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On the compact formulation of the derivation of a transfer matrix with respect to another matrix

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Abstract: A new operator is considered, allowing compact formulae and proofs in the context of the derivation of a transfer matrix with respect to another matrix. The problem of the parametric sensitivity matrix calculation is chosen for illustration. It consists in deriving a Multiple Input Multiple Output transfer function with respect to a parametric matrix and is central in robust control theory. Efficient algorithms may be straightforwardly got from the compact analytic formulae using the operator introduced.

Key-words: sensitivity, derivative, robust control

(Résumé : tsvp)

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Une expression compacte pour la dérivation d'une fonction de transfert par une matrice

Résumé : Dans ce papier, un nouvel opérateur mathématique est considéré, permettant des expressions compactes dans le contexte de la dérivation d'une matrice de fonction de transfert par rapport à une matrice. Le problème de la sensibilité paramétrique sert d'illustration. Il consiste en la dérivation d'une fonction de transfert à plusieurs entrées et plusieurs sorties par rapport à une matrice de paramètres. Ce problème est central en commande robuste, notamment pour la recherche de réalisations efficaces vis-à-vis de leur implantation numérique.

Mots clés : sensibilité, dérivation, commande robuste

1 Motivation

One important questioning in control theory is robustness. A property associated to a given system will be said to be robust if it is still satisfied when the system is slightly modified. Different properties may be considered, such as stability or say a certain level of performance measured, e.g. thanks to system norms. The problem is crucial in the theory of feedback, because the systems considered are (physical or mathematical) models which are representing the process with some approximations and uncertainties [10, 2]. Moreover, the feedback controller itself may be considered as an uncertain system, due to the inevitable approximation coming from the implementation. In particular, the use of computers introduces Finite Word Length quantification of the controller parameters [3, 7].

Whatever the case, the computation of the parametric sensitivity of MIMO transfer function is of particular interest. The problem involves the calculus of a matrix with respect to (w.r.t.) another matrix. But, as far the authors know, it exists no special techniques or special properties to simplify the expressions induced.

For example, let $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$ and $D \in \mathbb{R}^{p\times m}$ be four matrices defining the MIMO transfer function H_1 [8] :

$$H_1 : \begin{array}{ccc} \mathbb{C} & \rightarrow & \mathbb{C}^{p\times q} \\ z & \mapsto & C(zI_n - A)^{-1}B + D \end{array} \tag{1}$$

The sensitivity measure (in the context of FWL implementation) used (e.g. Gevers and Li [3]) is

$$M \triangleq \left\| \frac{\partial H_1}{\partial A} \right\|_2^2 + \left\| \frac{\partial H_1}{\partial B} \right\|_2^2 + \left\| \frac{\partial H_1}{\partial C} \right\|_2^2 + \left\| \frac{\partial H_1}{\partial D} \right\|_2^2 \tag{2}$$

where $\|.\|_2$ is the transfer function L_2-norm.
The analytic expression of $\frac{\partial H_1}{\partial A}$, $\frac{\partial H_1}{\partial B}$, $\frac{\partial H_1}{\partial C}$ and $\frac{\partial H_1}{\partial D}$ are easy to formulate in the SISO[1] case (when $p = q = 1$, so $H_1(z) \in \mathbb{C}$), but these expressions are less obvious in the MIMO case.

After having recalled general definitions and classical properties on matrix and transfer function derivatives in section 2, section 3 introduces a new operator $\circledast$ to simplify derivative expressions in the MIMO case. Finally, a more complicated example is solved in section 4, before conclusion in section 5.

2 Definitions

Let's introduce some interesting definitions about matrix and transfer function derivatives.

Definition 1 (Scalar derivative w.r.t. a matrix) *Let $X \in \mathbb{R}^{m\times n}$ be a matrix and $f : \mathbb{R}^{m\times n} \rightarrow \mathbb{C}$ a scalar function of X, differentiable w.r.t. each element of X.*

[1] *S*ingle *I*nput *S*ingle *O*utput

The derivative of f w.r.t. X is defined as the matrix $\frac{\partial f}{\partial X} \in \mathbb{C}^{m\times n}$ such

$$\left(\frac{\partial f}{\partial X}\right)_{i,j} \triangleq \frac{\partial f}{\partial X_{i,j}} \tag{3}$$

where $X_{i,j}$ is the (i,j) element of X.
This derivative defines the sensitivity of f w.r.t. X.

This definition can be extended to functions with values in $\mathbb{C}^{p\times l}$ as follow :

Definition 2 (Derivative of a matrix w.r.t. a matrix) *Let $X \in \mathbb{R}^{m\times n}$ be a matrix and $f : \mathbb{R}^{m\times n} \to \mathbb{C}^{p\times l}$ a function of X, where each component of f is differentiable w.r.t. each element of X.*
The derivative of f with respect to X is a matrix of $\mathbb{C}^{mp\times nl}$ partitioned in $m\times n$ matrix blocks of $\mathbb{C}^{p\times l}$. Each $(i,j)^{th}$ block is defined by

$$\frac{\partial f}{\partial X_{i,j}} \in \mathbb{C}^{p\times l} \tag{4}$$

Then

$$\frac{\partial f}{\partial X} \triangleq \begin{pmatrix} \frac{\partial f}{\partial X_{1,1}} & \frac{\partial f}{\partial X_{1,2}} & \cdots & \frac{\partial f}{\partial X_{1,n}} \\ \frac{\partial f}{\partial X_{2,1}} & \frac{\partial f}{\partial X_{2,2}} & \cdots & \frac{\partial f}{\partial X_{2,n}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f}{\partial X_{m,1}} & \frac{\partial f}{\partial X_{m,2}} & \cdots & \frac{\partial f}{\partial X_{m,n}} \end{pmatrix} \tag{5}$$

which can also be written as :

$$\frac{\partial f}{\partial X} = \sum_{r=1}^{m}\sum_{s=1}^{n} E_{r,s}^{m,n} \otimes \frac{\partial f}{\partial X_{r,s}} \tag{6}$$

where the matrices $E_{r,s}^{n,m}$ of $\mathbb{R}^{n\times m}$ are the elementary matrices *defined by*

$$\left(E_{r,s}^{n,m}\right)_{i,j} \triangleq \delta_{r,i}\delta_{s,j} \tag{7}$$

and $\otimes$ is the Kronecker product.

Remark 1 When X is a row vector and $f(X)$ a column vector, $\frac{\partial f}{\partial X}$ is the jacobian matrix of f.
Other definitions of the derivative of a matrix w.r.t. a matrix are sometimes used : $\frac{\partial A}{\partial X} \triangleq \frac{\partial \mathrm{Vec}(A)}{\partial \mathrm{Vec}(X)}$ or $\frac{\partial A}{\partial X} \triangleq \frac{\partial \mathrm{Vec}(A)}{\partial \mathrm{Vec}(X)^\top}$, in order to get a jacobian matrix. It exempts to consider blocks of matrices $\frac{\partial f}{\partial X_{i,j}}$. However some useful propositions (like proposition 1 and theorem 1) are more easy according to definition 2.

Remark 2 $\frac{\partial X}{\partial X}$, the derivative of $X \in \mathbb{R}^{p\times q}$ w.r.t. itself, is a constant matrix of $\mathbb{R}^{p^2\times q^2}$ such that :

$$\begin{aligned}\frac{\partial X}{\partial X} &= \sum_{r=1}^{p}\sum_{s=1}^{l}\left(E_{r,s}^{p,l}\otimes E_{r,s}^{p,l}\right) && (8)\\ &= \sum_{r=1}^{p}\sum_{s=1}^{l} E_{(r-1)p+r,(s-1)l+s}^{p^2,l^2} && (9)\end{aligned}$$

The transfer function sensitivity (SISO and MIMO case) is defined below

Definition 3 (Transfer function sensitivity) *Let $X \in \mathbb{R}^{m\times n}$ be a matrix and $H : \mathbb{C} \to \mathbb{C}^{p\times q}$ the transfer function which associate $H(z)$ to all $z \in \mathbb{C}$. $H(z)$ is supposed to be parametrized by X and to be differentiable w.r.t. each element of X whatever $z \in \mathbb{C}$. Finally, the sensitivity function of H w.r.t. X is the transfer function denoted by $\frac{\partial H}{\partial X}$, such that*

$$\frac{\partial H}{\partial X} : \begin{array}{ccc} \mathbb{C} & \to & \mathbb{C}^{pm\times qn} \\ z & \mapsto & \frac{\partial (H(z))}{\partial X}\end{array} \tag{10}$$

Remark 3 The subsequent properties on matrix derivatives also hold for transfer function, without any modifications.

The General Leibniz product rule for derivative of a product of matrices w.r.t. a matrix is the main property used when dealing with matrix derivative.

Proposition 1 *Let's consider $X \in \mathbb{R}^{k\times l}$, $Y \in \mathbb{R}^{l\times m}$ and $Z \in \mathbb{R}^{p\times l}$. The derivative of the product XY with respect to Z is*

$$\frac{\partial (XY)}{\partial Z} = \frac{\partial X}{\partial Z}(I_l \otimes Y) + (I_p \otimes X)\frac{\partial Y}{\partial Z} \tag{11}$$

Proof:
The proof can be found in [1]. ■

Remark 4 Proposition 1 applied to $\frac{\partial (YY^{-1})}{\partial Z}$ leads to

$$\frac{\partial \left(Y^{-1}\right)}{\partial Z} = -\left(I_p \otimes Y^{-1}\right)\frac{\partial Y}{\partial Z}\left(I_l \otimes Y^{-1}\right) \tag{12}$$

When considering the initial example H_1 (eq. (1)), it comes then ($\forall z \in \mathbb{C}$)

$$\begin{aligned}\frac{\partial H_1}{\partial A}(z) &= \left(I_n \otimes C(zI_n - A)^{-1}\right)\frac{\partial A}{\partial A}\left(I_n \otimes (zI_n - A)^{-1}B\right) && (13)\\ \frac{\partial H_1}{\partial B}(z) &= \left(I_n \otimes C(zI_n - A)^{-1}\right)\frac{\partial B}{\partial B} && (14)\\ \frac{\partial H_1}{\partial C}(z) &= \frac{\partial C}{\partial C}\left(I_n \otimes (zI_n - A)^{-1}B\right) && (15)\\ \frac{\partial H_1}{\partial D}(z) &= \frac{\partial D}{\partial D} && (16)\end{aligned}$$

The formulae given in equations (13) to (16) suffer from some drawbacks. First, they are expressed with constant matrices $\frac{\partial A}{\partial A}$, $\frac{\partial B}{\partial B}$, $\frac{\partial C}{\partial C}$ and $\frac{\partial D}{\partial D}$ that do not depends on A, B, C or D (they only depend on their dimensions). Secondly, their manipulations may be rather tedious when dealing with such complicate expressions as in section 4. Lastly, a more compact form is possible, as proposed in section 3.

3 A new operator for compact derivatives formulae

In order to simplify the expressions, the new operator $\circledast$ is proposed. Three propositions show how to use this operator in classical linear derivative problem.

Definition 4 *Let G and H be two transfer functions in $\mathbb{C}^{m\times p}$ and $\mathbb{C}^{l\times n}$.*
The operator $\circledast$ is defined by

$$G \circledast H \triangleq \mathrm{Vec}(G).\left[\mathrm{Vec}(H^\top)\right]^\top \tag{17}$$

where Vec *is the usual operator that vectorize a matrix. It corresponds to the product of each element of G with each element of H, in a particular order.*

This operator is used to state the main proposition of this paper, which encompass and simplify the Leibnitz rule of proposition 1 :

Theorem 1 *Let X be a matrix in $\mathbb{R}^{p\times l}$ and G, H be two transfer functions with values respectively in $\mathbb{C}^{m\times p}$ and $\mathbb{C}^{l\times n}$. G and H are supposed to be independent w.r.t. X. Then*

$$\frac{\partial(GXH)}{\partial X} = G \circledast H \tag{18}$$

$$\frac{\partial(GX^{-1}H)}{\partial X} = (GX^{-1}) \circledast (X^{-1}H) \tag{19}$$

Proof:
From proposition 1 and equation (9),

$$\frac{\partial(GXH)}{\partial X} = \sum_{r,s} (I_p \otimes G)\, E^{p^2,l^2}_{(r-1)p+r,(s-1)l+s}\, (I_l \otimes H) \tag{20}$$

Considering relation $\left(AE^{p^2,l^2}_{i,j}B\right) = A_{\bullet,i}B_{j,\bullet}$ where $A_{\bullet,i}$ denotes the i^{th} column of A and $B_{j,\bullet}$ the j^{th} row of B, then

$$\frac{\partial(GXH)}{\partial X} = \sum_{r,s} (I_p \otimes G)_{\bullet,(r-1)p+r}\, (I_l \otimes H)_{(s-1)l+s,\bullet} \tag{21}$$

The term $(I_p \otimes G)_{\bullet,(r-1)p+r}$ corresponds to

$$(I_p \otimes G)^\top_{\bullet,(r-1)p+r} = \Big(\begin{array}{ccccc} \boxed{\quad 0 \quad} & \dots & \boxed{\quad G_{\bullet,r} \quad} & \boxed{\quad 0 \quad} & \dots \end{array}\Big) \tag{22}$$

$$\uparrow r^{\text{th}} \text{ block}$$

so it is possible to write

$$(I_p \otimes G)_{\bullet,(r-1)p+r} (I_l \otimes H)_{(s-1)l+s,\bullet} = E_{r,s}^{p,l} \otimes (G_{\bullet,r}.H_{s,\bullet}) \tag{23}$$

$$= \begin{pmatrix} \boxed{0} & & \\ & \boxed{G_{\bullet,r}.H_{s,\bullet}} & \\ & & \boxed{0} \end{pmatrix} \tag{24}$$

$(r,s)^{\text{th}}$ block (size $m \times n$)

Finally, $(G_{\bullet,r}.H_{s,\bullet})_{i,j} = G_{i,r}.H_{s,j}$; so

$$\left(\frac{\partial(GXH)}{\partial X}\right)_{(r-1)m+i,(s-1)n+j} = G_{i,r}.H_{s,j} \tag{25}$$

and then,

$$\frac{\partial(GXH)}{\partial X} = \mathrm{Vec}(G).\left[\mathrm{Vec}(H^\top)\right]^\top \tag{26}$$

■

Using the previous result leads equations (13) to (16) to the simplified expressions :

$$\frac{\partial H_1}{\partial A} = \left(C(zI_n - A)^{-1}\right) \circledast \left((zI_n - A)^{-1}B\right) \tag{27}$$

$$\frac{\partial H_1}{\partial B} = \left(C(zI_n - A)^{-1}\right) \circledast I_n \tag{28}$$

$$\frac{\partial H_1}{\partial C} = I_n \circledast \left((zI_n - A)^{-1}B\right) \tag{29}$$

$$\frac{\partial H_1}{\partial D} = I_n \circledast I_n \tag{30}$$

Practically, the additional following properties are useful to simplify the derivative task and get compacter expressions. The proofs are trivial.

Proposition 2

$$(I_p \otimes G)(X \circledast Y)(I_l \otimes H) = (GX) \circledast (YH) \tag{31}$$

Proposition 3

$$\begin{pmatrix} A \circledast C & A \circledast D \\ B \circledast C & B \circledast D \end{pmatrix} = \begin{pmatrix} A & B \end{pmatrix} \circledast \begin{pmatrix} C \\ D \end{pmatrix} \tag{32}$$

4 Application to the case of closed-loop transfer function

In this section the problem consisting in deriving the redheffer product [9], and its specialization the lower linear fractional transformation of two transfer functions, is studied. The problem has an important practical interest in the context of robust control theory [10], when considering the model uncertainties of the process or even of the controller in the sense of FWL implementation [3].

Let's consider a plant $\mathcal{P}$ controlled by a controller $\mathcal{C}$ in a standard form [10] (see fig. 1). $W(k) \in \mathbb{R}^{p_1}$ and $Z(k) \in \mathbb{R}^{m_1}$ are the exogenous inputs and outputs (to control), whereas $U(k) \in \mathbb{R}^{p_2}$ and $Y(k) \in \mathbb{R}^{m_2}$ are the control and measure signals.

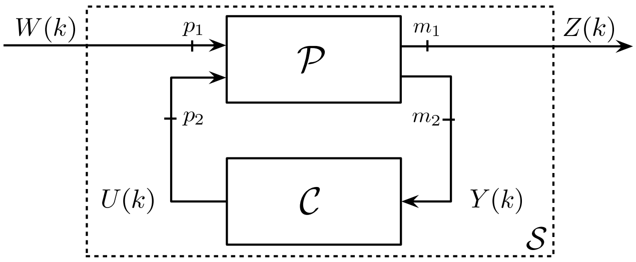

Figure 1: Closed-loop system considered

The plant $\mathcal{P}$ is defined by the recurrent relation

$$\left\{ \begin{array}{rcllll} X_\mathcal{P}(k+1) & = & AX_\mathcal{P}(k) & + & B_1W(k) & + & B_2U(k) \\ Z(k) & = & C_1X_\mathcal{P}(k) & + & D_{11}W(k) & + & D_{12}U(k) \\ Y(k) & = & C_2X_\mathcal{P}(k) & + & D_{21}W(k) \end{array} \right. \tag{33}$$

where $A \in \mathbb{R}^{n_\mathcal{P} \times n_\mathcal{P}}$, $B_1 \in \mathbb{R}^{n_\mathcal{P} \times p_1}$, $B_2 \in \mathbb{R}^{n_\mathcal{P} \times p_2}$, $C_1 \in \mathbb{R}^{m_1 \times n_\mathcal{P}}$, $C_2 \in \mathbb{R}^{m_2 \times n_\mathcal{P}}$, $D_{11} \in \mathbb{R}^{m_1 \times p_1}$, $D_{12} \in \mathbb{R}^{m_1 \times p_2}$, $D_{21} \in \mathbb{R}^{m_2 \times p_1}$. Note that the D_{22} term is null.
The controller is defined by

$$\left\{ \begin{array}{rcl} X(k+1) & = & A_Z X(k) + B_Z Y(k) \\ U(k) & = & C_Z X(k) + D_Z Y(k) \end{array} \right. \tag{34}$$

with $A_Z \in \mathbb{R}^{n \times n}$, $B_Z \in \mathbb{R}^{n \times m_2}$, $C_Z \in \mathbb{R}^{p_2 \times n}$ and $D_Z \in \mathbb{R}^{p_2 \times m_2}$.
The transfer function of the closed-loop system $\mathcal{S}$ is then (lower linear fractional transformation)

$$\bar{H} : z \rightarrow \bar{C} \left(zI_{n_\mathcal{P}+n} - \bar{A} \right)^{-1} \bar{B} + \bar{D} \tag{35}$$

with $\bar{A} \in \mathbb{R}^{n_{\mathcal{P}}+n \times n_{\mathcal{P}}+n}$, $\bar{B} \in \mathbb{R}^{n_{\mathcal{P}}+n \times p_1}$, $\bar{C} \in \mathbb{R}^{m_1 \times n_{\mathcal{P}}+n}$ and $\bar{D} \in \mathbb{R}^{m_1 \times p_1}$ and

$$\begin{aligned} \bar{A} &= \begin{pmatrix} A + B_2 D_Z C_2 & B_2 C_Z \\ B_Z C_2 & A_Z \end{pmatrix} & \bar{B} &= \begin{pmatrix} B_1 + B_2 D_Z D_{21} \\ B_Z D_{21} \end{pmatrix} \\ \bar{C} &= \begin{pmatrix} C_1 + D_{12} D_Z C_2 & D_{12} C_Z \end{pmatrix} & \bar{D} &= D_{11} + D_{12} D_Z D_{21} \end{aligned} \tag{36}$$

Last point, the matrices A_Z, B_Z, C_Z and D_Z depends on matrices J, K, L, M, N, P, Q, R and S ($J \in \mathbb{R}^{l \times l}$, $K \in \mathbb{R}^{n \times l}$, $L \in \mathbb{R}^{p_2 \times l}$, $M \in \mathbb{R}^{l \times n}$, $N \in \mathbb{R}^{l \times m_2}$, $P \in \mathbb{R}^{n \times n}$, $Q \in \mathbb{R}^{n \times m_2}$, $R \in \mathbb{R}^{p_2 \times n}$, $S \in \mathbb{R}^{p_2 \times m_2}$) that contain the exact coefficients for the realization of $\mathcal{C}$ [5], with

$$A_Z = K J^{-1} M + P, \qquad B_Z = K J^{-1} N + Q, \tag{37}$$

$$C_Z = L J^{-1} M + R, \qquad D_Z = L J^{-1} N + S. \tag{38}$$

Those parameters are grouped in a single matrix $Z \in \mathbb{R}^{l+n+p_2 \times l+n+m_2}$ as

$$Z \triangleq \begin{pmatrix} -J & M & N \\ K & P & Q \\ L & R & S \end{pmatrix} \tag{39}$$

In order to evaluate the sensitivity of $\bar{H}$ w.r.t Z (this sensitivity is linked to the good performance of the global scheme in the FWL context [3, 4, 6]), the problem is then to compute $\frac{\partial \bar{H}}{\partial Z}$ (or equivalently $\frac{\partial \bar{H}}{\partial J}$, $\frac{\partial \bar{H}}{\partial K}$, $\frac{\partial \bar{H}}{\partial L}$, ..., $\frac{\partial \bar{H}}{\partial S}$).

Proposition 4 *The sensitivity of $\bar{H}$ with respect to Z is given by*

$$\frac{\partial \bar{H}}{\partial Z} = \left[\bar{C} \left(zI - \bar{A} \right)^{-1} \bar{M}_1 + \bar{M}_2 \right] \circledast \left[\bar{N}_1 \left(zI - \bar{A} \right)^{-1} \bar{B} + \bar{N}_2 \right] \tag{40}$$

with

$$\bar{M}_1 = \begin{pmatrix} B_2 L J^{-1} & 0 & B_2 \\ K J^{-1} & I_n & 0 \end{pmatrix} \qquad \bar{N}_1 = \begin{pmatrix} J^{-1} N C_2 & J^{-1} M \\ 0 & I_n \\ C_2 & 0 \end{pmatrix} \tag{41}$$

$$\bar{M}_2 = \begin{pmatrix} D_{12} L J^{-1} & 0 & D_{12} \end{pmatrix} \qquad \bar{N}_2 = \begin{pmatrix} J^{-1} N D_{21} \\ 0 \\ D_{21} \end{pmatrix} \tag{42}$$

Proof:
Proposition 1 on equation (35) gives

$$\begin{aligned} \frac{\partial \bar{H}}{\partial Z} = \; & \left(I_{l+n+p_2} \otimes \bar{C}(zI - \bar{A})^{-1} \right) \frac{\partial \bar{A}}{\partial Z} \left(I_{l+n+m_2} \otimes (zI - \bar{A})^{-1} \bar{B} \right) \\ & + \left(I_{l+n+p_2} \otimes \bar{C}(zI - \bar{A})^{-1} \right) \frac{\partial \bar{B}}{\partial Z} \\ & + \frac{\partial \bar{C}}{\partial Z} \left(I_{l+n+m_2} \otimes (zI - \bar{A})^{-1} \bar{B} \right) + \frac{\partial \bar{D}}{\partial Z} \end{aligned} \tag{43}$$

Then, let's denote $\Theta = \begin{pmatrix} D_Z & C_Z \\ B_Z & A_Z \end{pmatrix}$, $\bar{A}$ can be reformulate by

$$\bar{A} = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} B_2 & 0 \\ 0 & I_n \end{pmatrix} \Theta \begin{pmatrix} C_2 & 0 \\ 0 & I_n \end{pmatrix} \tag{44}$$

So, with theorem 1

$$\frac{\partial \bar{A}}{\partial Z} = \left(I_{l+n+p_2} \otimes \begin{pmatrix} B_2 & 0 \\ 0 & I_n \end{pmatrix} \right) \frac{\partial \Theta}{\partial Z} \left(I_{l+n+m_2} \otimes \begin{pmatrix} C_2 & 0 \\ 0 & I_n \end{pmatrix} \right) \tag{45}$$

By similar process, it is obvious that

$$\frac{\partial \bar{B}}{\partial Z} = \left(I_{l+n+p_2} \otimes \begin{pmatrix} B_2 & 0 \\ 0 & I_n \end{pmatrix} \right) \frac{\partial \Theta}{\partial Z} \left(I_{l+n+m_2} \otimes \begin{pmatrix} D_{21} \\ 0 \end{pmatrix} \right) \tag{46}$$

$$\frac{\partial \bar{C}}{\partial Z} = \left(I_{l+n+p_2} \otimes \begin{pmatrix} D_{12} & 0 \end{pmatrix} \right) \frac{\partial \Theta}{\partial Z} \left(I_{l+n+m_2} \otimes \begin{pmatrix} C_2 & 0 \\ 0 & I_n \end{pmatrix} \right) \tag{47}$$

$$\frac{\partial \bar{D}}{\partial Z} = \left(I_{l+n+p_2} \otimes \begin{pmatrix} D_{12} & 0 \end{pmatrix} \right) \frac{\partial \Theta}{\partial Z} \left(I_{l+n+m_2} \otimes \begin{pmatrix} D_{21} \\ 0 \end{pmatrix} \right) \tag{48}$$

Then, the derivatives of Θ w.r.t. J, K, L, M, N, P, Q, R and S are

$$\begin{aligned}
\tfrac{\partial \Theta}{\partial J} &= -\begin{pmatrix} LJ^{-1} \\ KJ^{-1} \end{pmatrix} \circledast \begin{pmatrix} J^{-1}N & J^{-1}M \end{pmatrix} & & \\
\tfrac{\partial \Theta}{\partial K} &= \begin{pmatrix} 0 \\ I \end{pmatrix} \circledast \begin{pmatrix} J^{-1}N & J^{-1}M \end{pmatrix} & \tfrac{\partial \Theta}{\partial P} &= \begin{pmatrix} 0 \\ I \end{pmatrix} \circledast \begin{pmatrix} 0 & I \end{pmatrix} \\
\tfrac{\partial \Theta}{\partial M} &= \begin{pmatrix} LJ^{-1} \\ KJ^{-1} \end{pmatrix} \circledast \begin{pmatrix} 0 & I \end{pmatrix} & \tfrac{\partial \Theta}{\partial N} &= \begin{pmatrix} LJ^{-1} \\ KJ^{-1} \end{pmatrix} \circledast \begin{pmatrix} I & 0 \end{pmatrix} \\
\tfrac{\partial \Theta}{\partial L} &= \begin{pmatrix} I \\ 0 \end{pmatrix} \circledast \begin{pmatrix} J^{-1}N & J^{-1}M \end{pmatrix} & \tfrac{\partial \Theta}{\partial Q} &= \begin{pmatrix} 0 \\ I \end{pmatrix} \circledast \begin{pmatrix} I & 0 \end{pmatrix} \\
\tfrac{\partial \Theta}{\partial R} &= \begin{pmatrix} I \\ 0 \end{pmatrix} \circledast \begin{pmatrix} 0 & I \end{pmatrix} & \tfrac{\partial \Theta}{\partial S} &= \begin{pmatrix} I \\ 0 \end{pmatrix} \circledast \begin{pmatrix} I & 0 \end{pmatrix}
\end{aligned} \tag{49}$$

So, with proposition 3 :

$$\frac{\partial \Theta}{\partial Z} = \begin{pmatrix} LJ^{-1} & 0 & I_{p_2} \\ KJ^{-1} & I_n & 0 \end{pmatrix} \circledast \begin{pmatrix} J^{-1}N & J^{-1}M \\ 0 & I_n \\ I_{m_2} & 0 \end{pmatrix} \tag{50}$$

With property 2, $\frac{\partial \bar{A}}{\partial Z}$, $\frac{\partial \bar{B}}{\partial Z}$, $\frac{\partial \bar{C}}{\partial Z}$ and $\frac{\partial \bar{D}}{\partial Z}$ are obtained and lead to equation (40). ■

5 Conclusion

In order to simplify the calculus of derivative of matrices with respect to another matrix, the operator $\circledast$ has been introduced, and some important properties associated have been stated.

Its interest has been illustrated by application in robust control. Not only the expressions are made more compact, but also the related numerical computations of transfer function sensitivity are made more tractable. This has led in practical to a successful application in the context of FWL implementation [6].

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