\gamma(t+h)\right)\\&=\widetilde{A}_{\gamma(t)}\left(D_{\gamma(t)}(\gamma'(t))+\frac{1}{2}[D_{\gamma(t)}(\gamma'(t)),\gamma(t)]\right).\end{aligned}$$

For a differentiable path $\mathbf{y}_t$ in $\mathrm{A}(\mathbb{R}^2)$ we have introduced in (34) and (35) some paths α and β that corresponds indeed to the coordinates of the derivative of $\mathbf{y}$ in the basis $\{e_1,e_2,e_3\}$ and $\{\widetilde{e}_1,\widetilde{e}_2,\widetilde{e}_3\}$, in the sense that

$$\frac{\mathrm{d}\mathbf{y}_t}{\mathrm{d}t}=\sum_{i=1}^3\alpha^i(t)e_i(\mathbf{y}_t)=\sum_{i=1}^3\beta^i(t)\widetilde{e}_i(\mathbf{y}_t).$$

7. The rough paths and their integrals

7.1. **What are rough paths?** If $x\in\mathrm{G}(\mathbb{R}^2)$, then it is easily seen that for some universal constants c and c', $c\|x\|\leq|\log(x)|\leq c'\|x\|$, where $|\cdot|$ is the homogeneous norm we have defined on $\mathrm{A}(\mathbb{R}^2)$ by (16).

Definition 4. A *rough path* is a continuous path $\mathbf{x}$ with values in $\mathrm{T}_1(\mathbb{R}^2)$.

Denote by $\mathrm{C}^\alpha([0,T],\mathrm{T}_1(\mathbb{R}^2))$ the set of rough paths $\mathbf{x}:[0,T]\to\mathrm{T}_1(\mathbb{R}^2)$ such that

$$\|\mathbf{x}\|_\alpha\stackrel{\text{def}}{=}\sup_{0\leq s<t\leq T}\frac{\|\mathbf{x}_s^{-1}\otimes\mathbf{x}_t\|}{|t-s|^\alpha}$$

is finite.

A particular class of paths is the notion of geometric rough paths. The next definition follows from [FV06b], which correct some result of [Lyo98].

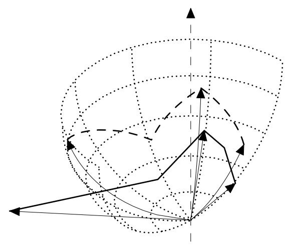


FIGURE 12. From the tangent plane $A(\mathbb{R}^2)$ at the point 1 (perpendicular to the vertical axis) to the manifold $G(\mathbb{R}^2)$: the paths $\mathbf{x}$ (dashed) and $\log(\mathbf{x})$ (plain).

Definition 5. A *geometric rough path* is a continuous path with values in $G(\mathbb{R}^2)$.

A *smooth rough path* is an element of the set

$$C^\infty([0,T];G(\mathbb{R}^2)) = \left\{ \exp(\mathbf{x}) \,\middle|\, \begin{array}{l} \mathbf{x} = x + \mathfrak{A}(x)[e_1,e_2] \\ \text{with } x \in C^\infty_{\mathrm{p}}([0,T];\mathbb{R}^2) \end{array} \right\}.$$

A *weak geometric p-rough path with Hölder control* is a path with values in $G(\mathbb{R}^2)$ which is $1/p$-Hölder continuous.

A *geometric p-rough path* is the closure of the set of smooth rough paths with respect to the $\|\cdot\|_{1/p}$-norm.

Remark 14. As discussed in Section 5.9, a rough path which is smooth is not necessarily a smooth rough path.

The space of weak geometric $1/\alpha$-rough paths with Hölder control is denoted by $C^\alpha([0,T];G(\mathbb{R}^2))$, while the space of $1/\alpha$-rough path with Hölder control is denoted by $C^{0,\alpha}([0,T];G(\mathbb{R}^2))$. This latter space is strictly included in $C^\alpha([0,T];G(\mathbb{R}^2))$. In addition $C^{0,\alpha}([0,T];G(\mathbb{R}^2))$ is a Polish space, while $C^\alpha([0,T];G(\mathbb{R}^2))$ is not a Polish space (this space is not separable: See [FV06b]). The difference between weak geometric p-rough paths and a geometric p-rough paths comes from an extension of the properties of Hölder continuous paths given in Remarks 2 and 3. For practical applications, the difference between weak geometric rough paths and geometric rough paths is not that important, if we are ready to weaken the Hölder norm (this is in general cost free).

Of course, there is a one-to-one correspondence between the paths with values in $G(\mathbb{R}^2)$ and the one with values in $A(\mathbb{R}^2)$: Since exp and log are Lipschitz continuous (see Lemma 18), we get easily the following lemma, which is illustrated by Figure 12.

Lemma 19. *A path* $\mathbf{x}$ *belongs to* $C^\alpha([0,T];G(\mathbb{R}^2))$ *if and only if* $\log(\mathbf{x})$ *belongs to* $C^\alpha([0,T];A(\mathbb{R}^2))$.

A path $\mathbf{y} = (a(t), b(t), c(t))_{t\in[0,T]}$ with value in $\mathrm{A}(\mathbb{R}^2)$ is then transformed into a path $\mathbf{x}_t = \exp(\mathbf{y}_t)$ with value in $\mathrm{G}(\mathbb{R}^2)$ by the relation

$$\mathbf{x}_t = a(t)e_1 + b(t)e_2 + \frac{1}{2}a(t)^2 e_1 \otimes e_1 + \frac{1}{2}b(t)^2 e_2 \otimes e_2 \\ + (a(t)b(t) + c(t))e_1 \otimes e_2 + (a(t)b(t) - c(t))e_2 \otimes e_1.$$

Similarly, a path $\mathbf{x}$ with values in $\mathrm{G}(\mathbb{R}^2)$ is transformed into a path $\mathbf{y}$ with values in $\mathrm{A}(\mathbb{R}^2)$ by setting $\mathbf{y}_t = \log(\mathbf{x}_t)$.

In addition, let us note that $\mathbf{x}_{s,t} \stackrel{\text{def}}{=} \mathbf{x}_s^{-1} \otimes \mathbf{x}_t = \exp((-\mathbf{y}_s) \boxplus \mathbf{y}_t)$ and then

$$\mathfrak{s}(\mathbf{x}_{s,t}) = \frac{1}{2}(x_t - x_s) \otimes (x_t - x_s)$$

where $x_t = a(t)e_1 + b(t)e_2$ is the path above which $\mathbf{x}$ lies.

Let us assume now that $\mathbf{y}$ belongs $\mathrm{C}^\alpha([0,T]; \mathrm{A}(\mathbb{R}^2))$ with $\alpha > 1/2$. We have seen in Lemma 7 that necessarily, $c(t) = \mathfrak{A}(x; 0, t)$. Hence, from (46),

$$\mathbf{x}_t = 1 + x_t + \mathfrak{A}(x; 0, t)[e_1, e_2] + \frac{1}{2}(x_t - x_0) \otimes (x_t - x_0). \tag{53}$$

As for $0 \leq s \leq t \leq T$,

$$\mathfrak{A}(x; s, t) = \frac{1}{2}\int_s^t (x_r^1 - x_s^1)\,\mathrm{d}x_r^2 - \frac{1}{2}\int_s^t (x_r^2 - x_s^2)\,\mathrm{d}x_r^1$$

$$\text{and } \frac{1}{2}(x_t^i - x_s^i)^2 = \int_s^t (x_r^i - x_s^i)\,\mathrm{d}x_r^i \text{ for } i = 1, 2,$$

we may rewrite (53) as

$$\mathbf{x}_t = 1 + x_t + \sum_{i,j=1,2} \left(\int_0^t (x_r^i - x_0^i)\,\mathrm{d}x_r^j \right) e_i \otimes e_j. \tag{54}$$

Let us note also that

$$\mathbf{x}_{s,t} \stackrel{\text{def}}{=} (-\mathbf{x}_s) \otimes \mathbf{x}_t = 1 + x_t - x_s + \sum_{i,j=1,2} \left(\int_s^t (x_r^i - x_s^i)\,\mathrm{d}x_r^j \right) e_i \otimes e_j.$$

This means that the terms of $\mathbf{x}_t$ in $\mathbb{R}^2 \otimes \mathbb{R}^2$ are the *iterated integrals of* x. When $\alpha < 1/2$, the difficulty comes from the fact that these iterated integrals are not canonically constructed. As the iterated integrals have some nice algebraic properties (see Section 8.2), we replace them by an object — a rough path — which shares the same algebraic properties, whose existence is not discussed in this article.

Let us end this Section with a result on paths that are not geometric. If $\mathbf{x}$ belongs to $\mathrm{C}^\alpha([0,T]; \mathrm{T}_1(\mathbb{R}^2))$ (with $\alpha \in (0,1]$) and $\mathbf{x}_t - 1 \in \mathrm{S}(\mathbb{R}^2)$ for all t, then

$$\|\mathbf{x}_s^{-1} \otimes \mathbf{x}_t\| = \sqrt{\frac{1}{2}|\mathbf{x}_t^2 - \mathbf{x}_s^2|} \leq \|\mathbf{x}\|_\alpha |t - s|^\alpha$$

with $\mathbf{x}_t = (1, \mathbf{x}^1_t, \mathbf{x}^2_t)$. This implies that $\mathbf{x}_t$ can be identified with a path in $\mathrm{C}^{2\alpha}([0,T];\mathbb{R}^3)$ (note that if $\alpha > 1/2$, then $\mathbf{x}$ is constant).

7.2. **Joining two points by staying in** $\mathrm{G}(\mathbb{R}^2)$**.** We have seen that the integral of a differential form f along a path $x : [0,T] \to \mathbb{R}^2$ may be written as the limit of the following scheme: we consider the family of dyadic partitions $\{t^n_k\}_{k=0,\dots,2^n}$ of $[0,T]$, and we construct approximations x^n of x such that $x_{t^n_k} = x^n_{t^n_k}$ for $k = 0,\dots,2^n$, and two successive points $x^n_{t^n_k}$ and $x^n_{t^n_{k+1}}$ are linked by a path that depends only on these two points. Then the integral $\mathfrak{I}(x)$ of f along x is defined as the limit of the integrals of f along x^n.

When x is a α-Hölder continuous path with values in $\mathbb{R}^2$ with $\alpha > 1/2$, then the "natural" family of approximation is given by piecewise linear approximations. If $\alpha \in (1/3, 1/2]$, we have seen that we need to replace x by a path $\mathbf{x}$ with values in $\mathrm{A}(\mathbb{R}^2)$ that projects onto x, and to construct x^n by joining two successive points $x^n_{t^n_k}$ and $x^n_{t^n_{k+1}}$ of x^n with some sub-Riemannian geodesics that is computed from $\mathbf{x}^n_{t^n_k}$ and $\mathbf{x}^n_{t^n_{k+1}}$. Such a path x^n is automatically lifted in a path $(x^n, \mathfrak{A}(x^n))$ in $\mathrm{C}^\alpha([0,T];\mathrm{A}(\mathbb{R}^2))$, and the integral $\mathfrak{I}(\mathbf{x})$ is defined as the limit of the $\mathfrak{I}(x^n)$.

Computations in Sections 6.1 and 7.5 has shown us that one may wish to work with piecewise linear approximations of paths of $\mathrm{C}^\alpha([0,t];\mathrm{A}(\mathbb{R}^2))$. For this, we have extended the differential form f to a differential form $\mathfrak{E}_{\mathrm{A}(\mathbb{R}^2)}(f)$ on $\mathrm{A}(\mathbb{R}^2)$. We have subsequently introduced some tensor space $\mathrm{T}(\mathbb{R}^2)$, as well as a Lie groups $\mathrm{G}(\mathbb{R}^2)$ and $\mathrm{T}_1(\mathbb{R}^2)$ whose Lie algebras are $\mathrm{A}(\mathbb{R}^2)$ and $\mathrm{T}_0(\mathbb{R}^2)$. We have also introduced in Section 6.8 an operator $\widetilde{D}_x : T_x\mathrm{T}_1(\mathbb{R}^2) \mapsto \mathrm{T}_0(\mathbb{R}^2)$ such that $\widetilde{D}_x(T_x\mathrm{G}(\mathbb{R}^2)) \subset \mathrm{A}(\mathbb{R}^2)$.

For a piecewise smooth path $\mathbf{x} : [0,T] \to \mathrm{G}(\mathbb{R}^2)$ with values that projects onto $x : [0,T] \to \mathbb{R}^2$, it is then natural to define

$$\mathfrak{L}(\mathbf{x}; 0, t) = \int_0^t \mathfrak{E}_{\mathrm{A}(\mathbb{R}^2)}(f)(x_s)\widetilde{D}_{\mathbf{x}_s}\left(\frac{\mathrm{d}\mathbf{x}_s}{\mathrm{d}s}\right)\mathrm{d}s \tag{55}$$

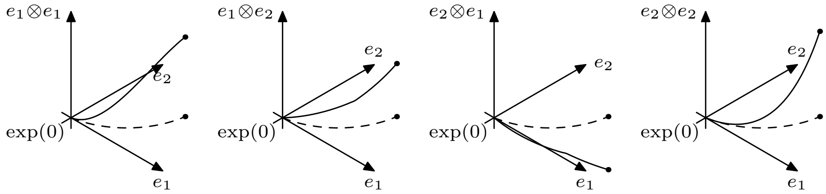
for $t \in [0,T]$, where $\mathfrak{E}_{\mathrm{A}(\mathbb{R}^2)}(f)$ has been defined in (37).

Remark 15. Note that here, we use the operator $\widetilde{D}_x$ to bring all the problems to the $\mathrm{T}_0(\mathbb{R}^2)$ identified with the tangent space $T_1\mathrm{T}_1(\mathbb{R}^2)$ at the point 1. If one wants to avoid this formulation, as we have seen it in Sections 6.7 and 6.8, one may defined $\mathfrak{E}_{\mathrm{A}(\mathbb{R}^2)}(f)$ as the differential form

$$\mathfrak{E}_{\mathrm{A}(\mathbb{R}^2)}(f)(x) = f_1(x)\widetilde{e}^1(x) + f_2(x)\widetilde{e}^2(x) + [f,f](x)\widetilde{e}^3(x), \tag{56}$$

where $\widetilde{e}^i(x)$ is the dual element of $\widetilde{e}_i(x)$ in $T_x\mathrm{T}_1(\mathbb{R}^2)$ for $i = 1,2,3$. Formula (55) may then be rewritten

$$\mathfrak{L}(\mathbf{x}; 0, t) = \int_0^t \mathfrak{E}_{\mathrm{A}(\mathbb{R}^2)}(f)(x_s)\frac{\mathrm{d}\mathbf{x}_s}{\mathrm{d}s}\,\mathrm{d}s.$$



(a) A sub-Riemannian geodesic in $\mathrm{G}(\mathbb{R}^2)$ as constructed from Section 5.9.

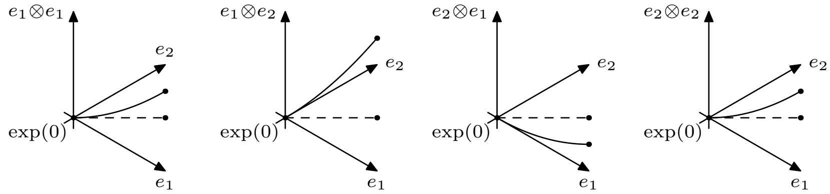


FIGURE 13. (b) The path $\varphi_{x,y}$ with $x = \exp(0)$ and $y = \exp((1, 1, 1))$.

Now, given a path $\mathbf{x} \in \mathrm{C}^\alpha([0, T]; \mathrm{G}(\mathbb{R}^2))$, we define the equivalent of the piecewise linear approximation $\mathbf{x}^n$ by using the curves constructed in Section 6.10 (see Figure 13 for an illustration. Note that unlike with the sub-Riemannian geodesics, $\mathbf{x}^n$ is not necessarily a smooth rough path, but it is a rough path which is smooth): set $\varphi_{a,b}(t) = a \otimes \exp(t \log(a^{-1} \otimes b))$ for $t \in [0, 1]$, and

$$\mathbf{x}^n_t = \varphi_{\mathbf{x}_{t^n_k}, \mathbf{x}_{t^n_{k+1}}} \left(\frac{t - t^n_k}{t^n_{k+1} - t^n_k} \right) \text{ for } t \in [t^n_k, t^n_{k+1}], \tag{57}$$

for $n \in \mathbb{N}^*$ and $t^n_k = Tk/2^n$, $k = 0, \dots, 2^n$.

Proposition 8. *For* $\mathbf{x} \in \mathrm{C}^\alpha([0, T]; \mathrm{G}(\mathbb{R}^2))$ *with* $\alpha > 1/3$*, let* $\mathbf{x}^n$ *be the path defined above by* (57)*. Then*

$$\mathfrak{I}(\log(\mathbf{x}); 0, t) = \lim_{n\to\infty} \mathfrak{L}(\mathbf{x}^n; 0, t)$$

uniformly in $t \in [0, T]$*.*

Proof. This follows from the computations of Sections 6.1 and 7.5, and from the definition of $\widetilde{D}_x$, since we have seen in Section 6.10 that $\widetilde{D}_{\varphi_{a,b}(t)}(\varphi'_{a,b}(t)) = \log(a^{-1} \otimes b)$ for $t \in [0, 1]$. □

As there is an identification between $\log(\mathbf{x})$ and $\mathbf{x}$, one can set for $\mathbf{x} \in \mathrm{C}^\alpha([0, T]; \mathrm{G}(\mathbb{R}^2))$, $\mathfrak{I}(\mathbf{x}) = \mathfrak{I}(\log(\mathbf{x}))$.

7.3. **A Riemann sum like definition.** We are now willing to give another definition of the integral in the spirit of Riemann sums, to get rid of the integrals between the successive times t^n_k and t^n_{k+1} for $k = 0, \dots, 2^n - 1$. For this, we use the Taylor development of f: For $x, y \in \mathbb{R}^2$ and $i = 1, 2$,

$$f_i(y^1, y^2) = f_i(x^1, x^2) + \sum_{j=1,2} \frac{\partial f_i}{\partial x_j}(x^1, x^2) z^j + \kappa^i_1(z)$$

with $|\kappa_1^i(z)| \leq \|f\|_{\mathrm{Lip}}|z|^{1+\gamma}$ and $z = y - x$. In addition,

$$[f,f](y^1,y^2) = [f,f](x^1,x^2) + \kappa_2(z) \text{ with } |\kappa_2(z)| \leq \|f\|_{\mathrm{Lip}}|z|^{\gamma-1}.$$

Let us set $\mathbf{x} \in \mathrm{C}^\alpha([0,T];\mathrm{G}(\mathbb{R}^2))$ with $\alpha > 1/3$ and $\mathbf{x}^n$ constructed as in Proposition 8. In addition, we define x and x^n by $x = \pi_{\mathbb{R}^2}(\mathbf{x})$ and $x^n = \pi_{\mathbb{R}^d}(\mathbf{x})$. Let us remark that $x^n = x^{\Pi^n}$, the piecewise linear interpolation of x. For $\Delta_n t = T2^{-n}$,

$$\left|\int_{t_k^n}^{t_{k+1}^n} [f,f](x_s^n)\mathfrak{a}(\mathbf{x}_{t_k^n,t_{k+1}^n})\frac{\mathrm{d}s}{\Delta_n t} - [f,f](x_{t_k^n})\mathfrak{a}(\mathbf{x}_{t_k^n,t_{k+1}^n})\right| \\ \leq \Delta_n t^{\alpha(1+\gamma)}\|\mathbf{x}\|_\alpha^{\alpha(1+\gamma)}\|f\|_{\mathrm{Lip}}.$$

In addition, with the Taylor formula,

$$(58)\quad \Big|\int_{t_k^n}^{t_{k+1}^n} f_i(x_s^n)(x_{t_{k+1}^n} - x_{t_k^n})\frac{\mathrm{d}s}{\Delta_n t} - f_i(x_{t_k^n})(x^i_{t_{k+1}^n} - x^i_{t_k^n}) \\ - \sum_{i,j=1,2}\frac{1}{2}\frac{\partial f_i}{\partial x_j}(x_{t_k^n})\pi_{e_j}(x_{t_{k+1}^n} - x_{t_k^n})\pi_{e_i}(x_{t_{k+1}^n} - x_{t_k^n})\Big| \\ \leq \Delta_n t^{\alpha(1+\gamma)}\|f\|_{\mathrm{Lip}}\|\mathbf{x}\|_\alpha^{1+\gamma}.$$

If $e^i(x)$ is the dual element of $e_i(x)$, we denote by $f(x)$ the linear operator $f = f_1(x)e^1(x) + f_2(x)e^2(x)$. If $e^i(x) \otimes e^j(x)$ is the dual element of $e_i(x) \otimes e_j(x)$ for $i,j = 1,2$, we denote by ∇f the linear operator

$$\nabla f(x) = \sum_{i,j=1,2}\frac{\partial f_i}{\partial x_j}(x)e^j(x) \otimes e^i(x)$$

so that with (45),

$$\frac{1}{2}\sum_{i,j=1,2}\frac{\partial f_i}{\partial x_j}(x_{t_k^n})\pi_{e_j}(x_{t_{k+1}^n} - x_{t_k^n})\pi_{e_i}(x_{t_{k+1}^n} - x_{t_k^n}) = \nabla f(x_{t_k^n})\mathfrak{s}(\mathbf{x}_{t_k^n,t_{k+1}^n}).$$

Hence, with (43), we deduce that

$$(59)\quad \int_{t_k^n}^{t_{k+1}^n} \mathfrak{E}_{\mathrm{A}(\mathbb{R}^2)}(f)(\mathbf{x}_s^n)\widetilde{D}_{\mathbf{x}_s^n}\left(\frac{\mathrm{d}\mathbf{x}_s^n}{\mathrm{d}s}\right)\mathrm{d}s \\ = f(x_{t_k^n})\pi_{\mathbb{R}^2}(\mathbf{x}_{t_k^n,t_{k+1}^n}) + \nabla f(x_{t_k^n})\pi_{\mathbb{R}^2\otimes\mathbb{R}^2}(\mathbf{x}_{t_k^n,t_{k+1}^n}) + \theta_k^n$$

with $|\theta_k^n| \leq \|f\|_{\mathrm{Lip}}\|\mathbf{x}\|_\alpha^{1+\gamma}\Delta_n t^{\alpha(1+\gamma)}$. Since $\alpha(\gamma+1) > 1$, $\lim_{n\to\infty}\sum_{k=0}^{2^n-1}|\theta_k^n| = 0$. We then define a differential form $\mathfrak{E}_{\mathrm{T}_1(\mathbb{R}^2)}(f)$ on $\mathrm{T}_1(\mathbb{R}^2)$ by

$$\mathfrak{E}_{\mathrm{T}_1(\mathbb{R}^2)}(f)(x) = \sum_{i=1,2} f_i(\pi_{\mathbb{R}^2}(x))e^i(x) + \sum_{i,j=1,2}\frac{\partial f_i}{\partial x_j}(\pi_{\mathbb{R}^2}(x))e^i(x) \otimes e^j(x),$$

With (59) and the property of the θ_k^n's, we get that, after having identified $\mathfrak{I}(\mathbf{x};0,T)$ with $\mathfrak{I}(\log(\mathbf{x});0,T)$ for $\mathbf{x}\in\mathrm{C}^\alpha([0,T];\mathrm{G}(\mathbb{R}^2))$,

$$\mathfrak{I}(\mathbf{x};0,T)=\lim_{n\to\infty}\sum_{k=0}^{2^n-1}\mathfrak{E}_{\mathrm{T}_1(\mathbb{R}^2)}(f)(x_{t_k^n})\mathbf{x}_{t_k^n,t_{k+1}^n}, \tag{60}$$

which is a Riemann sum like expression.

This means also that $\mathfrak{E}_{\mathrm{T}_1(\mathbb{R}^2)}(f)(x_{t_k^n})\mathbf{x}_{t_k^n,t_{k+1}^n}$ is a “good” approximation of $\mathfrak{I}(\mathbf{x};t_k^n,t_{k+1}^n)$.

7.4. **Another construction of the integral.** Let us assume that the functions f_1,f_2 take their values in the space $\mathbb{R}^m$ with $m>1$. For the sake of simplicity, we assume that $m=2$. The integral $\mathfrak{I}(\mathbf{x})=(\mathfrak{I}^1(\mathbf{x}),\mathfrak{I}^2(\mathbf{x}))$ becomes then a path in $\mathbb{R}^2$, and we are interested in construction its iterated integrals.

If x belongs to $\mathrm{C}^\alpha([0,T];\mathbb{R}^2)$ with $\alpha>1/2$, then $\mathfrak{I}(\mathbf{x})$ also corresponds to a Young integral and belongs to $\mathrm{C}^\alpha([0,T];\mathbb{R}^2)$. Hence, we use the natural lift in (54), which means that we have to define only $t\mapsto\mathfrak{A}(\mathfrak{I}(\mathbf{x};0,t);0,t)$, or equivalently, $\int_s^t\mathfrak{I}^i(\mathbf{x};s,r)\,\mathrm{d}\mathfrak{I}^j(\mathbf{x};s,r)$ for $i,j=1,2$.

We remark that, if $x_r=x_s+(r-s)(t-s)^{-1}(x_t-x_s)$,

$$\begin{aligned}\int_s^t\left(\int_s^r f_i^k(x_u)\,\mathrm{d}x^i\right)f_j^\ell(x_r)\,\mathrm{d}x_r^j&=f_i^j(x_s)f_j^\ell(x_s)\int_s^t(x_r^i-x_s^i)\,\mathrm{d}x_r^j\\&+\int_s^t\left(\int_s^r(f_i^k(x_r)-f_i^k(x_s))\,\mathrm{d}x_r^i\right)f_j^\ell(x_r)\,\mathrm{d}x_r^j\\&+\int_s^t f_i^k(x_s)(x_r^i-x_s^i)(f_j^\ell(x_r)-f_j^\ell(x_s))\,\mathrm{d}x_r^j.\end{aligned}$$

This suggests to take for an approximation of $\int_{t_k^n}^{t_{k+1}^n}\mathfrak{I}^i(\mathbf{x};t_k^n,s)\,\mathrm{d}\mathfrak{I}^j(\mathbf{x};t_k^n,s)$ the quantity

$$\mathbf{y}^{k,\ell}_{t_k^n,t_{k+1}^n}=\sum_{i,j=1,2}f_i^k(x_{t_k^n})f_k^\ell(x_{t_k^n})\mathbf{x}^{2,i,j}_{t_k^n,t_{k+1}^n},\ k,\ell=1,2.$$

With (60), we also set

$$\mathbf{y}^i_{t_k^n,t_{k+1}^n}=\mathfrak{E}_{\mathrm{T}_1(\mathbb{R}^2)}(f^i)(x_{t_k^n})\mathbf{x}_{t_k^n,t_{k+1}^n},\ i=1,2.$$

Let $\{\check{e}_1,\check{e}_2\}$ be the canonical basis of $\mathbb{R}^2$ and $\{\check{e}^1,\check{e}^2\}$ be its dual basis, which we distinguish from $\{e_1,e_2\}$ to refer to the space in which f takes its values. Then we introduce the differential form $\mathfrak{E}_{\mathrm{T}_1(\mathbb{R}^2),\mathrm{T}_1(\mathbb{R}^2)}(f)$ with value in $\mathrm{T}_1(\mathbb{R}^2)$ defined by, for $z\in\mathrm{T}_1(\mathbb{R}^2)$,

$$\begin{aligned}\mathfrak{E}_{\mathrm{T}_1(\mathbb{R}^2),\mathrm{T}_1(\mathbb{R}^2)}(f)(z)=1&+\mathfrak{E}_{\mathrm{T}_1(\mathbb{R}^2)}(f^1)(z)\check{e}^1+\mathfrak{E}_{\mathrm{T}_1(\mathbb{R}^2)}(f^2)(z)\check{e}^2\\&+\sum_{i,j=1,2}f^i(\pi_{\mathbb{R}^2}(z))f^j(\pi_{\mathbb{R}^2}(z))\check{e}^1\otimes\check{e}^2,\end{aligned}$$

or in the more compact form,

$$\mathfrak{E}_{\mathrm{T}_1(\mathbb{R}^2)}(f)(z) = \mathfrak{E}_{\mathrm{T}_1(\mathbb{R}^2)}(f)(z) + f(\pi_{\mathbb{R}^2}(z)) \otimes f(\pi_{\mathbb{R}^2}(z))$$

with $f = f^1\check{e}^1 + f^2\check{e}^2$. Hence, in order to approximate $\mathfrak{I}(\mathbf{x}; s, t)$ and its iterated integral, we may then set

$$\mathbf{y}_{s,t} = \mathfrak{F}(f, \mathbf{x}; s, t) \stackrel{\text{def}}{=} \mathfrak{E}_{\mathrm{T}_1(\mathbb{R}^2),\mathrm{T}_1(\mathbb{R}^2)}(f)(\mathbf{x}_s)\mathbf{x}_{s,t} \tag{61}$$

and set, for $t \in (t^n_{\underline{M}(t,n)-1}, t^n_{\underline{M}(t,n)}]$ and $s \in [t^n_{\overline{M}(s,n)}, t^n_{\overline{M}(s,n)})$,

$$\mathfrak{I}^n(\mathbf{x}; s, t) \stackrel{\text{def}}{=} \mathfrak{F}(\mathbf{x}; s, t^n_{\overline{M}(s,n)}) \otimes \left(\bigotimes_{k=\overline{M}(s,n)}^{\underline{M}(t,n)-1} \mathfrak{F}(\mathbf{x}; t^n_k, t^n_{k+1}) \right) \otimes \mathfrak{F}(\mathbf{x}; t^n_{\underline{M}(t,n)}, t). \tag{62}$$

Finally, we set

$$\mathfrak{I}(\mathbf{x}; s, t) = \lim_{n\to\infty} \mathfrak{I}^n(\mathbf{x}; s, t) \tag{63}$$

when this limit exists.

In the definition of $\mathfrak{F}(\mathbf{x})$, we have assumed that $\mathbf{x}$ is a path with values in $\mathrm{G}(\mathbb{R}^2)$. Indeed, this definition may be extended to paths with values in $\mathrm{T}_1(\mathbb{R}^2)$. In addition, let us note that if $\mathbf{x}$ takes its values in $\mathrm{G}(\mathbb{R}^2)$, then $\mathfrak{F}(\mathbf{x}; s, t) \in \mathrm{G}(\mathbb{R}^2)$. The analysis of $\mathfrak{F}(\mathbf{x}; s, t)$ for $\mathbf{x}$ in $\mathrm{S}(\mathbb{R}^2)$ is performed in Section 7.5.

We will see below that the integral defined by (62)-(63) satisfies the relation

$$\mathfrak{I}(\mathbf{x}; s, t) = \mathfrak{I}(\mathbf{x}; s, r) \otimes \mathfrak{I}(\mathbf{x}; r, t),\ \forall 0 \leq s \leq r \leq t \leq T, \tag{64}$$

which means that $t \in [0, T] \mapsto \mathfrak{I}(\mathbf{x}; 0, t)$ is a path with values in $\mathrm{T}_1(\mathbb{R}^2)$ and $\mathfrak{I}(\mathbf{x}; s, t)$ represents its increments.

But $\mathfrak{I}^n(\mathbf{x})$ does not satisfy (64), unless s, r, t belong to $\{t^n_k\}_{k=0,\dots,2^n-1}$.

The next results are borrowed from [LQ02, Section 3.2, p. 40] or from [Lyo98, Section 3.1, p. 273].

Definition 6. A function $\mathbf{y}_{s,t}$ from $\Delta_+ = \{(s, t) \in [0, T]^2 | 0 \leq s \leq t \leq T\}$ to $\mathrm{T}_1(\mathbb{R}^2)$ is an *almost rough path* if there exists some constants $C > 0$ and $\theta > 1$ such that

$$\|\mathbf{y}_{s,t} - \mathbf{y}_{s,r} \otimes \mathbf{y}_{r,t}\|_\star \leq C|t - s|^\theta,\ \forall 0 \leq s \leq r \leq t \leq T.$$

where $\|\cdot\|_\star$ is the norm defined by $\|x\|_\star = \max\{|x^1|, |x^2|\}$.

An almost rough path is the "basic brick" for constructing a rough path. We give a proof of the next theorem in Section C in the appendix.

Theorem 2. *Let* $\mathbf{y} : \Delta_+ \to \mathrm{T}_1(\mathbb{R}^2)$ *be an almost rough path such that* $\|\mathbf{y}_{s,t}\| \leq C|t - s|^\alpha$ *for* $\alpha \in (1/3, 1]$ *and* $C > 0$. *Set*

$$\mathbf{y}^n_{s,t} \stackrel{def}{=} \mathbf{y}_{s,\overline{M}(s,n)} \otimes \left(\bigotimes_{k=\overline{M}(s,n)}^{\underline{M}(t,n)-1} \mathbf{y}_{t^n_k,t^n_{k+1}} \right) \otimes \mathbf{y}_{\underline{M}(t,n),t},\ \forall (s, t) \in \Delta_+.$$

Then there exists a unique path $\mathbf{z}$ *in* $\mathrm{C}^\alpha([0,T];\mathrm{T}_1(\mathbb{R}^2))$ *and a sequence* $(K_n)_{n\in\mathbb{N}}$ *decreasing to* 0 *such that*

$$\|\mathbf{z}_{s,t}-\mathbf{y}^n_{s,t}\|_\star \leq K_n|t-s|^\theta.$$

If $\mathbf{y}$ *is an almost rough path in* $\mathrm{G}(\mathbb{R}^2)$*, then* $\mathbf{z}$ *is a weak geometric rough path with* α*-Hölder control.*

In addition, if $\mathbf{y}$ *and* $\mathbf{y}'$ *are both almost rough paths with*

$$|\pi_{\mathbb{R}^2}(\mathbf{y}_{s,t}-\mathbf{y}'_{s,t})| \leq \varepsilon|t-s|^\alpha,\ |\pi_{\mathbb{R}^2\otimes\mathbb{R}^d}(\mathbf{y}_{s,t}-\mathbf{y}'_{s,t})| \leq \varepsilon|t-s|^{2\alpha}$$

for all $(s,t)\in\Delta_+$*, then the corresponding rough paths* $\mathbf{z}$ *and* $\mathbf{z}'$ *satisfy*

$$|\pi_{\mathbb{R}^d}(\mathbf{z}_{s,t}-\mathbf{z}'_{s,t})| \leq K(\varepsilon)|t-s|^\alpha,\ |\pi_{\mathbb{R}^d\otimes\mathbb{R}^d}(\mathbf{z}_{s,t}-\mathbf{z}'_{s,t})| \leq K(\varepsilon)|t-s|^{2\alpha}$$

for some function $K(\varepsilon)$ *decreasing to* 0 *as* $\varepsilon\to 0$ *that depends only on* T*,* α *and* θ*.*

The existence of $\mathfrak{I}(\mathbf{x})$ in (63) as a (weak geometric) rough path when $\mathbf{x}$ is a (weak geometric) rough path is then justified by the next proposition and the application of Theorem 2.

Roughly speaking, the proof follows the same line as the one of the Young integral: the reader is referred to [Lyo98, Section 3.2.2, p. 289], [LQ02, Section 5.2, p. 117], [Lej03, Section 3] or [LV06].

Proposition 9. *For* $\mathbf{x}\in\mathrm{C}^\alpha([0,T];\mathrm{T}_1(\mathbb{R}^2))$ *with* $\alpha\in(1/3,1]$*, the function* $(s,t)\in\Delta_+\mapsto\mathfrak{F}(\mathbf{x};s,t)$ *is an almost rough path. In addition, if* $\mathbf{x}\in\mathrm{C}^\alpha([0,T];\mathrm{G}(\mathbb{R}^2))$*, then* $\mathfrak{F}(\mathbf{x};s,t)$ *belongs to* $\mathrm{G}(\mathbb{R}^2)$*. Hence,* $\mathfrak{I}(\mathbf{x})$ *given by* (63) *exists and belongs to* $\mathrm{C}^\alpha([0,T];\mathrm{T}_1(\mathbb{R}^2))$ *(resp.* $\mathrm{C}^\alpha([0,T];\mathrm{G}(\mathbb{R}^2))$*) if* $\mathbf{x}\in\mathrm{C}^\alpha([0,T];\mathrm{T}_1(\mathbb{R}^2))$ *(resp.* $\mathrm{C}^\alpha([0,T];\mathrm{G}(\mathbb{R}^2))$*).*

We have already seen that the integral $\mathfrak{I}(\mathbf{x})$ lies above the integral we constructed in Section 6 using some approximation of $\mathbf{x}$. With Theorem 2, we have not only the continuity of $\mathbf{x}\mapsto\mathfrak{I}(\mathbf{x})$, but we also get that it is a locally Lipschitz map under a stronger assumption on f and we are not bound in using the $\|\cdot\|_\beta$ norm with $\beta<\alpha$ while working with α-Hölder paths. In addition, we may consider any family of partitions whose mesh decreases to zero (see Remark 4 or the proof of Theorem 5 in Appendix C).

We introduce a new norm $\|\cdot\|_{\star,\alpha}$ on $\mathrm{C}^\alpha([0,T];\mathrm{T}_1(\mathbb{R}^d))$, which is not equivalent to $\|\cdot\|_\alpha$ but which generates the same topology: for $\mathbf{x}\in\mathrm{C}^\alpha([0,T];\mathrm{T}_1(\mathbb{R}^d))$,

$$\|\mathbf{x}\|_{\star,\alpha} = \sup_{0\leq s<t\leq T}\max\left\{\frac{|\pi_{\mathbb{R}^d}(\mathbf{x}_{s,t})|}{(t-s)^\alpha},\frac{|\pi_{\mathbb{R}^d\otimes\mathbb{R}^d}(\mathbf{x}_{s,t})|}{(t-s)^{2\alpha}}\right\}.$$

In the next theorem, we summarize Proposition 9 as well as some continuity results.

Theorem 3. *If* $f\in\mathrm{Lip}(\gamma;\mathbb{R}^2\to\mathbb{R}^2)$ *with* $\alpha(\gamma+1)>1$ *and* $\alpha>1/3$*, then the limit of* $(\mathfrak{I}^n(\mathbf{x};s,t))_{n\in\mathbb{N}}$ *in* (63) *exists and is unique. Besides,* $\mathfrak{I}$ *maps continuously* $(\mathrm{C}^\alpha([0,T];\mathrm{T}_1(\mathbb{R}^2)),\|\cdot\|_{\star,\alpha})$ *to* $(\mathrm{C}^\alpha([0,T];\mathrm{T}_1(\mathbb{R}^2)),\|\cdot\|_{\star,\alpha})$*.*

If f is of class $\mathrm{C}^2(\mathbb{R}^2;\mathbb{R}^2)$ with a κ-Hölder continuous second-order derivative with $\alpha(\kappa+2)>1$, then $\mathfrak{I}$ is locally Lipschitz continuous.

In addition, if $\mathbf{x}$ is a smooth rough path, then

$$\mathfrak{I}(\mathbf{x};0,t)=\exp\left(\int_0^t f(x_s)\,\mathrm{d}x_s+\mathfrak{A}(\mathfrak{I}(\mathbf{x};0,t);0,t)[\check{e}_1,\check{e}_2]\right)$$

and $\mathfrak{I}(\mathbf{x})$ is also a smooth rough path.

Hence, for $\mathbf{x}\in\mathrm{C}^{0,\alpha}([0,T];\mathrm{G}(\mathbb{R}^2))$, then there exists a sequence of paths $\mathbf{x}^n\in\mathrm{C}^\infty_{\mathrm{p}}([0,T];\mathrm{G}(\mathbb{R}^2))$ converges to $\mathbf{x}$ in $\|\cdot\|_\alpha$, then $\mathfrak{I}(\mathbf{x})=\lim_{n\to\infty}\mathfrak{I}(\mathbf{x}^n)$ and $\mathfrak{I}(\mathbf{x}^n)$ is a smooth rough path and $\mathfrak{I}(\mathbf{x})$ belongs to $\mathrm{C}^{0,\alpha}([0,T];\mathrm{G}(\mathbb{R}^2))$.

Now, if $\mathbf{x}$ is only a weak geometric $1/\alpha$-rough path with Hölder control, then we have seen that $\mathbf{x}$ may be approximated by some smooth rough paths $\mathbf{x}^n$ in the β-Hölder norm $\|\cdot\|_\beta$ with $\beta<\alpha$. Hence, $\mathfrak{I}(\mathbf{x}^n)$ converges to $\mathfrak{I}(\mathbf{x})$ in $\|\cdot\|_\beta$ with $\beta<\alpha$. Anyway, $\mathfrak{I}(\mathbf{x})$ belongs to $\mathrm{C}^\alpha([0,T];\mathrm{G}(\mathbb{R}^2))$.

We then deduce the following stability result.

Corollary 5. *If $\mathfrak{I}$ is defined by Theorem 3, then $\mathfrak{I}$ maps $\mathrm{C}^\alpha([0,T];\mathrm{G}(\mathbb{R}^2))$ into $\mathrm{C}^\alpha([0,T];\mathrm{G}(\mathbb{R}^2))$ and $\mathrm{C}^{0,\alpha}([0,T];\mathrm{G}(\mathbb{R}^2))$ into $\mathrm{C}^{0,\alpha}([0,T];\mathrm{G}(\mathbb{R}^2))$.*

We end this section with a lemma similar to Lemma 11.

Lemma 20. *For any $\mathbf{x}\in\mathrm{C}^\alpha([0,T];\mathrm{T}_1(\mathbb{R}^2))$, $\mathfrak{I}(\mathbf{x};s,t)=\mathfrak{I}(\mathbf{x}_{|[s,t]})$ for all $0\leq s<t\leq T$.*

Proof. If $\mathbf{x}\in\mathrm{C}^\alpha([0,T];\mathrm{G}(\mathbb{R}^2))$, then the proof of this Lemma is similar to the one of Lemma 11. If $\mathbf{x}\in\mathrm{C}^\alpha([0,T];\mathrm{T}_1(\mathbb{R}^2))$, then the results at the end in Section 7.5 allows us to conclude in the same way. □

7.5. **Integral along a path living in the tensor space.** Now, we consider $\mathbf{x}\in\mathrm{C}^\alpha([0,T];\mathrm{T}_1(\mathbb{R}^2))$ with $\alpha\in(1/3,1/2)$. What can be said about $\mathfrak{I}(\mathbf{x})$? From Proposition 7, one may decompose $\mathbf{x}_t$ as the sum $\mathbf{x}_t=\mathbf{y}_t+\mathbf{z}_t$ with $\mathbf{y}=\Upsilon_{\mathrm{a}}(\mathbf{x})$ and $\mathbf{z}=\Upsilon_{\mathrm{s}}(\mathbf{x})$. In addition, $(\mathbf{y},\mathbf{z})$ belongs to $\mathrm{C}^\alpha([0,T];\mathrm{G}(\mathbb{R}^2)\times\mathrm{S}(\mathbb{R}^2))$, where the homogeneous norm on $\mathrm{G}(\mathbb{R}^2)\times\mathrm{S}(\mathbb{R}^2)$ has been defined by (52). In particular, this implies that $\pi_{e_i\otimes e_j}(\mathbf{z})$ belongs to $\mathrm{C}^{2\alpha}([0,T];\mathbb{R})$, *i.e.*, each of its component is 2α-Hölder continuous. In addition, for $(s,t)\in\Delta_+$, $\mathfrak{E}_{\mathrm{T}_1(\mathbb{R}^2),\mathrm{T}_1(\mathbb{R}^2)}(f)(\mathbf{x}_s)(\mathbf{y}_t-\mathbf{y}_s)$ belongs to $\mathrm{G}(\mathbb{R}^2)$, while

$$\begin{aligned}\mathfrak{E}_{\mathrm{T}_1(\mathbb{R}^2),\mathrm{T}_1(\mathbb{R}^2)}(f)(\mathbf{x}_s)(\mathbf{z}_t-\mathbf{z}_s)&=\sum_{k,i,j=1,2}\frac{\partial f_i^k}{\partial x_j}(x_s)\pi_{e_i\otimes e_j}(\mathbf{z}_t-\mathbf{z}_s)\check{e}_k\\&\quad+\sum_{k,\ell,i,j=1,2}f_i^k(x_s)f_j^\ell(x_s)\pi_{e_i\otimes e_j}(\mathbf{z}_t-\mathbf{z}_s)\check{e}_k\otimes\check{e}_\ell.\end{aligned}$$

Since $\pi_{e_i\otimes e_j}(\mathbf{z}_t-\mathbf{z}_s)=\pi_{e_j\otimes e_i}(\mathbf{z}_t-\mathbf{z}_s)$, we get that $\mathfrak{E}_{\mathrm{T}_1(\mathbb{R}^2),\mathrm{T}_1(\mathbb{R}^2)}(f)(\mathbf{x}_s)(\mathbf{z}_t-\mathbf{z}_s)$ belongs to $\mathbb{R}^2\oplus\mathrm{S}(\mathbb{R}^2)$. Besides, for $t\in[0,T]$,

$$\begin{aligned}\lim_{n\to\infty}\sum_{k=0}^{\underline{M}(t,n)}\pi_{\mathbb{R}^2}\big(\mathfrak{E}_{\mathrm{T}_1(\mathbb{R}^2),\mathrm{T}_1(\mathbb{R}^2)}(f)(\mathbf{x}_{t_k^n})(\mathbf{z}_{t_{k+1}^n}-\mathbf{z}_{t_k^n})\big)\\ =\sum_{k=1,2}\check{e}^k\int_0^t\frac{\partial f_1^k}{\partial x_1}(x_s)\,\mathrm{d}\pi_{e_1\otimes e_1}(z_s)+\check{e}^k\int_0^t\frac{\partial f_2^k}{\partial x_2}(x_s)\,\mathrm{d}\pi_{e_2\otimes e_2}(z_s)\\ +\check{e}^k\int_0^t\left(\frac{\partial f_1^k}{\partial x_2}(x_s)+\frac{\partial f_2^k}{\partial x_1}(x_s)\right)\,\mathrm{d}\pi_{e_1\otimes e_2}(z_s)\end{aligned}$$

which we can write in the more compact form $\int_0^t\nabla f(x_s)\,\mathrm{d}\mathbf{z}_s$.

In addition, if $\{\alpha_k\}_{k=0,\dots,m}$ and $\{\beta_k\}_{k=0,\dots,m}$ belongs to $\mathrm{T}_0(\mathbb{R}^2)$, then

$$\begin{aligned}\bigotimes_{k=0}^m(1+\alpha_k+\beta_k)=\bigotimes_{k=1}^m(1+\alpha_k)+\sum_{k=0,\dots,m}\alpha_k^1\otimes\left(\sum_{\ell=k+1,\dots,m}\beta_\ell^1\right)\\ +\sum_{k=0,\dots,m}\beta_k^1\otimes\left(\sum_{\ell=k+1,\dots,m}\alpha_\ell^1\right)+\sum_{k=0,\dots,m}\beta_k^1\otimes\left(\sum_{\ell=k+1,\dots,m}\beta_\ell^1\right)\end{aligned}$$

with $\alpha_k^1=\pi_{\mathbb{R}^2}(\alpha_k)$ and $\beta_k^1=\pi_{\mathbb{R}^2}(\beta_k)$. We remark that

$$\sum_{k=0,\dots,m}\beta_k^1\otimes\left(\sum_{\ell=k+1,\dots,m}\alpha_\ell^1\right)=\sum_{\ell=1}^m\sum_{k=0}^\ell\alpha_k\otimes\beta_\ell,$$

and

$$\sum_{k=0,\dots,m}\beta_k^1\otimes\left(\sum_{\ell=k+1,\dots,m}\beta_\ell^1\right)=\frac{1}{2}\left(\sum_{k=0}^m\beta_k\right)\otimes\left(\sum_{k=0}^m\beta_k\right).$$

We set

$$1+\alpha_k=\mathfrak{F}(f,\mathbf{y};t_k^n,t_{k+1}^n)\text{ and }\beta_k=\mathfrak{E}_{\mathrm{T}_1(\mathbb{R}^d)}(f)(\mathbf{x}_{t_k^n})(\mathbf{z}_{t_{k+1}^n}-\mathbf{z}_{t_k^n}).$$

In addition, $\sum_{k=0,\dots,\underline{M}(t,n)}\alpha_k^1$ converges to $\pi_{\mathbb{R}^2}(\mathfrak{I}(\mathbf{x};0,t))$, while $\sum_{k=0,\dots,\underline{M}(t,n)}\beta_k^1$ converges to $\int_0^t\nabla f(x_s)\,\mathrm{d}\mathbf{z}_s$. We also remark that if $\beta_k^2=\pi_{\mathbb{R}^2\otimes\mathbb{R}^2}(\beta_k)$, then

$$\sum_{k=0}^{\underline{M}(t,n)}\beta_k^2\xrightarrow[n\to\infty]{}\int_0^t f(x_s)\otimes f(x_s)\,\mathrm{d}\mathbf{z}_s.$$

By combining all the facts and using techniques similar to the one in [LQ02, Section 3.3.3, p. 56] or in [LV06], since the components of $\int_0^t\nabla f(x_s)\,\mathrm{d}\mathbf{z}_s$ are 2α-Hölder continuous, we can get that

$$\bigotimes_{k=0}^{\underline{M}(t,n)-1}\mathfrak{F}(f,\mathbf{x};t_k^n,t_{k+1}^n)\xrightarrow[n\to\infty]{}\int_0^t f(x_s)\,\mathrm{d}\mathbf{y}_s+\mathfrak{K}(\mathbf{y},\mathbf{z};0,t)$$

with

$$\mathfrak{K}(\mathbf{y},\mathbf{z};0,t) = \int_0^t \nabla f(y_s)\,\mathrm{d}\mathbf{z}_s + \int_0^t f(y_s)\otimes f(y_s)\,\mathrm{d}\mathbf{z}_s + \sum_{k,\ell=1,2} \check{e}_k\otimes\check{e}_\ell \int_0^t \nabla f^k(y_s)\left(\int_0^s f^\ell(y_s)\,\mathrm{d}\mathbf{y}_s\right)\mathrm{d}\mathbf{z}_s + \sum_{k,\ell=1,2} \check{e}_k\otimes\check{e}_\ell \int_0^t f^k(y_s)\left(\int_0^s \nabla f^\ell(y_s)\,\mathrm{d}\mathbf{z}_s\right)\mathrm{d}\mathbf{y}_s + \frac{1}{2}\left(\int_0^t \nabla f(y_s)\,\mathrm{d}\mathbf{z}_s\right)\otimes\left(\int_0^t \nabla f(y_s)\,\mathrm{d}\mathbf{z}_s\right). \tag{65}$$

In the previous expression, we have to remember that $\mathbf{x}$ and $\mathbf{y}$ lives above the same path $x = y$.

Thus, if for each $n\in\mathbb{N}$, $\mathbf{z}^n$ belongs to $\mathrm{C}^\infty_{\mathrm{p}}([0,T];\mathrm{S}(\mathbb{R}^2))$ and converges to $\mathbf{z}$, while $\mathbf{y}^n\in\mathrm{C}^\infty_{\mathrm{p}}([0,T];\mathrm{G}(\mathbb{R}^2))$ converges to $\mathbf{y}$, one get that $\mathbf{x}^n = \mathbf{y}^n + \mathbf{z}^n$ converges to $\mathrm{C}^\alpha([0,T];\mathrm{T}_1(\mathbb{R}^2))$ and

$$\mathfrak{I}(\mathbf{x}) = \lim_{n\to\infty}(\mathfrak{I}(\mathbf{y}^n) + \mathfrak{K}(\mathbf{y}^n,\mathbf{z}^n)),$$

where the limit is in $\mathrm{C}^\beta([0,T];\mathrm{T}_1(\mathbb{R}^2))$ for all $\beta<\alpha$. Of course, both $\mathfrak{K}(\mathbf{y}^n,\mathbf{z}^n)$ and $\mathfrak{I}(\mathbf{y}^n)$ correspond to integrals of differential forms along piecewise smooth paths, and hence to ordinary integrals.

Yet it has to be note the following fact: If $\mathbf{x}\in\mathrm{C}^\alpha([0,T];\mathrm{T}_1(\mathbb{R}^2))$ but $\mathbf{x}\notin\mathrm{C}^\alpha([0,T];\mathrm{G}(\mathbb{R}^2))$, then it is not possible to find a family $(\mathbf{x}^n)_{n\in\mathbb{N}}$ of smooth rough paths such that $\mathfrak{I}(\mathbf{x}^n)$ converges to $\mathfrak{I}(\mathbf{x})$. This means that $\mathfrak{I}(\mathbf{x})$ cannot be approximated by the ordinary integrals $\mathfrak{I}(\mathbf{x}^n)$. This motivates our definition of geometric rough paths. However, using the decomposition of $\mathrm{T}_1(\mathbb{R}^2)$ as $\mathrm{G}(\mathbb{R}^2)\times\mathrm{S}(\mathbb{R}^2)$, it is then possible to interpret any α^{-1}-rough path as a geometric $(1/\alpha,2/\alpha)$-rough path in the sense defined in [LV06].

7.6. **On geometric rough paths lying above the same path.** We have seen in Lemma 6 that if $\mathbf{x}$ and $\mathbf{y}$ are two paths in $\mathrm{C}^\alpha([0,T];\mathrm{A}(\mathbb{R}^2))$ with $\alpha\in(1/3,1/2)$ and lying above the same path taking its values in $\mathbb{R}^2$ (*i.e.*, $\pi_{\mathbb{R}^2}(\mathbf{x}) = \pi_{\mathbb{R}^2}(\mathbf{y})$), then there exists a path $\varphi\in\mathrm{C}^{2\alpha}([0,T];\mathbb{R})$ such that $\mathbf{x} = \mathbf{y} + \varphi[e_1,e_2]$. In addition, $(-\mathbf{x}_s)\boxplus\mathbf{x}_t = (-\mathbf{y}_s)\boxplus\mathbf{y}_t + (\varphi_t-\varphi_s)[e_1,e_2]$. Now, if we lift $\mathbf{x}$ and $\mathbf{y}$ as paths in $\mathrm{C}^\alpha([0,T];\mathrm{G}(\mathbb{R}^2))$ by $\widehat{\mathbf{x}}_t = \exp(\mathbf{x}_t)$ and $\widehat{\mathbf{y}}_t = \exp(\mathbf{y}_t)$, we deduce that there exists $\psi\in\mathrm{C}^\alpha([0,T];\mathrm{T}_0(\mathbb{R}^2))$ such that $\widehat{\mathbf{x}}_t = \widehat{\mathbf{y}}_t + \psi_t$ and in addition, $\widehat{\mathbf{x}}_s^{-1}\otimes\widehat{\mathbf{x}}_t = \widehat{\mathbf{y}}_s^{-1}\otimes\widehat{\mathbf{y}}_t + \psi_t - \psi_s$. This path is given by

$$\psi_t = \varphi_t e_1\otimes e_2 - \varphi_t e_2\otimes e_1 = \varphi_t[e_1,e_2].$$

Each component of ψ is 2α-Hölder continuous. Using the map $\mathfrak{K}$ previously defined by (65), we get that

$$\mathfrak{I}(\mathbf{x}) = \mathfrak{I}(\mathbf{y}) + \mathfrak{K}(\mathbf{y},\psi).$$

Finally, using the fact that ψ is anti-symmetric, setting $[f,f] = \check{e}_1[f^1,f^1] + \check{e}_2[f^2,f^2]$, we get that

$$\begin{aligned}\mathfrak{K}(\mathbf{y},\psi;s,t) &= \int_s^t [f,f](y_s)\,\mathrm{d}\varphi_s \\ &\quad + \sum_{k,\ell=1,2} \check{e}_k\otimes\check{e}_\ell \int_0^t [f^k,f^k](y_s)\left(\int_0^s f^\ell(y_s)\,\mathrm{d}\mathbf{y}_s\right)\mathrm{d}\varphi_s \\ &\quad + \sum_{k,\ell=1,2} \check{e}_k\otimes\check{e}_\ell \int_0^t f^\ell(y_s)\left(\int_0^s [f^k,f^k](y_s)\,\mathrm{d}\varphi_s\right)\mathrm{d}\mathbf{y}_s \\ &\qquad + \frac{1}{2}\left(\int_0^t [f,f](y_s)\,\mathrm{d}\varphi_s\right)\otimes\left(\int_0^t [f,f](y_s)\,\mathrm{d}\varphi_s\right)\end{aligned}$$

for all $0 \leq s \leq t \leq T$.

If $[f,f] = 0$, we deduce that $\mathfrak{K}(\mathbf{y},\psi) = 0$ and then that $\mathfrak{I}(\mathbf{y}) = \mathfrak{I}(\mathbf{x})$. In other words, any rough path lying above the same path x gives rise to the same integral.

With the results in [LV07], which asserts that it is always possible to lift a path $x \in \mathrm{C}^\alpha([0,T];\mathbb{R}^2)$ to a path $\mathbf{x} \in \mathrm{C}^\alpha([0,T];\mathrm{G}(\mathbb{R}^2))$ when $\alpha \in (1/3,1/2]$, this means that if $[f,f] = 0$, one may define $\mathfrak{I}$ only on $\mathrm{C}^\alpha([0,T];\mathbb{R}^2)$ for $\alpha \in (1/3,1/2]$, but the continuity of $\mathfrak{I}$ remains an open question.

8. Variations in the construction of the integral

8.1. Case of a path living in a d-dimensional space. The case of a space of dimension d is not harder to treat than the case of $d = 2$. For this, one has only to consider the area between the components grouped by pairs.

The tensor space $\mathrm{T}(\mathbb{R}^d)$ becomes then the space $\mathrm{T}(\mathbb{R}^d) = \mathbb{R}\oplus\mathbb{R}^d\oplus(\mathbb{R}^d\otimes\mathbb{R}^d)$ whose basis is, if $\{e_1,\dots,e_d\}$ is a basis of $\mathbb{R}^d$,

$$1, e_1, \dots, e_d, e_1\otimes e_1, e_1\otimes e_2, \dots, e_d\otimes e_d,$$

Hence, $\mathrm{T}(\mathbb{R}^d)$ is a space of dimension $1 + d + d^2$.

The space $\mathrm{A}(\mathbb{R}^2)$ is a space of dimension $d + d(d-1)/2$, whose basis is given by

$$\{e_i | i = 1,\dots,d\} \cup \{[e_i,e_j] | i \neq j,\ i,j = 1,\dots,d\}$$

with $[e_i,e_j] = e_i\otimes e_j - e_j\otimes e_i$. The space $\mathrm{A}(\mathbb{R}^2)$ is then $\mathbb{R}^d\oplus[\mathbb{R}^d,\mathbb{R}^d]$, where $[\mathbb{R}^d,\mathbb{R}^d] = \big\{[x,y]\big| x,y\in\mathbb{R}^d\big\}$.

The applications exp and log are defined as previously:

$$\exp(x) = 1 + x + \frac{1}{2}x\otimes x \text{ for } x \in \mathrm{A}(\mathbb{R}^d)$$

$$\text{and } \log(1+x) = x - \frac{1}{2}x\otimes x \text{ for } x \in \mathrm{T}(\mathbb{R}^d),\ \pi_1(x) = 0.$$

The space $\mathrm{G}(\mathbb{R}^d) = \exp(\mathrm{A}(\mathbb{R}^d))$ is a subgroup of $(\mathrm{T}_1(\mathbb{R}^d), \otimes)$, where $\mathrm{T}_1(\mathbb{R}^d) = \left\{x \in \mathrm{T}(\mathbb{R}^d) \middle| \pi_1(x) = 1\right\}$, and $(\mathrm{A}(\mathbb{R}^d), [\cdot,\cdot])$ is the Lie algebra of $(\mathrm{G}(\mathbb{R}^d), \otimes)$. It may also be identified with its tangent plane at any point.

A smooth path x in $\mathbb{R}^d$ is then lifted into a path $\widehat{\mathbf{x}}$ in $\mathrm{A}(\mathbb{R}^d)$ by

$$\widehat{\mathbf{x}}_t = x_t + \sum_{i,j=1,\dots,d,\ i<j} \mathfrak{A}((x^i, x^j); 0, t)[e_i, e_j],$$

where (x^i, x^j) is the two dimensional path composed of the i-th and j-th component of x. Let us remark that $\mathfrak{A}((x^i, x^j)) = -\mathfrak{A}((x^j, x^i))$ and $\mathfrak{A}((x^i, x^i)) = 0$.

The path $\widehat{\mathbf{x}}$ is then lifted into a path $\mathbf{x}$ in $\mathbb{R}^d$ by $\mathbf{x} = \exp(\widehat{\mathbf{x}})$, and thus

$$\mathbf{x}_t = 1 + x_t + \sum_{i,j=1}^{d} \int_0^t (x_s^j - x_0^j)\, \mathrm{d}x_s^i e_j \otimes e_i.$$

The symmetric part $\mathfrak{s}(\mathbf{x})$ of $\pi_{\mathbb{R}^d\otimes\mathbb{R}^d}(\mathbf{x})$ is

$$\mathfrak{s}(\mathbf{x}_t) = \frac{1}{2}(x_t - x_0) \otimes (x_t - x_0)$$

while the anti-symmetric part $\mathfrak{a}(\mathbf{x})$ of $\pi_{\mathbb{R}^d\otimes\mathbb{R}^d}(\mathbf{x})$ is

$$\mathfrak{a}(\mathbf{x}_t) = \sum_{i,j=1}^{d} \mathfrak{A}((x^i, x^j); 0, t) e_i \otimes e_j = \sum_{\substack{i=1,\dots,d \\ i<j}} \mathfrak{A}((x^i, x^j); 0, t)[e_i, e_j].$$

Hence, all the previous notions and results are easily extended to this case.

Finally, note that the theory of rough paths may also be applied to the infinite dimensional case (see [LLQ02] for example).

8.2. **Using iterated integrals.** We saw in Sections 7.1 and 8.1 that a path $x \in \mathrm{C}^\alpha([0,T]; \mathbb{R}^d)$ with $\alpha > 1/2$ may be naturally lifted as a path $\mathbf{x}$ in $\mathrm{G}(\mathbb{R}^d)$ with

$$\mathbf{x}_t = 1 + x_t + \sum_{i,j=1}^{d} \left(\int_0^t (x_r^i - x_0^i)\, \mathrm{d}x_r^j\right) e_i \otimes e_j.$$

The term $K^{i,j}(x; 0, t) = \pi_{e_i\otimes e_j}(\mathbf{x}_t)$ is called an *iterated integral* of x.

Fix $d \geq 1$ and consider the tensor space $\mathrm{T}^\infty(\mathbb{R}^d)$ defined by

$$\mathrm{T}^\infty(\mathbb{R}^d) = \mathbb{R} \oplus \mathbb{R}^d \oplus (\mathbb{R}^d \otimes \mathbb{R}^d) \oplus (\mathbb{R}^d \otimes \mathbb{R}^d \otimes \mathbb{R}^d) \oplus \cdots,$$

and, for a smooth path $x : [0,T] \to \mathbb{R}^d$, the iterated integrals

$$K^{i_1,\dots,i_\ell}(x; 0, t) = \int_0^t \int_0^{t_1} \cdots \int_0^{t_{\ell-1}} \mathrm{d}x_{t_\ell}^{i_1} \cdots \mathrm{d}x_{t_1}^{i_\ell}$$

for each integer ℓ and each $(i_1,\dots,i_\ell)\in\{1,\dots,d\}^\ell$. It was noted first by K.T. Chen in the 50's [Che58, Che57] that the formal power series

$$\Psi(x;0,t)=\sum_{\ell\geq 0}\sum_{(i_1,\dots,i_\ell)\in\{1,\dots,d\}^\ell} K^{i_1,\dots,i_\ell}(x;0,t)e_{i_1}\otimes\cdots\otimes e_{i_\ell}$$

in $\mathrm{T}^\infty(\mathbb{R}^d)$ provides an algebraic way to encode the geometric object which is the path x. For that, $\Psi(x;0,t)$ is sometimes called the *signature* of the path.

With the tensor product $\otimes$, $\mathrm{T}^\infty(\mathbb{R}^d)$ remains a group, and thus if $x:[0,T]\to\mathbb{R}^d$ and $y:[0,S]\to\mathbb{R}^d$ are two smooth paths, $\Psi(x\cdot y;0,T+S)=\Psi(x;0,T)\otimes\Psi(y;0,S)$. In addition, if $\overline{x}$ is the path $\overline{x}_t=x_{T-t}$, then $\Psi(\overline{x};T)=\Psi(x;T)^{-1}$. The signature characterizes x in the sense that there is a one-to-one equivalence[3] between the algebraic object $\Psi(x)$ and the geometric object x in $\mathrm{C}^\infty_{\mathrm{p}}([0,T];\mathbb{R}^d)$ (see also [HL06] for some extension).

Let $[x,y]$ be the Lie bracket $[x,y]=x\otimes y-y\otimes x$. Let us denote by $\mathrm{A}^\infty(\mathbb{R}^d)$ the subset of $\mathrm{T}^\infty(\mathbb{R}^d)$ defined by

$$\mathrm{A}^\infty(\mathbb{R}^d)=\mathbb{R}^d\oplus[\mathbb{R}^d,\mathbb{R}^d]\oplus[\mathbb{R}^d,[\mathbb{R}^d,\mathbb{R}^d]]\oplus\cdots.$$

This subset is stable under the application of the Lie bracket $[\cdot,\cdot]$. The tensor space $\mathrm{T}^\infty(\mathbb{R}^d)$ is the universal Lie algebra of $\mathrm{A}^\infty(\mathbb{R}^d)$ (see [Reu93] for example). One may then define two maps $\exp:\mathrm{A}^\infty(\mathbb{R}^d)\to\mathrm{T}^\infty_1(\mathbb{R}^d)$ and $\log:\mathrm{T}^\infty_1(\mathbb{R}^d)$, where $\mathrm{T}^\infty_1(\mathbb{R}^d)$ is the subset of $\mathrm{T}^\infty(\mathbb{R}^d)$ such that $\pi_{\mathbb{R}}(x)=1$ which are given by

$$\exp(x)=1+x+\frac{1}{2}x\otimes x+\frac{1}{6}x\otimes x\otimes x+\cdots,$$
$$\log(1+x)=x-\frac{1}{2}x\otimes x+\frac{1}{3}x\otimes x\otimes x-\cdots.$$

In particular, if $\mathrm{G}^\infty(\mathbb{R}^d)=\exp(\mathrm{A}^\infty(\mathbb{R}^d))$, then $(\mathrm{G}^\infty(\mathbb{R}^d),\otimes)$ is a closed subgroup of $(\mathrm{T}^\infty_1(\mathbb{R}^d),\otimes)$. In addition, exp is one-to-one from $\mathrm{A}^\infty(\mathbb{R}^d)$ to $\mathrm{G}^\infty(\mathbb{R}^d)$, and log is its inverse.

One of the striking result from K.T. Chen, which uses some properties of the iterated integrals, is that $\Psi(x;0,t)$ belongs to $\mathrm{G}^\infty(\mathbb{R}^d)$, or equivalently, with the Baker-Campbell-Hausdorff-Dynkin formula, that $\log(\Psi(x;0,t))$ belongs to $\mathrm{A}^\infty(\mathbb{R}^d)$.

This approach proved to be very useful, since it allows us to consider equation driven by smooth paths or differential equations in an algebraic setting, and allows us formal computations. Numerous topics in control theory uses this point of view (see for example [Fli81, Isi95, Kaw98]). It way also used in the stochastic context to deal with flow of Stochastic Differential Equations (see for example [Yam79, FNC82, BA89, Cas93]... or the book [Bau04]).

[3] In fact, this equivalence is not exactly one-to-one, unless one eliminates paths such that, one some time interval, x goes from a point a to a point b and then back to a by reversing the path.

For some integer k, we may truncate Ψ by considering that $e_{i_1} \otimes \cdots \otimes e_{i_\ell} = 0$ for all $\ell > k$. For such a truncated power series $\Psi_k(x)$ we still get the relationship $\Psi_k(x \cdot y; 0, T+S) = \Psi_k(x; 0, T) \otimes \Psi_k(y; 0, S)$. In particular, we deduce that

$$\Psi_k(x_{|[s,t]}; s, t) = \Psi_k(x_{|[s,r]}; s, r) \otimes \Psi_k(x_{|[r,t]}; r, t)$$

for all $0 \leq s \leq r \leq t \leq T$. With $k = 2$, we get exactly that our natural lift of $\mathbf{x}_t = \Psi_2(x; 0, t)$ satisfies the relationship $\mathbf{x}_{s,t} = \mathbf{x}_{s,r} \otimes \mathbf{x}_{r,t}$.

Thus, given a path $\mathbf{x}$ in $\mathrm{T}_1(\mathbb{R}^2)$, one can think of $\pi_{e_i \otimes e_j}(\mathbf{x}_t)$ as the iterated integrals of x^j against x^i. Of course, one knows that for irregular paths, there is no canonical way to define them (think of Brownian motion trajectories). Anyway, for weak geometric rough paths, this iterated integrals are approximated by iterated integrals of some smooth paths.

We may now present another heuristic argument to derive the expression of $\mathfrak{F}(f, \mathbf{x}; s, t)$ and then (62). This argument is the historical one (see [Lyo98, Lej03, LQ02, LCL07]). Consider a smooth path $x : [0, T] \to \mathbb{R}^d$ and a smooth function $f = (f_1, \dots, f_d)$. Then using a Taylor development, one gets that

$$\begin{aligned}\sum_{i=1}^{d} \int_s^t f_i(x_r) \,\mathrm{d}x_r^i &= \sum_{i=1}^{d} f_i(x_s)(x_t^i - x_s^i) \\ &\quad + \sum_{\ell \geq 1} \sum_{(i_1, \dots, i_\ell) \in \{1, \dots, d\}^\ell} \frac{\partial^\ell f_i}{\partial x_{i_1} \cdots \partial x_{i_\ell}}(x_0) K^{i_\ell, \dots, i_1, i}(x; s, t) \\ &= \mathfrak{E}_{\mathrm{T}^\infty(\mathbb{R}^d)}(f)(x_s)\Psi(x; s, t)\end{aligned}$$

with, for $z \in \mathbb{R}^d$,

$$\mathfrak{E}_{\mathrm{T}^\infty(\mathbb{R}^d)}(f)(z) = \sum_{\ell \geq 0} \sum_{(i_1, \dots, i_\ell \in \{1, \dots, d\}^\ell} \frac{\partial^\ell f_i}{\partial x_{i_1} \cdots \partial x_{i_\ell}}(z) e^{i_1} \otimes \cdots \otimes e^{i_\ell}.$$

In the usual case, we keep only the first term $\sum_{i=1}^d f_i(x_s)(x_t^i - x_s^i)$ as an approximation of $\sum_{i=1}^d \int_s^t f_i(x_r) \,\mathrm{d}x_r^i$, and we use it as the term in a Riemann sum. Keeping higher order terms has no influence, since $K^{i_1, \dots, i_\ell}(x; s, t) \leq (1/\ell!) \|x'\|_\infty^\ell (t-s)^\ell$.

The idea is then to keep enough terms, if x is α-Hölder continuous and we get an object $\mathbf{x}^{(k)}$ having the same algebraic properties as $\Psi_k(x; s, t)$ for some integer k, to get a Riemann sum that converges. In [Lyo98, LQ02], T. Lyons and his co-authors proved that the number of terms shall be $k = \lfloor 1/\alpha \rfloor$. In particular, from $\mathbf{x}^{(k)}$, it is possible to reconstruct an object living in $\mathrm{T}^\infty(\mathbb{R}^d)$ and equal to $\Psi(x)$ when x is smooth and possessing the same algebraic properties as $\Psi(x)$.

For $k = 2$ and using the path $\mathbf{x}$ as the object $\mathbf{x}^{(2)}$, we get the expression (62).

8.3. **Paths with quadratic variation.** For the Brownian motion or a semi-martingale, one knows how to construct several integrals — the major ones are the Itô and the Stratonovich integrals — whose difference depends on the fact that their trajectories have finite quadratic variation.

With the theory of rough paths, we can indeed construct a pathwise equivalent theory of the Itô integral. For this, we need the path to have a quadratic variation.

Definition 7. Given $\alpha \in (1/3, 1/2]$, a path $x \in \mathrm{C}^\alpha([0,T];\mathbb{R}^2)$ has a *quadratic variation* if there exists a process $\mathfrak{Q}(x) \in \mathrm{C}^\alpha([0,T];\mathrm{S}(\mathbb{R}^2))$ such that $\xi_0 = 0$ and, if $z^{\otimes 2} = z \otimes z$ for $z \in \mathbb{R}^2$,

$$\mathfrak{Q}_n(x;t) = \frac{t - t^n_{\underline{M}(t,n)}}{t^n_{\underline{M}(t,n)+1} - t^n_{\underline{M}(t,n)}} (x_{t^n_{\underline{M}(t,n)+1}} - x_{t^n_{\underline{M}(t,n)}})^{\otimes 2} + \sum_{k=0}^{\underline{M}(t,n)-1} (x_{t^n_{k+1}} - x_{t^n_k})^{\otimes 2}$$

and $\mathfrak{Q}(x) = \lim_{n\to\infty} \mathfrak{Q}_n(x)$ where the limit holds in $\mathrm{C}^\alpha([0,T];\mathrm{S}(\mathbb{R}^2))$.

Remark 16. Note that with the norm we use, this means that the components of $\mathfrak{Q}(x)$ are 2α-Hölder continuous.

Remark 17. If $x \in \mathrm{C}^\alpha([0,T];\mathbb{R}^2)$ with $\alpha > 1/2$, then it is easily seen that necessarily, $\mathfrak{Q}(x;t) = 0$ for $t \in [0,T]$.

Trajectories of the Brownian motion and of Hölder continuous martingales present this feature (see [Sip93, CL05]).

Thus, a natural expression for the equivalent of the Itô integral consists in considering the path $\mathbf{x}^n$ defined in (57), and to set

$$\mathfrak{D}(\mathbf{x}^n;0,t) = \sum_{k=0 \text{ s.t. } t^n_k \leq t} \mathfrak{E}_{\mathrm{A}(\mathbb{R}^2)}(f)(\mathbf{x}^n_{t^n_k}) \frac{\mathrm{d}\mathbf{x}^n(t^n_k)}{\mathrm{d}t} \Delta_n t$$

where $\mathfrak{E}_{\mathrm{A}(\mathbb{R}^2)}(f)(\mathbf{x}^n_{t^n_k})$ has been defined by (55). This construction differs from (55), since

$$\sum_{k=0 \text{ s.t. } t^n_k \leq t} \mathfrak{E}_{\mathrm{A}(\mathbb{R}^2)}(f)(\mathbf{x}_{t^n_k}) \frac{\mathrm{d}\mathbf{x}^n(t^n_k)}{\mathrm{d}t} \Delta_n t = \sum_{k=0 \text{ s.t. } t^n_i \leq t} \int_{t^n_i}^{t^n_{i+1}} \mathfrak{E}_{\mathrm{A}(\mathbb{R}^2)}(f)(\mathbf{x}_{t_k}) \log(\mathbf{x}_{t^n_k,t^n_{k+1}}) \,\mathrm{d}s.$$

A comparison with (59) leads to

$$\mathfrak{D}(\mathbf{x}^n;0,t) = \sum_{k=0 \text{ s.t. } t^n_i \leq t} \mathfrak{F}(f,\mathbf{x},t^n_k,t^n_{k+1}) - \nabla f(x_{t^n_k})\mathfrak{s}(\mathbf{x}_{t^n_k,t^n_{k+1}}).$$

If x has a quadratic variation $\mathfrak{Q}(x)$, then the component of $\mathfrak{Q}(x)$ are 2α-Hölder continuous. In addition, the components of ∇f belongs to the space $\mathrm{Lip}(\gamma-1;\mathbb{R}^2\to\mathbb{R}^2)$. Hence, since $\mathfrak{Q}_n(x)$ converges to $\mathfrak{Q}(x)$ and

$$\left|\nabla f(x_{t_k^n})\mathfrak{s}(\mathbf{x}_{t_k^n,t_{k+1}^n}) - \int_{t_k^n}^{t_{k+1}^n} \nabla f(x_s)\,\mathrm{d}\,\mathfrak{Q}_n(x;s)\right| \\ \leq \Delta_n t^{\alpha(1+\gamma)}\|f\|_{\mathrm{Lip}}\|\mathbf{x}\|_\alpha^{\alpha(1+\gamma)},$$

we easily get the convergence of the last term to the Young integral defined by $\frac{1}{2}\int_0^T \nabla f(x_r)\,\mathrm{d}\,\mathfrak{Q}(x;r)$.

Thus, the limit of $\mathfrak{D}(\mathbf{x};0,t)$ is $\mathfrak{I}(\mathbf{x};0,t) - \int_0^t \frac{1}{2}\nabla f(x_s)\,\mathrm{d}\,\mathfrak{Q}(x_s)$ for $t\in[0,T]$. The integral $\mathfrak{D}(\mathbf{x})$ is constructed is the same at the first level as if we have used the $(1/\alpha,2/\alpha)$-Hölder continuous rough path $(\mathbf{x},-\frac{1}{2}\,\mathfrak{Q}(x))$ (see [LV06]).

8.4. **Link with stochastic integrals.** Itô and Stratonovich integrals are defined as limit in probability of Riemann sum. On the other hand, the rough path theory gives a pathwise definition of the integral, but the price to pay is to add a supplementary information. Is there some link between the two integrals?

Let B be a d-dimensional Brownian motion (a semi-martingale may be used as well). A natural way to construct a rough path $\mathbf{B}$ lying above B is to set

$$\pi_{e_i\otimes e_j}(\mathbf{B}_t) = \int_0^t (B_r^i - B_0^i)\circ\,\mathrm{d}B_r^j.$$

for $i,j=1,\dots,d$. For the construction of $\mathbf{B}$ as a rough path, see for example [Sip93, LQ02, Lej03, CL05]. The process $\log(\mathbf{B})$ is called the *Brownian motion on the Heisenberg group*, and add been widely studied (See references in Section B).

The continuity result of the rough path integral and the Wong-Zakai theorem allows us to identify the integral $\mathfrak{I}(\mathbf{B};0,T)$ with the Stratonovich integral given by

$$\int f(B_s)\circ\,\mathrm{d}B_s = \lim_{n\to\infty}\sum_{k=0}^{2^n-1}\frac{1}{2}(f(B_{t_{k+1}^n})+f(B_{t_k^n}))(B_{t_{k+1}^n}-B_{t_k^n})$$

where the limit is a limit in probability. We will see here that there is another relationship between the two integrals without invoking this continuity result, and that the construction of the Stratonovich and Itô integrals (although under stronger condition on the function f than the one required by the "classical" theory) can be deduced from the rough paths theory.

The theory of rough paths also gives a better intuitive understanding of the counter-examples to the Wong-Zakai theorem (see [McS72, IW89] for SDEs and [LL06a] in the context of rough paths).

The projection on $\mathbb{R}^d$ of $\mathfrak{I}(\mathbf{B};0,T)$ is given by

$$\begin{aligned}(66)\quad &\pi_{\mathbb{R}^d}(\mathfrak{I}(\mathbf{B};0,T))\\ &\quad=\lim_{n\to\infty}\sum_{k=0}^{2^n-1}\Big(f(B_{t_k^n})(B_{t_{k+1}^n}-B_{t_k^n})+\nabla f(B_{t_k^n})\pi_{\mathbb{R}^d\otimes\mathbb{R}^d}(\mathbf{B}_{t_k^n,t_{k+1}^n})\Big)\end{aligned}$$

which we rewrite using $\mathfrak{a}$ and $\mathfrak{s}$ as

$$\begin{aligned}\pi_{\mathbb{R}^d}(\mathfrak{I}(\mathbf{B};0,T))=\lim_{n\to\infty}\sum_{k=0}^{2^n-1}\Big(&f(B_{t_k^n})(B_{t_{k+1}^n}-B_{t_k^n})+\nabla f(B_{t_k^n})\mathfrak{a}(\mathbf{B}_{t_k^n,t_{k+1}^n})\\ &+\nabla f(B_{t_k^n})\mathfrak{s}(\mathbf{B}_{t_k^n,t_{k+1}^n})\Big).\end{aligned}$$

But we have seen that

$$f(B_{t_k^n})(B_{t_{k+1}^n}-B_{t_k^n})+\nabla f(B_{t_k^n})\mathfrak{s}(\mathbf{B}_{t_k^n,t_{k+1}^n})\approx\int_{t_k^n}^{t_{k+1}^n}f(B_s^{\Pi^n})\,\mathrm{d}B_s^{\Pi^n}$$

with B^{Π^n} the piecewise linear approximation of B along the dyadic partition Π^n, and $\approx$ meaning that the difference between the two terms is less than $C2^{-n\theta}$ with $\theta>1$.

On the other hand, using the relation $f(x)-f(y)=\int_0^1\nabla f(x+\tau(y-x))(y-x)\,\mathrm{d}\tau$ and the change of variable $\tau'=2^n\tau$, we get that for $k=0,\dots,2^n$,

$$\begin{aligned}&\sum_{i=1}^d(f_i(B_{t_{k+1}^n})-f_i(B_{t_k^n}))(B^i_{t_{k+1}^n}-B^i_{t_k^n})\\ &\qquad=\int_{t_k^n}^{t_{k+1}^n}\sum_{i,j=1}^d\frac{\partial f_i}{\partial x_j}(B_s^{\Pi^n})(B^j_{t_{k+1}^n}-B^j_{t_k^n})(B^i_{t_{k+1}^n}-B^i_{t_k^n})2^n\,\mathrm{d}s\\ &\qquad\qquad\approx\nabla f(B_{t_k^n})(B_{t_{k+1}^n}-B_{t_k^n})\otimes(B_{t_{k+1}^n}-B_{t_k^n}).\end{aligned}$$

With (45), $\mathfrak{s}(\mathbf{B}_{s,t})=\frac{1}{2}(B_t-B_s)\otimes(B_t-B_s)$. This implies that

$$\sum_{i=1}^d\frac{1}{2}(f_i(B_{t_{k+1}^n})-f_i(B_{t_k^n}))(B^i_{t_{k+1}^n}-B^i_{t_k^n})\approx\nabla f(B_{t_k^n})\mathfrak{s}(\mathbf{B}_{t_k^n,t_{k+1}^n}).$$

On the other hand, let us remark that if $M_k=\mathfrak{a}(\mathbf{B}_{t_k^n,t_{k+1}^n})$, then $(\sum_{\ell=0}^k M_\ell)_{k=0,\dots,2^n}$ forms a martingale with respect to $(\mathcal{F}_k)_{k=0,\dots,2^n}$, where $(\mathcal{F}_t)_{t\geq0}$ is the filtration of the Brownian motion. In addition,

$$\mathbb{E}[(M_k)^2]\leq\frac{6T^2}{2^{2n}}.$$

Hence,

$$\mathbb{E}\left[\left(\sum_{k=0}^{2^n-1}\nabla f(B_{t_k^n})\mathfrak{a}(\mathbf{B}_{t_k^n,t_{k+1}^n})\right)^2\right]\leq\frac{6T^2}{2^n}\|\nabla f\|_\infty$$

and the latter term converges to 0 in probability. The convergence of the Stratonovich integral in probability follows from the last convergence and the almost sure convergence of the rough path approximation given in (66).

Regarding the Itô integrals, we lift the Brownian motion B as a Brownian motion $\mathbf{B}'$ with $\pi_{\mathbb{R}^2}(\mathbf{B}') = B$ and

$$\pi_{e_i\otimes e_j}(\mathbf{B}'_t) = \int_0^t (B_r^i - B_0^i)\,\mathrm{d}B_r^j = \int_0^t (B_r^i - B_0^i)\circ\,\mathrm{d}B_r^j - \frac{1}{2}\delta_{i,j}t.$$

However, let us note that the anti-symmetric part $\mathfrak{a}(\mathbf{B}')$ is equal to the anti-symmetric part of $\mathfrak{a}(\mathbf{B})$. Indeed, due to the Wong-Zakai theorem [IW89], $\mathbf{B}$ is a geometric rough path, while $\mathbf{B}'$ is not a geometric rough path. From the previous computations, we get easily that

$$\pi_{\mathbb{R}^d}(\mathfrak{I}(\mathbf{B}';0,T)) = \mathfrak{I}_{\mathbb{R}^d}(\mathfrak{I}(\mathbf{B};0,T)) - \frac{1}{2}\sum_{i=1}^d \int_0^T \frac{\partial f_i}{\partial x_i}(B_s)\,\mathrm{d}s = \int_0^T f(B_s)\,\mathrm{d}B_s$$

and thus $\mathbf{B}'$ gives rise to the Itô integral. The effect of the bracket terms $t \mapsto \langle B^i, B^j\rangle_t = \delta_{i,j}t$ on $\mathfrak{I}(\mathbf{B}')$ with respect to $\mathfrak{I}(\mathbf{B})$ in studied in Section 7.5.

9. Solving a differential equations

The theory of rough paths may be applied to solve differential equations, since one can transform integrals into differential equations using a fixed point principle. Indeed, as noted in Section 8.2, most of the ideas from the rough paths theory comes from the developments around iterated integrals as a way to deal formally with ordinary differential equations. Thus, the algebraic structures we used were introduced in the context of differential equations, not integrals (see for example [Mag54, Che57, Str87]... and also [Yam79, FNC82, BA89, Cas93]... on Stratonovich stochastic differential equations).

We wish now to consider the following differential equation

$$y_t = y_0 + \int_0^t g(y_s)\,\mathrm{d}x_s, \tag{67}$$

where x is an irregular path. We assume that x lives in $\mathbb{R}^d$, and y lives in $\mathbb{R}^m$. We denote by $\{e_1,\dots,e_d\}$ (resp. $\{\overline{e}_1,\dots,\overline{e}_m\}$) the canonical basis of $\mathbb{R}^d$ (resp. $\mathbb{R}^m$). If one wishes to interpret this integral as a rough path, one has first to transform the vector field

$$g(z) = \sum_{\substack{i=1,\dots,d\\k=1,\dots,m}} \overline{e}_k g_i^k(z)\frac{\partial}{\partial x_i}$$

into a differential form h which is integrated along a path (x, y) living in $\mathbb{R}^d \oplus \mathbb{R}^m$. For this, the natural extension is

$$h(z, z') = \sum_{\substack{i=1,\dots,d\\k=1,\dots,m}} \overline{e}_k g_i^k(z') e^i + \sum_{i=1}^{d} e_i \cdot e^i,\ z \in \mathbb{R}^d,\ z' \in \mathbb{R}^m.$$

Hence, if x is smooth and (67) has a smooth solution y,

$$(x_t, y_t) = (x_0, y_0) + \int_0^t h(x_s, y_s)\,\mathrm{d}(x_s, y_s) = (x_0, y_0) + \int_{(x,y)_{|[0,t]}} h.$$

In order to deal with an irregular path x, the last integral will be defined as a rough path, which means that we shall consider a rough path $\mathbf{z}$ living above (x, y), in the tensor space $\mathrm{T}_1(\mathbb{R}^d \oplus \mathbb{R}^m)$. We have also to extend the differential form h. For $(z, z') \in \mathbb{R}^d \oplus \mathbb{R}^m$, define by $\mathfrak{E}_{\mathrm{T}_1(\mathbb{R}^d\oplus\mathbb{R}^m)}(h)(z, z')$ the linear form on $\mathrm{T}_0(\mathbb{R}^d \oplus \mathbb{R}^m)$ by

$$\begin{aligned}\mathfrak{E}_{\mathrm{T}_1(\mathbb{R}^d\oplus\mathbb{R}^m)}(h)(z, z') &= h(z, z') + \sum_{\substack{i=1,\dots,d\\k,\ell=1,\dots,m}} \overline{e}_k \frac{\partial g_i^k}{\partial x_\ell}(z') \overline{e}^\ell \otimes e^i \\ &+ \sum_{\substack{k,\ell=1,\dots,m\\i,j=1,\dots,d}} \overline{e}_k \otimes \overline{e}_\ell g_i^k(z') g_j^\ell(z') e^i \otimes e^j + \sum_{i,j=1,\dots,d} e_i \otimes e_j \cdot e^i \otimes e^j \\ &+ \sum_{\substack{k=1,\dots,m\\i,j=1,\dots,d}} \overline{e}_k \otimes e_j g_i^k(z) e^i \otimes e^j + \sum_{\substack{k=1,\dots,m\\i,j=1,\dots,d}} e_i \otimes \overline{e}_k g_j^k(z) e^i \otimes e^j.\end{aligned}$$

We then use Remark 15 to transform this linear form into a differential form on $\mathrm{T}_1(\mathbb{R}^d \oplus \mathbb{R}^m)$. The idea is now to apply a Picard iteration scheme. Define by $\mathfrak{I}$ the integral with respect to the differential form h. If $\mathbf{z}^0$ is a rough path in $\mathrm{C}^\alpha([0, T]; \mathrm{T}_1(\mathbb{R}^d \oplus \mathbb{R}^m))$ lying above (x, y^0) for some path $y^0 \in \mathrm{C}^\alpha([0, T]; \mathbb{R}^m))$ and $\pi_{\mathrm{T}_1(\mathbb{R}^d)}(\mathbf{z}^0) = \mathbf{x}$, then set recursively $\mathbf{z}^{k+1} = \mathfrak{I}(\mathbf{z}^k)$. The problem is to study the convergence of $(\mathbf{z}^k)_{k\in\mathbb{N}}$.

Definition 8. A solution of (67) is a rough path $\mathbf{z}$ living in $\mathrm{T}_1(\mathbb{R}^d \oplus \mathbb{R}^m)$ with $\mathbf{z}_0 = (x_0, y_0, 0)$ and such that $\mathfrak{I}(\mathbf{z}; s, t) = \mathbf{z}_{s,t}$ for all $0 \leq s \leq t \leq T$ and $\pi_{\mathrm{T}_1(\mathbb{R}^d)}(\mathbf{z}) = \mathbf{x}$.

Let us start our study by the following observation: from the choice of h, $\pi_{\mathrm{T}_1(\mathbb{R}^d)}(\mathbf{z}^k)$ is equal to $\mathbf{x}$, whatever k. In addition, to compute $\mathbf{z}^{k+1}$, we need $\mathbf{x} = \pi_{\mathrm{T}_1(\mathbb{R}^d)}(\mathbf{z}^k)$, $\pi_{\mathbb{R}^m}(\mathbf{z}^k)$ and $\pi_{\mathbb{R}^m\otimes\mathbb{R}^d}(\mathbf{z}^k)$. If $\mathbf{z}^k$ lies above (x, y^k), the last term corresponds to the iterated integrals of y^k against x.

For proofs, the reader is referred to [Lyo98, Section 4.1, p. 296], [LQ02, Chapter 6, p. 148] and to [LV06].

Theorem 4. *Let* $\mathbf{x}$ *be a rough path in* $\mathrm{C}^\alpha([0,T];\mathrm{T}_1(\mathbb{R}^d))$*. Let* $g_1,\dots,g_d$ *be vector fields on* $\mathbb{R}^m$ *with a bounded derivatives which are* κ*-Hölder continuous with* $\alpha(2+\kappa)>1$*. Then there exists at least one solution to* (67) *in* $\mathrm{C}^\alpha([0,T];\mathrm{T}_1(\mathbb{R}^d\oplus\mathbb{R}^m))$*.*

If $g_1,\dots,g_d$ *are vector fields on* $\mathbb{R}^m$ *that are two times differentiable with, for* $i,j,k=1,\dots,d$*,* $\partial_{x_j} g_i$ *which is bounded and* $\partial^2_{x_k,x_j} g_i$ *which is bounded and* κ*-Hölder continuous with* $\alpha(2+\kappa)>1$*, then the solution of* (67) *is unique and* $\mathbf{x}\mapsto\mathbf{z}$ *is continuous from* $(\mathrm{C}^\alpha([0,T];\mathrm{T}_1(\mathbb{R}^d)),\|\cdot\|_\alpha)$ *to* $(\mathrm{C}^\alpha([0,T];\mathrm{T}_1(\mathbb{R}^d\oplus\mathbb{R}^m),\|\cdot\|_\alpha)$*.*

Remark 18. The map $\mathbf{x}\mapsto\mathbf{z}$ is called the *Itô map*. Its differentiability is studied in [LQ98, LQ02], in [LL06b] (for $\alpha>1/2$) and in [FV08].

Here again, because of the continuity of $\mathbf{x}\mapsto\mathbf{z}$, we get that if $\mathbf{x}$ belongs to $\mathrm{C}^\alpha([0,T];\mathrm{G}(\mathbb{R}^d))$, then $\mathbf{z}\in\mathrm{C}^\alpha([0,T];\mathrm{G}(\mathbb{R}^d\oplus\mathbb{R}^m))$ and if $\mathbf{x}$ belongs to $\mathrm{C}^{0,\alpha}([0,T];\mathrm{G}(\mathbb{R}^d))$, then $\mathbf{z}\in\mathrm{C}^{0,\alpha}([0,T];\mathrm{G}(\mathbb{R}^d\oplus\mathbb{R}^m))$.

Finally, the solution of (67) may also be interpreted using an Euler scheme, as in [FV08], following [Dav07]. In addition, A.M. Davie proved in [Dav07] that there exists a unique solution of g_i are of class C^2, and that the solution may not be unique of g_i has only Hölder continuous derivatives.

Appendix A. Carnot groups and homogeneous gauges and norms

Let $(\mathrm{G},\times)$ be a Lie group, and $(\mathrm{g},[\cdot,\cdot])$ be its Lie algebra G is a *Carnot group of step* k [Mon02, Bau04] if for some positive integer k, $\mathrm{g}=\mathrm{V}_1\oplus\mathrm{V}_2\oplus\cdots\oplus\mathrm{V}_k$ — this decomposition being called a *stratification* — with

$$[\mathrm{V}_1,\mathrm{V}_i]=\mathrm{V}_{i+1}\text{ for }i=1,\dots,k-1\text{ and }[\mathrm{V}_1,\mathrm{V}_k]=\{0\},$$

where $[\mathrm{V}_i,\mathrm{V}_j]=\{[x,y]|x\in\mathrm{V}_i,y\in\mathrm{V}_j\}$. A Carnot group is naturally equipped with a *dilatation* operator $\delta_\lambda(x)=(\lambda^{\alpha_1}x^1,\cdots,\lambda^{\alpha_k}x^k)$ with $x^i\in\exp(\mathrm{V}^i)$ and some positive real numbers $\alpha_1,\dots,\alpha_k$, where exp is the map from g to G. This dilatation operator shall verify $\delta_\lambda(x\times y)=\delta_\lambda(x)\times\delta_\lambda(y)$. If the dimension of V_1 is finite, the real number $N=\alpha_1\dim(\mathrm{V}_1)+\cdots+\alpha_k\dim(\mathrm{V}_k)$ is called the *homogeneous dimension*.

On G equipped with a dilatation operator δ, an *homogeneous gauge* is a continuous function which maps x into a non-negative real number $\|x\|$ such that $\|x\|=0$ if and only if x is the neutral elements of G, and for all $\lambda\in\mathbb{R}$, $\|\delta_\lambda(x)\|=|\lambda|\cdot\|x\|$.

An homogeneous gauge is an *homogeneous norm* if $\|x^{-1}\|=\|x\|$ for all $x\in\mathrm{G}$. In addition, this homogeneous norm is said to be *sub-additive* if $\|x\times y\|\leq\|x\|+\|y\|$ for all $x,y\in\mathrm{G}$.

If V_1 is of finite dimension, then a homogeneous norm always exists [FS82]. For this, equip the Lie algebra g with the Euclidean norm $|\cdot|$ and denote by exp the canonical diffeomorphism from g to G. For

$x \in \mathrm{g}$, let $r(x)$ be the smallest positive real such that $|\delta_{r(x)}x| = 1$, which exists, since $|\delta_r x|$ is increasing from $[0, +\infty)$ to $[0, +\infty)$. Then, for $y \in \mathrm{G}$, $\|y\| = 1/r(\exp^{-1} y)$ defines a symmetric homogeneous norm.

Two homogeneous gauges $\|\cdot\|$ and $\|\cdot\|'$ are said to be equivalent if for some constants C and C', $C\|x\|' \leq \|x\| \leq C'\|x\|'$ for all $x \in \mathrm{G}$.

Proposition 10 ([Goo77]). *If the dimension of* V_1 *is finite, then all the homogeneous gauges are equivalent. In addition, for a homogeneous gauge* $\|\cdot\|$, *there exists some constant* C *and* C' *such that* $\|x^{-1}\| \leq C\|x\|$ *and*

$$|x \times y| \leq C'(|x| + |y|),$$

for all $x, y \in \mathrm{G}$.

Proof. If $\exp^{-1}(x)$ is decomposed as $y_1, \dots, y_k$ with $y_i \in \mathrm{V}_i$ for $x \in \mathrm{G}$, then set $|x|' = \sum_{i=1}^k |y_i|^{1/i}$, where $|\cdot|$ denotes the Euclidean norm on each of the finite-dimensional vector space V_i. It is easily verified that $|x|$ is a homogeneous gauge. Let $\|\cdot\|$ be another homogeneous gauge. Set $\varphi(x) = \|x\|/|x|'$. Then φ and $1/\varphi$ are continuous on $\mathrm{G} \setminus \{1\}$, where 1 is the neutral element of G. As $\{x \in \mathrm{G} \,|\, |x|' = 1\}$ is compact, we easily get that φ and $1/\varphi$ are bounded, and then that for some constants C and C', $C \leq \|x\| \leq C'$ when $|x|' = 1$. This implies that $\|\cdot\|$ and $|\cdot|'$ are equivalent by using the dilatation $\delta_{1/|x|'}$ for a general x.

The other results are proved in a similar way by using $\varphi(x) = \|x^{-1}\|/\|x\|$ and $\varphi(x, y) = \|x \times y\|/(\|x\| + \|y\|)$. □

It follows that any homogeneous gauge can be transformed in an equivalent homogeneous norm by setting $\|x\|' = \|x\| + \|x^{-1}\|$.

The notion a *Lipschitz* function is then extended to homogeneous gauges.

Definition 9. If $(\mathrm{G}, \times)$ and $(\mathrm{G}', \times)$ are two nilpotent Carnot groups with homogeneous gauges $\|\cdot\|$ and $\|\cdot\|'$, then $f : \mathrm{G} \to \mathrm{G}'$ is said to be *Lipschitz* if for some constant C,

$$\|f(x)^{-1} \times f(y)\|' \leq C\|x^{-1} \times y\|$$

for all $x, y \in \mathrm{G}$.

The group $(\mathrm{A}(\mathbb{R}^2), \boxplus)$ (and thus $(\mathrm{G}(\mathbb{R}^2), \otimes)$) is obviously a Carnot group of step 2 with $\mathrm{V}_1 = \mathbb{R}^2$ and $\mathrm{V}_2 = [\mathbb{R}^2, \mathbb{R}^2]$, and $\delta_\lambda(x) = (\lambda x^1, \lambda^2 x^2)$. Its homogeneous dimension is 4. Homogeneous norms and gauges are easily constructed. It is sufficient to consider $\|x\| = |x^1| + \sqrt{|x^2|}$, $\|x\| = \max\{|x^1|, \sqrt{|x^2|}\}$ either on $\mathrm{A}(\mathbb{R}^2)$ or $\mathrm{G}(\mathbb{R}^2)$. Of course, if $\|\cdot\|$ is a homogeneous gauge on $\mathrm{A}(\mathbb{R}^2)$, then $\|\cdot\|'$ defined by $\|x\|' = \|\log(x)\|$ is a homogeneous gauge on $\mathrm{G}(\mathbb{R}^2)$.

Appendix B. The Brownian motion on the Heisenberg group

We have seen in Section 8.4 that the Brownian motion is naturally lifted as a rough path and then that the integrals correspond to the usual Itô or Stratonovich integrals.

The tangent plane of $\mathrm{A}(\mathbb{R}^2)$ may be identified with $\mathrm{A}(\mathbb{R}^2)$, and we denote by ∂_x, ∂_y and ∂_z the basis of $T_x\mathrm{A}(\mathbb{R}^2)$ at a point x which is deduced from the canonical coordinates e_1, e_2 and $[e_1, e_2]$.

Let V^1, V^2 and V^3 be the left invariant vector fields that goes through 0 and that coincide respectively with ∂_x, ∂_y and ∂_z at this point.

For example, for $a \in \mathrm{A}(\mathbb{R}^2)$ and all $x \in \mathrm{A}(\mathbb{R}^2)$ and smooth function f on $\mathrm{A}(\mathbb{R}^2)$, $V^i f(a \boxplus x) = V^i f \circ L_a(x)$ where $L_a(x) = a \boxplus x$ for $i = 1, 2, 3$. We have seen in Section 6.12 that the V^i's are decomposed in the basis $\{\partial_x, \partial_y, \partial_z\}$ as

$$V^1 = \partial_x - \frac{1}{2} y \partial_z,\ V^2 = \partial_y + \frac{1}{2} x \partial_y \text{ and } V^3 = \partial_z.$$

We remark that $[V^1, V^2] = V^3$ and $[V^i, V^j] = 0$ in all the other cases. The tangent plane at any point of $\mathrm{A}(\mathbb{R}^2)$ is then equipped with a scalar product $\langle\langle \cdot, \cdot \rangle\rangle$ such that $\langle\langle V^i, V^j \rangle\rangle = \delta_{i,j}$ for $i, j = 1, 2, 3$, *i.e.,* for which $\{V^1, V^2, V^3\}$ forms an orthonormal basis. With this scalar product, $\mathrm{A}(\mathbb{R}^2)$ becomes a Riemannian manifold.

Let $B = (B^1, B^2)$ be a two dimensional Brownian motion, and $B^n = (B^{n,1}, B^{n,2})$ for $n = 1, 2, \dots$ be a family of piecewise linear approximation of B along a family of deterministic partitions whose mesh decreases to 0.

We then consider $\mathbf{X}$ the solution of the Stratonovich SDE

$$\mathbf{X}_t = \int_0^t V^1(\mathbf{X}_s) \circ \mathrm{d}B^1_s + \int_0^t V^2(\mathbf{X}_s) \circ \mathrm{d}B^2_s$$

as well as the solutions $\mathbf{X}^n$ of the ordinary differential equations

$$\mathbf{X}^n_t = \int_0^t V^1(\mathbf{X}^n_s) \circ \mathrm{d}B^{1,n}_s + \int_0^t V^2(\mathbf{X}^n_s) \circ \mathrm{d}B^{2,n}_s.$$

Using the decomposition of the V^i on the coordinates $\{\partial_x, \partial_y, \partial_z\}$, we get that

$$\mathbf{X}_t = B^1_t e_1 + B^2_t e_2 + \mathfrak{A}(B^1, B^2; 0, t)[e_1, e_2]$$

where

$$\mathfrak{A}(B^1, B^2; 0, t) = \frac{1}{2} \int_0^t B^1_s \circ \mathrm{d}B^2_s - \frac{1}{2} \int_0^t B^2_s \circ \mathrm{d}B^1_s$$

is the Lévy area of (B^1, B^2). The process $\mathbf{X}$, we have already mentioned in Section 8.4 is the *Brownian motion on the Heisenberg group.* Similarly, we get that

$$\mathbf{X}^n_t = B^{1,n}_t e_1 + B^{2,n}_t e_2 + \mathfrak{A}(B^{1,n}, B^{2,n}; 0, t)[e_1, e_2]$$

and one knows from the Wong-Zakai theorem [IW89] that $\mathbf{X}^n$ converges in probability to $\mathbf{X}$ (with a dyadic partition, we get an almost sure convergence in the α-Hölder norm for any $\alpha < 1/2$ [Sip93, CL05]). Let us note that the piecewise smooth curves $\mathbf{X}^n$ are horizontal curves, so that in this case, the natural approximation of $\mathbf{X} \in \mathrm{C}^\alpha([0,T];\mathrm{A}(\mathbb{R}^2))$ is provided by the piecewise linear approximations of (B^1, B^2) naturally lifted as paths in $\mathrm{A}(\mathbb{R}^2)$. Many processes shares this property : see for example [CQ02, CL05, Lej06].

This is a special case of a Brownian motion in a Lie group. Its short time behavior and its density have been already widely studied: see for example [Gav77, A+81, Bis84, BA89, Bau04], ... From the Hörmander theorem, as $\{V^1, V^2, [V^1, V^2]\}$ spans the tangent space at any point, one knows that for any $t > 0$, $\mathbf{X}_t$ has a density on the three dimensional space $\mathrm{A}(\mathbb{R}^2)$, although it is constructed from a two dimensional Brownian motion. The infinitesimal generator of $\mathbf{X}$ is

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}(V^1)^2 + \frac{1}{2}(V^2)^2 \\ &= \frac{1}{2}\partial_x^2 + \frac{1}{2}\partial_y^2 + \frac{1}{2}x\partial_{zy}^2 - \frac{1}{2}y\partial_{zx}^2 + \frac{1}{8}(x^2+y^2)\partial_z^2.\end{aligned}$$

This is an hypo-elliptic generator.

Appendix C. From almost rough paths to rough paths

C.1. **Theorems and proofs.** In this Section we prove Theorem 2 on almost rough paths, which we rewrite in a more general setting than with Hölder continuous norms.

We set $\Delta_+ = \{(s,t) \in [0,T]^2 \,|\, 0 \le s \le t \le T\}$. A *control* is a function $\omega : \Delta_+ \to \mathbb{R}_+$ such that ω is continuous, ω is super-additive, *i.e.,*

$$\forall 0 \le s < t < u \le T,\ \omega(s,t) + \omega(t,u) \le \omega(s,u)$$

and $\omega(t,t) = 0$ for all $t \in [0,T]$. If ω is super-additive and $\theta \ge 1$, then ω^θ is also super-additive.

We recall that for $x = (\xi, x^1, x^2)$ in $\mathrm{T}_\xi(\mathbb{R}^d)$ with $\xi = 0$ or $\xi = 1$, we have defined $\|x\| = \max\{|x^1|, \sqrt{\frac{1}{2}|x^2|}\}$. We also set $\|x\|_\star = \max\{|x^1|, |x^2|\}$. These two norms are not equivalent, but they define the same topology.

For a continuous path $\mathbf{x}$ with values in $\mathrm{T}_1(\mathbb{R}^2)$, we introduce the norms

$$\|\mathbf{x}\|_{p,\omega} = \sup_{0 \le s < t \le T} \frac{\|\mathbf{x}_{s,t}\|}{\omega(s,t)^{1/p}}$$

and

$$\|\mathbf{x}\|_{\star,p,\omega} = \sup_{0 \le s < t \le T} \max\left\{ \frac{|\mathbf{x}^1_{s,t}|}{\omega(s,t)^{1/p}}, \frac{|\mathbf{x}^2_{s,t}|}{\omega(s,t)^{2/p}} \right\}$$

with $\mathbf{x}^1 = \pi_{\mathbb{R}^d}(\mathbf{x})$ and $\mathbf{x}^2 = \pi_{\mathbb{R}^d \otimes \mathbb{R}^d}(\mathbf{x})$. Let us note that $\|\mathbf{x}\|_{\star,p,\omega}$ is finite if and only if $\|\mathbf{x}\|_{p,\omega}$ is finite. Hence, we denote by $\mathrm{C}^{p,\omega}([0,T];\mathrm{T}_1(\mathbb{R}^d))$

the space of continuous paths with values in $\mathrm{T}_1(\mathbb{R}^d)$ for which $\|\mathbf{x}\|_{p,\omega}$ (or equivalently $\|\mathbf{x}\|_{\star,p,\omega}$ is finite. We rewrite the first part of Theorem 2 with a control ω.

Remark 19. The case of α-Hölder continuous paths corresponds to $\omega(s,t) = t-s$ and $p = 1/\alpha$. All the results we gave about the existence of the integral, solving a differential equation, ... may be written using a control $\omega(s,t)$ instead of $\omega(s,t) = t-s$ and the appropriate norms $\|\cdot\|_{p,\omega}$ and $\|\cdot\|_{\star,p,\omega}$. Similarly, we are not bound to use dyadic partitions, although some results may be related to dyadic partitions (see for example [CL05] for an application to semi-martingales), and it is in generally computationally more simple.

Theorem 5. *Let $(\mathbf{x}_{s,t})_{(s,t)\in\Delta_+}$ be a family of elements of $\mathrm{T}_1(\mathbb{R}^d)$ such that for some $\theta > 1$, $K > 0$,*

$$\|\mathbf{x}\|_{p,\omega} < +\infty \text{ and } \|\mathbf{x}_{s,t} - \mathbf{x}_{s,r}\otimes\mathbf{x}_{r,t}\|_\star \le K\omega(s,t)^\theta \tag{68}$$

for all $(s,t)\in\Delta_+$. We call such a family an almost rough path *controlled by ω.*

Then there exists a rough path $\mathbf{y}$ in $\mathrm{C}^{p,\omega}([0,T];\mathrm{T}_1(\mathbb{R}^d))$ such that

$$\|\mathbf{y}_{s,t} - \mathbf{x}_{s,t}\|_\star \le C\omega(s,t)^\theta \tag{69}$$

for some constant C that depends only on K, θ, p, $\omega(0,T)$ and $\|\mathbf{x}\|_{\star,p,\omega}$. In addition, $\mathbf{y}$ is unique up to the value of $\mathbf{y}_0$.

In addition, if $\mathbf{x}_{s,t}$ belongs to $\mathrm{G}(\mathbb{R}^d)$ for any $0\le s\le t\le T$, then $\mathbf{y}$ is a weak geometric rough path with p-variation controlled by ω.

We give two proofs of this theorem. The first proof concerns the general case, and is taken from [Lyo98]. The other proof is a simpler proof in the case $\omega(s,t) = t-s$, which is adapted from [FdLPM08]. For integrals, where $\mathbf{x}_{s,t} = f(z_s)\mathbf{z}_{s,t}$ for some rough path $\mathbf{z}$ of finite p-variation, one can find some increasing, continuous function $\varphi : [0,T]\to\mathbb{R}_+$ such that $\mathbf{z}\circ\varphi$ is Hölder continuous (See [CG98] and [CL05] for an example of application in the context of rough paths), so that in many cases, one can consider that $\omega(s,t) = t-s$ (as the integral of a differential form along a path in insensitive to change of time).

Proof. Let us remark first that if $\alpha^{(n)} = \bigotimes_{i=1}^n (1+\alpha_i)$ with $\alpha_i \in \mathrm{T}_0(\mathbb{R}^d)$, then

$$\alpha^{(n)} = 1 + \sum_{i=1}^n \alpha_i + \sum_{i=1}^n \sum_{j=i+1}^n \alpha_i\otimes\alpha_j.$$

Hence, if $\overline{\alpha}^{(n)} = \bigotimes_{i=1}^n (1+\overline{\alpha}_i)$ with $\overline{\alpha}_k = \alpha_k + \zeta$ for some $k\in\{1,\dots,n\}$ and $\overline{\alpha}_i = \alpha_i$, $i\neq k$, then

$$\overline{\alpha}^{(n)} = \alpha^{(n)} + \zeta + \zeta\otimes\sum_{j=k+1}^n \alpha_i + \sum_{i=1}^{k-1}\alpha_i\otimes\zeta. \tag{70}$$

Let us set for a partition $\pi = \{t_k\}_{k=1}^{n+1}$ of $[s,t]$ with $t_1 = s$ and $t_{n+1} = t$,

$$\mathbf{x}_{s,t}^{(\pi)} = \bigotimes_{k=1}^{n} \mathbf{x}_{t_k,t_{k+1}}. \tag{71}$$

We set $\mathbf{x}_{s,t}^{1,(\pi)} = \pi_{\mathbb{R}^d}(\mathbf{x}_{s,t}^{(\pi)})$ and $\mathbf{x}_{s,t}^{1} = \pi_{\mathbb{R}^d}(\mathbf{x}_{s,t})$. Let $\{t_i\}$ be some point of π (except s, t), and set $\widehat{\pi} = \pi \setminus \{t_i\}$. Then

$$\mathbf{x}_{s,t}^{1,(\pi)} - \mathbf{x}_{s,t}^{1,(\widehat{\pi})} = \mathbf{x}_{t_{i_1},t_i}^{1} + \mathbf{x}_{t_i,t_{i+1}}^{1} - \mathbf{x}_{t_{i-1},t_{i+1}}^{1}.$$

For a partition $\pi = \{t_i\}_{i=1,\dots,n+1}$ with $n+1$ points in $[s,t]$ and $t_1 = s$, $t_{n+1} = t$, we pick a point t_i such that $\omega(t_{i-1}, t_{i+1}) \leq 2\omega(s,t)/n$. This is possible if $n > 3$ thanks to Lemma 2.2.1 [Lyo98, p. 244]. Then

$$|\mathbf{x}_{s,t}^{1,(\pi)} - \mathbf{x}_{s,t}^{1,(\widehat{\pi})}| \leq K \frac{2^\theta}{n^\theta} \omega(s,t)^\theta.$$

If n has 3 elements $\{t_1, t_2, t_3\}$ with $t_1 = s$ and $t_3 = s$, then $|\mathbf{x}_{s,t}^{1,(\pi)} - \mathbf{x}_{s,t}| \leq K\omega(s,t)^\theta$. Thus, by summing from $k = 1, \dots, n$ by choosing carefully an element of the partition to suppress, we get that

$$|\mathbf{x}_{s,t}^{1,(\pi)} - \mathbf{x}_{s,t}^{1}| \leq 2^\theta \zeta(\theta) K \omega(s,t)^\theta \tag{72}$$

with $\zeta(\theta) = \sum_{n\geq 1} 1/n^\theta$. This is true for any partition π, whatever its size.

Let us consider now a sequence of partitions π^n of $[0,T]$ whose meshes decrease to 0. We set $\pi^n[s,t] = (\pi^n \cap [s,t]) \cup \{s,t\}$. Then for any $(s,t) \in \Delta_+$, $(\mathbf{x}_{s,t}^{1,(\pi^n[s,t])})_{n\in\mathbb{N}}$ has a convergent subsequence.

One can extract a subsequence $(n_k)_{k\in\mathbb{N}}$ such that $(\mathbf{x}_{s,t}^{1,(\pi^{n_k}[s,t])})_{k\in\mathbb{N}}$ converges for any $(s,t) \in \Delta_+$, $s,t \in \mathbb{Q}$. We denote by $y_{s,t}$ one of the possible limits for $(s,t) \in \Delta_+$, $s,t \in \mathbb{Q}$. With set $K_1 = K 2^\theta \zeta(\theta)$ and with (72), we get that

$$|y_{s,t} - \mathbf{x}_{s,t}^{1}| \leq K_1 \omega(s,t)^\theta. \tag{73}$$

As ω is continuous and $\mathbf{x}_{s,t}^{1}$ converges to 0 as $|t-s| \to 0$, we may extend y by continuity on Δ_+.

In addition, for $0 \leq s < r < t \leq T$ and $r \in \pi^n$, then

$$\mathbf{x}_{s,t}^{1,(\pi^n[s,t])} = \mathbf{x}_{s,r}^{1,(\pi^n[s,r])} + \mathbf{x}_{r,t}^{1,(\pi^n[r,t])}.$$

Choosing the partitions π^n such that $\pi^n \subset \pi^{n+1}$ and $\pi^n \subset \mathbb{Q}$ for each $\mathbb{Q}$, we get that, by passing to the limit for $r \in \pi^{n_{k_0}}$ for some k_0 and $s,t \in \mathbb{Q}$, we get that $y_{s,t} = y_{s,r} + y_{r,t}$. Using the continuity of y, this is true for any $0 \leq s < r < t \leq T$. We define $y_t = y_{0,t}$ and remark that $y_{s,t} = y_t - y_s$.

Now, let us consider another function z on $[0,T]$ with values in $\mathbb{R}^d$ and satisfying $|z_t - z_s - \mathbf{x}_{s,t}^{1}| \leq 2^\theta \zeta(\theta) K \omega(s,t)^\theta$ for all $(s,t) \in \Delta_+$.

Since

$$|(y_t - y_s) - (z_t - z_s)| \leq |(y_r - y_s) - \mathbf{x}_{r,s}^{1}| + |(z_r - z_s) - \mathbf{x}_{r,s}^{1}|$$

for $r \in [s,t]$, $(s,t) \in \Delta_+$,

$$|(y_t - y_s) - (z_t - z_s)| \le 2K_1\omega(s,t)^\theta.$$

Thus, $\widehat{y}_t = y_t - z_t$ is controlled by ω^θ with $\theta > 1$ and is necessarily constant. Otherwise,

$$|\widehat{y}_t - \widehat{y}_0| \le \sum_{k=0,\dots,2^n-1,\ t_k^n \le t} |\widehat{y}_{t_{k+1}^n} - \widehat{y}_{t_k^n}| \le 2K_1\omega(s,t) \sup_{k=0,\dots,2^n-1} \omega(t_k^n, t_{k+1}^n)^{\theta-1}$$

and this converges to 0.

We have now to construct the second level of the rough path. For this purpose, we set $\mathbf{z}_{s,t} = 1 + y_t - y_s + \pi_{\mathbb{R}^d\otimes\mathbb{R}^d}(\mathbf{x}_{s,t})$, and, for a partition π with s and t as endpoints, we define $\mathbf{z}_{s,t}^{(\pi)}$ as $\mathbf{x}_{s,t}^{(\pi)}$ in (71) with $\mathbf{z}$ instead of $\mathbf{x}$. Let us note that $\mathbf{z}$ is also an almost rough path, since

$$\begin{aligned}\mathbf{z}_{s,t} - \mathbf{z}_{s,r} \otimes \mathbf{z}_{r,t} = \mathbf{x}_{s,t}^2 - \mathbf{x}_{s,r}^2 - \mathbf{x}_{r,t}^2 - \mathbf{x}_{s,r}^1 \otimes \mathbf{x}_{r,t}^1 \\ - (\mathbf{z}_{s,r}^1 - \mathbf{x}_{r,t}^1) \otimes \mathbf{z}_{r,t}^1 + \mathbf{z}_{s,r}^1 \otimes (\mathbf{z}_{r,t}^1 - \mathbf{x}_{r,t}^1)\end{aligned}$$

and therefore with (73),

$$\|\mathbf{z}_{s,t} - \mathbf{z}_{s,r} \otimes \mathbf{z}_{r,t}\|_\star \le K_2\omega(s,t)^\theta$$

where $K_2 = \left\{K + 2K_1(K_1 + \|\mathbf{x}\|_{\star,p,\omega})\omega(0,T)^{1/p}\right\}\omega(s,t)^\theta$.

For $0 \le s < t \le T$ and $\pi = \{t_i\}_{i=1,\dots,n+1}$ a partition of $[s,t]$ with $n+1$ points and $t_1 = s$, $t_{n+1} = t$, then for some $i \in \{2,\dots,n\}$ and $\widehat{\pi} = \pi \setminus \{t_i\}$,

$$\|\mathbf{z}_{s,t}^{(\pi)} - \mathbf{z}_{s,t}^{(\widehat{\pi})}\|_\star \le K_2\omega(t_{i-1}, t_{i+1})^\theta.$$

One may choose t_i such that $\omega(t_{i-1}, t_{i+1}) \le 2\omega(s,t)/n$. Hence, as previously,

$$\|\mathbf{z}_{s,t}^{(\pi)} - \mathbf{z}_{s,t}\|_\star \le 2^\theta\zeta(\theta)K_2\omega(s,t)^\theta. \tag{74}$$

Then, the same arguments apply and one can show that for all $(s,t) \in \Delta_+$, there exists $\mathbf{y}_{s,t} \in \mathrm{T}_1(\mathbb{R}^d)$ such that $\pi_{\mathbb{R}^d}(\mathbf{y}_{s,t}) = y_t - y_s$, where y was the function previously defined at the first level, for all $0 \le s \le r \le t \le T$, $\mathbf{y}_{s,t} = \mathbf{y}_{s,r} \otimes \mathbf{y}_{r,t}$ and $\|\mathbf{y}_{s,t} - \mathbf{x}_{s,t}\|_\star \le K_3\omega(s,t)^\theta$ with $K_3 = K_2 2^\theta\zeta(\theta)$. In particular, $\mathbf{y}$ is continuous on Δ_+ and $t \mapsto \mathbf{y}_{0,t}$ is a rough path in $\mathrm{C}^{p,\omega}([0,T];\mathrm{T}_1(\mathbb{R}^d))$ lying above y.

Let $\widehat{\mathbf{y}}$ be another rough path in $\mathrm{C}^{p,\omega}([0,T];\mathrm{T}_1(\mathbb{R}^d))$ lying above y and such that $\|\widehat{\mathbf{y}}_{s,t} - \mathbf{x}_{s,t}\|_\star \le K_3\omega(s,t)^\theta$. Hence,

$$\|\mathbf{y}_{s,t} - \widehat{\mathbf{y}}_{s,t}\|_\star \le |\mathbf{y}_{s,r}^2 - \mathbf{z}_{s,r}^2| + |\mathbf{y}_{r,t}^2 - \mathbf{z}_{r,t}^2| + |\widehat{\mathbf{y}}_{s,r}^2 - \mathbf{z}_{s,r}^2| + |\widehat{\mathbf{y}}_{r,t}^2 - \mathbf{z}_{r,t}^2|$$

for all $0 \le s \le r \le t \le T$. As previously, it follows that $\mathbf{y}_{s,t} = \widehat{\mathbf{y}}_{s,t}$ for all $(s,t) \in \Delta_+$.

This proves that $\mathbf{y}$ is unique up to to an additive constant.

The question is now to know whether or not $\mathbf{y}$ is also the limit of $(\mathbf{x}^{(\pi^n)})_{n\in\mathbb{N}}$ for a family of partitions $(\pi^n)_{n\in\mathbb{N}}$ whose meshes decrease to 0.

With the notation from the beginning of the proof, if $\{\alpha_i\}_{i=1,\dots,n}$ is a family of elements in $\mathrm{T}_0(\mathbb{R}^d)$ and $\{\eta_i\}_{i=1,\dots,n}$ belongs to $\mathbb{R}^d$, then

$$\bigotimes_{i=1}^{n}(1+\alpha_i+\eta_i)=\bigotimes_{i=1}^{n}(1+\alpha_i)+\sum_{i=1}^{n}\eta_i+\sum_{i=1}^{n-1}\eta_i\otimes\sum_{j=i+1}^{n}\alpha_j \\ +\sum_{i=1}^{n-1}\alpha_i\otimes\sum_{j=i+1}^{n}\eta_j+\sum_{i=1}^{n-1}\eta_i\otimes\sum_{j=i+1}^{n}\eta_j.$$

Now, let us set $\alpha_i=\mathbf{x}_{t_i,t_{i+1}}$ and $\eta_i=\mathbf{y}^1_{t_i,t_{i+1}}$ for some partition $\pi=\{t_i\}_{i=1,\dots,n+1}$ of $[s,t]$. Then for some constant C_1,

$$\left|\sum_{i=1}^{n}\eta_i\right|\leq\sum_{i=1}^{n}C_1\omega(t_i,t_{i+1})^\theta\leq C\omega(0,T)\sup_{i=1,\dots,n}\omega(t_i,t_{i+1})^{\theta-1}.$$

This last term converges to 0. Finally, let us remark that

$$\sum_{i=1}^{n-1}\alpha_i\otimes\sum_{j=i+1}^{n}\eta_j=\sum_{j=2}^{n}\left(\sum_{i=1}^{j-1}\alpha_i\right)\otimes\eta_j.$$

But from (72), for $k\in\{2,\dots,n\}$,

$$\left|\sum_{i=1}^{k}\alpha_i\right|=|\mathbf{x}^{1,(\pi\cap[s,t_k])}_{s,t_k}|\leq K_1\omega(s,t)^\theta+\|\mathbf{x}\|_{\star,p,\omega}\omega(s,t)^{1/p}.$$

It follows that for some constant C_2 depending only on $\|\mathbf{x}\|_{\star,p,\omega}$, K_1, $\omega(0,T)$, θ and p that

$$\left|\sum_{j=2}^{n}\left(\sum_{i=1}^{j-1}\alpha_i\right)\otimes\eta_j\right|\leq C_2\omega(s,t)\sup_{i=2,\dots,n}\omega(t_i,t_{i+1})^{\theta-1}.$$

Similarly,

$$\left|\sum_{i=1}^{n-1}\eta_i\otimes\sum_{j=i+1}^{n}\alpha_j\right|\leq C_2\omega(s,t)\sup_{i=1,\dots,n-1}\omega(t_i,t_{i+1})^{\theta-1}$$

and

$$\left|\sum_{i=1}^{n-1}\eta_i\otimes\sum_{j=i+1}^{n}\eta_j\right|\leq K_1\omega(s,t)^2\sup_{i=1,\dots,n}\omega(t_i,t_{i+1})^{2\theta-2}.$$

It follows then that for some constant C_3 depending on C_2, K_1, θ and $\omega(0,T)$,

$$\|\mathbf{x}^{(\pi)}_{s,t}-\mathbf{z}^{(\pi)}_{s,t}\|_\star\leq C_3\omega(s,t)\sup_{i=1,\dots,n}\omega(t_i,t_{i+1})^{\theta-1}.$$

This proves that if $(\pi^n)_{n\in\mathbb{N}}$ is a family of partitions whose meshes converge to 0 as $n\to\infty$, then $\mathbf{x}^{(\pi^n)}_{s,t}$ converges to $\mathbf{y}_{s,t}$. In addition, combined with (73) and (74), this gives (69).

The last assertion of this theorem follows from the fact that $\mathbf{x}^{(\pi)}$ belongs to $\mathrm{G}(\mathbb{R}^d)$ if $\mathbf{x}_{s,t}$ belongs to $\mathrm{G}(\mathbb{R}^d)$, which is a closed subgroup of $\mathrm{T}_1(\mathbb{R}^d)$. □

Proof of Theorem 5 (Alternative proof for $\omega(s,t) = K_1(t-s)$). Let us define a distance on $\mathrm{T}_1(\mathbb{R}^d)$ by $d(x,y) = \|x-y\|_\star$. Let us note that

$$d(x\otimes z, y\otimes z) \leq d(x,y)(1+\|z\|_\star) \tag{75}$$

$$\text{and } d(z\otimes x, z\otimes z) \leq d(x,y)(1+\|z\|_\star) \tag{76}$$

for all $x,y,z\in\mathrm{T}_1(\mathbb{R}^d)$.

For $0\leq s\leq t\leq T$, let us set $r=(t+s)/2$, $\mathbf{x}^0_{s,t}=\mathbf{x}_{s,t}$ and recursively,

$$\mathbf{x}^{n+1}_{s,t} = \mathbf{x}^n_{s,r}\otimes\mathbf{x}^n_{r,t}.$$

By the triangular inequality,

$$d(\mathbf{x}^{n+2}_{s,t},\mathbf{x}^{n+1}_{s,t}) \leq d(\mathbf{x}^{n+1}_{s,r}\otimes\mathbf{x}^{n+1}_{r,t},\mathbf{x}^{n+1}_{s,r}\otimes\mathbf{x}^{n}_{r,t}) + d(\mathbf{x}^{n+1}_{s,r}\otimes\mathbf{x}^{n}_{r,t},\mathbf{x}^{n}_{s,r}\otimes\mathbf{x}^{n}_{r,t}).$$

With (75) and (76),

$$\begin{aligned} d(\mathbf{x}^{n+2}_{s,t},\mathbf{x}^{n+1}_{s,t}) \leq d(\mathbf{x}^{n+1}_{r,t},\mathbf{x}^{n}_{r,t})(1+\|\mathbf{x}^{n+1}_{s,r}\|_\star)& \\ + d(\mathbf{x}^{n+1}_{s,r},\mathbf{x}^{n}_{s,r})(1+\|\mathbf{x}^{n}_{r,t}\|_\star)&. \end{aligned} \tag{77}$$

Let us set

$$V_n(\tau) = \sup_{0\leq s\leq t\leq s+\tau} d(\mathbf{x}^{n+1}_{s,t},\mathbf{x}^n_{s,t}) \text{ and } h_n(\tau) = \sup_{0\leq s\leq t\leq s+\tau}\|\mathbf{x}^n_{s,t}\|_\star.$$

From (77),

$$V_{n+1}(\tau) \leq (2+h_n(\tau/2)+h_{n+1}(\tau/2))V_n(\tau/2)$$

Let us choose $2<\kappa<2^\theta$. As $V_0(\tau)=K(t-s)^\theta$, the quantity

$$\overline{V}(\tau) = \sum_{k=0}^{+\infty}\kappa^n V_0(\tau/2^n)$$

is finite. We remark that

$$h_{n+1}(\tau) \leq h_n(\tau)+V_n(\tau) \leq h_0(\tau)+\overline{V}(\tau).$$

Let us fix τ_0 such that $1+h_0(\tau_0)\overline{V}(\tau_0) < \kappa/2$. This is possible since $h_0(\tau)$ and $\overline{V}(\tau)$ converges to 0 as τ decreases to 0. Assume that

$$V_n(\tau) \leq \kappa^n V_0(\tau/2^n) \text{ for } \tau\leq\tau_0 \tag{78}$$

For $\tau\leq\tau_0$, $2+h_n(\tau/2)+h_{n+1}(\tau/2)\leq\kappa$ and then $V_{n+1}(\tau)\leq\kappa^{n+1}V_0(\tau/2^{n+1})$. Then, (78) is true for any $n\in\mathbb{N}$ and $\sum_{n\geq 0}V_n(\tau)\leq\overline{V}(\tau)$ converges. This means that $(\mathbf{x}^n_{s,t})_{n\in\mathbb{N}}$ is a Cauchy sequence for all $(s,t)\in\Delta_+$ such that $t-s\leq\tau$.

Let us denote by $\mathbf{y}_{s,t}$ the limit of $(\mathbf{x}^n_{s,t})_{n\in\mathbb{N}}$, which is continuous in s and t. This limit satisfies $\mathbf{y}_{s,t}=\mathbf{y}_{s,r}\otimes\mathbf{y}_{r,t}$ with $r=(t+s)/2$. In addition, $d(\mathbf{y}_{s,t},\mathbf{x}_{s,t})\leq C(t-s)^\theta$ for some constant C. We extend $\mathbf{y}_{s,t}$ to $(s,t)\in\Delta_+$ by setting $\mathbf{y}_{s,t}=\mathbf{y}_{t^m_0,t^m_1}\otimes\cdots\otimes\mathbf{y}_{t^m_{2m-1},t^m_{2m}}$ for the partition $t_i^m = s + i(t-s)2^{-m}$, $i = 0, \dots, 2^m$ when m is large enough so that $(t-s) \leq \tau_0 2^m$. We easily get that $\mathbf{y}$ is *mid-point additive*, that $\mathbf{y}$ does not depend on m, is $\mathbf{y}_{s,t} = \mathbf{y}_{s,r} \otimes \mathbf{y}_{r,t}$ for $r = (t+s)/2$ and satisfies $d(\mathbf{y}_{s,t}, \mathbf{x}_{s,t}) \leq C'(t-s)^\theta$ for $(s,t) \in \Delta_+$ with possibly another constant C'.

Let us now prove that $\mathbf{y}$ is unique. If $\mathbf{z}$ be another function from $\Delta_+ \to \mathrm{T}_1(\mathbb{R}^d)$ which satisfies

$$\mathbf{z}_{s,t} = \mathbf{z}_{s,r} \otimes \mathbf{z}_{r,t} \text{ for } r = (t+s)/2 \text{ and } d(\mathbf{z}_{s,t}, \mathbf{x}_{s,t}) \leq C'(t-s)^\theta \tag{79}$$

for some $C' > 0$ and any $(s,t) \in \Delta_+$. For $(s,t) \in \Delta_+$ and $r = (t+s)/2$,

$$\begin{aligned} d(\mathbf{y}_{s,t}, \mathbf{z}_{s,t}) &\leq d(\mathbf{y}_{s,t}, \mathbf{y}_{s,r} \otimes \mathbf{z}_{r,t}) + d(\mathbf{y}_{s,r} \otimes \mathbf{z}_{r,t}, \mathbf{z}_{s,t}) \\ &\leq d(\mathbf{y}_{r,t}, \mathbf{z}_{r,t})(1 + \|\mathbf{y}_{s,r}\|_\star) + d(\mathbf{y}_{s,r}, \mathbf{z}_{s,r})(1 + \|\mathbf{z}_{r,t}\|_\star) \\ &\leq \kappa(\tau/2) W(\tau/2), \end{aligned}$$

where $W(\tau) = \sup_{s \leq t \leq s+\tau} d(\mathbf{y}_{s,t}, \mathbf{z}_{s,t})$ and

$$\kappa(\tau) = 2 + \sup_{0 \leq t-s \leq \tau} \|\mathbf{y}_{s,t}\|_\star + \sup_{0 \leq t-s \leq \tau} \|\mathbf{z}_{s,t}\|.$$

Thus, $W(\tau) \leq \kappa W(\tau/2)$. Now, let us note that

$$W(\tau) \leq \sup_{s \leq t \leq s+\tau} \big(d(\mathbf{y}_{s,t}, \mathbf{x}_{s,t}) + d(\mathbf{z}_{s,t}, \mathbf{x}_{s,t}) \big) \leq 2C\tau^\theta.$$

Then, if $\tau < \tau_0$ with $\kappa(\tau_0) < 2^\theta$,

$$W(\tau) \leq \kappa(\tau_0)^n W(\tau/2^n) \leq \frac{C\kappa(\tau_0)^n \tau^\theta}{2^{(n+1)(\theta-1)}} \xrightarrow[n \to \infty]{} 0,$$

which means that $W(\tau) = 0$ for $\tau \in [0, \tau_0]$. Using the fact that both $\mathbf{y}$ and $\mathbf{z}$ are mid-point additive, we get that $\mathbf{y}_{s,t} = \mathbf{z}_{s,t}$ for all $(s,t) \in \Delta_+$ and that $\mathbf{y}$ is unique.

Now, let us fix $(s,t) \in \Delta_+$ and $n \in \mathbb{N}$. Set

$$\mathbf{z}_{s,t} = \mathbf{y}_{t_0^n, t_1^n} \otimes \cdots \otimes \mathbf{y}_{t_{n-1}^n, t_n^n}$$

for $t_i^n = s + (t-s)i/n$. Let us note that for $r = (t+s)/2$,

$$\mathbf{z}_{s,t} = \mathbf{z}_{s,r} \otimes \mathbf{z}_{r,t} \text{ for } s = \frac{t+s}{2},\ (s,t) \in \Delta_+.$$

It follows that

$$\begin{aligned} d(\mathbf{z}_{s,t},\mathbf{x}_{s,t}) &\leq d\left(\bigotimes_{i=0}^{n-1}\mathbf{y}_{t_i^n,t_{i+1}^n},\mathbf{y}_{t_0^n,t_1^n}\otimes\mathbf{x}_{t_1^n,t_2^n}\right)\\ &\quad+d(\mathbf{y}_{t_0^n,t_1^n}\otimes\mathbf{x}_{t_1^n,t_2^n},\mathbf{x}_{t_0^n,t_1^n}\otimes\mathbf{x}_{t_1^n,t_2^n})+d(\mathbf{x}_{t_0^n,t_1^n}\otimes\mathbf{x}_{t_1^n,t_2^n},\mathbf{x}_{t_0^n,t_1^n})\\ &\leq d\left(\bigotimes_{i=1}^{n-1}\mathbf{y}_{t_i^n,t_{i+1}^n},\mathbf{x}_{t_1^n,t_2^n}\right)(1+\|\mathbf{y}_{t_0^n,t_1^n}\|_\star)\\ &\quad+d(\mathbf{y}_{t_0^n,t_1^n},\mathbf{x}_{t_0^n,t_1^n})(1+\|\mathbf{x}_{t_1^n,t_2^n}\|_\star)+K|t-s|^\theta\\ &\leq C_1 d\left(\bigotimes_{i=1}^{n-1}\mathbf{y}_{t_i^n,t_{i+1}^n},\mathbf{x}_{t_1^n,t_2^n}\right)+K|t-s|^\theta+C_2\frac{|t-s|^\theta}{n^\theta}\end{aligned}$$

for some constants C_1 and C_2 that depend only on T, K and K_1. Applying the same computation recursively leads to

$$d(\mathbf{z}_{s,t},\mathbf{x}_{s,t})\leq C_3|t-s|^\theta$$

for some constant C_3 that depends on K, T, K_1 and n. We have previously proved that any function $\mathbf{z}:\Delta_+\to\mathrm{T}_1(\mathbb{R}^d)$ which satisfies (79) is equal to $\mathbf{y}$, so that $\mathbf{y}_{s,t}=\bigotimes_{i=1}^{n-1}\mathbf{y}_{t_i^n,t_{i+1}^n}$. Then, $\mathbf{y}_{s,t}=\mathbf{y}_{s,s+p(t-s)}\otimes\mathbf{y}_{s+p(t-s),t}$ for all $p\in\mathbb{Q}$. From the continuity of $(s,t)\in\Delta_+\mapsto\mathbf{y}_{s,t}$, we deduce that $\mathbf{y}_{s,r}\otimes\mathbf{y}_{r,t}=\mathbf{y}_{s,t}$ for any $r\in[s,t]$, $(s,t)\in\Delta_+$. □

Theorem 6. *Let* $\mathbf{x}$ *and* $\mathbf{y}$ *be two almost rough paths satisfying both* (68) *with the same constants* K *and* θ.

(i) *Assume that there exists an* $\varepsilon>0$ *such that*

$$\|\mathbf{x}-\mathbf{y}\|_{\star,p,\omega}\leq\varepsilon.$$

Then there exists some function $\varepsilon\mapsto K(\varepsilon)$ *that depends only on* K, θ, p, $\omega(0,T)$, $\|\mathbf{x}\|_{\star,p,\omega}$ *and* $\|\mathbf{y}\|_{\star,p,\omega}$ *such that the two rough paths* $\widetilde{\mathbf{x}}$ *and* $\widetilde{\mathbf{y}}$ *associated to* $\mathbf{x}$ *and* $\mathbf{y}$ *by Theorem 5 satisfy*

$$\|\widetilde{\mathbf{x}}-\widetilde{\mathbf{y}}\|_{*,p,\omega}\leq K(\varepsilon)$$

with $K(\varepsilon)$ *to* 0 *as* $\varepsilon\to 0$.

(ii) *If in addition for all* $(s,t)\in\Delta_+$,

$$\|\mathbf{x}_{s,t}-\mathbf{x}_{s,r}\otimes\mathbf{x}_{r,t}-(\mathbf{y}_{s,t}-\mathbf{y}_{s,r}\otimes\mathbf{y}_{r,t})\|_\star\leq\varepsilon\omega(s,t)^\theta,$$

then $K(\varepsilon)=K'\varepsilon$ *for some constant* K' *depends only on* K, θ, p, $\omega(0,T)$, $\|\mathbf{x}\|_{\star,p,\omega}$ *and* $\|\mathbf{y}\|_{\star,p,\omega}$.

Proof. We prove first the statement (ii) of this theorem. We use the same notations as previously. For a partition $\pi=\{t_i\}_{i=1,\dots,n}$ of $[s,t]$ with $t_1=s$, $t_{n+1}=t$, we consider $\mathbf{x}_{s,t}^{(\pi)}$ and $\mathbf{y}_{s,t}^{(\pi)}$ as above.

We pick a point t_i in π such that $\omega(t_{i-1},t_{i+1})\leq 2\omega(s,t)/n$. Hence, for

$$\xi=\mathbf{x}_{t_{i-1},t_i}\otimes\mathbf{x}_{t_i,t_{i+1}}-\mathbf{x}_{t_{i-1},t_{i+1}}-\mathbf{y}_{t_{i-1},t_i}\otimes\mathbf{y}_{t_i,t_{i+1}}+\mathbf{y}_{t_{i-1},t_{i+1}},$$

we get that, with (70),

$$\|\mathbf{x}^{(\pi)}_{s,t} - \mathbf{x}^{(\widehat{\pi})}_{s,t} - (\mathbf{y}^{(\pi)}_{s,t} - \mathbf{y}^{(\widehat{\pi})}_{s,t})\|_\star \\ \leq \|\xi\|_\star \left(1 + \sum_{j=1,\dots,n\ j\neq i} |\mathbf{x}^1_{t_j,t_{j+1}} - \mathbf{y}^1_{t_j,t_{j+1}}|\right)$$

where $\mathbf{x}^1_{s,t}$ (resp. $\mathbf{y}^1_{s,t}$) is the projection of $\mathbf{x}_{s,t}$ (resp. $\mathbf{y}_{s,t}$) on $\mathbb{R}^d$.

With (72), we get that for some constant C that depends only on K, θ, p, $\omega(0,T)$, $\|\mathbf{x}\|_{p,\omega}$ and $\|\mathbf{y}\|_{p,\omega}$,

$$\sum_{j=1,\dots,n\ j\neq i} |\mathbf{x}^1_{t_j,t_{j+1}} - \mathbf{y}^1_{t_j,t_{j+1}}| \leq |\mathbf{x}^{1,(\pi)}_{s,t}| + |\mathbf{y}^{1,(\pi)}_{s,t}| \\ \leq (C\omega(s,t)^{\theta-1/p} + \|\mathbf{y}\|_{\star,p,\omega} + \|\mathbf{x}\|_{\star,p,\omega})\omega(s,t)^{1/p}.$$

Thus, for some constant K,

$$\|\mathbf{x}^{(\pi)}_{s,t} - \mathbf{x}^{(\widehat{\pi})}_{s,t} - (\mathbf{y}^{(\pi)}_{s,t} - \mathbf{y}^{(\widehat{\pi})}_{s,t})\|_\star \leq \varepsilon \frac{K}{n^\theta}\omega(s,t)^\theta.$$

It follows that by removing successively all the points of π carefully,

$$\|\mathbf{x}^{(\pi)}_{s,t} - \mathbf{x}_{s,t} - (\mathbf{y}^{(\pi)}_{s,t} - \mathbf{y}_{s,t})\|_\star \leq \varepsilon\zeta(\theta)K\omega(s,t)^\theta.$$

As we have seen that $\mathbf{x}^{(\pi)}$ and $\mathbf{y}^{(\pi)}$ converges to $\widetilde{\mathbf{x}}_{s,t}$ and $\widetilde{\mathbf{y}}_{s,t}$, we deduce that

$$\|\widetilde{\mathbf{x}}_{s,t} - \mathbf{x}_{s,t} - (\widetilde{\mathbf{y}}_{s,t} - \mathbf{y}_{s,t})\|_\star \leq \varepsilon\zeta(\theta)K\omega(s,t)^\theta.$$

The result is then easily deduced.

Now, to prove the statement (i), we have just to remark that for some $1/\theta < \eta < 1$,

$$\|\mathbf{x}_{s,t} - \mathbf{x}_{s,r}\otimes\mathbf{x}_{r,t} - \mathbf{y}_{s,t} + \mathbf{y}_{s,r}\otimes\mathbf{y}_{r,t}\|_\star \\ \leq 2^{\eta-1}(\|\mathbf{x}_{s,t} - \mathbf{x}_{s,r}\otimes\mathbf{x}_{r,t}\|_\star^\eta + \|\mathbf{y}_{s,t} - \mathbf{y}_{s,r}\otimes\mathbf{y}_{r,t}\|_\star^\eta) \\ \times(\|\mathbf{x}_{s,t} - \mathbf{y}_{s,t}\|_\star + \|\mathbf{x}_{s,r}\otimes\mathbf{x}_{r,t} - \mathbf{y}_{s,r}\otimes\mathbf{y}_{r,t}\|_\star)^{1-\eta} \\ \leq C\omega(s,t)^{\eta\theta+(1-\eta)/p}\varepsilon^{1-\eta}$$

for some constant C that depends only on η, θ, $\omega(0,T)$. and then to apply the result of (ii) by replacing ε by $\varepsilon^{1-\eta}$ and θ by $\eta\theta$. □

C.2. **An algebraic interpretation.** We give now an algebraic interpretation of this construction, which is strongly inspired by the one given by M. Gubinelli in [Gub04].

Let us consider the sets

$$\Delta_1 = [0,T],\ \Delta_2 = \{(s,t) | 0 \leq s \leq t \leq T\} \\ \text{and } \Delta_3 = \{(s,r,t) | 0 \leq s \leq r \leq t \leq T\},$$

as well as $\mathcal{C}_i$ be the set of functions from Δ_i to $\mathrm{T}_1(\mathbb{R}^d)$ for $i = 1,2,3$.

Let us introduce the operator from $\mathcal{C}_1 \cup \mathcal{C}_2$ to $\mathcal{C}_2 \cup \mathcal{C}_3$ defined by

$$\delta(\mathbf{x})_{s,t} = \mathbf{x}_s^{-1} \otimes \mathbf{x}_t,\ (s,t) \in \Delta_2,\ \mathbf{x} \in \mathcal{C}_1,$$
$$\delta(\mathbf{x})_{s,r,t} = \mathbf{x}_{s,t} - \mathbf{x}_{s,r} \otimes \mathbf{x}_{r,t},\ (s,r,t) \in \Delta_3,\ \mathbf{x} \in \mathcal{C}_2,$$

so that δ maps $\mathcal{C}_i$ to $\mathcal{C}_{i+1}$, $i = 1, 2$. Let us note that if $\mathbf{x} \in \mathcal{C}_1$, then $\delta(\delta(\mathbf{x})) = 0$, so that the range $\mathrm{Range}(\delta_{|\mathcal{C}_1})$ of $\delta_{|\mathcal{C}_1}$ is contained in the Kernel $\mathrm{Ker}(\delta_{|\mathcal{C}_2})$ of $\delta_{|\mathcal{C}_2}$. Indeed, we get a better result.

Lemma 21. *The range of $\delta_{|\mathcal{C}_1}$ is equal to the kernel of $\delta_{|\mathcal{C}_2}$, and δ is injective from $\mathcal{C}_1(x)$ into $\mathcal{C}_2$ where $\mathcal{C}_1(x)$ is the set of paths $\mathbf{x}$ in $\mathcal{C}_1$ with $\mathbf{x}_0 = x$ for $x \in \mathrm{T}_1(\mathbb{R}^2)$. In particular, when restricted to $\mathrm{Range}(\delta_{\mathcal{C}_1(x)})$, δ is invertible.*

Proof. We have already seen the inclusion of $\mathrm{Range}(\delta_{|\mathcal{C}_1})$ in $\mathrm{Ker}(\delta_{|\mathcal{C}_2})$.

Now, let $\mathbf{x} \in \mathcal{C}_2$ be in $\mathrm{Ker}(\delta_{|\mathcal{C}_2})$. Then set $\mathbf{y}_t = \mathbf{x}_{0,t}$. As $\delta(\mathbf{x})_{0,s,t} = \mathbf{y}_t - \mathbf{y}_s \otimes \mathbf{x}_{s,t} = 0$, we get that $\mathbf{x}_{s,t} = \mathbf{y}_s^{-1} \otimes \mathbf{y}_t$ and thus $\mathbf{x} = \delta(\mathbf{y})$. This proves the result.

If two paths $\mathbf{x}$ and $\mathbf{y}$ are distinct in $\mathcal{C}_1(x)$, then $\delta(\mathbf{x})_{0,t} = x^{-1} \otimes \mathbf{x}_t$ is different from $\delta(\mathbf{y})_{0,t} = x^{-1} \otimes \mathbf{y}_t$ and δ is injective from $\mathcal{C}_1(x)$ into $\mathcal{C}_2$. □

Given a rough path $\mathbf{x}$, which then belongs to $\mathcal{C}_1$ and a differential form f, the integral $\mathfrak{I}(\mathbf{x}) = \int f(x)\,\mathrm{d}\mathbf{x}$ is also a path in $\mathcal{C}_1(0)$. The idea is then to consider an approximation of $\mathfrak{I}(\mathbf{x}; s, t)$ for $t - s$ small, and to project it on the range of $\delta_{|\mathcal{C}_1(x)}$. Of course, the approximation of $\mathfrak{I}(\mathbf{x}; s, t)$ shall be close enough to the range of $\delta_{|\mathcal{C}_1(x)}$.

For $p \geq 1$ and $\theta > 1$, we define the distance $d_{\theta,\omega}$ on $\mathcal{C}_2$ by

$$D_{\star,\theta,\omega}(\mathbf{x}, \mathbf{y}) = \sup_{(s,t) \in \Delta^+} \frac{\|\mathbf{x}_{s,t} - \mathbf{y}_{s,t}\|_\star}{\omega(s,t)^\theta}.$$

To simplify the notation, we extend $\delta_{|\mathcal{C}_2}$ as a function defined on $\Delta^2 \times [0, T]$ by setting $\delta(\mathbf{x})_{s,r,t} = 1$ if $r \notin [s, t]$. For a fixed $r \in [0, T]$, $\delta_{\cdot,r,\cdot}(\mathbf{x})$ is then a function in $\mathcal{C}_2$.

Theorem 5 is the rewritten the following way.

Theorem 7. *For $K, K' > 0$ and $\theta > 1$, we denote by $B(K, K', \theta, p, \omega)$ the subsets of functions $\mathbf{x} \in \mathcal{C}_2$ for which*

$$\|\mathbf{x}\|_{\star,p,\omega} \leq K \text{ and } \sup_{r \in [0,T]} D_{\star,p,\omega}(\delta(\mathbf{x})_{\cdot,r,\cdot}, 0) \leq K'.$$

Then to any $\mathbf{x}$ in $B(K, K', \theta, p, \omega)$ is associated a unique element $\widehat{\mathbf{x}}$ $\mathrm{Ker}(\delta_{|\mathcal{C}_2})$. In addition, for some constants C_1 and C_2 that depends only on K, K', θ, p and $\omega(0, T)$,

$$\|\widehat{\mathbf{x}}\|_{\star,p,\omega} \leq C_1 \text{ and } D_{\star,p,\omega}(\mathbf{x}, \widehat{\mathbf{x}}) \leq C_2.$$

In addition, if the distance $\Theta_{\star,p,\theta,\omega}$ is defined on $\cup_{K,K'>0} B(K, K', \theta, p, \omega)$ by

$$\Theta_{\star,p,\theta,\omega}(\mathbf{x}, \mathbf{y}) = \max\{\|\mathbf{x} - \mathbf{y}\|_{\star,p,\omega}, d_{\star,\theta,\omega}(\mathbf{x}, \mathbf{y})\},$$

then this map $\Pi : \mathbf{x} \mapsto \widehat{\mathbf{x}}$ is locally Lipschitz with respect to $\Theta_{\star,p,\theta,\omega}$.

From the definition, an almost rough path $\mathbf{x}$ of p-variation controlled by ω belongs to $\cup_{K,K'>0}B(K,K',\theta,p,\omega)$. It is then "projected" on an element $\Pi(\mathbf{x})$ in $\mathcal{C}_2$ in the kernel of $\delta_{\mathcal{C}_2}$, which is also equal to the image of $\delta_{\mathcal{C}_1(1)}$. The reciprocal image of $\Pi(\mathbf{x})$ gives then a rough path in $\mathrm{C}^{p,\omega}([0,T];\mathrm{T}_1(\mathbb{R}^d))$.

Given an element f in $\mathrm{Lip}(\gamma;\mathbb{R}^d\to\mathbb{R}^m)$ with $\gamma>p-1$, the map $\mathfrak{F}(f,\mathbf{x})$ defined by (61) define an element of $\mathcal{C}_2$. The integral $\mathfrak{I}$ may then be defined as the composition of the maps

$$\mathfrak{I}=\delta^{-1}_{|\mathcal{C}_1(1)}\circ\Pi\circ\mathfrak{F}(f,\cdot),$$

which corresponds to the construction given in Section 7.4.

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