Perturbation analysis of an M/M/1 queue in a diffusion random environment

Christine Fricker, Fabrice Guillemin, Philippe Robert

PERTURBATION ANALYSIS OF AN M/M/1 QUEUE IN A DIFFUSION RANDOM ENVIRONMENT

CHRISTINE FRICKER, FABRICE GUILLEMIN, AND PHILIPPE ROBERT

Abstract. We study in this paper an $M/M/1$ queue whose server rate depends upon the state of an independent Ornstein-Uhlenbeck diffusion process $(X(t))$ so that its value at time t is $\mu\phi(X(t))$, where $\phi(x)$ is some bounded function and $\mu > 0$. We first establish the differential system for the conditional probability density functions of the couple $(L(t), X(t))$ in the stationary regime, where $L(t)$ is the number of customers in the system at time t. By assuming that $\phi(x)$ is defined by $\phi(x) = 1 - \varepsilon((x \wedge a/\varepsilon) \vee (-b/\varepsilon))$ for some positive real numbers a, b and ε, we show that the above differential system has a unique solution under some condition on a and b. We then show that this solution is close, in some appropriate sense, to the solution to the differential system obtained when ϕ is replaced with $\Phi(x) = 1 - \varepsilon x$ for sufficiently small ε. We finally perform a perturbation analysis of this latter solution for small ε. This allows us to check at the first order the validity of the so-called reduced service rate approximation, stating that everything happens as if the server rate were constant and equal to $\mu(1 - \varepsilon\mathbb{E}(X(t)))$.

1. Introduction

We consider in this paper an $M/M/1$ queue with a server rate varying in time. We specifically assume that the server rate at time t is equal to $\mu\phi(X(t))$ for some function ϕ and some auxiliary process $(X(t))$. Throughout this paper, we shall assume that the modulating process $(X(t))$ is an Ornstein-Uhlenbeck process with mean $m > 0$, drift coefficient $\alpha > 0$, and diffusion coefficient $\sigma > 0$. This process satisfies Itô's stochastic equation

$$dX(t) = -\alpha(X(t) - m)dt + \sigma dB(t), \tag{1}$$

where $(B(t))$ is a standard Brownian motion. The stationary distribution of the process $(X(t))$ is a normal distribution with mean m and variance $\sigma^2/(2\alpha)$; the associated probability density function is defined on the whole of $\mathbb{R}$ and is given by

$$n(x) \stackrel{\text{def.}}{=} \frac{1}{\sigma}\sqrt{\frac{\alpha}{\pi}} \exp\left(-\frac{\alpha(x-m)^2}{\sigma^2}\right). \tag{2}$$

Throughout this paper, we shall assume that the Ornstein-Uhlenbeck process is stationary.

If $L(t) = j$ denotes the number of customers in the $M/M/1$ queue and $X(t) = x$ at time t, then the transitions of the process $(L(t))$ are given by

$$j \to \begin{cases} j+1 & \text{with rate} \quad \lambda, \\ j-1 & \text{with rate} \quad \mu\phi(X(t)). \end{cases}$$

Key words and phrases. $M/M/1$ queue, Self-Adjoint Operators, Perturbation Analysis, Power Series Expansion, Reduced Service Rate.

In the following, we shall assume that the condition $\rho \stackrel{\text{def.}}{=} \lambda/\mu < \mathbb{E}(\phi(X(0))) \leq 1$ is satisfied so that it is straightforward to show the existence of a stationary probability distribution for the Markov process $(X(t), L(t))$; see Meyn and Tweedie [12] for example.

The study of the above system is motivated by the problem of bandwidth sharing in telecommunication networks and the coexistence on the same transmission links of elastic traffic, which adapts to the level of congestion of the network by achieving a fair sharing of the available bandwidth, and unresponsive traffic, which consumes bandwidth without taking care of other traffic. See for instance [11] for a discussion about bandwidth sharing in packet networks. The choice of an Ornstein–Uhlenbeck process as modulating process is natural for several reasons: mathematically this is a standard "typical" diffusion process with an equilibrium distribution and secondly it can be seen as a centered approximation of the number of jobs of an $M/M/\infty$ queue (the unresponsive traffic), see for example Borovkov [4] or Iglehart [6] or Chapter 6 of Robert [17].

One of the objectives of this paper is to investigate the so-called Reduced Service Rate (RSR) property for which the system would behave as if the server rate were equal to the mean value $\mu\mathbb{E}(\phi(X_0))$. Even though some results can be established for arbitrary perturbation functions $\phi(x)$, we shall pay special attention in the following to the case when the function $\phi(x)$ has the form

$$\phi(x) = 1 - \varepsilon((x \wedge (a/\varepsilon)) \vee (-b/\varepsilon)) \tag{3}$$

for some small $0 < \varepsilon < 1$ and real numbers $0 < a < 1$ and $b > 0$, where we use the notation $a \vee b = \max(a, b)$ and $a \wedge b = \min(a, b)$. The choice of the bounded perturbation function is discussed at the end of the paper.

As it will be seen, one of the important technical problems encountered in the perturbation analysis is the existence of a reasonably smooth density probability function for the couple $(X(t), L(t))$ in the stationary regime. Conditions on ε for ensuring the existence and the uniqueness of a density probability function will be established in the following via Hilbertian analysis. More precisely, let $p_j(x)$ denote the stationary probability density function that the process $(L(t))$ is in state j and the process $(X(t))$ is in state x and let P denote the vector whose jth component is $p_j(x)/\rho^j$. In a fist step, we show that P is solution to an equation of the type

$$\Omega f + V(\phi) f = 0, \tag{4}$$

where Ω is a selfadjoint second order differential operator on $\mathcal{D}'(\mathbb{R})^{\mathbb{N}}$, $\mathcal{D}'(\mathbb{R})$ denoting the set of distributions in $\mathbb{R}$. Unfortunately, the operator $V(\phi)$ is not selfadjoint so that Kato's perturbation theory for selfadjoint operators cannot be applied. Nevertheless, we prove that the above equation has a unique non null smooth solution $P \in C^2(\mathbb{R})^{\mathbb{N}}$ for sufficiently small ε. In addition, we prove that when replacing $\phi(x)$ with $\Phi(x) = 1 - \varepsilon x$, we obtain an equation of the type

$$\Omega f + \varepsilon V(\Phi) f = 0, \tag{5}$$

which has a unique non null smooth solution for sufficiently small ε, $V(\Phi)$ being independent of ε. Denoting this solution by g, we prove that P and g are close to each other for some adequate norm when ε is small. We then perform a power series expansion in ε of g and we determine the radius of convergence of this series. By explicitly computing the two first terms of the series, this eventually enables us to prove the validity of the reduced service rate approximation at the first order for the system.

The problem considered in this paper falls into the framework of queueing systems with time varying server rate, which have been studied in the queueing literature in many different situations. In Núñez-Queija and Boxma [15], the authors consider a queueing system where priority is given to some flows driven by Markov Modulated Poisson Processes (MMPP) with finite state spaces and the low priority flows share the remaining server capacity according to the processor sharing discipline. By assuming that arrivals are Poisson and service times are exponentially distributed, the authors solve the system via a matrix analysis. Similar models have been investigated in Núñez-Queija [13, 14] by still using the quasi-birth and death process associated with the system and a matrix analysis. The integration of elastic and streaming flows has been studied by Delcoigne et al. [5], where stochastic bounds for the mean number of active flows have been established. More recently, priority queueing systems with fast dynamics, which can be described by means of quasi birth and death processes, have been studied via a perturbation analysis of a Markov chain by Altman et al [1]. A probabilistic analysis of these queues with varying service rate has been presented in Antunes et al. [2, 3]. Our point of view in this paper is completely different since a functional analysis approach is used to tackle the perturbation analysis. The key difficulty for the case considered in the present paper is that the associated Markov chain has an infinite state space.

This paper is organized as follows: In Section 2, we establish the basic system of partial differential equations for the joint probability density functions of the process $(X(t), L(t))$. We recall in Section 3 some basic results on the generators of the Ornstein-Uhlenbeck process and the occupation process in an $M/M/1$ queue. In Section 4 it is proved that this system has a unique solution with convenient regularity properties in an adequate Hilbert space when ε is sufficiently small. In Section 5, ~~we carry out a perturbation analysis for the perturbation function $\Phi(x) = 1 - \varepsilon x$~~, we show that when replacing $\phi(x)$ with $\Phi(x) = 1 - \varepsilon x$, the corresponding differential system has also a smooth solution in the underlying Hilbert space when ε is sufficiently small. We then prove that the solutions to the differential systems for ϕ defined by Equation (3) and Φ are close to each other in some appropriate sense. By expanding the solution of the second differential system in power series of ε, we show that at the first order the so-called Reduced Service Rate property for the original system holds; the subsequent terms of the associated expansion are also expressed. Some concluding remarks are presented in Section 6.

2. Fundamental differential problem

2.1. **Notation and differential system.** The goal of this section is to establish the fundamental differential system for the conditional probability density functions $p_j(x)$, $j \geq 0$, in the stationary regime, where $p_j(x)$ is the probability that the process $(L(t))$ is in state j knowing that the Ornstein-Uhlenbeck process $(X(t))$ is in state x. As long as we do not have proved regularity results, these functions have to be considered in the sense of distributions, i.e., for all $j \geq 0$, $p_j(x) \in \mathcal{D}'(\mathbb{R})$, where $\mathcal{D}'(\mathbb{R})$ is the set of distributions in $\mathbb{R}$. The distributions $p_j(x)$, $j \geq 0$, are formally defined as follows: for every infinitely differentiable function with compact support

$\varphi(x)$ (denoted, for short, $\varphi \in C_0^\infty(\mathbb{R})$)

$$\int_{\mathbb{R}} p_j(x)n(x)\varphi(x)dx = -\int_{\mathbb{R}} \varphi'(x)\mathbb{P}(X(0) \leq x, L(0) = j)dx,$$

where $n(x)$ is the normal distribution given by Equation (2).

Throughout this paper, we shall use the following notation. The functional space

$$L^2(\mathbb{R}, n) = \left\{ f : \mathbb{R} \to \mathbb{R} : \int_{\mathbb{R}} f(x)^2 n(x)dx < \infty \right\}$$

is a Hilbert space equipped with the scalar product defined for $f, g \in L^2(\mathbb{R}, n)$ by

$$(f, g)_2 = \int_{\mathbb{R}} f(x)g(x)n(x)dx$$

and the norm of an element $f \in L^2(\mathbb{R}, n)$ is $\|f\|_2 = \sqrt{(f, g)_2}$. If $\mathcal{H}$ is a separable Hilbert space equipped with the scalar product $(.,.)_{\mathcal{H}}$ and associated norm $\|.\|_{\mathcal{H}}$, we define the Hilbert space

$$L^2(\mathbb{R}, n; \mathcal{H}) = \left\{ (f_j(x), j \geq 0) \in L^2(\mathbb{R}; n)^{\mathbb{N}} : \int_{\mathbb{R}} \|f(x)\|^2_{\mathcal{H}} n(x)dx < \infty \right\}$$

equipped with the scalar product

$$(f, g) = \int_{\mathbb{R}} (f(x), g(x))_{\mathcal{H}} n(x)dx.$$

Finally, let $\ell^2(\rho)$ be the Hilbert space composed of those sequences $(c_j, j \geq 0)$ taking values in $\mathbb{R}$ and such that $\sum_{j=0}^\infty c_j^2 \rho^j < \infty$, and equipped with the scalar product defined by: if $c = (c_j)$ and $d = (d_j)$ in $\ell^2(\rho)$, $(c, d)_\rho = \sum_{j=0}^\infty c_j d_j \rho^j$; the associated norm is defined by: for $c \in \ell^2(\rho)$, $\|c\|_\rho = \sqrt{(c, c)_\rho}$. Let e_j denote the sequence with all entries equal to 0 except the jth one equal to 1. The family $(e_j, j \geq 0)$ is a basis for $\ell^2(\rho)$. The space $\ell^2_1(\rho)$ denotes the subspace of $\ell^2(\rho)$ spanned by the vectors e_j for $j \geq 1$.

In a first step, we determine the infinitesimal generator of the Markov process $(X(t), L(t))$ taking values in $\mathbb{R} \times \mathbb{N}$. We specifically have the following result.

Lemma 1. *The process $(X(t), L(t))$ is a Markov process in $\mathbb{R} \times \mathbb{N}$ with infinitesimal generator $\mathcal{G}$ defined by*

$$(6) \quad \mathcal{G}f(x, j) = \frac{\sigma^2}{2} \frac{\partial^2 f}{\partial x^2}(x, j) - \alpha(x - m)\frac{\partial f}{\partial x}(x, j) \\ + \lambda \left(f(x, j + 1) - f(x, j)\right) + \mu\phi(x)\mathbb{1}_{\{j>0\}} \left(f(x, j - 1) - f(x, j)\right),$$

for every function $f(x, j)$ from $\mathbb{R} \times \mathbb{N}$ in $\mathbb{R}$, twice differentiable with respect to the first variable.

Proof. According to Equation (1), the infinitesimal generator of an Ornstein-Uhlenbeck process applied to some twice differentiable function g on $\mathbb{R}$ is given by

$$(7) \qquad Hg = \frac{\sigma^2}{2} \frac{\partial^2 g}{\partial x^2}(x) - \alpha(x - m)\frac{\partial g}{\partial x}(x).$$

The second part of Equation (6) corresponds to the infinitesimal generator of the number of customers in a classical $M/M/1$ queue with arrival rate λ and service rate $\mu\phi(x)$, when the Ornstein-Uhlenbeck process is in state x. □

For finite sequences $f=(f_j(x))$ of infinitely differentiable functions with compact support, the equation for the invariant measure for the Markov process $(L(t),X(t))$ is given by $\sum_{j\geq 0}\int_{\mathbb{R}} \mathcal{G}f(x,j)p_j(x)n(x)dx=0$, that is,

$$\sum_{j\geq 0}\int_{-\infty}^{\infty}\Bigg(\frac{\sigma^2}{2}\frac{d^2f_j}{dx^2}-\alpha(x-m)\frac{df_j}{dx} \\ +\mu\mathbb{1}_{\{j>0\}}\phi(x)f_{j-1}(x)-(\lambda+\mu\phi(x)\mathbb{1}_{\{j>0\}})f_j(x)+\lambda f_{j+1}(x)\Bigg)p_j(x)n(x)dx=0.$$

Via integration by parts, we obtain for every finite sequence $f=(f_j(x))$ of infinitely differentiable functions with compact support

$$\sum_{j\geq 0}\Bigg(\int_{-\infty}^{\infty}\left(\frac{\sigma^2}{2}\frac{d^2P_j}{dx^2}-\alpha(x-m)\frac{dP_j}{dx}\right)f_j(x)n(x)dx \\ +\int_{-\infty}^{\infty}\left(\mu\mathbb{1}_{\{j>0\}}\phi(x)f_{j-1}(x)-(\lambda+\mu\phi(x)\mathbb{1}_{\{j>0\}})f_j(x)+\lambda f_{j+1}(x)\right)p_j(x)n(x)dx\Bigg) \\ =0$$

and then

$$\sum_{j\geq 0}\int_{-\infty}^{\infty}\Bigg(\frac{\sigma^2}{2}\frac{d^2P_j}{dx^2}-\alpha(x-m)\frac{dP_j}{dx} \\ +\mu\mathbb{1}_{\{j>0\}}P_{j-1}(x)-(\lambda+\mu\phi(x)\mathbb{1}_{\{j>0\}})P_j(x)+\lambda\phi(x)P_{j+1}(x)\Bigg)f_j(x)n(x)dx=0.$$

This implies the following result.

Proposition 1. *The family $(P_j(x)\stackrel{\text{def}}{=}p_j(x)/\rho^j, j\geq 0)\in\mathcal{D}'(\mathbb{R})^{\mathbb{N}}$ is solution ~~in the sense of distributions~~ to the following infinite differential system: for $j\geq 0$,*

$$\frac{\sigma^2}{2}\frac{d^2P_j}{dx^2}-\alpha(x-m)\frac{dP_j}{dx} \\ +\mu\mathbb{1}_{\{j>0\}}P_{j-1}(x)-(\lambda+\mu\phi(x)\mathbb{1}_{\{j>0\}})P_j(x)+\lambda\phi(x)P_{j+1}(x)=0. \tag{8}$$

2.2. **Additional properties.** For the system considered in this paper, we have $\mu\phi(x)>\mu(1-a)$ for all $x\in\mathbb{R}$ and $0<a<1$. Classical stochastic ordering arguments imply that the process $(L(t))$ is stochastically dominated for the strong ordering sense by the queuing process of the $M/M/1$ queue with input rate λ and service rate $\mu(1-a)$. Hence, if the solution $(P_j(x),j\geq 0)$ of the infinite differential system (8) is related to the conditional probability density functions $(p_j(x),j\geq 0)$ of the couple $(X(0),L(0))$ as $P_j(x)=p_j(x)/\rho^j$ for all $j\geq 0$, then

$$\forall x\in\mathbb{R},\forall j\geq 0,\quad P_j(x)\leq\frac{1}{(1-a)^j}. \tag{9}$$

If $a<1-\sqrt{\rho}$, it is easily checked that for all $x\in\mathbb{R}$, the sequence $(P_j(x),j\geq 0)$ is in the Hilbert space $\ell^2(\rho)$.

In addition, for all $j\geq 0$

$$\int_{\mathbb{R}}p_j(x)^2n(x)dx\leq\mathbb{P}(L(0)=j)<\infty, \tag{10}$$

since $p_j(x)=\mathbb{P}(L(0)=j\mid X(0)=x)\leq 1$. It follows that for all $j\geq 0$, the function $p_j(x)$ should be in the space $L^2(\mathbb{R},n)$.

Hence, if the solution $(P_j(x), j \geq 0)$ of the infinite differential system (8) is related to the conditional probability density functions $(p_j(x), j \geq 0)$ of the couple $(X(0), L(0))$ as specified above, then $P_j(x) \in L^2(\mathbb{R}, n)$ for all $j \geq 0$. From inequality (10), we also deduce that if $a < 1 - \sqrt{\rho}$,

$$\sum_{j=0}^{\infty} \int_{\mathbb{R}} \left(\frac{p_j(x)}{\rho^j}\right)^2 n(x) dx \rho^j < \infty$$

since $\mathbb{P}(L(0) = j) \leq \rho^j/(1-a)^j$.

If follows from the above remarks that to show the regularity of the conditional probability density functions $p_j(x)$ for $j \geq 0$ under the assumption $a < 1 - \sqrt{\rho}$, we are led to prove that the differential systems admits a unique regular solution in the space $L^2(\mathbb{R}, n; l^2(\rho))$.

In the next section, we review some properties of the operators associated with the generators of the Ornstein-Uhlenbeck process and the Markov process describing the number of customers in an $M/M/1$ queue.

3. Some results on the operators associated with the generators of the Ornstein-Uhlenbeck process and the $M/M/1$ queue

It is well known in the literature (see for instance [19]) that the operator H defined by Equation (7) is selfadjoint in the Hilbert space $L^2(\mathbb{R}, n)$. The eigenvalues of this operator are the numbers $-\alpha j$, $j \geq 0$, and the normalized eigenvector associated with the eigenvalue $-\alpha j$ is the function h_j given by

$$h_j(x) = \frac{1}{\sqrt{2^j j! \sqrt{\pi}}} H_j(\sqrt{2\alpha}(x-m)/\sigma), \tag{11}$$

where $H_j(x)$ is the jth Hermite polynomial. The sequence $(h_j, j \geq 0)$ is an orthonormal basis of $L^2(\mathbb{R}, n)$. The domain of the operator H is the set

$$D(H) = \left\{f \in H^2(\mathbb{R}, n) : x^2 f \in L^2(\mathbb{R}, n)\right\},$$

where $H^2(\mathbb{R}, n)$ is the Sobolev space defined as follows:

$$H^2(\mathbb{R}, n) = \{f \in C^1(\mathbb{R}) : f,\ f' \in L^2(\mathbb{R}, n) \text{ and the weak derivative } f'' \in L^2(\mathbb{R}, n)\}.$$

~~While the operator H is well known in the literature, less information is available on t~~ The operator A associated with the Markov process describing the number of customers in an $M/M/1$ queue and defined in $\ell^2(\rho)$ by the infinite matrix

$$A = \begin{pmatrix} -\lambda & \lambda & 0 & . & . & . \\ \mu & -(\lambda+\mu) & \lambda & 0 & . & . \\ 0 & \mu & -(\lambda+\mu) & \lambda & 0 & . \\ 0 & 0 & \mu & -(\lambda+\mu) & \lambda & . \\ . & . & . & . & . & . \end{pmatrix}. \tag{12}$$

has already been studied in the technical literature, notably in [7] (see also [8]). From these references, we know that the operator A is selfadjoint. Associated with the operator A is the operator A_1 defined in $\ell^2_1(\rho)$ by the infinite matrix given by

$$A_1 = \begin{pmatrix} -(\lambda+\mu) & \lambda & 0 & . & . & . \\ \mu & -(\lambda+\mu) & \lambda & 0 & . & . \\ 0 & \mu & -(\lambda+\mu) & \lambda & 0 & . \\ 0 & 0 & \mu & -(\lambda+\mu) & \lambda & . \\ . & . & . & . & . & . \end{pmatrix}. \tag{13}$$

Note that the above matrix is the generator of the Markov process describing the number of customers in an $M/M/1$ queue and absorbed at state 0 (see [7]). Finally, let $A_1^{[N]}$ denote the truncated operator associated with the finite matrix

$$A_1^{[N]} = \begin{pmatrix} -(\lambda+\mu) & \lambda & 0 & . & . & . \\ \mu & -(\lambda+\mu) & \lambda & 0 & . & . \\ 0 & \mu & -(\lambda+\mu) & \lambda & 0 & . \\ 0 & 0 & 0 & \ddots & \ddots & . \\ . & . & . & . & \mu & -\mu \end{pmatrix}. \tag{14}$$

The above matrix is the generator of the Markov process describing the number of customers in the finite capacity $M/M/1/N$ queue and absorbed at state 0. (See [9] for a related model.) In the following, we recall the spectral properties of the operators A, A_1 and $A_1^{[N]}$; see [7, 8, 9] for details

Lemma 2. *The operator A in $\ell^2(\rho)$ is bounded and symmetric, and then selfadjoint. In particular, for all $f \in \ell^2(\rho)$,*

$$0 \leq (-Af, f)_\rho \leq \mu(1+\sqrt{\rho})^2 \|f\|^2_\rho, \tag{15}$$

which implies that the operator $-A$ is monotonic (i.e., $(-Af,f) \geq 0$ for all $f \in \ell^2(\rho)$). There exists a unique normalized measure $d\psi(z)$, referred to as spectral measure, whose support is the spectrum $\sigma(A)$ of operator A, and a family of spaces $\{\mathcal{H}_z(\rho)\}$, $z \in \sigma(A)$, such that

- *the Hilbert space $\ell^2(\rho)$ is equal to the direct sum of the spaces $\mathcal{H}_z(\rho)$, i.e., every $f \in \ell^2(\rho)$ can be decomposed into a family $(f_z, z \in \sigma(A))$, where $f_z \in \mathcal{H}_z(\rho)$ and $\int \|f_z\|^2_\rho d\psi(z) < \infty$. Moreover,*
$$(f,g)_\rho = \int (f_z, g_z)_\rho d\psi(z).$$
- *The operator A is such $(Af)_z = z f_z$ for $z \in \sigma(A)$, where $(Af)_z$ is the projection of (Af) on the space $\mathcal{H}_z(\rho)$.*

The spectral measure $d\psi(x)$ is specifically given by

$$\int h(x) d\psi(x) = (1-\rho)h(0) - \frac{\sqrt{\rho}}{\pi} \int_{-\mu(1+\sqrt{\rho})^2}^{-\mu(1-\sqrt{\rho})^2} \frac{h(x)}{x} \sqrt{1 - \left(\frac{x+\lambda+\mu}{2\sqrt{\lambda\mu}}\right)^2}\, dx, \tag{16}$$

for any smooth function h.

The spectrum of the operator A is $\sigma(A) = [-\mu(1+\sqrt{\rho})^2, -\mu(1-\sqrt{\rho})^2] \cup \{0\}$. The operator A has a unique eigenvalue equal to 0, and the eigenspace $\mathcal{H}_0(\rho)$ is spanned by the vector e with all components equal to 1. For $z \in (-(\sqrt{\lambda}+\sqrt{\mu})^2, -(\sqrt{\lambda}-$

$\sqrt{\mu})^2)$, the space $\mathcal{H}_z(\rho)$ is the vector space spanned by the vector $Q(z)$, whose components $Q_j(z)$, $j\geq 0$ are defined by the following recursion:

$$\begin{cases} Q_0(z)=1, \quad Q_1(z)=(z+\lambda)/\lambda \\ \mu Q_{j+1}(z)-(z+\lambda+\mu)Q_j(z)+\mu Q_{j-1}(z)=0, \quad j\geq 1. \end{cases} \tag{17}$$

The vectors $(Q(z))$ for $z\in(-(\sqrt{\lambda}+\sqrt{\mu})^2, -(\sqrt{\mu}-\sqrt{\lambda})^2)$ form an orthogonal family with weight function $d\psi(z)$: for all j,k,

$$\int_{\mathbb{R}} Q_j(x)Q_k(x)d\psi(x)=\frac{1}{\rho^j}\delta_{j,k},$$

where $\delta_{j,k}$ is the Kronecker symbol, equal to 1 if $j=k$ and 0 if $j\neq k$.

Note the polynomials $Q_j(x)$ appearing in the above result are known as perturbed Chebyshev polynomials in the literature on orthogonal polynomials [18]. For the operator A_1, we have the following result, where we use Chebyshev polynomials of the second kind $(U_n(x))$ defined by the recursion

$$\begin{cases} U_0(x)=1, \quad Q_1(x)=2x \\ U_{j+1}(x)=2xU_j(x)-U_{j-1}(x), \quad j\geq 1. \end{cases} \tag{18}$$

Lemma 3. *The operator A_1 in the subspace $\operatorname{span}(e_j, j\geq 1)$ is bounded and symmetric, and then selfadjoint. In particular, for all $f\in\operatorname{span}(e_j, j\geq 1)$,*

$$\mu(1-\sqrt{\rho})^2\|f\|_\rho^2 \leq (-A_1 f, f)_\rho \leq \mu(1+\sqrt{\rho})^2\|f\|_\rho^2. \tag{19}$$

The associated normalized spectral measure $d\psi_1(z)$ is given by

$$d\psi_1(x)=\frac{2}{\pi}\sqrt{1-\left(\frac{x+\lambda+\mu}{2\sqrt{\lambda\mu}}\right)^2}\,\mathbb{1}_{\{x\in(-\mu(1+\sqrt{\rho})^2,-\mu(1-\sqrt{\rho})^2)\}}\frac{dx}{2\sqrt{\lambda\mu}}. \tag{20}$$

The spectrum of the operator A_1 is diffuse (there are no eigenvalues) and equal to the interval $\sigma(A_1)=[-\mu(1+\sqrt{\rho})^2, -\mu(1-\sqrt{\rho})^2]$. The Hilbert space $\operatorname{span}(e_j, j\geq 1)$ is equal to the direct sum of the spaces $\mathcal{H}_z^{(1)}(\rho)$, for $z\in(-\mu(1+\sqrt{\rho})^2, -\mu(1-\sqrt{\rho})^2)$, where the space $\mathcal{H}_z^{(1)}(\rho)$ is the vector space spanned by the vector $Q^{(1)}(z)$, whose components $Q_j^{(1)}(z)$, $j\geq 0$ are defined by the following recursion:

$$\begin{cases} Q_0^{(1)}(z)=1, \quad Q_1^{(1)}(z)=(z+\lambda+\mu)/\lambda \\ \mu Q_{j+1}^{(1)}(z)-(z+\lambda+\mu)Q_j^{(1)}(z)+\mu Q_{j-1}^{(1)}(z)=0, \quad j\geq 1. \end{cases} \tag{21}$$

The vectors $(Q^{(1)}(z))$ for $z\in(-(\sqrt{\lambda}+\sqrt{\mu})^2, -(\sqrt{\mu}-\sqrt{\lambda})^2)$ form an orthogonal family with weight function $d\psi_1(z)$. The polynomials $(Q_j^{(1)}(z))$ are related to Chebyshev polynomials as follows: for $j\geq 0$,

$$Q_j^{(1)}(z)=\frac{1}{\rho^{j/2}}U_j\left(\frac{z+\lambda+\mu}{2\sqrt{\lambda\mu}}\right).$$

Finally, for the operator $A_1^{[N]}$, we have the following result.

Lemma 4. *The operator $A_1^{[N]}$ is symmetric (and then selfadjoint) in the vector space* $\mathrm{span}(e_1,\dots,e_N)$ *equipped with the scalar product induced by* $(.,.)_\rho$. *The eigenvalues of the operator $A_1^{[N]}$ are the solutions to the polynomial equation*

$$Q^{(1)}_{N+1}(-x;1) = Q^{(1)}_N(-x;1),$$

where the polynomials $Q^{(1)}_j(x)$ are defined by the recursion (21). *The eigenvalues are denoted by $-x_j^{[N]}$, $j=1,\dots,N$ with $x_1^{[N]} < x_2^{[N]} < \dots < x_N^{[N]}$. The vectors $Q^{(1,N)}(x_j)$, $j=1,\dots,N$, form an orthogonal basis of* $\mathrm{span}(e_1,\dots,e_N)$, *where $Q^{(1,N)}(x_j)$ is the vector with the kth component equal to $Q^{(1)}_k(x_j)$, $k,j=1,\dots,N$, where the polynomials $(Q^{(1)}_k(z))$ are defined by Equation* (21).

The operator A naturally induces in $L^2(\mathbb{R},n;l^2(\rho))$ an operator that we still denote by A. The same property is valid for the operators A_1 and $A_1^{[N]}$ in the spaces $L^2(\mathbb{R},n;\mathrm{span}(e_j,j\geq 1))$ and $L^2(\mathbb{R},n;\mathrm{span}(e_j,j=1,\dots,N))$, respectively. Similarly, the operator H induces in $L^2(\mathbb{R},n;l^2(\rho))$ an operator that we still denote by H and which is defined as follows: for $f\in L^2(\mathbb{R},n;l^2(\rho))$, Hf is the element with the jth component equal to Hf_j. This operator also induces in $L^2(\mathbb{R},n;\mathrm{span}(e_j,j\geq 1))$ and $L^2(\mathbb{R},n;\mathrm{span}(e_j,j=1,\dots,N))$ operators denoted by H_1 and $H_1^{[N]}$, respectively. The operators A, A_1, $A_1^{[N]}$, H, H_1 and $H_1^{[N]}$ are clearly selfadjoint in the spaces where they are defined.

With the above definitions, the fundamental differential system (8) reads

$$(H+A)f + Vf = 0, \tag{22}$$

where the operator V is defined by: for $f\in L^2(\mathbb{R},n;l^2(\rho))$, $Vf = (\phi(x)-1)Bf$, where B is the operator associated with the infinite matrix

$$B = \begin{pmatrix} 0 & \lambda & 0 & . & . & . & . \\ 0 & -\mu & \lambda & 0 & . & . & . \\ 0 & 0 & -\mu & \lambda & 0 & . & . \\ 0 & 0 & 0 & -\mu & \lambda & 0 & . \\ . & . & . & . & . & . & . \end{pmatrix}.$$

In the notation of Equation (4), we have $\Omega = H + A$ and $V(\varepsilon) = -\varepsilon((x\wedge(a/\varepsilon))\vee(-b/\varepsilon))B$.

The matrices

$$B_1 = \begin{pmatrix} -\mu & \lambda & 0 & . & . & . & . \\ 0 & -\mu & \lambda & 0 & . & . & . \\ 0 & 0 & -\mu & \lambda & 0 & . & . \\ 0 & 0 & 0 & -\mu & \lambda & 0 & . \\ . & . & . & . & . & . & . \end{pmatrix}.$$

and

$$B_1^{[N]} = \begin{pmatrix} -\mu & \lambda & 0 & . & . & . \\ 0 & -\mu & \lambda & 0 & . & . \\ 0 & 0 & \ddots & \ddots & \ddots & . \\ . & . & . & . & 0 & -\mu \end{pmatrix}$$

define the operators B_1 and $B_1^{[N]}$ in the spaces $L^2(\mathbb{R},n;\mathrm{span}(e_j,j\geq 1))$ and $L^2(\mathbb{R},n;\mathrm{span}(e_j,j=1,\dots,N))$, respectively. It is straightforwardly checked that the operator B, B_1 and $B_1^{[N]}$ are bounded with a norm less than or equal to

$\mu(1+\sqrt{\rho})$. With the operators B_1 and $B_1^{[N]}$ are associated the operators V_1 and $V_1^{[N]}$ induced by V in the spaces $L^2(\mathbb{R}, n; \text{span}(e_j, j \geq 1))$ and $L^2(\mathbb{R}, n; \text{span}(e_j, j = 1, \ldots, N))$, respectively.

In the next section, we prove that the differential system (8) (or equivalently Equation (22)) has a unique solution in $L^2(\mathbb{R}, n; \ell^2(\rho))$.

4. Existence and uniqueness of a solution

When we refer to the existence of a density probability density function satisfying the differential system (8), we think of a vector $(P_j(x))$ such that every function $P_j(x)$ is twice continuously differentiable over $\mathbb{R}$ (i.e., $P_j(x) \in C^2(\mathbb{R})$ for all $j \geq 0$). But, this differential system may have a solution, which is in $L^2(\mathbb{R}, n; \ell^2(\rho))$ but with components not in $C^2(\mathbb{R})$. In the following, we prove that the differential system (8) has a unique solution in $L^2(\mathbb{R}, n; \ell^2(\rho))$ and then we show at the end of the section that the components of the solution are $C^2(\mathbb{R})$ functions.

If $f \in L^2(\mathbb{R}, n; \ell^2(\rho))$ is solution to Equation (22), then

$$(H_1 + A_1)f^1 + V_1 f^1 = -\mu e_1(f_0), \tag{23}$$

where f^1 is the projection of f on the space $L^2(\mathbb{R}, n, \text{span}(e_j, j \geq 1))$ and $e_1(f_0)$ is the element of $L^2(\mathbb{R}, n, \text{span}(e_j, j \geq 1))$ with the first component equal to $f_0(x)$ and all other components equal to 0.

The operator $(H_1 + A_1)$ is self-adjoint and invertible. It is straightforwardly checked that the norm

$$\|(H_1 + A_1)^{-1}\| \stackrel{def}{=} \sup\{\left\|(H_1 + A_1)^{-1} f\right\| : f \in L^2(\mathbb{R}, n, \text{span}(e_j, j \geq 1)), \|f\| = 1\}$$

is such that

$$\|(H_1 + A_1)^{-1}\| \leq \frac{1}{\mu(1-\sqrt{\rho})^2},$$

since by Lemma 3 and the monotonicity of the operator $-H_1$, we have for all f in $L^2(\mathbb{R}, n, \text{span}(e_j, j \geq 1))$

$$(-(H_1 + A_1)f, f) \geq (-H_1 f, f) + (-A_1 f, f) \geq -\mu(1-\sqrt{\rho})^2 \|f\|^2.$$

In addition, if $f \in L^2(\mathbb{R}, n; \ell^2(\rho))$ is a solution to Equation (22), then the function $\sum_{j=0}^{\infty} \rho^j f_j(x) \in L^2(n)$ since by Schwarz inequality

$$\int_{-\infty}^{\infty} \left| \sum_{j=0}^{\infty} \rho^j f_j(x) \right|^2 n(x)dx \leq \frac{1}{1-\rho} \int_{-\infty}^{\infty} \sum_{j=0}^{\infty} |f_j(x)|^2 \rho^j n(x)dx = \frac{1}{1-\rho} \|f\|^2 < \infty.$$

By summing all the lines of Equation (22), we see that the function $\sum_{j=0}^{\infty} \rho^j f_j(x)$ has to be solution to the equation $Hg = 0$ in $L^2(\mathbb{R}, n)$.

By using the above observations, we prove the existence and uniqueness of a non trivial solution in $L^2(\mathbb{R}, n; \ell^2(\rho))$ to Equation (22) by showing the following results:

1. If ε satisfies some condition, Equation (23) has a solution for all $f_0 \in L^2(n)$; if (f_j) is the solution, we set

$$K f_0 = \sum_{j=1}^{\infty} \rho^j f_j \in L^2(\mathbb{R}, n). \tag{24}$$

(2) If f is a non trivial solution to Equation (22), then f_0 is a non trivial solution to equation

$$f_0 + Kf_0 = 1. \tag{25}$$

(3) Equation (25) has a unique non trivial solution.

To prove the last point, we intend to use the Fredholm alternative, which requires that the operator K is compact.

Lemma 5. *Under the condition*

$$a \vee b < \frac{(1-\sqrt{\rho})^2}{1+\sqrt{\rho}}, \tag{26}$$

where $a \vee b = \max(a, b)$, Equation (23) has a unique solution in $L^2(\mathbb{R}, n; \ell_1^2(\rho))$.

Proof. Equation (23) can be rewritten as

$$(\mathbb{I} + (H_1 + A_1)^{-1}V_1)f = -\mu(H_1 + A_1)^{-1}e_1(f_0).$$

Since V_1 is bounded so that for all $f \in L^2(\mathbb{R}, n; \ell_1^2(\rho))$, $|(V_1 f, f)| \leq \mu(a \vee b)(1 + \sqrt{\rho})\|f\|^2$, we deduce that the operator $(H_1 + A_1)^{-1}V_1$ is bounded with a norm less than or equal to $(a \vee b)(1 + \sqrt{\rho})/(1 - \sqrt{\rho})^2$. Under Condition (26), the norm of operator $(H_1 + A_1)^{-1}V_1$ is less than 1 and we then deduce that $(\mathbb{I} + (H_1 + A_1)^{-1}V_1)$ is invertible [16] and Equation (23) has a unique solution in $L^2(\mathbb{R}, n; \ell_1^2(\rho))$. □

It is worth noting that under Condition (26), we have $a < (1 - \sqrt{\rho})$. Moreover, the above result ensures that the operator K is well defined by Equation (24). Now, we prove that under the same condition, the operator K is monotonic.

Lemma 6. *Under Condition (26), the operator K is monotonic, which implies that there exists at most one non trivial solution to Equation (25).*

Proof. We first note that from Equation (24), we have

$$(Kf_0, f_0)_2 = (-\mu(H_1 + A_1 + V_1)^{-1}e_1(f_0), e(f_0)),$$

where $e(f_0)$ is the vector with all entries equal to f_0. The above equation can be rewritten as

$$(Kf_0, f_0)_2 = (-\mu(\mathbb{I} + A_1^{-1}(H_1 + V_1))^{-1}A_1^{-1}e_1(f_0), e(f_0)).$$

Since $-\mu A_1^{-1}e_1(f_0) = e(f_0)$, we have

$$(Kf_0, f_0)_2 = ((\mathbb{I} + A_1^{-1}(H_1 + V_1))^{-1}e(f_0), e(f_0)) \tag{27}$$

and hence $(Kf_0, f_0)_2 \geq 0$. Indeed, for all $f \in L^2(\mathbb{R}, n; \ell_1^2(\rho))$

$$\begin{aligned} ((\mathbb{I} + A_1^{-1}H_1 + A_1^{-1}V_1)f, f) &= ((\mathbb{I} + A_1^{-1}H_1)f, f) + (A_1^{-1}V_1 f, f) \\ &\geq \left(1 - \frac{(a \vee b)(1 + \sqrt{\rho})}{(1 - \sqrt{\rho})^2}\right)\|f\|^2, \end{aligned}$$

where we have used the fact that the operator $A_1^{-1}H_1$ is monotonic, $\|V_1\| \leq \mu(a \vee b)(1 + \sqrt{\rho})$, and $\|A_1^{-1}\| \leq 1/(\mu(1 - \sqrt{\rho})^2)$. The above inequality implies that

$$\begin{aligned} &((\mathbb{I} + A_1^{-1}H_1 + A_1^{-1}V_1)^{-1}f, f) \\ &\qquad \geq \left(1 - \frac{(a \vee b)(1 + \sqrt{\rho})}{(1 - \sqrt{\rho})^2}\right)\|(\mathbb{I} + A_1^{-1}H_1 + A_1^{-1}V_1)^{-1}f\|^2 \geq 0, \end{aligned}$$

and Inequality (27) follows. □

We now turn to the compactness of the operator K. The major difficulty comes from the fact that the operator $(H_1 + A_1)^{-1}$ is not compact, since we know that the spectrum of this self-adjoint operator is not discrete. However, by truncating the infinite matrix defined by Equation (12), we can introduce compact operators and subsequently prove that the operator K is compact. Let us fix some $N > 0$. We first prove the following technical lemma.

Lemma 7. *For $N > 0$, the operator $(H_1^{[N]} + A_1^{[N]})^{-1}$ in $L^2(\mathbb{R}, n; \mathrm{span}(e_1, \dots, e_N))$ is compact.*

Proof. By using Lemma 4 and the orthonormal basis (h_n) is $L^2(\mathbb{R}, n)$, we know that the family $e_{j,k}(x) = h_k(x)Q^{(1,N)}(x_j)$ for $k \geq 0$ and $j = 1, \dots, N$ forms an orthogonal basis of $L^2(\mathbb{R}, n; \mathrm{span}(e_1, \dots, e_N))$. In particular, we have

$$(H_1^{[N]} + A[N]_1)^{-1} e_{j,k}(x) = -\frac{1}{k\alpha + x_j^{[N]}} e_{j,k}(x).$$

The operator $(H_1^{[N]} + A_1^{[N]})$ then appears as the norm limit as $M \to \infty$ of the finite rank operators $(H_1^{[N,M]} + A_1^{[N,M]})^{-1}$ defined in the vector space $\mathrm{span}(e_{j,k}(x), j = 1, \dots, N, k = 0, \dots, M)$ by

$$(H_1^{[N,M]} + A_1^{[N,M]})^{-1} e_{j,k}(x) = -\frac{1}{\alpha k + x_j^{[N]}} e_{j,k}(x)$$

and the result follows. □

Lemma 8. *Under Condition* (26)*, the operator K is compact.*

Proof. Let us consider a bounded sequence (f_0^i) in $L^2(\mathbb{R}, n)$. (Without loss of generality, we assume that $\|f_0^i\|_2 = 1$.) Since the operator $(H_1^{[N]} + A_1^{[N]})^{-1} V_1^{[N]}$ is bounded with a norm less than or equal to $(a \vee b)\mu(1 + \sqrt{\rho})/x_1^{[N]}$, and the operator $(H_1^{[N]} + A_1^{[N]})^{-1}$ is compact, we deduce that the operator $(H_1^{[N]} + A_1^{[N]} + V_1^{[N]})^{-1}$ is compact.

Let $f^i = (f_j^i(x))$ denote the vector $-\mu(H_1 + A_1 + V_1)^{-1} e_1(f_0^i)$. Since the operator $(H_1 + A_1 + V_1)^{-1}$ is bounded with a norm less than or equal to $1/(\mu((1 - \sqrt{\rho})^2 - a \vee b\varepsilon(1 + \sqrt{\rho})))$, the vector (f^i) is such that

$$\sqrt{\sum_{j=1}^{N} \|f_j^i\|_2^2 \rho^j} \leq \|f^i\| \leq \frac{1}{((1 - \sqrt{\rho})^2 - a \vee b(1 + \sqrt{\rho}))}. \tag{28}$$

We have

$$\begin{pmatrix} f_1^i \\ \vdots \\ f_N^i \end{pmatrix} = (H_1^{[N]} + A_1^{[N]} + V_1^{[N]})^{-1} \begin{pmatrix} -\mu f_0^i \\ 0 \\ \vdots \\ 0 \\ -\lambda\phi f_{N+1}^i + \lambda f_N^i \end{pmatrix}$$

In view of inequality (28), the sequence appearing in the right hand side of the above equation is bounded. Hence, since the operator $(H_1^{[N]} + A_1^{[N]} + V_1^{[N]})^{-1}$ is compact, it is possible to extract a sub-sequence (f^{i_k}) such that

$$(H_1^{[N]}+A_1^{[N]}+V_1^{[N]})^{-1}\begin{pmatrix} -\mu f_0^{i_k} \\ 0 \\ \vdots \\ 0 \\ -\lambda\phi f_{N+1}^{i_k}+\lambda f_N^{i_k} \end{pmatrix} \to \begin{pmatrix} f_1^\infty \\ f_2^\infty \\ \vdots \\ f_{N-1}^\infty \\ f_N^\infty \end{pmatrix}$$

as $k\to\infty$ in $L^2(\mathbb{R},n;\text{span}(e_1,\dots,e_N))$, where the vector appearing in the right hand side of the above equation is in $L^2(\mathbb{R},n;\text{span}(e_1,\dots,e_N))$.

In particular, we have $f_N^{i_k}\to f_N^\infty$ in $L^2(\mathbb{R},n)$ as $k\to\infty$. This implies that

$$\begin{pmatrix} f_{N+1}^{i_k} \\ f_{N+2}^{i_k} \\ \vdots \end{pmatrix} = (H_1^{[N]}+A_1^{[N]}+V_1^{[N]})^{-1}\begin{pmatrix} -\mu f_N^{i_k} \\ 0 \\ \vdots \end{pmatrix}$$

$$\to (H_1^{[N]}+A_1^{[N]}+V_1^{[N]})^{-1}\begin{pmatrix} -\mu f_N^{\infty} \\ 0 \\ \vdots \end{pmatrix}$$

as $k\to\infty$ in $L^2(\mathbb{R},n,\ell_1^2(\rho))$, since the operator $(H_1^{[N]}+A_1^{[N]}+V_1^{[N]})^{-1}$ is bounded. We set

$$\begin{pmatrix} f_{N+1}^{\infty} \\ f_{N+2}^{\infty} \\ \vdots \end{pmatrix} = (H_1^{[N]}+A_1^{[N]}+V_1^{[N]})^{-1}\begin{pmatrix} -\mu f_N^{\infty} \\ 0 \\ \vdots \end{pmatrix}$$

The vector with the jth component equal to f_j^∞ is in $L^2(\mathbb{R},n,\ell^2(\rho))$ and we have $Kf_0^{i_k}\to\sum_{j=1}^\infty \rho^j f_j^\infty$ in $L^2(\mathbb{R},n)$ as $k\to\infty$. We then deduce that from every bounded sequence (f_0^i) in $L^2(\mathbb{R},n)$, we can extract a sub-sequence $(f_0^{i_k})$ such that $Kf_0^{i_k}$ is converging in $L^2(\mathbb{R},n)$. The operator K is hence compact. □

By using the above lemmas and the Fredholm alternative, we can state the following result.

Proposition 2. *Under Condition* (26)*, Equation* (25) *has a unique solution. This establishes that Equation* (22) *has a unique non trivial solution in* $L^2(\mathbb{R},n,\ell^2(\rho))$ *if Condition* (26) *is satisfied.*

The above result has been established for the perturbation function $\phi(x)=1-\varepsilon((x\wedge(a/\varepsilon))\vee(-(b/\varepsilon)))$ and we have exploited the fact that the function $|1-\phi(x)|$ is bounded by $a\vee b$. In fact, it is possible to prove a similar result when ϕ is replaced with $\Phi(x)=1-\varepsilon x$. Let us define the operator W in $L^2(\mathbb{R},n;\ell^2(\rho))$ by: if $f=(f_j(x),j\geq 0)$

$$Wf=-\varepsilon xBf.$$

Note that the domain of W is given by

$$D(W)=\left\{(f_j(x),j\geq 0)\in L^2(\mathbb{R},n)^{\mathbb{N}}:\int_{\mathbb{R}}\sum_{j=0}^\infty x^2f_j(x)^2n(x)dx<\infty\right\}.$$

In the notation of Equation (5), we have $V(\Phi)=-xB$.

We can then state the following result, whose proof is given in Appendix A.

Proposition 3. *Under the condition*

$$2\varepsilon\frac{(1+\sqrt{\rho})}{(1-\sqrt{\rho})^2}\left(m+\frac{\sigma}{\sqrt{\alpha}}\right)<1, \tag{29}$$

the equation

$$(H+A+W)f=0 \tag{30}$$

has a unique solution in $L^2(\mathbb{R},n,\ell^2(\rho))$.

To prove the existence and the uniqueness of the solutions to Equations (22) and (30), we have only supposed that the components of the solutions are in $D(H)$, in particular the components are in $H^2(\mathbb{R},n)$. But, by examining the differential systems satisfied by the different components, notably by taking into account the continuity of the functions $\phi(x)$ and $\Phi(x)$, these components are clearly in $C^2(\mathbb{R})$.

To conclude this section, let us mention that the solution to the differential system (8) have the following probabilistic interpretation: for all $j\geq 0$, in the stationary regime $P_j(x)=\mathbb{P}(L(t)=j\mid X(t)=x)/\rho^j$. In particular, we have for all $j\geq 0$

$$0\leq P_j(x)\leq 1 \tag{31}$$

and for all $x\in\mathbb{R}$

$$\sum_{j=0}^{\infty}P_j(x)\rho^j=1. \tag{32}$$

5. Perturbation analysis

The goal of this section is to prove the following Reduced Service Rate approximation.

Theorem 1. *For sufficiently small* ε, *the first order expansion of the generating function of the stationary distribution of* $(L(t))$ *is given by*

$$\mathbb{E}\left(u^{L(t)}\right)=\frac{1-\rho}{1-\rho u}-\frac{\rho(1-u)}{(1-\rho u)^2}m\varepsilon+o(\varepsilon).$$

Therefore, $\mathbb{E}(u^{L(t)})\sim\mathbb{E}(u^{L_\varepsilon})$, where L_ε has the stationary distribution of the number of customers in an $M/M/1$ queue when the server rate is $1-\varepsilon m$. This shows a principle of reduced service rate approximation, i.e., everything happens as if the server rate were fixed equal to $1-m\varepsilon$.

To show the above result, we proceed as follows:

(1) We compare the solutions to Equations (22) and (30) when Conditions (26) and (29) are satisfied. In particular, we compute an upper bound for the norm of their difference.
(2) We develop the solution g to Equation (30) in power series expansion of ε. In particular, we explicitly compute the two first terms.
(3) We finally prove Theorem 1.

Throughout this section, we denote by P and g the solutions to Equations (22) and (30) in $L^2(\mathbb{R},n;\ell^2(\rho))$, respectively.

5.1. **Comparison of the solutions P and g.** The solutions to Equations (22) and (30) when they exist are close to each other when ε is small. We specifically have the following result.

Proposition 4. *Assume that Conditions* (26) *and* (29) *are satisfied. These solutions P and g are such that*

$$\|P-g\| \leq D(\varepsilon), \tag{33}$$

where the function $D(\varepsilon)$ is given by

$$D(\varepsilon)=M\left(1+\sqrt{\frac{\rho}{1-\rho}}+\lambda\sqrt{\frac{\rho}{1-\rho}}M\right)\Delta(\varepsilon) \tag{34}$$

with

$$\begin{aligned} M &= \frac{1}{\mu(1-\sqrt{\rho})^2}\left(1+\frac{(1+\sqrt{\rho})}{(1-\sqrt{\rho})^2}\left(m+\frac{\sigma}{\sqrt{\alpha}}\right)\right), \\ \Delta(\varepsilon)^2 &= (\mu^2+3\lambda\mu)\int_{\mathbb{R}}\left((\varepsilon x-a)^2\mathbb{1}_{\{x\geq a/\varepsilon\}}+(\varepsilon x+b)^2\mathbb{1}_{\{x\leq -b/\varepsilon\}}\right)n(x)dx. \end{aligned}$$

The function $D(\varepsilon)$ is $O\left(\varepsilon^{5/2}\exp(-\alpha(a\wedge b)^2/(2\sigma^2\varepsilon^2))\right)$ when $\varepsilon\to 0$.

Proof. The vector P is such that

$$(H+W+A)P+(V-W)P=0$$

and hence, if P^1 denotes the projection of P on the space $\mathrm{span}(e_j, j\geq 1)$,

$$(H_1+A_1+W_1)P^1=(W_1-V_1)P^1-\mu e_1(P_0)$$

and then

$$P^1=(H_1+A_1+W_1)^{-1}(W_1-V_1)P^1-\mu(H_1+A_1+W_1)^{-1}e_1(P_0). \tag{35}$$

It follows that

$$\begin{aligned}\sum_{j=1}^{\infty}P_j(x)\rho^j \equiv (P^1,e^1)_\rho &= ((H_1+A_1+W_1)^{-1}(W_1-V_1)P^1,e^1)_\rho \\ &\quad -\mu((H_1+A_1+W_1)^{-1}e_1(P_0),e^1)_\rho.\end{aligned}$$

We have

$$-\mu((H_1+A_1+W_1)^{-1}e_1(P_0),e^1)_\rho=K'g_0,$$

where the operator K' is defined as the operator K (defined by Equation (24)) but by replacing V with W. In addition, since P satisfies $(P,e)_\rho=1$, we come up with the conclusion that P_0 verifies

$$1-P_0=((H_1+A_1+V_1)^{-1}(W_1-V_1)P^1,e^1)_\rho+K'P_0.$$

Since $g_0+K'g_0=1$, we obtain

$$(P_0-g_0)+K'(P_0-g_0)=((H_1+A_1+W_1)^{-1}(W_1-V_1)P^1,e^1)_\rho.$$

Since the operator K' is monotonic, the above equation implies that

$$\|P_0-g_0\|_2\leq\sqrt{\frac{\rho}{1-\rho}}\|(H_1+A_1+W_1)^{-1}\|\|(W_1-V_1)P^1\|.$$

The norm $\|(W_1-V_1)P^1\|$ is given by

$$\|(W_1-V_1)P^1\|^2 = \sum_{j=1}^{\infty}\int_{\mathbb{R}}\left((\varepsilon x-a)^2\mathbb{1}_{\{x\geq a/\varepsilon\}}+(\varepsilon x+b)^2\mathbb{1}_{\{x\leq -b/\varepsilon\}}\right)(\mu P_j(x)-\lambda P_{j+1}(x))^2\rho^j n(x)dx$$

and by using Equation (31), we obtain

$$\|(W_1-V_1)P^1\|^2 \leq (\mu^2+3\lambda\mu)\int_{\mathbb{R}}\left((\varepsilon x-a)^2\mathbb{1}_{\{x\geq a/\varepsilon\}}+(\varepsilon x+b)^2\mathbb{1}_{\{x\leq -b/\varepsilon\}}\right)n(x)dx$$

Simple computations show that

$$\begin{aligned}\int_{\mathbb{R}}(\varepsilon x-a)^2\mathbb{1}_{\{x\geq a/\varepsilon\}}n(x)dx &\sim \frac{5\varepsilon^5\sigma^5}{8\alpha^{5/2}a^3\sqrt{\pi}}e^{-\frac{\alpha a^2}{\varepsilon^2\sigma^2}},\\ \int_{\mathbb{R}}(\varepsilon x+b)^2\mathbb{1}_{\{x\leq -b/\varepsilon\}}n(x)dx &\sim \frac{5\varepsilon^5\sigma^5}{8\alpha^{5/2}b^3\sqrt{\pi}}e^{-\frac{\alpha b^2}{\varepsilon^2\sigma^2}},\end{aligned}$$

when $\varepsilon\to 0$. The term $\|(W_1-V_1)P^1\|$ is hence $O(\exp(\varepsilon^{5/2}\exp(-\alpha(a\wedge b)^2/(2\sigma^2\varepsilon^2)))$ when $\varepsilon\to 0$. In addition, since by using Equation (61) in Appendix A

$$\|(H_1+A_1+W_1)^{-1}\|\leq \frac{1}{\mu(1-\sqrt{\rho})^2}\left(1+\frac{(1+\sqrt{\rho})}{(1-\sqrt{\rho})^2}\left(m+\frac{\sigma}{\sqrt{\alpha}}\right)\right),$$

we deduce that $\|f_0-g_0\|_2$ is dominated by a term, which is $O(\exp(\varepsilon^{5/2}\exp(-\alpha(a\wedge b)^2/(2\sigma^2\varepsilon^2)))$ when $\varepsilon\to 0$. Finally, by using the fact that $(H^1+A_1+W_1)g = -\mu e_1(g_0)$, we deduce from Equation (35) that

$$P^1-g^1 = (H_1+A_1+W_1)^{-1}(W_1-V_1)P^1-\mu(H_1+A_1+W_1)^{-1}e_1(P_0-g_0),$$

which implies that

$$\|P^1-g^1\|\leq \|(H_1+A_1+W_1)^{-1}\|\|(W_1-V_1)P^1\|+\lambda\|(H_1+A_1+W_1)^{-1}\|\|P_0-g_0\|_2$$

and hence $\|P^1-g^1\|$ is dominated by a term which is $O(\exp(\varepsilon^{5/2}\exp(-\alpha(a\wedge b)^2/(2\sigma^2\varepsilon^2)))$ when $\varepsilon\to 0$. □

5.2. **Power series expansion of the solution g.**

5.2.1. *Notation.* We assume that the solution g to Equation (30) can be uniquely decomposed as a power series expansion of the form

$$g = g^{(0)}+\varepsilon g^{(1)}+\varepsilon^2 g^{(2)}+\ldots., \tag{36}$$

where $g^{(i)}\in L^2(\mathbb{R},n;\ell^2(\rho))$ for $i\geq 0$. In addition, to facilitate the computations, we shall consider the generating function $g_u(x)=\sum_{j=0}^{\infty}g_j(x)u^j\rho^j$, which is an element of $L^2(\mathbb{R},n;\ell^2(1/\rho))$. Indeed, if g is written in the form

$$g=\sum_{j=0}^{\infty}c_j(u)h_j(x),$$

where $c_j(u)=\sum_{m=0}^{\infty}c_{j,m}u^m$ with $(c_{j,m},m\geq 0)\in\ell^2(\rho)$, then $g_u(x)$ can be written as

$$g_u(x)=\sum_{j=0}^{\infty}C_j(u)h_j(x),$$

with $C_j(u) = \sum_{m=0}^{\infty} C_{j,m} u^m = c_j(\rho u)$. Since $(c_{j,m}, m \geq 0) \in \ell^2(\rho)$, $(C_{j,m}, m \geq 0) \in \ell^2(1/\rho)$. Finally, we have $\|g\|^2 = \sum_{j=0}^{\infty} \|c_j\|_\rho^2 = \sum_{j=0}^{\infty} \|C_j\|_{1/\rho}^2$.

The generating function $g_u(x)$ will be expanded as

$$g_u(x) = g_u^{(0)}(x) + \varepsilon g_u^{(1)}(x) + \varepsilon^2 g_u^{(2)}(x) + \cdots , \tag{37}$$

where $g_u^{(i)} \in L^2(\mathbb{R}, n)$ for all $i \geq 0$. The function $g_u^{(0)}(x)$ corresponds to the case $\varepsilon = 0$ and is given by

$$g_u^{(0)}(x) = \frac{c(g)}{1 - \rho u}, \tag{38}$$

where $c(g)$ is the normalizing constant.

In the following, we prove that the elements $g^{(i)}$ have to satisfy a recurrence relation of the form $g^{(i)} = \Theta(x g^{(i-1)})$ for $i \geq 1$ and for some linear operator Θ whose norm is finite.

In the following, we assume that the expansion (37) is valid and we investigate the conditions which have to be satisfied by the elements $g^{(i)}$. In a first step, we prove the following property satisfied by the functions $(g_u^{(i)}(x))$.

Lemma 9. *For $i \geq 0$, the vector $g^{(i)}$ is in $L_i^2(\mathbb{R}, n; \ell^2(\rho))$, where $L_i^2(\mathbb{R}, n; \ell^2(\rho))$ is the sub-space of $L^2(\mathbb{R}, n; \ell^2(\rho)$ composed of those elements $(f_j(x))$ such that $f_j(x) \in \mathrm{span}(h_0, \ldots, h_i)$ for all $j \geq 0$, the functions h_j being defined by Equation (11); the function $g_u^{(i)}(x)$ in Expansion (37) hence satisfies for $N > i$*

$$\lim_{x \to \pm\infty} \frac{1}{x^N} g_u^{(i)}(x) = 0. \tag{39}$$

Proof. The proof is by mathematical induction. The result is true for $i = 0$ since $g^{(0)} = c(g)e$ (e being the vector with all components equal to 1).

If the result is true for i. From Equation (30), we have

$$(H + A)g^{(i+1)} = -W g^{(i)}.$$

By using the recurrence relation satisfied by Hermite polynomials [10]

$$H_{j+1}(x) - 2xH_j(x) + 2iH_{j-1}(x) = 0, \tag{40}$$

it is easily checked that the image by the multiplication by x of $\mathrm{span}(h_0, \ldots, h_i)$ is $\mathrm{span}(h_0, \ldots, h_{i+1})$. Therefore, since by assumption $g^{(i)}$ belongs to $L_i^2(\mathbb{R}, n; \ell^2(\rho))$, we immediately deduce from the uniqueness of the decomposition on the basis $(h_i e_j, i \geq 0, j \geq 0)$ of the Hilbert space $L^2(\mathbb{R}, n; \ell^2(\rho))$ and the selfadjointness of the operator $H + A$, that $g^{(i+1)}$ is in $L_{i+1}^2(\mathbb{R}, n; \ell^2(\rho))$ and the result follows. □

5.2.2. *First order term.* In a first step, we pay special attention to the derivation of the first order term because it gives the basic arguments to derive higher order terms. Moreover, the explicit form of the first order term will be used to examine the validity of the reduced service rate approximation (see Theorem 1).

On the basis of the domination property given by Lemma 9, we explicitly compute the function $g_u^{(1)}(x)$. From Equation (30), it is easily checked that the function

$g_u^{(1)}(x)$ satisfies the equation

$$\begin{aligned}\frac{\sigma^2}{2}\frac{\partial^2 g_u^{(1)}}{\partial x^2} - \alpha(x-m)\frac{\partial g_u^{(1)}}{\partial x} + \alpha\nu(u)g_u^{(1)}(x) \\ = \mu\left(\frac{1}{u}-1\right)\left(g_0^{(1)}(x) - x(g_0^{(0)}(x) - g_u^{(0)}(x))\right) \\ = \mu\left(\frac{1}{u}-1\right)\left(g_0^{(1)}(x) + x\frac{\rho u c(g)}{(1-\rho u)}\right)\end{aligned} \tag{41}$$

where the constant $\nu(u)$ is given by

$$\nu(u) = \frac{\mu(1-u)(1-\rho u)}{\alpha u}.$$

In a first step, we search for a particular solution to the ordinary differential equation

$$\frac{\sigma^2}{2}\frac{\partial^2 \xi_u}{\partial x^2} - \alpha(x-m)\frac{\partial \xi_u}{\partial x} + \alpha\nu(u)\xi_u(x) = x\frac{\rho\mu(1-u)c(g)}{(1-\rho u)}$$

of the form

$$\xi_u(x) = a(u) + b(u)x.$$

Straightforward manipulations show that

$$b(u) = \frac{\rho\mu(1-u)(1-\rho)}{\alpha(\nu(u)-1)c(g)} \quad \text{and} \quad a(u) = -\frac{m}{\nu(u)}b(u).$$

Noting that $\xi_0(x) \equiv 0$, it follows that if we write $g_u^{(1)}(x) = \xi_u(x) + \psi_u(x)$, then the function $\psi_u(x)$ is solution to the equation

$$\frac{\sigma^2}{2}\frac{\partial^2 \psi_u}{\partial x^2} - \alpha(x-m)\frac{\partial \psi_u}{\partial x} + \alpha\nu(u)\psi_u(x) = \mu\left(\frac{1}{u}-1\right)\psi_0(x). \tag{42}$$

By using the domination property of Lemma 9, we can determine the form of the function $\psi_0(x)$.

Lemma 10. *The function $\psi_0(x)$ is given by*

$$\psi_0(x) = c_0 + c_1\frac{\sqrt{\alpha}(x-m)}{\sigma}$$

for some constants c_0 and c_1.

Proof. By introducing the function $k_u(x)$ defined by

$$k_u(x) = \exp\left(-\frac{\alpha(x-m)^2}{2\sigma^2}\right)\psi_u(x) \tag{43}$$

and then the change of variable

$$z = \frac{\sqrt{\alpha}(x-m)}{\sigma}, \tag{44}$$

Equation (42) becomes

$$\frac{\partial^2 k_u}{\partial z^2} + (2\nu(u)+1-z^2)k_u(z) = \frac{2\mu}{\alpha}\left(\frac{1}{u}-1\right)k_0(z). \tag{45}$$

The homogeneous equation reads

$$\frac{\partial^2 k_u}{\partial z^2} + (2\nu(u)+1-z^2)k_u = 0,$$

which solutions are parabolic cylinder functions (see Lebedev [10] for details). Two independent solutions $v_1(u;z)$ and $v_2(u;z)$ of this homogeneous equation are given in terms of Hermite functions as

$$v_1(u;z) = e^{-z^2/2} H_{\nu(u)}(z) \quad \text{and} \quad v_2(u;z) = e^{z^2/2} H_{-\nu(u)-1}(iz). \tag{46}$$

The Wronskian $\mathcal{W}$ of these two functions is given by

$$\mathcal{W}(z) = e^{-(\nu+1)\pi i/2}.$$

By using the method of variation of parameters, the solution to Equation (45) is given by

$$\begin{aligned} k_u(z) &= \gamma_1(u) v_1(u;z) + \gamma_2(u) v_2(u;z) \\ &\quad - \frac{2\mu}{\alpha}\left(\frac{1}{u} - 1\right) e^{(\nu+1)\pi i/2} \int_0^z \left[v_1(u;y)v_2(u;z) - v_1(u;z)v_2(u;y)\right] k_0(y)\, dy, \end{aligned}$$

where $\gamma_1(u)$ and $\gamma_2(u)$ are constants, which depend upon u.

The function $\psi_u(x)$ enjoys the same domination property as function $g_u^{(1)}(x)$, given by Lemma 9. Hence, for $N > 1$

$$\lim_{z\to\pm\infty} \frac{1}{z^N} e^{z^2/2} k_u(z) = 0. \tag{47}$$

From Lebedev [10], we have the following asymptotic estimates

$$H_\nu(z) \sim (2z)^\nu \left[\sum_{k=0}^{n} \frac{(-1)^k}{k!}(-\nu)_{2k}(2z)^{-2k} + O(|z|^{-2n-2})\right] \tag{48}$$

when $|z| \to \infty$ and $|\arg z| \leq 3\pi/4 - \delta$ for some $\delta > 0$. Moreover, when $z \to -\infty$

$$H_\nu(z) \sim \begin{cases} \dfrac{\sqrt{\pi}}{\Gamma(-\nu)}|z|^{-\nu-1}e^{z^2}\left[\displaystyle\sum_{k=0}^{n}\frac{1}{k!}(\nu+1)_{2k}(2z)^{-2k} + O(|z|^{-2n-2})\right], & \nu \notin \mathbb{N} \\ (2z)^\nu, \quad \nu \in \mathbb{N}. \end{cases}$$

The above asymptotic estimates and Lemma (9) imply that for $u \in (0,1)$ such that $\nu(u) \in \mathbb{N}$ with $\nu > 1$, we have

$$\begin{aligned} \gamma_1(u) &= -\frac{2\mu}{\alpha}\left(\frac{1}{u}-1\right)e^{(\nu+1)\pi i/2}\int_0^\infty v_2(u;y)k_0(y)\,dy \\ &= -\frac{2\mu}{\alpha}\left(\frac{1}{u}-1\right)e^{(\nu+1)\pi i/2}\int_0^{-\infty} v_2(u;y)k_0(y)\,dy \end{aligned}$$

and

$$\begin{aligned} \gamma_2(u) &= \frac{2\mu}{\alpha}\left(\frac{1}{u}-1\right)e^{(\nu+1)\pi i/2}\int_0^\infty v_1(u;y)k_0(y)\,dy \\ &= \frac{2\mu}{\alpha}\left(\frac{1}{u}-1\right)e^{(\nu+1)\pi i/2}\int_0^{-\infty} v_1(u;y)k_0(y)\,dy. \end{aligned}$$

The latter equation implies that for all $n > 1$

$$\int_{-\infty}^{\infty} e^{-y^2/2} k_0(y) H_n(y)\, dy = 0, \tag{49}$$

where $H_n(x)$ is the nth Hermite polynomial. By Property (47), the function $y \to \exp(y^2/2)k_0(y)$ is in $L^2(\mathbb{R}, \exp(-y^2)\,dy)$. Since Hermite polynomials form an orthogonal basis in this Hilbert space, Equation (49) entails that the function $y \to \exp(y^2/2)k_0(y)$ is orthogonal to all Hermite polynomials H_n with $n > 1$ and then that this function belongs to the vector space spanned by H_0 and H_1. Hence, function $k_0(z)$ should be of the form

$$k_0(z) = (c_0 + c_1 z)e^{-z^2/2}$$

for some constants c_0 and c_1 and the result follows. □

By using the above lemma, we are now able to establish the expression of $g_u^{(1)}(x)$.

Proposition 5. *The function $g_u^{(1)}(x)$ is given by*

$$g_u^{(1)}(x) = \frac{c(g)(u_1 - 1)(\tilde{u}_1 - \rho u)(u-1)}{u_1\tilde{u}_1(1-\rho)(u-\tilde{u}_1)(1-\rho u_1)(1-\rho u)^2} m + \frac{c(g)(1-u)}{(u-\tilde{u}_1)(1-\rho u_1)(1-\rho u)} x, \tag{50}$$

where u_1 and $\tilde{u}_1$ are the two real solutions to the quadratic equation

$$\rho u^2 - \left(1 + \rho + \frac{\alpha}{\mu}\right) u + 1 = 0$$

with $0 < u_1 < 1 < \tilde{u}_1$

Proof. By taking into account Lemma 10, the function $K_u(z)$ defined by

$$K_u(x) = g_u^{(1)}(x) \exp\left(-\frac{\alpha(x-m)^2}{2\sigma^2}\right)$$

and the change of variable (44), satisfies the equation

$$\frac{\partial^2 K_u}{\partial z^2} + (2\nu(u) + 1 - z^2)K_u(z) = \frac{2\mu}{\alpha}\left(\frac{1}{u} - 1\right)\left(c_0 + c_1 z + \frac{\rho u c(g)}{1-\rho u}\left(\frac{\sigma z}{\sqrt{\alpha}} + m\right)\right) e^{-z^2/2}. \tag{51}$$

We search for a particular solution of the form

$$K_u(z) = (a(u) + b(u)z)\, e^{-z^2/2}.$$

Straightforward computations yield

$$a(u) = \frac{1}{(1-\rho u)}\left(c_0 + \frac{\rho u c(g)}{1-\rho u} m\right),$$

$$b(u) = \frac{(1-u)}{\rho(u-u_1)(u-\tilde{u}_1)}\left(c_1 + \frac{\rho\sigma u c(g)}{\sqrt{\alpha}(1-\rho u)}\right).$$

It follows that the general solution to the above equation can be written as

$$K_u(z) = (a(u) + b(u)z)\, e^{-z^2/2} + \gamma_1(u)v_1(u;z) + \gamma_2(u)v_2(u;z), \tag{52}$$

where the functions v_1 and v_2 are defined by Equation (46) and the constants $\gamma_1(u)$ and $\gamma_2(u)$ depend upon u.

By differentiating once Equation (52) with respect to z and using the fact that the Wronskian of the functions $v_1(u;z)$ and $v_2(u;z)$ is $\exp[(\nu(u)+1)\pi i/2]$, we can easily express $\gamma_1(u)$ and $\gamma_2(u)$ by means of $K_u(z)$, $a(u)$, and $b(u)$. This shows that $\gamma_1(u)$ and $\gamma_2(u)$ are analytic in the open unit disk deprived of the points 0 and u_1. From the asymptotic properties satisfied by the functions v_1 and v_2, we know that

$\gamma_1(u) = 0$ and $\gamma_2(u) = 0$ for u such that $\nu(u) > 1$. It follows that $\gamma_1(u) \equiv \gamma_2(u) \equiv 0$ for $|u| < 1$.

By using the fact that $g_u^{(1)}(x)$ has to be analytic in variable u in the unit disk, we necessarily have

$$c_1 = -\frac{\rho\sigma u_1 c(g)}{\sqrt{\alpha}(1-\rho u_1)}$$

and then,

$$b(u) = \frac{\sigma c(g)(1-u)}{\sqrt{\alpha}(u-\tilde{u}_1)(1-\rho u_1)(1-\rho u)}.$$

Moreover, since $g_1^{(1)}(x) \equiv 0$, we have

$$c_0 = -\frac{\rho c(g) m}{1-\rho}$$

and then,

$$a(u) = \frac{\rho(u-1)c(g)}{(1-\rho)(1-\rho u)^2} m. \tag{53}$$

By using the expressions of $a(u)$ and $b(u)$, the result follows. □

5.2.3. *Higher order terms.* We assume that $g_u^{(i)}(x)$ can be expressed as

$$g_u^{(i)}(x) = \sum_{j=0}^{i} c_{i,j}(u) h_j(x), \tag{54}$$

where the function h_j is defined by Equation (11) and the coefficients $c_{i,j}$ are analytic functions in variable u. This assumption will be justified a posteriori. From previous sections, this representation is valid for $i = 0, 1$. If it is valid for $i-1$, then the function $g_u^{(i)}(x)$, $i \geq 1$, satisfies the equation

$$\frac{\sigma^2}{2}\frac{\partial^2 g_u^{(i)}}{\partial x^2} - \alpha(x-m)\frac{\partial g_u^{(i)}}{\partial x} + \alpha\nu(u) g_u^{(i)}(x) = \mu\left(\frac{1}{u}-1\right)\left(g_0^{(i)}(x) - x(g_0^{(i-1)}(x) - g_u^{(i-1)}(x))\right). \tag{55}$$

First note that by using the recurrence relation (40) satisfied by Hermite polynomials, it is easily checked that

$$x(g_u^{(i-1)}(x) - g_0^{(i-1)}(x)) = \sum_{j=0}^{i} d_{i,j}(u) h_j(x),$$

where

$$d_{i,i}(u) = \frac{\sigma\sqrt{i}}{2\sqrt{\alpha}}(c_{i-1,i-1}(u) - c_{i-1,i-1}(0)),$$

and for $0 \leq j \leq i-1$,

$$d_{j,i}(u) = \frac{\sigma\sqrt{j}}{2\sqrt{\alpha}}(c_{i-1,j-1}(u) - c_{i-1,j-1}(0)) + m(c_{i-1,j}(u) - c_{i-1,j}(0)) + \frac{\sqrt{j+1}\sigma}{\sqrt{\alpha}}(c_{i-1,j+1}(u) - c_{i-1,j+1}(0)).$$

By using the above notation, we have the following result.

Proposition 6. *The coefficients $c_{i,j}$ appearing in the representation* (54) *of* $g_u^{(i)}(x)$ *are recursively defined as follows: we have*

$$c_{0,0}(u) = \frac{1-\rho}{1-\rho u},$$

and for $i \geq 1$,

$$\begin{aligned} c_{i,0}(u) &= \frac{d_{i,0}(u) - d_{i,0}(1)}{1-\rho u}, \\ c_{i,j}(u) &= \frac{\mu}{\alpha}\left(\frac{1}{u}-1\right)\frac{d_{i,j}(u) - d_{i,j}(u_j)}{\nu(u) - j} \quad 1 \leq j \leq i, \end{aligned}$$

where for $j \geq 1$, u_j and $\tilde{u}_j$ are the two real solutions to the quadratic equation $\nu(u) = j$, i.e.

$$\rho u^2 - \left(1 + \rho + \frac{j\alpha}{\mu}\right)u + 1 = 0$$

with $0 < u_j < 1 < \tilde{u}_j$.

Proof. As in the previous section, we first search for a solution to the equation

$$\frac{\sigma^2}{2}\frac{\partial^2 \xi_u^{(i)}}{\partial x^2} - \alpha(x-m)\frac{\partial \xi_u^{(i)}}{\partial x} + \alpha\nu(u)\xi_u^{(i)}(x) \\ = \mu\left(\frac{1}{u}-1\right)x(g_u^{(i-1)}(x) - g_0^{(i-1)}(x)).$$

Assuming that the function $\xi_u^{(i)}(x)$ is of the form

$$\xi_u^{(i)}(x) = \sum_{k=0}^{i} \delta_{i,j}(u) h_j(x),$$

we have, by using the fact that the functions $h_j(x)$ are eigenfunctions of the operator H associated with the eigenvalues $-\alpha j$ and that these functions are linearly independent, for $j = 0, \dots, i$,

$$\delta_{i,j} = \frac{\mu}{\alpha}\left(\frac{1}{u}-1\right)\frac{d_{j,k}(u)}{\nu(u)-j}.$$

It is easily checked that $\xi_0^{(i)}(x) \equiv 0$. We can then decompose $g_u^{(i)}(x)$ as as

$$g_u^{(i)}(x) = \psi_u^{(i)}(x) + \xi_u^{(i)}(x),$$

where the function $\psi_u^{(i)}(x)$ is solution to the equation

$$\frac{\sigma^2}{2}\frac{\partial^2 \psi_u^{(i)}}{\partial x^2} - \alpha(x-m)\frac{\partial \psi_u^{(i)}}{\partial x} + \nu(u)\psi_u^{(i)}(x) = \mu\left(\frac{1}{u}-1\right)\psi_0^{(i)}(x).$$

By using the same arguments as in the proof of Lemma 10, we can easily show that $\psi_0^{(i)}(x)$ has the form

$$\psi_0^{(i)}(x) = \sum_{j=0}^{i} c_j h_j(x),$$

where the coefficients $c_j \in \mathbb{C}$ for $j = 0, ..., i$. It follows that the function $g_u^{(i)}(x)$ is solution to the ordinary differential equation

$$\frac{\sigma^2}{2}\frac{\partial^2 g_u^{(i)}}{\partial x^2} - \alpha(x-m)\frac{\partial g_u^{(i)}}{\partial x} + \alpha\nu(u) g_u^{(i)}(x) = \mu\left(\frac{1}{u}-1\right)\sum_{j=0}^{i}(c_j + d_{i,j}(u))h_j(x).$$

By using the same arguments as in the proof of Proposition 5, we come up with the conclusion that $g_u^{(i)}(x)$ is of the form (54) with the coefficients $c_{i,j}(u)$ given by

$$c_{i,j}(u) = \frac{\mu}{\alpha}\left(\frac{1}{u}-1\right)\frac{c_j + d_{i,j}(u)}{\nu(u)-j}.$$

Since the function $g_u^{(i)}(x)$ has to be analytic in the open unit disk, we have for $j \geq 1$

$$c_j = -d_{i,j}(u_j)$$

In addition, since $g_1^{(i)}(x) \equiv 0$, we have $c_0 = -d_{i,j}(1)$. □

The normalizing constant $c(g)$ is chosen such that

$$\int_{-\infty}^{\infty} g_1(x)n(x)dx = 1.$$

From the above analysis, we see that $g_1^{(i)}(x) \equiv 0$ for $i \geq 1$ so that $c(g) = 1-\rho$.

5.2.4. *Radius of convergence.* In this section, we examine under which conditions the expansion (36) defines an element of $L^2(\mathbb{R}, n, \ell^2(\rho))$. In a first step, note that as a consequence of Proposition 6, the function $g_u^{(i)}(x)$ can be written as

$$g_u^{(i)}(x) = x\Theta\left(g_u^{(i-1)}(x)\right) = \Theta\left(xg_u^{(i-1)}(x)\right)$$

where the operator Θ is defined in $L^2(\mathbb{R}, n; \ell^2(1/\rho))$ as follows: for an element $f \in L^2(\mathbb{R}, n; \ell^2(1/\rho))$ represented as

$$f_u(x) \stackrel{def}{=} \sum_{j=0}^{\infty} c_j(u)h_j(x),$$

the element $F = \Theta f$ is defined by

$$F_u(x) = \sum_{j=0}^{\infty}\mu\left(\frac{1}{u}-1\right)\frac{c_j(u)-c_j(u_j)}{\nu(u)-j}h_j(x),$$

where we set $u_0 = 1$ and $\tilde{u}_0 = 1/\rho$.

It is easily checked that for $j \geq 1$, $0 < u_j < 1 < 1/\sqrt{\rho} < \tilde{u}_j$. Moreover, the function $c_j(u)$ appearing in the expression of f_u is analytic in the disk $D_\rho = \{z : |z| < 1/\sqrt{\rho}\}$ and continuous in the closed disk $\overline{D}_\rho = \{z : |z| \leq 1/\sqrt{\rho}\}$ for $j \geq 0$. Similarly, for all $j \geq 0$, the function

$$u \to \frac{\mu}{\alpha}\left(\frac{1}{u}-1\right)\frac{c_j(u)-c_j(u_j)}{\nu(u)-j}$$

is analytic in D_ρ and continuous in $\overline{D}_\rho$. With the above notation, we can state the main result of this section.

Proposition 7. *The operator* Θ *is bounded and if* $\varepsilon < 1/(m\|\Theta\|)$, *where* $\|\Theta\|$ *denotes the norm of* Θ, *then the sequence defined by Equation* (36) *(or equivalently by Equation* (37)*) is in* $L^2(\mathbb{R}, n; \ell^2(\rho))$.

Proof. Let $f \in L^2(\mathbb{R}, n; \ell^2(1/\rho))$ be defined by the function

$$f_u(x) = \sum_{j=0}^{\infty} c_j(u) h_j(x).$$

For $(c_j) \in \ell^2(1/\rho)$ associated with the generating function

$$c(u) = \sum_{j=0}^{\infty} c_j u^j,$$

we have

$$\|c\|_{1/\rho}^2 = \frac{1}{2\pi} \int_0^{2\pi} \left| c\left(\frac{1}{\sqrt{\rho}} e^{i\theta}\right) \right|^2 d\theta.$$

Let us moreover define the sequence $(\tilde{c}_j)$ associated with the generating function

$$\tilde{c}(u) = \frac{\mu}{\alpha}\left(\frac{1}{u} - 1\right) \frac{c(u) - c(u_j)}{\nu(u) - j}.$$

Assume first that $j \geq 1$, then

$$\tilde{c}(u) = \frac{1}{\rho}(1-u)\frac{1}{u - \tilde{u}_j}\frac{c(u) - c(u_j)}{u - u_j}.$$

and then

$$\|\tilde{c}\|_{1/\rho}^2 \leq \frac{1}{\rho^2}\left(1 + \frac{1}{\sqrt{\rho}}\right)^2 \frac{1}{(\tilde{u}_j - 1/\sqrt{\rho})^2} \frac{1}{2\pi} \int_0^{2\pi} \left| \frac{c(e^{i\theta}/\sqrt{\rho}) - c(u_j)}{e^{i\theta}/\sqrt{\rho} - u_j} \right|^2 d\theta.$$

Simple manipulations show that

$$\frac{1}{2\pi} \int_0^{2\pi} \left| \frac{c(e^{i\theta}/\sqrt{\rho}) - c(u_j)}{e^{i\theta}/\sqrt{\rho} - u_j} \right|^2 d\theta \leq \|c\|_\rho^2 \frac{1}{(1/\sqrt{\rho} - u_j)^2} \left(1 + \sqrt{\frac{1}{1 - \rho u_j^2}}\right)^2.$$

It follows that $\|\tilde{c}\|_{1/\rho} \leq \kappa_j \|c\|_{1/\rho}$, where

$$\begin{aligned}\kappa_j &= \frac{1}{\rho}\left(1 + \frac{1}{\sqrt{\rho}}\right) \frac{1}{(\tilde{u}_j - 1/\sqrt{\rho})(1/\sqrt{\rho} - u_j)} \left(1 + \sqrt{\frac{1}{1 - \rho u_j^2}}\right) \\ &= \frac{1 + \sqrt{\rho}}{(1 - \sqrt{\rho})^2 + \frac{n\alpha}{\mu}} \left(1 + \sqrt{\frac{1}{1 - \rho u_j^2}}\right).\end{aligned}$$

It is easily checked that the sequence (κ_j) for $n \geq 1$ is decreasing.

When $j = 0$, we define

$$\tilde{c}(u) = \frac{\mu}{\alpha}\left(\frac{1}{u} - 1\right) \frac{c(u) - c(1)}{\nu(u)} = \frac{c(u) - c(1)}{1 - \rho u}.$$

It is then easily checked that $\|\tilde{c}\|_{1/\rho} \leq \kappa_0 \|c\|_{1/\rho}$, where

$$\kappa_0 = \frac{1}{1 - \sqrt{\rho}} \left(1 + \sqrt{\frac{1}{1 - \rho}}\right).$$

Define $\kappa = \max\{\kappa_0, \kappa_1\}$. The above computations show that for all $f \in \ell^2(1/\rho)$, $\|\Theta f\| \leq \kappa \|f\|$. It follows that the operator Θ is bounded; its norm is denoted by $\|\Theta\| \stackrel{\text{def.}}{=} \inf\{c > 0 : \forall f \in \mathcal{H},\ \|\Theta f\| \leq c\|f\|\}$. The above computations show that

$$\|\Theta\| \leq \frac{1 + \sqrt{\rho}}{(1 - \sqrt{\rho})^2} \left(1 + \sqrt{\frac{1}{1 - \rho}}\right). \tag{56}$$

From the above computations, we deduce that

$$\|g^{(i)}\| \leq \|\Theta\|^i \|c^{(0)*i}\|$$

where the sequence $c^{(0)*i}$ is associated with the function

$$\frac{1-\rho}{1-\rho u} x^i.$$

Straightforward computations show that

$$\|c^{(0)*i}\|^2 = \left(\frac{\sigma}{2\sqrt{\alpha}}\right)^{2i} H_{2i}\left(\frac{\sqrt{\alpha}m}{\sigma}\right), \tag{57}$$

where $H_i(x)$ is the ith Hermite polynomial. Using the asymptotic estimate (48), we have

$$\|c^{(0)*i}\| \sim m^i$$

when $i \to \infty$. It follows that $\|c^{(i)}\| \leq a_i$ with $a_i \sim (\|\Theta\| m)^i$ as i tends to infinity. It follows that the sequence defined by the expansion (37) is convergent in $L^2(\mathbb{R}, n; \ell^2(\rho))$ if $\varepsilon \|\Theta\| m < 1$. □

5.3. **Proof of Theorem 1.** We assume that Conditions (26) and (29) are satisfied. By observing that $\mathbb{E}\left(u^{L(t)}\right) = (P, U)$, we have for $u \leq 1$

$$\left|\mathbb{E}\left(u^{L(t)}\right) - (g, U)\right| \leq \|P - g\| \|U\| = \frac{1}{1-\rho u}\|P - g\| \leq \frac{1}{1-\rho u} D(\varepsilon)$$

where U is the vector of $L^2(\mathbb{R}, n, \ell^2(\rho))$ with the jth component equal to u^j and $D(\varepsilon)$ is defined by Equation (34).

From the power series expansion of g, we have

$$\left|(g, U) - (g^{(0)}, U) - \varepsilon (g^{(1)}, U)\right| \leq \sum_{j=2}^{\infty} \varepsilon^j \|g^{(i)}\| \|U\| \leq \frac{1}{1-\rho u} \sum_{j=2}^{\infty} \kappa(j) \varepsilon^j,$$

with

$$\kappa(j) = \left(\frac{1+\sqrt{\rho}}{(1-\sqrt{\rho})^2}\left(1 + \sqrt{\frac{1}{1-\rho}}\right)\right)^j \left(\frac{\sigma}{2\sqrt{\alpha}}\right)^{2i} H_{2i}\left(\frac{\sqrt{\alpha}m}{\sigma}\right),$$

where we have used Equations (56) and (57). We clearly have

$$\begin{aligned}
(g^{(0)}, U) &= \int_{-\infty}^{\infty} g_u^{(0)}(x) n(x) dx = \frac{1-\rho}{1-\rho u} \\
(g^{(1)}, U) &= \int_{-\infty}^{\infty} g_u^{(1)}(x) n(x) dx = a(u) = -\frac{\rho(1-u)}{(1-\rho u)^2} m,
\end{aligned}$$

where $a(u)$ is defined by Equation (53). Theorem 1 then follows.

To complete the analysis, note that the generating function of L_ε, the stationary number of customers in an $M/M/1$ with input rate λ and service rate $\mu(1 - m\varepsilon)$, is for $|u| < 1$ and when $\varepsilon < (1-\rho)/m$

$$\mathbb{E}\left(u^{L_\varepsilon}\right) - (g^{(0)}, U) - (g^{(1)}, U)\varepsilon = \frac{\rho(1-u)m^2}{(1-\rho-m\varepsilon)(1-\rho)^2}\varepsilon^2$$

Hence, by gathering these relations and by taking $u = 1$, we obtain an uniform bound for the difference between $\mathbb{E}\left(u^{L(t)}\right)$ and $\mathbb{E}\left(u^{L_\varepsilon}\right)$ for $u \in [0,1]$.

$$\sup_{0\le u\le 1}\left|\mathbb{E}\left(u^{L(t)}\right) - \mathbb{E}\left(u^{L_\varepsilon}\right)\right| \le E_B \stackrel{\text{def.}}{=} \frac{1}{1-\rho}D(\varepsilon) + \frac{1}{1-\rho}\sum_{j=2}^{\infty}\kappa(j)\varepsilon^j + \frac{2\rho m^2}{(1-\rho-m\varepsilon)(1-\rho)^2}\varepsilon^2 \tag{58}$$

Below are some numerical experiences on the role of ε and σ on the bound E_B for $a = 1/2$, $b = 1$ $m = 1$, $\lambda = 7$, $\alpha = 1$ and $\mu = 10$. It is reasonably low for small values of ε, it seems to be quite sensitive on the values of the parameter σ as the figures below show.

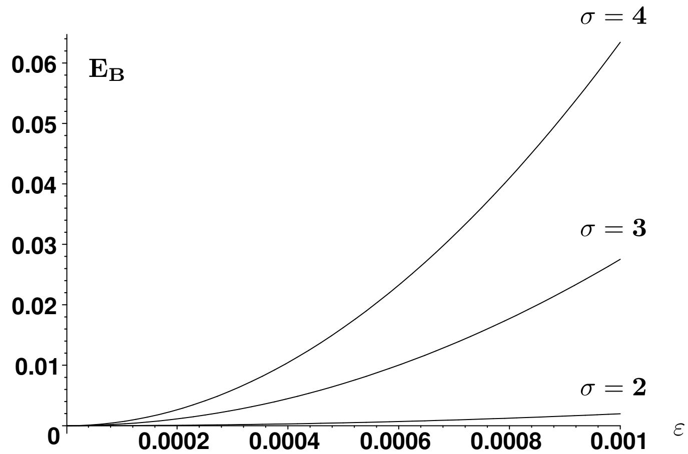

FIGURE 1. The bound $\varepsilon \to E_B$ of Relation (58) for $\sigma = 2, 3, 4$.

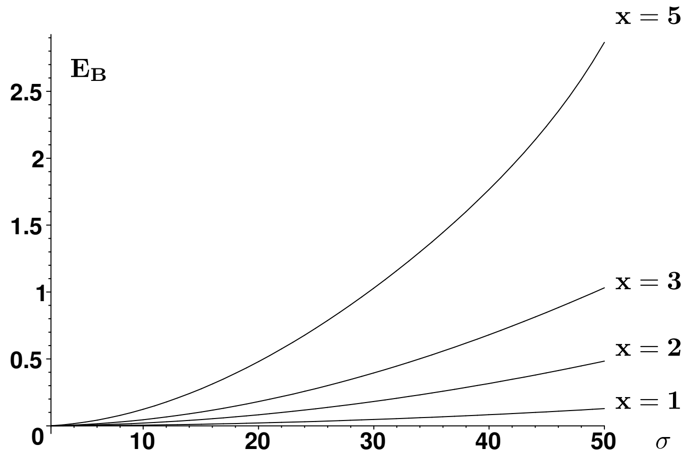

FIGURE 2. The bound $\sigma \to E_B$ of Relation (58) for $\varepsilon = x \cdot 10^{-4}$ with $x = 1, 2, 3, 5$.

6. Concluding remarks

The perturbation analysis performed in this paper has allowed us to prove the validity of the so-called reduced service rate approximation for the system considered under some specific conditions. Such an approximation is very important from a practical point of view because each type of traffic can be considered in isolation, the impact of unresponsive traffic on elastic traffic is only via the mean value.

The results presented in this paper have been obtained for a particular form of the perturbation function $\phi(x)$. Of course, the same approach could be extended to more complicated perturbation functions of the form $\Phi(x) = 1 - \varepsilon p(x)$ for some function $p(x)$. The key point consists of determining how the operator corresponding to the multiplication by $p(x)$ acts on the basic functions $h_j(x)$ for $j \geq 0$ (defined by Equation (11)). For computing explicit expressions, however, the main difficulty is in solving the differential equations satisfied by the coefficients of the expansion. When $p(x)$ is a polynomial, a particular solution to the equations similar to Equations (41) and (55) is obtained in the form of a polynomial times the function $\exp(-\alpha(x-m)^2/\sigma^2)$ and in that case, explicit computations can be carried out.

The perturbation function $\phi(x)$ defined by Equation (3) corresponds to the case when unresponsive flows have a peak bit rate ε much smaller than the transmission capacity of the link. The results of this paper show that the reduced service rate approximation yields in some conditions accurate results for the performance of elastic flows.

Appendix A. Proof of Proposition 3

To prove Proposition 3, we proceed as for the proof of Proposition 2. We first show that the operator $(H_1 + A_1)^{-1} W_1$ is bounded, where W_1 is the restriction to $\ell_1^2(\rho)$ of the operator W. (Note that the operator associated with the multiplication by $\Phi(x) - 1$ is not bounded in $L^2(\mathbb{R}, n)$.) For this purpose we use the fact that an element of $f = (f_j(x)) \in L^2(\mathbb{R}, n; \ell_1^2(\rho))$ can be decomposed as

$$f = \sum_{j=1}^{\infty} \sum_{k=0}^{\infty} c_{j,k} h_k(x) e_j.$$

and the squared norm is

$$\|f\|^2 = \sum_{j=1}^{\infty} \sum_{k=0}^{\infty} c_{j,k}^2 \rho^j.$$

By using the above decomposition, we have

$$Bf = \sum_{j=1}^{\infty} \sum_{k=0}^{\infty} (-\mu c_{j,k} + \lambda c_{j+1,k}) h_k(x) e_j$$

By using the recurrence relation satisfied by Hermite polynomials

$$xH_k(x) = \frac{1}{2} H_{k+1}(x) + kH_{k-1}(x), \tag{59}$$

we deduce that

$$xh_k(x) = \frac{\sigma}{\sqrt{2\alpha}} \left(\sqrt{\frac{k+1}{2}} h_{k+1}(x) + \sqrt{\frac{k}{2}} h_{k-1}(x) \right) + mh_k(x).$$

and then

$$W_1 f = \varepsilon \sum_{j=1}^{\infty}\sum_{k=0}^{\infty} C_{j,k} h_k(x) e_j,$$

where

$$C_{j,k} = m\left(\mu c_{j,k} - \lambda c_{j+1,k}\right) + \frac{\sigma}{\sqrt{2\alpha}}\left(\mathbb{1}_{\{k>0\}}\sqrt{\frac{k}{2}}\left(\mu c_{j,k-1} - \lambda c_{j+1,k-1}\right) + \sqrt{\frac{k+1}{2}}\left(\mu c_{j,k+1} - \lambda c_{j+1,k+1}\right)\right).$$

From the above relations, we have

$$(H_1 + A_1)^{-1} W_1 f = \varepsilon\left(\sum_{j=1}^{\infty} C_{j,0}(H_1+A_1)^{-1} h_0(x) e_j + \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} C_{j,k}(H_1+A_1)^{-1} h_k(x) e_j\right) \tag{60}$$

For the first term in the right hand side of Equation (60), we have

$$\left\|\sum_{j=1}^{\infty} C_{j,0}(H_1+A_1)^{-1} h_0(x) e_j\right\| \leq \left\|(H_1+A_1)^{-1}\right\| \left\|\sum_{j=1}^{\infty} C_{j,0} h_0(x) e_j\right\|$$
$$= \left\|(H_1+A_1)^{-1}\right\| \sqrt{\sum_{j=1}^{\infty} C_{j,0}^2 \rho^j} \leq \mu(1+\sqrt{\rho})\left(m + \frac{\sigma}{2\sqrt{\alpha}}\right)\left\|(H_1+A_1)^{-1}\right\| \|f\|,$$

since

$$\sum_{j=1}^{\infty} C_{j,0}^2 \rho^j \leq \left(m\sqrt{\sum_{j=1}^{\infty}(\mu c_{j,0} - \lambda c_{j+1,0})^2 \rho^j} + \frac{\sigma}{2\sqrt{\alpha}}\sqrt{\sum_{j=1}^{\infty}(\mu c_{j,1} - \lambda c_{j+1,1})^2 \rho^j}\right)^2$$

together with the inequalities

$$\sqrt{\sum_{j=1}^{\infty}(\mu c_{j,0} - \lambda c_{j+1,0})^2 \rho^j} \leq \left(\mu\sqrt{\sum_{j=1}^{\infty} c_{j,0}^2 \rho^j} + \sqrt{\lambda\mu}\sqrt{\sum_{j=1}^{\infty} c_{j+1,0}^2 \rho^{j+1}}\right) \leq \mu(1+\sqrt{\rho})\|f\|,$$

and

$$\sqrt{\sum_{j=1}^{\infty}(\mu c_{j,1} - \lambda c_{j+1,1})^2 \rho^j} \leq \mu(1+\sqrt{\rho})\|f\|.$$

For the second term in the right hand side of Equation (60), since the operator H is invertible on the space $\mathrm{span}(h_k, k \geq 1)$, we have

$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} C_{j,k}(H_1+A_1)^{-1} h_k(x) e_j = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} C_{j,k}(\mathbb{I} + H_1^{-1}A_1)^{-1} H_1^{-1} h_k(x) e_j$$

and then, by using the fact that $\|(\mathbb{I} + H_1^{-1}A_1)^{-1}\| \leq 1$, we obtain

$$\left\|\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} C_{j,k}(H_1+A_1)^{-1} h_k(x) e_j\right\| \leq \sqrt{\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\left(\frac{C_{j,k}}{k}\right)^2 \rho^j}.$$

From the inequality

$$\sqrt{\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\left(\frac{C_{j,k}}{k}\right)^2\rho^j}$$
$$\leq m\sqrt{\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\frac{\left(\mu c_{j,k}-\lambda c_{j+1,k}\right)^2}{k^2}\rho^j}+\frac{\sigma}{2\sqrt{\alpha}}\sqrt{\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\frac{\left(\mu c_{j,k-1}-\lambda c_{j+1,k-1}\right)^2}{k}\rho^j}$$
$$+\frac{\sigma}{2\sqrt{\alpha}}\sqrt{\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\frac{(k+1)\left(\mu c_{j,k+1}-\lambda c_{j+1,k+1}\right)^2}{k^2}\rho^j}$$

we deduce that

$$\sqrt{\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\left(\frac{C_{j,k}}{k}\right)^2\rho^j}\leq \mu(1+\sqrt{\rho})\left(m+\frac{\sigma}{\sqrt{\alpha}}\right)\|f\|$$

Finally, by using the fact that $\|(H_1+A_1)^{-1}\|\leq 1/(\mu(1-\sqrt{\rho})^2)$, we come up with the conclusion that

$$\|(H_1+A_1)^{-1}W_1 f\|\leq 2\varepsilon\frac{(1+\sqrt{\rho})}{(1-\sqrt{\rho})^2}\left(m+\frac{\sigma}{\sqrt{\alpha}}\right)\|f\|. \tag{61}$$

The operator $(H_1+A_1)^{-1}W_1$ is hence bounded. Under Condition (29), the norm of this operator is less than 1 and we can adapt word by word the proof of Proposition 2.

References

[1] E. Altman, K. Avrachenkov, and R. R. Núñez-Queija. Perturbation analysis for denumerable markov chains with application to queueing models. Advances in Applied Probability, 36(3):839–853, 2004.

[2] N. Antunes, C. Fricker, F. Guillemin, and P. Robert. Integration of streaming services and TCP data transmission in the Internet. Performance Evaluation, 62(1-4):263–277, October 2005.

[3] N. Antunes, C. Fricker, F. Guillemin, and P. Robert. Perturbation analysis of a variable $M/M/1$ queue: A probabilistic approach. Advances in Applied Probability, 38(1):263–283, 2006.

[4] A. A. Borovkov. Limit laws for queueing processes in multichannel systems. Sibirsk. Mat. Ž., 8:983–1004, 1967.

[5] F. Delcoigne, A. Proutière, and G. Régnié. Modelling integration of streaming and data traffic. In ITC specialist seminar on IP traffic, Würzburg, Germany, July 2002.

[6] D. Iglehart. Weak convergence of compound stochastic process. I. Stochastic Processes Appl., 1:11–31; corrigendum, ibid. 1 (1973), 185–186, 1973.

[7] S. Karlin and J.L. McGregor. Many server queueing processes with Poisson input and exponential service times. Pacific J. Math., 8:87–118, 1958.

[8] S. Karlin and J.L. McGregor. Random walks. Illinois J. Math, 3:66–81, 1959.

[9] S. Karlin and J.L. McGregor. Ehrenfest urn model. J. Appli. Probab., 2:352–376, 1965.

[10] N.N. Lebedev. Special functions and their applications. Dover Publications, 1972.

[11] L. Massoulié and J. Roberts. Bandwidth sharing: Objectives and algorithms. In INFOCOM'99. Eighteenth Annual Joint Conference of the IEEE Computer and Communications Societies, pages 1395–1403, 1999.

[12] S. Meyn and R. Tweedie. Markov chains and stochastic stability. Communications and control engineering series. Springer, 1993.

[13] R. Núñez-Queija. Sojourn times in a processor sharing queue with service interruptions. Queueing Systems, 34:351–386, 2000.

[14] R. Núñez-Queija. Sojourn times in non-homogeneous QBD processes with processor sharing. Stoch. Mod., pages 61–92, 2001.
[15] R. Núñez-Queija and O.J. Boxma. Analysis of a multi-server queueing model of ABR. J. Appl. Math. Stoch. An., 11:339–354, 1998.
[16] M. Reed and B. Simon. Methods of Modern Mathematical Physics, Vol. 2: Fourier Analysis, Self-Adjointness. Academic Press, New York, 1975.
[17] P. Robert. Stochastic Networks and Queues, volume 52 of Stochastic Modelling and Applied Probability Series. Springer, New-York, June 2003.
[18] G. Sansigre and G. Valent. A large family of semi-classical polynomials: the perturbed Chebyshev. J. Comput. Appl. Math., 57:271–281, 1995.
[19] E. Schaumburg. Estimation of Markov processes with Lévy type generators. Preprint available at http://www.kellogg.northwestern.edu/faculty/schaumburg/htm/ResearchPapers/mle.pd, 2005.

(Christine Fricker) INRIA Paris — Rocquencourt, Domaine de Voluceau, 78153 Le Chesnay, France
E-mail address: Christine.Fricker@inria.fr

(Fabrice Guillemin) France Telecom R&D, CORE/CPN, 22300 Lannion, France
E-mail address: Fabrice.Guillemin@orange-ftgroup.com

(Philippe Robert) INRIA Paris — Rocquencourt, Domaine de Voluceau, 78153 Le Chesnay, France
E-mail address: Philippe.Robert@inria.fr
URL: http://www-rocq.inria.fr/~robert