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Rapport de recherche

Facial non-repetitive edge-colouring of plane graphs

Frédéric Havet [*] , Stanislav Jendroľ [‡] , Roman Soták [‡], Erika Škrabuľáková [‡]

Abstract: A sequence $r_1, r_2, \ldots, r_{2n}$ such that $r_i = r_{n+i}$ for all $1 \leq i \leq n$, is called a *repetition*. A sequence S is called *non-repetitive* if no *block* (i.e. subsequence of consecutive terms of S) is a repetition. Let G be a graph whose edges are coloured. A trail is called *non-repetitive* if the sequence of colours of its edges is non-repetitive. If G is a plane graph, a *facial non-repetitive edge-colouring* of G is an edge-colouring such that any *facial trail* (i.e. trail of consecutive edges on the boundary walk of a face) is non-repetitive. We denote $\pi'_f(G)$ the minimum number of colours of a facial non-repetitive edge-colouring of G. In this paper, we show that $\pi'_f(G) \leq 8$ for any plane graph G. We also get better upper bounds for $\pi'_f(G)$ in the cases when G is a tree, a plane triangulation, a simple 3-connected plane graph, a hamiltonian plane graph, an outerplanar graph or a Halin graph. The bound 4 for trees is tight.

Key-words: non-repetive edge-colouring, Thue chromatic number, Thue chromatic index, planar graphs

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Arête-coloration facialement non-répétitive des graphes plans

Résumé : Une suite $r_1, r_2, \ldots, r_{2n}$ telle que $r_i = r_{n+i}$ pour tout $1 \le i \le n$, est appelée une *répétition*. Une suite S est *non-répétitive* si aucun *bloc* (i.e. sous-suite de termes consécutifs de S) n'est une répétition. Soit G un graphe dont les arêtes sont colorées. Un parcours est *non-répétitif* si la suite des couleurs de ses arêtes est non-répétitive. Si G est un graphe plan, une *arête-coloration facialement non-répétitive* est une arête-coloration telle que tout *parcours facial* (i.e. parcours d'arêtes consécutives sur la frontière d'une face) est non-répétitif. On note $\pi'_f(G)$ le nombre minimum de couleurs d'une arête-coloration facialement non-répétitive de G. Dans ce rapport, nous montrons que $\pi'_f(G) \le 8$ pour tout graphe plan G. Nous donnons également de meilleures bornes supérieures pour $\pi'_f(G)$ lorsque G est un arbre, un graphe plan triangulé, un graphe plan simple et 3-connexe, un graphe plan hamiltonien, une graphe planaire extérieur ou un graphe de Halin. En particulier, pour les arbres nous donnons la borne supérieure de 4 qui est la meilleure possible.

Mots-clés : arête-coloration non-répétitive, nombre Thue-chromatique, indice Thue-chromatique, graphes planaires

1   Introduction

A sequence $r_1, r_2, \ldots, r_{2n}$ such that $r_i = r_{n+i}$ for all $1 \leq i \leq n$, is called a *repetition*. A sequence S is called *non-repetitive* if no *block* (i.e. subsequence of consecutive terms of S) is a repetition. The celebrated Thue Theorem [18] states that arbitrarily long non-repetitive sequences can be formed using only three symbols. Alon et al. [1] introduced a natural generalization of Thue's sequences for edge-colouring of graphs. An edge-colouring φ of a graph G is *non-repetitive* if the sequence of colours on any path in G is non-repetitive. The *Thue chromatic index* of G, denoted $\pi'(G)$, is the minimum number of colours of a non-repetitive edge-colouring of G. Thue's sequences show that the Thue chromatic index of a path is at most 3. Actually, $\pi'(P_n) = 3$, for all $n \geq 5$, as it is easy to see that any sequence of length 4 on two symbols contains a repetition. An immediate corollary is that the Thue chromatic index of a cycle is at most 4. In [6], Currie showed that $\pi'(C_n) = 4$ only for $n \in \{5, 7, 9, 10, 14, 17\}$. For other values of $n > 5$, $\pi'(C_n) = 3$. Note that every non-repetitive edge-colouring is a proper edge-colouring. So for simple graphs $\pi'(G) \geq \chi'(G) \geq \Delta(G)$. Alon et al. [1] showed that $\pi'(G) \leq c\Delta(G)^2$ for some absolute constant c and conjecture that $\pi'(G) \leq c'\Delta(G)$ for some absolute constant c'. They proved this conjecture for several classes of graphs: the complete graph K_n on n vertices satisfies $\pi'(K_n) \leq 2n$ and for any tree T, $\pi'(T) \leq 4(\Delta(T) - 1)$.

In this paper, we study a variant of non-repetitive edge-colouring for plane graphs which is less constrained. We use the standard terminology according to Bondy and Murty [3] except for few notations defined throughout this article. However we recall some frequently used terms.

Let G be a plane graph. A *facial trail* in G is a trail made of consecutive edges of the boundary walk of some face. A *facial non-repetitive edge-colouring* of G is an edge-colouring of G such that any facial trail is non-repetitive. The *facial Thue chromatic index* of G, denoted $\pi'_f(G)$, is the minimum number of colours of a facial non-repetitive edge-colouring of G. Note that the facial Thue chromatic index depends on the embedding of the graph. For example, the graph depicted in Figure 1 has different facial Thue chromatic index depending on its embedding. With the embedding on the left, its facial Thue chromatic index is at least 4 as one of its face is a 5-cycle and with the embedding on the right its facial Thue chromatic index is 3 as shown by the depicted facial non-repetitive edge 3-colouring. Note in addition that in general a facial non-repetitive edge-colouring is not proper. See for example the facial non-repetitive 3-edge-colouring of the graph on the right in Figure 1.

In the following, all the graphs we will consider come along with an embedding in the plane. The *(geometric) dual* of the plane graph $G = (V, E, F)$ is the graph $G^* = (V^*, E^*, F^*)$ with the following property: for each face α of G there is a vertex α^* of G^*; two vertices α^* and β^* are joined by an edge e^* of G^* if and only if they are incident to the same edge e in G. The edge e^* of G^* is associated with the edge e in G and more generally the set of edges $X \subseteq E(G)$ is associated with $X^* = \{e^* : e \in X\}$. It is well known that the dual of G^* is G and that G^* is also a plane graph.
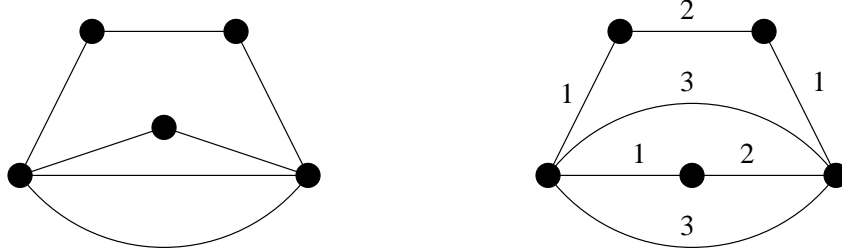
Figure 1: Two embeddings of the same graph with different facial Thue chromatic index

Let G be a 2-edge-connected plane graph. Then its dual G^* is loopless. So a proper edge-colouring of G^* exists. Let a proper edge-colouring of G^* be given, then assigning to each edge e of G the colour of its corresponding edge in G^* produces a facial non-repetitive edge-colouring of G. Hence the facial Thue chromatic index of G is at most the chromatic index of its dual.

Proposition 1. *Let G be a 2-edge-connected plane graph. Then $\pi'_f(G) \leq \chi'(G^*)$.*

All the upper bounds on $\chi'(G^*)$ applies to $\pi'_f(G)$. Shannon's Theorem [16] implies $\pi'_f(G) \leq \left\lceil \frac{3}{2}\Delta(G^*) \right\rceil$ and Vizing–Gupta Theorem [19, 9] implies that $\pi'_f(G) \leq \Delta(G^*) + \mu(G^*)$ where $\mu(G^*)$ is the maximum of the edges multiplicities in G^*. For some special classes of 2-edge-connected plane graphs $\chi'(G^*) = \pi'_f(G)$. This is in particular the case for plane triangulations.

Corollary 2. *Let G be a plane triangulation. Then $\pi'_f(G) = 3$.*

Proof. As G is a triangulation, its dual G^* is a 2-connected cubic plane graph. As shown by Tait in 1880 (see [17, 20]) the 3-edge-colourability of such graphs is equivalent to the Four Colour Theorem ([15, 20]). Hence, by Proposition 1, $\pi'_f(G) \leq \chi'(G^*) \leq 3$.

Now every face of G is a triangle and requires at least 3 colours in a facial non-repetitive edge-colouring. So $\pi'_f(G) = 3$.   □

However the chromatic index of a general (plane) graph G^* is not bounded as it is at least its maximum degree $\Delta(G^*)$. In the Section 4, we show that the facial Thue chromatic index of a plane graph is bounded by 8. We get tight upper bounds for trees (Section 3), simple 3-connected graphs and Halin graphs (Section 4). In the Section 5, we improve this upper bounds when G is hamiltonian and thus derive bounds for outerplanar graphs. In the final Section 6 we discuss our results and formulate an open problem.

2   Lemmas on Thue sequences

To be able to prove the theorems below we need some preliminary lemmas.

For a sequence of symbols $S = a_1, a_2, \ldots, a_n$ with $a_i \in \mathbb{A}$, for all $1 \leq k \leq l \leq n$, we denote by $S_{k,l}$ the block $a_k, a_{k+1}, \ldots, a_l$.

Lemma 3. *Let $A = a_1, a_2, \ldots, a_m$ be a non-repetitive sequence with $a_i \in \mathbb{A}$ for all $i = 1, 2, \ldots, m$. Let $B^i = b_1^i, b_2^i, \ldots, b_{m_i}^i$, $0 \leq i \leq r + 1$ be non-repetetive sequences with $b_j^i \in \mathbb{B}$ for all $i = 0, 1, \ldots, r+1$ and $j = 1, 2, \ldots, m_i$. If $\mathbb{A} \cap \mathbb{B} = \emptyset$ then $S = B^0, A_{1,n_1}, B^1, A_{n_1+1,n_2}, \ldots, B^r, A_{n_r+1,m}, B^{r+1}$ is a non-repetitive sequence.*

Proof. Suppose by way of contradiction that $R = r_1, \ldots, r_{2t}$ is a repetitive block of S, i.e. $r_k = r_{k+t}$ for every $k = 1, 2, \ldots, t$. R could be a subsequence neither of the sequence A nor of any sequence B^i because they are non-repetitive. Hence R consists of some blocks from A and some blocks from several consecutive B^i's, $i \in \{0, 1, \ldots, r+1\}$ too. If we delete from R all blocks of B^i, $i \in \{0, 1, \ldots, r+1\}$, then the obtained sequence R' is repetitive. This implies that A is repetitive, which is a contradiction. $\square$

The second lemma is an immediate implication of the existence of Thue sequences.

Lemma 4. *Every cycle admits a non-repetitive edge-colouring in $\{1, 2, 3, 4\}$ such that exactly one edge is coloured 4.*

Proof. Let C be a cycle. Colour an edge e of C with 4. $C \setminus e$ is a path and can be non-repetitively edge-coloured with $\{1, 2, 3\}$ using a Thue sequence. It is easy to see that such an edge-colouring of C is non-repetitive. $\square$

3 Trees

An *internal vertex of a tree* is a vertex, that is not a leaf. An *almost even tree* is a tree where every internal vertex, except at most one, is of even degree. An edge incident with a leaf is called *pending edge*.
Note that in a tree any facial trail is in fact a facial path.

Lemma 5. *Let S be a subdivision of a graph $K_{1,3}$. Then $\pi'_f(S) = 3$.*

Proof. In [4] Breštar et al. showed that if a tree T admits a non-repetitive 3-edge-colouring, then there exists a non-repetitive 3-edge-colouring of the tree T' obtained from T by subdividing each edge with 12 vertices. Using the above fact recursively on $K_{1,3}$, we obtain that, for each subdivision S of $K_{1,3}$, $\pi'(S) = 3$ and consequently $\pi'_f(S) = 3$. $\square$

Theorem 6. *Let T be an almost even tree. Then $\pi'_f(T) \leq 3$.*

Proof. We prove the result by induction on the number of leaves of T.

If T has two leaves then it is a path. Thus, the statement follows from the existence of arbitrary long non-repetitive sequence over three symbols (see [18]). If T has three leaves, then it is a subdivision of $K_{1,3}$ and by Lemma 5 we are done.

Suppose now that T has more than three leaves. Let R be the smallest subtree of T containing all the vertices of degree at least 3 in T. Then all the leaves of R have degree at least 3 in T. Moreover, because T is almost even, there is a leaf v of R such that $deg_T(v) \geq 4$. Now $T - v$ is a forest of $deg_T(v)$ trees among which at most one is not a path.

If $deg_T(v) \geq 5$, let P_0, P_1, P_2 and P_3 be four paths of $T - v$ which are consecutive in the clockwise order around v. Let T_1 be the tree obtained from T by extending each of the four P_i into a path Q_i of length equal to the maximal length of the P_j, $1 \leq j \leq 4$. Clearly, T_1 is a supertree of T having the same number of leaves. Let T_2 be the tree obtained from T_1 by removing Q_1 and Q_2. Then T_2 has two leaves less than T_1 and T. So, by the induction hypothesis, it admits a facial non-repetitive 3-edge-colouring. Colouring the edges of P_1 (resp. P_2) with the same pattern as the edges of P_3 (resp. P_0), we obtain a facial non-repetitive 3-edge-colouring of T.

If $deg_T(v) = 4$, the proof is very similar to the one when $deg_T(v) \geq 5$. We consider three paths P_1, P_2 and P_3 of $T - v$ which are consecutive in the clockwise order around v. Then P_0 is the path of T whose union with v and P_1 forms a maximal facial path of T. The rest of the proof is then identical. $\quad\square$

Theorem 7. *Let T be a tree and let K be a subtree of T which is almost even. Then there exists a facial non-repetitive 4-edge-colouring of T that uses only 3 colours on the edges of K.*

Proof. We proceed by induction on the number of internal vertices of odd degree in the tree T.

If there are less than two such vertices then T is almost even tree and we are done by Theorem 6.

Suppose now that T has at least two internal odd degree vertices. Consider a maximal (with respect to inclusion) almost even subtree S of T, which contains the tree K as a subgraph. The maximality of S implies that for every vertex $v \in V(S)$, $deg_S(v) \geq deg_T(v) - 1$ and that the leaves of S are also the leaves of T. In addition, exactly one internal vertex with odd degree in T has all its incident edges in S. By Theorem 6, there exists a facial non-repetitive edge-colouring φ of S that uses colours from the set $\{1, 2, 3\}$.

Let W be the boundary walk of the unique face of T beginning and finishing in the same fixed leaf of the tree S. Let $e_1, e_2, \ldots, e_k$ be the edges of $E(T) \setminus E(S)$ having one endvertex in $V(S)$, the indices being taken in the order of appearance on W. (Clearly k is the number of internal odd degree vertices of S minus 1.) By maximality of S, those edges are pairwise disjoint. For $1 \leq i \leq k$, let T_i be the tree of $T \setminus e_i$ not intersecting S. Each edge e_i appears twice in W. Let us denote $e_{i,1}$ and $e_{i,2}$ (resp. $e_{i,3}$ and $e_{i,4}$) the edges preceding and succeeding the first (resp. second) appearance of e_i along W. Clearly $e_{i,1}$ and $e_{i,4}$ are in $E(S)$ while $e_{i,2}$ and $e_{i,3}$ are not. Note that if the endvertex of e_i which is not in $V(S)$ has degree at most 2, then $e_{i,2} = e_{i,3}$. Moreover if e_i is a pending edge then $e_i = e_{i,2} = e_{i,3}$ (in this case T_i consists of one vertex only).

Let us now extend the colouring φ in the following way: For all $1 \leq i \leq k$, we set $\varphi(e_i) = 4$. We then colour the edges of T_i. Let K_i be the subtree of T_i

induced by the edges of T_i that are in a facial path containing e_i in T. Then K_i is either a path or a subdivision of $K_{1,3}$ and thus is almost even. Hence, by the induction hypothesis, there is a facial non-repetitive edge-colouring φ_i of T_i with colours in $\{1, 2, 3, 4\}$ such that no edge of K_i is assigned 4. Free to permute the colours, we may assume that $\varphi_i(e_{i,2}) \neq \varphi(e_{i-1,4})$ and $\varphi_i(e_{i,3}) \neq \varphi(e_{i+1,1})$ (the indices should be taken modulo k).

Let us show that the union of φ, $\varphi_1, \ldots, \varphi_k$ is a facial non-repetitive edge-colouring of T. Let P be a facial path of T. Then it contains at most two edges of $\{e_1, \ldots, e_k\}$.

If P has no edge in this set then it is entirely contained in either S or one of the T_i's. So it is not repetitive since $\varphi_1, \ldots, \varphi_k$ and φ are facially non-repetitive.

If P contains exactly one edge in $\{e_1, \ldots, e_k\}$, then this edge is the unique one of P coloured 4. Hence P is not repetitive.

If P contains two edges in $\{e_1, \ldots, e_k\}$, then they are consecutives: there exists $1 \leq i < k$ such that edges e_i and e_{i+1} are in $E(P)$. These edges are the only ones in P being coloured with 4. But since $\varphi(e_{i,4}) \neq \varphi_{i+1}(e_{i+1,2})$ and $\varphi(e_{i,3}) \neq \varphi_{i+1}(e_{i+1,1})$, P is not repetitive. $\qquad\square$

Proposition 8. *Let T be the tree depicted in Figure 2. Then $\pi'_f(T) = 4$.*
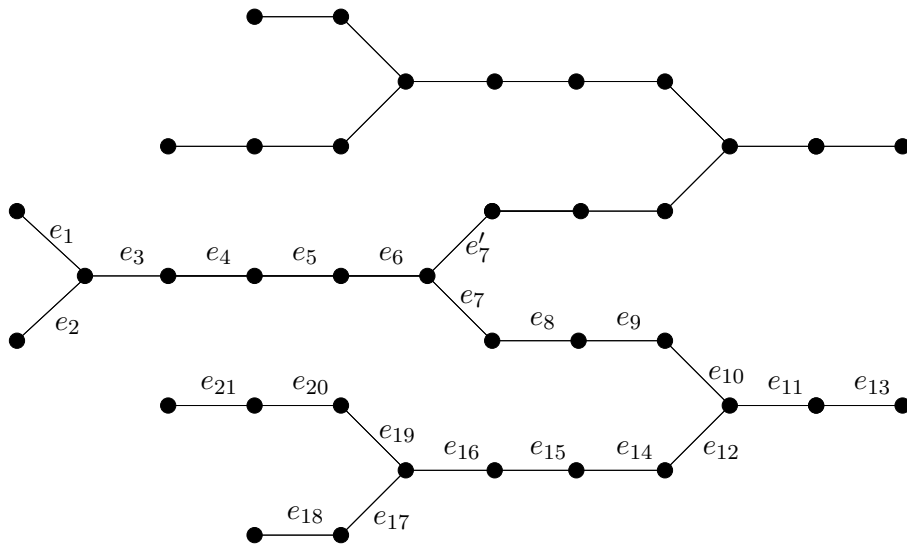

Figure 2: A tree with facial Thue chromatic index 4

Proof. Let us show that T has no facial non-repetitive edge-colouring with 3 colours. Suppose by way of contradiction that such an edge-colouring c exists with colour set $\{1, 2, 3\}$. Without loss of generality, we may assume that $c(e_i) = i$ for $1 \leq i \leq 3$. Moreover, by symmetry, we may assume that $c(e_4) = 2$. Then

$c(e_5) = 1$ otherwise (e_2, e_3, e_4, e_5) would be repetitive. We have $c(e_6) \neq 2$ otherwise one edge f of $\{e_7, e_7'\}$ is coloured 1 and (e_4, e_5, e_6, f) is repetitive. Thus $c(e_6) = 3$. Now $c(e_7') = 1$ otherwise $(e_1, e_3, e_4, e_5, e_6, e_7')$ would be repetitive. So $c(e_7) = 2$. Moreover because $(e_3, e_4, e_5, e_6, e_7, e_8)$ is not repetitive, $c(e_8) = 3$ and because (e_6, e_7, e_8, e_9) is not repetitive, $c(e_9) = 1$. With similar arguments, we deduce that $c(e_{10}) = 2$, $c(e_{11}) = 1$, $c(e_{12}) = 3$, $c(e_{13}) = 3$, $c(e_{14}) = 2$, $c(e_{15}) = 1$, $c(e_{16}) = 3$, $c(e_{17}) = 1$, $c(e_{18}) = 2$, $c(e_{19}) = 2$, $c(e_{20}) = 3$ and $c(e_{21}) = 1$. But then $(e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}, e_{12}, e_{14}, e_{15}, e_{16}, e_{19}, e_{20}, e_{21})$ is repetitive.   □

Denote by $\pi_f'(\mathcal{T})$ the maximum of $\pi_f'(T)$ over all trees T. Theorem 7 says that $\pi_f'(\mathcal{T}) \leq 4$. Together with Proposition 8, this gives

Theorem 9. $\pi_f'(\mathcal{T}) = 4$.

4   General upper bounds

The Lemma 10 appears in [3], p. 258. For the sake of completeness we provide a simple proof of it. We need the lemma in the following formulation.

Lemma 10. *Let G be a connected plane graph and T^* be a spanning tree of its dual G^*. Let T be a subgraph of G with edge set $E(T)$ associated to the edge set of T^*. Then $G \setminus E(T)$ is a tree.*

Proof. Clearly $|E(T)| = |E(T^*)| = |V(G^*)| - 1 = |F(G)| - 1$. With Euler's Formula $(|E(G)| + 2 = |V(G)| + |F(G)|)$ it implies $|E(G)| - |E(T)| = |V(G)| - 1$. It is enough to show that $G \setminus E(T)$ is connected. If not, then $E(T)$ contains an edge cut set which corresponds to a cycle in T^* - a contradiction.
Therefore $G \setminus E(T)$ is connected on $|V(G)| - 1$ edges and thus $G \setminus E(T)$ is a tree, see [3].   □

Theorem 11. *Let G be a connected plane graph. Then $\pi_f'(G) \leq 8$.*

Proof. Denote by T^* a spanning tree of G^*. Let T be the subgraph of G induced by the edges associated to the edges of T^*. By Lemma 10, $G \setminus E(T) = T_0$ is a tree and hence there is a facial non-repetitive edge-colouring $\varphi(T_0)$ using colours from $\{1, 2, 3, 4\}$.

Let us colour the edges of $E(T)$ with the four new colors $\{5, 6, 7, 8\}$. Let us root the tree T^* in some of its leaf f_0. For every face $f \neq f_0$ of G, we denote by e_f the edge shared by f and its parent in the rooted tree T^*. We colour the (uncoloured) edges of the faces of G successively from the root f_0 to the leaves of T^*. First, we assign to the unique uncoloured edge incident with f_0 the colour 5. Then at each step, we shall colour the uncoloured edges of a face f with colours in $\{5, 6, 7, 8\} \setminus c_f$, where c_f is a colour of the edge e_f received by e_f when colouring the parent of f. So far, all the coloured edges of f are those in $E(T_0)$ which are coloured by φ except e_f which is coloured with $c_f \in \{5, 6, 7, 8\}$. Let $e_1, e_2, \ldots, e_k$ be uncoloured edges incident with f in clokwise order around

f starting at e_f. (Note that no edge from the set $\{e_1, e_2, \ldots, e_k\}$ is a bridge of G. Otherwise if, say, e_i is a bridge then the associated edge e_i^* in G^* is a loop. But no loop can appear in T^*.) We colour the e_i's non-repetitively with three colours of $\{5, 6, 7, 8\} \setminus \{c_f\}$. Doing so, no trail on the boundary of f is a repetition because the colour of e_f appears only once around f and the colouring of the edges of $f \setminus e_f$ is non-repetitive by Lemma 3. □

Theorem 12. *Let G be a simple 3-connected plane graph.*
Then $\pi_f'(G) \leq 7$.

Proof. The dual graph G^* of the graph G is a simple 3-connected plane graph (see [3], p. 258). By Barnette's Theorem [2] G^* contains a spanning tree T^* of maximum degree $\Delta(T^*) \leq 3$. To prove that there is a facial non-repetitive colouring φ of G with 7 colours we proceed analogously as in the proof of Theorem 11. First we colour the edges of $G \setminus E(T)$ with colours from the set $\{1, 2, 3, 4\}$ analogously to the proof of Theorem 7. After the colouring φ, each face has all its edges coloured except at most three. To colour the remaining edges of T it is sufficient to use three new colours from $\{5, 6, 7\}$ in the same way as in the proof of Theorem 11. The details are left to the reader. □

Motivated by the proof of Theorem 12 one can prove the following

Theorem 13. *Let G be a connected plane graph and let its dual contains a Hamilton path H^*. Then $\pi_f'(G) \leq 6$.*

Proof. To prove that there is a facial non-repetitive colouring φ of G with 6 colours we proceed analogously as in the proof of Theorem 11. First we colour the edges of $G \setminus E(H)$ with colours from the set $\{1, 2, 3, 4\}$ non-repetitively as in the proof of Theorem 7. Here $E(H)$ is the set of edges corresponding to those of H^* in G^*. After this colouring, each face has all its edges coloured except at most two. To colour the remaining edges of H it is sufficient to use two new colours 5 and 6 in the same way as in the proof of Theorem 11. The details are left to the reader. □

A *Halin graph* is a plane graph which is an union of a tree T of order $p \geq 3$ having no vertices of degree 2 and a cycle C connecting the leaves of T in a cyclic order determining by a plane embedding of T. It is easy to see that the dual of any Halin graph contains a Hamilton path. Applying Theorem 13 we get the following.

Corollary 14. *Let G be a Halin graph. Then $\pi_f'(G) \leq 6$.*

5   Plane hamiltonian graphs

A *Hamilton cycle* in a graph G is a cycle containing all the vertices. A graph G containing a Hamilton cycle is said to be *hamiltonian*. The aim of this section is to give better upper bounds on the facial Thue chromatic index of plane graph, provided that it is hamiltonian.

Let G be a plane hamiltonian graph and H a Hamilton cycle in G. We denote by G_1 (resp. G_2) the subgraph of G induced by H and the edges of G inside (resp. outside) of H. Evidently G_i, $i = 1, 2$, is a 2-connected outerplanar graph. Recall that a plane graph is *outerplanar* if all its vertices are incident to the outer face.

The *weak dual* G^w of a planar graph G is the graph obtained from G^* by deleting the vertex corresponding to the outer face of G (and all the edges incident with it). Fleischner et al. [8] showed that if G is outerplanar then G^w is a forest and that it is a tree if G is 2-connected. For $i = 1, 2$, we denote T_i the weak dual of G_i. The edges of G_i not on H are called *chords*.

Theorem 15. *Let G be a hamiltonian plane graph and H a Hamilton cycle of G. Then $\pi'_f(G) \leq 7$.*
Moreover, if $\max\{\Delta(T_1), \Delta(T_2)\} \leq 4$ then $\pi'_f(G) \leq 6$
and if $\max\{\Delta(T_1), \Delta(T_2)\} \leq 2$ then $\pi'_f(G) \leq 5$.

Proof. Let us first colour non-repetitively the edges of H with $\{1, 2, 3, 4\}$ so that exactly one edge e_0 is coloured with 4. This is possible according to Lemma 4.

Next we colour the chords of G_1. To do so we root the tree T_1 in f_0 – the face of G_1 incident to e_0. For every face f, let e_f be the edge shared by f and its parent if $f \neq f_0$ or be e_0 if $f = f_0$.

We now colour the chords of the faces of G_1 successively form the root f_0 to the leaves of T_1. At each step, we shall colour the uncoloured chords of a face f with colours in $\{4, 5, 6, 7\}$. So far, all the coloured edges of f are those in $H \setminus \{e_0\}$ which are coloured in $\{1, 2, 3\}$ except e_f which is coloured $c_f \in \{4, 5, 6, 7\}$. Let $e_1, e_2, \ldots, e_k$ be uncoloured chords incident to f in clokwise order around f starting at e_f. We colour the e_i with a non-repetitive sequence with the three colour of $\{4, 5, 6, 7\} \setminus \{c_f\}$. Doing so, no trail on the boundary of f is a repetition because the colour of e_f appears only once around f and $f \setminus e_f$ is non-repetitive by Lemma 3.

We proceed analogously to colour the chords of G_2 to obtain a facial non-repetitive edge-colouring of G.

If $\max\{\Delta(T_1), \Delta(T_2)\}$ is at most 4 (resp. 2), we proceed in a similar way as above. But at each step we want the face f of G_1 or G_2 in consideration to have at most 3 (resp. 1) non-coloured edges. This will guarantee that 2 colours (resp. 1 colour) are enough for a non-repetitive colouring of the uncoloured chords. So the set $\{4, 5, 6\}$, (resp. $\{4, 5\}$) is enough to colour all the chords. If $|V(G)| \notin \{5, 7, 9, 10, 14, 17\}$ then colouring non-repetitively the outer face with $\{1, 2, 3\}$ and taking f_1 to be a leaf of T_1 (or T_2), we have the desired property. The cases when $|V(G)| \in \{5, 7, 9, 10, 14, 17\}$ are left to reader. $\square$

An almost identical proof in which the Hamilton cycle H is replaced by the facial walk of the outer face yields the result for 2-edge-connected outerplanar graphs in which every inner face has a chord.

The bound of Theorem 15 applies for 2-connected outerplanar graphs.

Theorem 16. *Let G be a 2-connected outerplanar graph. Then $\pi'_f(G) \leq 7$. Moreover denoting by T be the weak dual of G the following hold:*

(i)   *if G is a cycle and $|V(G)| = 2$, then $\pi'_f(G) = 2$;*
(ii) *if G is a cycle and $|V(G)| \notin \{2, 5, 7, 9, 10, 14, 17\}$, then $\pi'_f(G) = 3$;*
(iii) *if G is a cycle and $|V(G)| \in \{5, 7, 9, 10, 14, 17\}$, then $\pi'_f(G) = 4$;*
(iv) *if $\Delta(T) \leq 2$, then $\pi'_f(G) \leq 5$;*
(v)   *if $\Delta(T) \leq 4$, then $\pi'_f(G) \leq 6$.*

6   Discussion

In this paper, we gave upper bounds for the facial Thue chromatic index of various classes of plane graphs. However most of these upper bounds are certainly not tight. Hence, for each of these classes $\mathcal{C}$, it would be nice to find its facial Thue chromatic index $\pi'_f(\mathcal{C}) = \max\{\pi'_f(G), G \in \mathcal{C}\}$ especially for the class $\mathcal{P}$ of plane graphs.

Problem 17. What is the value of $\pi'_f(\mathcal{P})$? In other words, what is the smallest integer k such that $\pi'_f(G) \leq k$ for all plane graphs G?

We believe that the following is true

Conjecture 18. Let G be a 3-connected plane graph. Then $\pi'_f(G) \leq 6$.

There are many generalisations of non-repetitive sequences and for most of them it also make sense to study their graph theoretical variants. In principle, any property of sequences can be translated into a property of graphs, via coloured trails. In particular, one may take any avoidable pattern (see [7]) and study its behaviour on graphs. In another direction, one may look at vertex-colouring instead of edge-colouring. The *Thue chromatic number* of G, denoted $\pi(G)$, is the minimum number of colours of a non-repetitive vertex-colouring of G. Several papers study non-repetitive colourings, see e.g. [1, 4, 5, 10, 11, 12, 13, 14]. A major conjecture in this area is due to Grytczuk [11].

Conjecture 19 (Grytczuk [11]). There exists an absolute constant k such that any planar graph has a non-repetitive vertex k-colouring.

Certainly, it is equally interesting to study the facial version of all the above-mentionned variants. In particular, showing that the *facial Thue chromatic number* of a plane graph is bounded would be a first step towards Conjecture 19.

7   Acknowledgement

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