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New bijective links on planar maps via orientations

Éric Fusy

ABSTRACT. This article presents new bijections on planar maps. At first a bijection is established between bipolar orientations on planar maps and specific "transversal structures" on triangulations of the 4-gon with no separating 3-cycle, which are called irreducible triangulations. This bijection specializes to a bijection between rooted non-separable maps and rooted irreducible triangulations. This yields in turn a bijection between rooted loopless maps and rooted triangulations, based on the observation that loopless maps and triangulations are decomposed in a similar way into components that are respectively non-separable maps and irreducible triangulations. This gives another bijective proof (after Wormald's construction published in 1980) of the fact that rooted loopless maps with n edges are equinumerous to rooted triangulations with n inner vertices.

1. Introduction

Planar maps (connected graphs equipped with a planar embedding) are a rich source of structural correspondences and enumerative results. For counting purpose, all planar maps (shortly called maps) are here assumed to be rooted, that is, they have a marked oriented edge with the outer face on its left. As discovered by Tutte in the 60's [**Tut63**], many families of maps (eulerian, triangulations, quadrangulations...) have strikingly simple counting sequences, which are reminiscent of formulas for families of plane trees. Tutte's method is based on generating functions. At first a recursive decomposition of maps by root edge deletion is translated to a recurrence on the counting sequence, and then to a functional equation satisfied by the associated generating function. Solving this equation yields a formula for the counting sequence (by coefficient extraction). Tutte's method has the advantage of being quite automatic, but requires an involved machinery —the quadratic method— to solve equations on generating functions [**Tut73**].

By now there is a bijective approach, introduced by Schaeffer in his thesis [**Sch98**], to prove counting formulas for many map families. The main ingredients are as follows: given a family of map $\mathcal{M}$, one searches a *regular* combinatorial structure —typically an orientation with simple outdegree conditions— specific to the maps in $\mathcal{M}$, which gives a way to associate with each map of $\mathcal{M}$ a tree with simple degree conditions. Conversely the map is recovered by performing local operations on the tree so as to close the faces one by one. The bijection ensures that the counting sequence of $\mathcal{M}$ is equal to the counting sequence of the associated tree family, which has typically a closed formula involving binomial coefficients.

These enumerative techniques (Tutte's recursive method or Schaeffer's bijective approach) yield thus an extensive table of counting formulas for families of maps. By examining such a table, one notices that seemingly unrelated map families are equinumerous. This asks for bijective explanations as direct as possible to understand the underlying structural correspondences. On a few instances there already exist very simple bijective constructions. Such bijections are of a type different from the above mentioned bijective constructions from trees. Indeed, they are from map to map and operate directly on the embedding in a simple local way. Let us mention the classical *duality* construction between maps with i vertices and j faces and maps with j vertices and i faces, the *radial mapping* between maps with n edges and 4-regular maps with n vertices, and the so-called *trinity mapping* between bipartite 3-regular maps with $2n$ vertices and eulerian maps with n edges (the two latter constructions can be traced back to Tutte [**Tut63**]). We have observed two further coincidences:
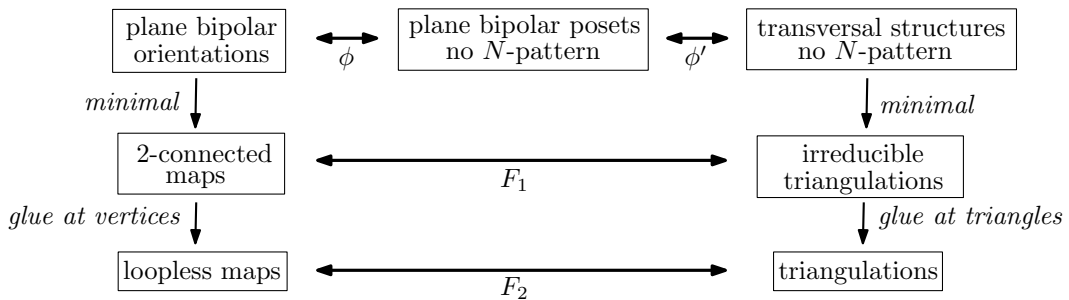
FIGURE 1. Relations between the combinatorial structures and map families considered in the article. Bijective links are indicated by double arrows. Below each double arrow is indicated the name of the mapping for the left-to-right arrow.

(i): non-separable maps with n edges are equinumerous to so-called irreducible triangulations (triangulations of the 4-gon with no separating triangle) with $n + 3$ vertices,

(ii): loopless maps with n edges are equinumerous to triangulations with $n + 3$ vertices.

In this article, we describe new bijective constructions for proving (i) and (ii). To wit, the bijection for (i) can be seen as a direct map-to-map correspondence. But it borrows ideas from the above mentioned bijective constructions of maps from trees. At first we associate a combinatorial structure with each of the two map-families: plane bipolar orientations for non-separable maps and transversal structures for irreducible triangulations (these are defined in Section 2). We show in Section 3 that these structures are closely related: to each plane bipolar orientation O is associated in a bijective way a so-called N-avoiding plane bipolar poset $P = \phi(O)$, which yields in turn a transversal structure $X = \phi'(P)$; the mapping $\Phi = \phi' \circ \phi$ is bijective onto so-called N-avoiding transversal structures.

Then, the correspondence Φ specialises to a bijection, described in Section 4, between non-separable maps and irreducible triangulations. The bijection, denoted by F_1, is based on the two following properties:

- each non-separable map has a *minimal* plane bipolar orientation and each irreducible triangulation has a *minimal* transversal structure;
- the correspondence Φ matches the plane bipolar orientations that are minimal on their non-separable maps with the transversal structures that are minimal on their irreducible triangulations.

As described in Section 5, the bijection F_1 yields in turn a bijection, denoted F_2, for proving (ii). We make use of two classical decompositions: a loopless map is decomposed into a non-separable map, called the core, and a collection of components that are smaller loopless maps; whereas a triangulation is decomposed into an irreducible triangulation, called the core, and a collection of components that are smaller triangulations. The key observation is that the two decompositions are parallel, for a convenient choice of the size parameters. The bijection F_2 is thus specified recursively: the cores are matched by the bijection F_1, and the components are matched recursively by F_2. Let us mention that another bijective proof of (ii) has been described by Wormald [**Wor80**], still recursive but based on different principles (in [**Wor80**], an isomorphism is established between the generating tree of loopless maps and the generating tree of triangulations with an additional catalytic variable), see the discussion after Theorem 19. To sum up, the whole bijective scheme of the article is shown in Figure 1.

2. Preliminaries

2.1. Planar maps. A *planar map*, shortly called a map, is a connected unlabelled planar graph embedded in the plane with no edge-crossings, the embedding being considered up to continuous deformation. Loops and multiple edges are allowed. In addition to the vertices and edges of the graph embedded, a map has *faces*, which are the maximal connected areas of the plane split by the embedding. The unbounded face is called the outer face, the other ones are called inner faces. Edges and vertices are said to be inner or outer whether they are incident to the outer face or not. A *corner* of a map is a triple (e, v, e') where v is a vertex and e and e' are two consecutive edges in clockwise order around v.

A map is *rooted* by distinguishing and orienting an edge, called the *root*, with the condition that the root has the outer face on its right (equivalently, it is a map with a distinguished corner incident to the outer
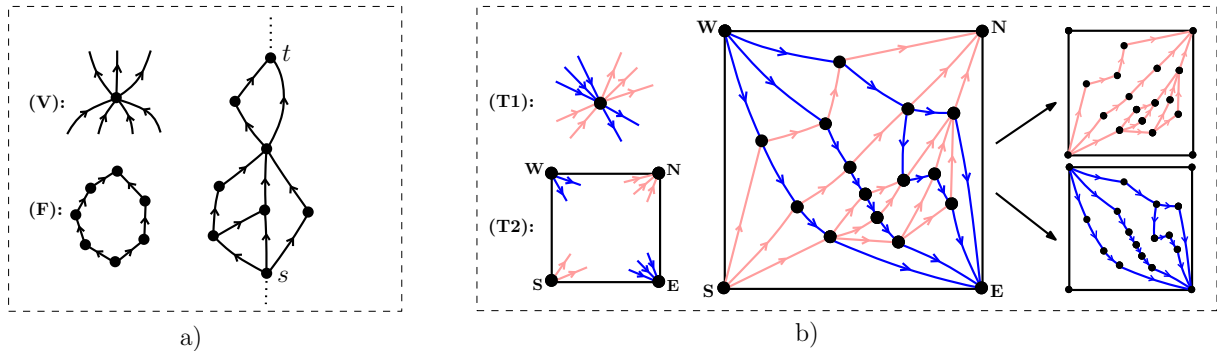
FIGURE 2. (a) Plane bipolar orientations: local conditions and an example. (b) Transversal structures: local conditions and an example.

face). The origin of the root is called the *root vertex*. In this article we will consider the following families of (rooted) planar maps:

- *Loopless maps.* A loopless is a map such that each edge has two distinct extremities. Multiple edges are allowed.
- *non-separable maps.* A map is non-separable if it is loopless and the deletion of any one vertex does not disconnect the map. Multiple edges are allowed.
- *Triangulations.* A triangulation is a map with no loop nor multiple edges and with all faces of degree 3. These correspond to maximal planar graphs embedded in the plane.
- *Irreducible triangulations.* A triangulation of the 4-gon is a map with no loop nor multiple edges, with a quadrangular outer face and triangular inner faces. An irreducible triangulation of the 4-gon, shortly called irreducible triangulation, is a triangulation of the 4-gon such that the interior of any 3-cycle is a face.

2.2. Plane bipolar orientations. A *bipolar orientation* on a connected graph G is an acyclic orientation with a unique source (vertex with only outgoing edges) denoted by s and a unique sink (vertex with only ingoing edges) denoted by t. Equivalently, the partial order induced on the vertices by the orientation has a unique minimum and a unique maximum. Bipolar orientations constitute the natural combinatorial structure characterising 2-connectivity. Indeed, as is well known [**dFOdMR95**], a graph G with two marked vertices s and t admits a bipolar orientation with source s and sink t if and only if G is non-separable upon connecting s and t by an edge.

The source and the sink of the bipolar orientation are also called the *poles* or the *special vertices*. The other vertices are said to be *non-special*. A *plane bipolar orientation* is a bipolar orientation on a planar map M such that the source and the sink are outer vertices of M. It is convenient when considering plane bipolar orientations to draw two half-lines starting respectively from s and t and reaching into the outer face. The outer face is thus split into two unbounded faces, which are called the *special faces*; looking from s to t, the one on the left is called the *left special face* and the one on the right is called the *right special face*. As described in [**dFOdMR95, FFNO08**], plane bipolar orientations have the nice property that they are characterised by two simple local properties, one around vertices and one around faces, see Figure 2(a):

- **(V):** Around each non-special vertex, the edges form a nonempty interval of outgoing edges and a nonempty interval of ingoing edges.
- **(F):** The contour of each inner face f is made of two oriented paths with same origin s_f and same end t_f. The two special faces are each bordered by a path that goes from s to t.

Given Property (V), we define the *left lateral face* of a non-special vertex v as the face in the corner between the last ingoing edge and the first outgoing edge in clockwise order around v; define similarly the *right lateral face* of v as the face in the corner between the last outgoing edge and the first ingoing edge in clockwise order around v. Given an inner face f, the vertices s_f and t_f are respectively called the *source* and the *sink* of f. The path from s_f to t_f that has the exterior of f on its left is called the *left lateral path* of f and is denoted $P_\ell(f)$; the path from s_f to t_f that has the exterior of f on its right is called the *right lateral path*

of f and is denoted $P_r(f)$. The last edge of $P_\ell(f)$ is called the *topleft edge* of f and the first edge of $P_r(f)$ is called the *bottomright edge* of f. The vertices of $P_\ell(f)\backslash\{s_f, t_f\}$ are called *left lateral vertices* of f, and the vertices of $P_r(f)\backslash\{s_f, t_f\}$ are called *right lateral vertices* of f. The paths bordering the left special face and the right special face are called respectively the *left outer path* and the *right outer path*.

It is well known that a plane bipolar orientation O induces a partial order on the edge-set: $e \le e'$ if and only if there exists an oriented path passing by e before passing by e'. In other words $\le$ is the transitive closure of the binary relation: $e \prec e'$ if and only if there is a vertex v such that e is ingoing at v and e' is outgoing at v.

Due to a classical duality relation satisfied by plane bipolar orientations [**dFOdMR95**], there is a partial order on the face-set and another partial order, called dual, on the edge-set. Precisely, given two faces f and f' (special or not), write $f \prec_F f'$ if there is an edge of O with f on its left and f' on its right; the transitive closure of $\prec_F$ is a partial order, called the *left-to-right order* on the faces of O. And the dual order on the edge-set, denoted $\le^*$, is defined as the transitive closure of the relation: $e \prec^* e'$ (with e and e' edges of O) if and only if there exists a face f of O such that e is in the left lateral path and e' is in the right lateral path of f.

Given a plane bipolar orientation with no multiple edges, a transitive edge is an edge whose two extremities are connected by an oriented path of length at least 2. A *plane bipolar poset* is a plane bipolar orientation with at least 3 vertices, with no multiple edges, and with no transitive edges. The terminology refers to the fact that a plane bipolar poset is a planarly embedded Hasse diagram representing a poset.

The following property is easily checked from Condition (F):

FACT 1. *A plane bipolar orientation with at least 3 vertices is a plane bipolar poset if and only if the two lateral paths of each inner face have length at least 2.*

An important remark to be used later is that each inner face f of a plane bipolar poset has at least one lateral vertex on each side: the last vertex of $P_\ell(f)\backslash\{s_f, t_f\}$ is called the *topleft lateral vertex* of f and the first vertex of $P_r(f)\backslash\{s_f, t_f\}$ is called the *bottomright lateral vertex* of f. Given a plane bipolar poset, a *N-pattern* is a triple (e_1, e_2, e_3) of edges such that e_1 and e_2 have the same origin v, e_2 and e_3 have the same end v', e_1 follows e_2 in clockwise order around v, and e_3 follows e_2 in clockwise order around v'. The edge e_2 is called the *central edge* of the N-pattern. A $\mathcal{N}$-*pattern* is defined similarly, upon replacing clockwise by counterclockwise. Plane bipolar posets with no N-pattern are said to be *N-avoiding*; these play an important role in the bijections to be given next.

2.3. Transversal structures. Transversal structures play a similar part for irreducible triangulations as plane bipolar orientations for non-separable maps.

Let us give the precise definition. Given a rooted irreducible triangulation T, denote by N, E, S, W the outer vertices of T in clockwise order around the outer face, starting from the origin of the root. A *transversal structure* of T is an orientation and a bicoloration of the inner edges of T, say each inner edge is red or blue, such that the following conditions are satisfied, see Figure 2(b) [1].

(T1): The edges incident to an inner vertex of T form in clockwise order: a nonempty interval of outgoing red edges, a nonempty interval of outgoing blue edges, a nonempty interval of ingoing red edges, and a nonempty interval of ingoing blue edges.

(T2): The edges incident to N, E, S, and W are respectively ingoing red, ingoing blue, outgoing red, and outgoing blue.

Transversal structures were introduced by He [**He93**] under the name of regular edge-labellings. They were further investigated by the author [**Fus09**] and have many applications in graph drawing: straight-line drawing [**Fus09**], visibility drawing [**KH97**], rectangular layouts [**KH97**]. Transversal structures characterise triangulations of the 4-gon that are irreducible. Indeed, irreducibility is necessary, and each irreducible triangulation admits a transversal structure [**Fus09**]. Moreover, transversal structures are closely related to plane bipolar orientations by the following property:

FACT 2. *Let X be a transversal structure on an irreducible triangulation T having at least one inner vertex. Then the oriented map X_r formed by the red edges and the vertices of $T\backslash\{W, E\}$ is a plane bipolar poset, called the* red bipolar poset *of X. Similarly, the oriented map X_b formed by the blue edges and the vertices of $T\backslash\{S, N\}$ is a plane bipolar poset, called the* blue bipolar poset *of X.*

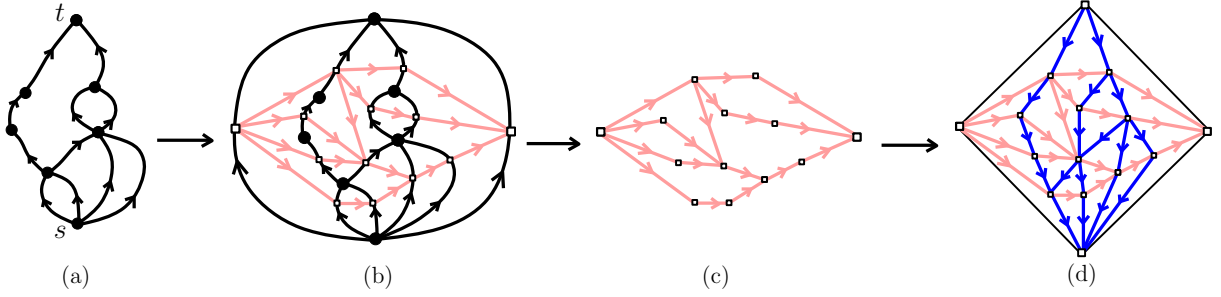[1]In all figures, red edges are light and blue edges are dark.

FIGURE 3. (a) A plane bipolar orientation, (c) the associated N-avoiding plane bipolar poset, (d) the associated N-avoiding transversal structure.
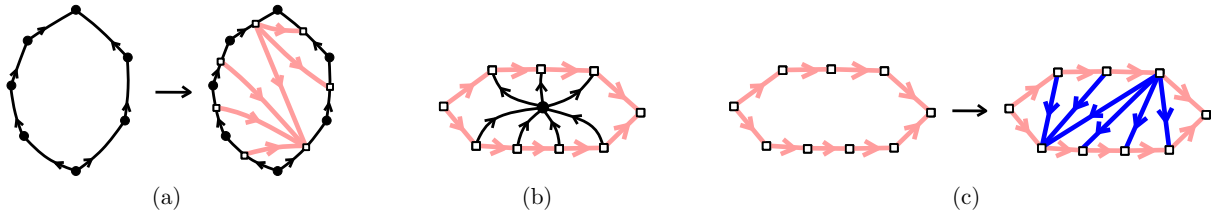

FIGURE 4. (a) Insertion of the edges of $P = \phi(O)$ inside a face of O. (b) Configuration of an inner face of P around the associated non-special vertex of O. (c) Insertion of the blue edges of $X = \phi'(P)$ inside a face of P.

PROOF. The fact that X_{r} and X_{b} are plane bipolar orientations was proved in [**Fus09**] (the main argument is that a monocoloured circuit of X would have a monocolored chordal path of the other color inside, which is not possible by Condition T1). Clearly in each inner face f of X_{r}, the two lateral paths are not both of length 1, as T has no multiple edge. Hence, f has at least one lateral vertex v on one side (left or right), so f has at least one blue edge inside by Condition (T1). But, by condition (T1) again, such a blue edge goes from a left lateral vertex of f to a right lateral vertex of f. Hence both lateral paths of f have length greater than 1, so X_{r} is a plane bipolar poset by Fact 1. □

A transversal structure is called N-*avoiding* if both its red and its blue bipolar posets are N-avoiding.

3. Bijections between plane bipolar orientations and N-avoiding structures

This section covers the top-line of the diagram shown in Figure 1. We present a bijective mapping Φ from plane bipolar orientations to N-avoiding transversal structures; Φ is the composition of two bijections ϕ and ϕ': ϕ maps a plane bipolar orientation to an N-avoiding plane bipolar poset, and ϕ' completes an N-avoiding plane bipolar poset into an N-avoiding transversal structure.

3.1. From plane bipolar orientations to N-avoiding plane bipolar posets. Let O be a plane bipolar orientation. We associate with O an oriented planar map $P = \phi(O)$ as follows, see Figure 3(a)-(c).

- *Add two outer edges.* Add two edges ℓ and r going from the source to the sink of O, so that the whole map is contained in the 2-cycle delimited on the left by ℓ and on the right by r. The augmented bipolar orientation is denoted $\widetilde{O}$.
- *Insert the vertices of P.* A vertex of P, depicted in white, is inserted in the middle of each edge of $\widetilde{O}$. The vertex inserted in the edge ℓ is denoted s_P, and the vertex inserted in the edge r is denoted t_P.
- *Insert the edges of P.* Edges of P are planarly inserted in the interior of each inner face f of $\widetilde{O}$ so as to create the following adjacencies, see Figure 4(a): the vertices of P in the left lateral path of f are connected to the vertex of P in the bottomright edge of f; and the vertices of P in the right lateral path of f are connected to the vertex of P in the topleft edge of f. The inserted edges are directed from the left lateral path to the right lateral path of f.

LEMMA 3. *Let O be a plane bipolar orientation with n edges. Then the oriented planar map $P = \phi(O)$ is an N-avoiding plane bipolar poset with $n+2$ vertices. The source of P is the vertex s_P and the sink of P is the vertex t_P.*

When P and the augmented bipolar orientation $\widetilde{O}$ are superimposed, there is exactly one non-special vertex v of O in each inner face f of P: the outgoing edges of v are incident to the left lateral vertices of f and the ingoing edges of v are incident to the right lateral vertices of f, see Figure 4(b).

PROOF. Let us first prove that P is acyclic. Given two vertices v and v' of P, write $v \prec_P v'$ if there is an edge of P from v to v'. Let e and e' be the edges of O that respectively correspond to v and v'. Note that $v \prec_P v'$ implies $e \prec^* e'$. Hence the transitive closure of $\prec_P$ corresponds to a suborder of $\leq^*$, so P is acyclic. To prove bipolarity, observe that, for each vertex v of P distinct from s_P and t_P, the corresponding edge e of O has one inner face of $\widetilde{O}$ on each side. Hence, by definition of ϕ, v has at least one ingoing edge in the face on the left of e (possibly more than one if e is the bottomright edge of that face) and at least one outgoing edge in the face of $\widetilde{O}$ on the right of e (possibly more than one if e is the topleft edge of that face).

Next, the rules for inserting the edges of P easily imply that each non-special vertex v of O gives rise to an inner face f_v of P such that the edges of P outgoing at v are incident to the left lateral vertices of f_v and the edges of P ingoing at v are incident to the right lateral vertices of f_v, see Figure 4(b). In addition the left lateral path of f_v has length $\text{Outdeg}(v)+1$ and the right lateral path of f_v has length $\text{Indeg}(v)+1$. Hence every inner face of P of the form f_v has its two lateral paths of length greater than 1.

Claim. The number of non-special vertices of O is equal to the number of inner faces of P.

Proof of the claim. Write $V(M)$, $E(M)$ and $F(M)$ for the sets of vertices, edges, and inner faces of a map M and write $\deg(f)$ for the number of edges on the contour of an inner face f. By definition of ϕ, $|V(P)| = |E(\widetilde{O})|$ and $E(P) = \sum_{f \in F(\widetilde{O})} (\deg(f) - 1) = 2|E(\widetilde{O})| - 2 - |F(\widetilde{O})|$. By the Euler relation, $|F(P)| = |E(P)| - |V(P)| + 1$, so $|F(P)| = |E(\widetilde{O})| - |F(\widetilde{O})| - 1$, which is equal to $(|V(\widetilde{O})| - 2)$ again by the Euler relation. Hence, the number of inner faces of P is equal to the number of non-special vertices of O. $\triangle$

The claim ensures that every inner face of P is of the form f_v for some non-special vertex v of O. Hence every inner face of P has its two lateral paths of lengths greater than 1, so P is a plane bipolar poset by Fact 1.

Finally, the fact that P has no N-pattern is due to the two following observations: i) two edges of P with same origin are inside the same face of O, and two edges of P with same end are inside the same face of O, hence any N-pattern must be inside a face of O; ii) there is no N-pattern of P inside a face of O. $\square$

As we prove next, the mapping ϕ is in fact a bijection and has an explicit simple inverse.

Given a bipolar poset P (N-avoiding or not), let $O = \psi(P)$ be the oriented planar map defined as follows:

- *Insert the vertices of O.* One vertex of O is inserted in each face of P. The vertex inserted in the right special face of P is denoted s and the vertex inserted in the left special face of P is denoted t.
- *Insert the edges of O.* Each non-special vertex v of P gives rise to an edge of O that goes from the vertex of O inside the right lateral face of v to the vertex of O inside the left lateral face of v.

The correspondence between non-special vertices of O and inner faces of P, as stated in Lemma 3 and shown in Figure 4(b), ensures that ψ is exactly the procedure to recover a plane bipolar orientation from its image $P = \phi(O)$. In other words:

LEMMA 4. *For each plane bipolar orientation O, $\psi(\phi(O)) = O$.*

Next we prove that ψ is also the right inverse of ϕ when restricted to N-avoiding structures.

LEMMA 5. *For any plane bipolar poset P, the oriented map $O = \psi(P)$ is a plane bipolar orientation. In addition $\phi(O) = P$ if P is N-avoiding.*

PROOF. Let v, v' be two vertices of O, and let f, f' be the faces of P corresponding respectively to v and v'. Clearly, if there is an oriented path of edges of O going from v to v', then f is smaller than f' for the left-to-right order on the faces of P. Hence, O is acyclic. In addition, each vertex $v \neq \{s, t\}$ of O has indegree equal to the number of left lateral vertices and outdegree equal to the number of right lateral vertices of the corresponding inner face of P. Hence, s is the only source and t the only sink of O, so O is bipolar.

Now assume that P is N-avoiding, and let us now show that $\phi(O) = P$. Recall that the first step of ϕ is to add two edges from s to t, yielding an augmented map $\widetilde{O}$. Consider the oriented map M obtained by superimposing $\widetilde{O}$ and P, that is, there is a vertex of P in the middle of each edge of $\widetilde{O}$ and a vertex of $\widetilde{O}$ inside each face of P, see Figure 3(b). Call P-vertices and P-edges the vertices and edges of M that come from P, and call O-vertices and O-edges the vertices and edges of M that come from $\widetilde{O}$. Note that a P-edge corresponds to an edge of P, but an O-edge corresponds to a half-edge of $\widetilde{O}$.

Let v be a P-vertex. By definition of ψ (and by Property (V) of plane bipolar orientations), the edges of M incident to v form in clockwise order: a nonempty interval of outgoing P-edges, one ingoing O-edge, a nonempty interval of ingoing P-edges, and one outgoing O-edge. Hence any P-edge inside an inner face f of $\widetilde{O}$ must go from a P-vertex in the left lateral path of f to a P-vertex in the right lateral path of f. Thus the P-edges inside f are naturally ordered from down to top; in this order, two consecutive edges are said to be *meeting* if they share a vertex and *parallel* otherwise. By a simple counting argument, the number of P-edges inside f is $\deg(f) - 1 - p(f)$, where $p(f)$ is the number of parallel consecutive P-edges inside f and $\det(f)$ is the degree of f. As in the proof of Lemma 3, write $V(M)$, $E(M)$, and $F(M)$ for the sets of vertices, edges, and inner faces of a map M. From the above discussion, the total number of P-edges satisfies

$$(1) \qquad |E(P)| = \sum_{f \in F(\widetilde{O})} \deg(f) - 1 - p(f) = 2|E(\widetilde{O})| - 2 - |F(\widetilde{O})| - \sum_{f \in F(\widetilde{O})} p(f).$$

Moreover, $|E(P)| = |V(P)| + |F(P)| - 1$ by the Euler relation, $|V(P)| = |E(\widetilde{O})|$ by definition of ϕ, and $|F(P)| = |V(\widetilde{O})| - 2$ by definition of ψ. Hence $|E(P)| = |E(\widetilde{O})| + |V(\widetilde{O})| - 3$, which is equal to $2|E(\widetilde{O})| - |F(\widetilde{O})| - 2$ again by the Euler relation. Hence, from the first expression (1) of $|E(P)|$ obtained above, we conclude that $\sum_{f \in F(\widetilde{O})} p(f) = 0$, so all the $p(f)$ are zero, so that all consecutive P-edges in any inner face of $\widetilde{O}$ are meeting. As P has no N-pattern and all consecutive edges are meeting, the only possible configuration for the P-edges inside a face of $\widetilde{O}$ is the configuration shown in Figure 4(a). In other words, the P-edges are those inserted when applying the mapping ϕ to O. Hence $\phi(O) = P$. □

From Lemma 4 and Lemma 5 we obtain our first bijection:

PROPOSITION 6. *For $n \geq 1$ and $i \geq 0$, the mapping ϕ is a bijection between plane bipolar orientations with n edges and i non-special vertices, and N-avoiding plane bipolar posets with n non-special vertices and i inner faces. The inverse mapping of ϕ is ψ.*

REMARK. As recalled in Section 2.2, any bipolar orientation O on a graph $G = (V, E)$ (planar or not) gives rise to a partial order $P = \beta(O)$ on E, called edge-poset, according to the precedence order of the edges along oriented paths (observe that no embedding is needed to define this poset). It is well known that the mapping β is indeed a bijection between bipolar orientations on graphs and so called *N-free posets*, which are posets with no induced N in the Hasse diagram [**HM87**], that is, no quadruple (v_1, v_2, v_3, v_4) of vertices such that the only adjacencies on these vertices are: an edge from v_1 to v_3, an edge from v_2 to v_4 and an edge from v_1 to v_4. Moreover, it is possible to enrich this construction so as to take a planar embedding into account: the planar embedding of the plane bipolar orientation turns to a so-called 2-realizer of the associated poset [**dFdM96**].

Our mapping ϕ shares some resemblance with this classical construction, but there are significant differences. First, the bipolar poset obtained from ϕ might have induced N in the form of $\bigwedge$-patterns (due to the embedding, an induced N can appear either as an N-pattern or as an $\bigwedge$-pattern). Second, our bijection has the nice feature that it operates directly on the embedding, in both directions (ϕ and ψ). The mapping ϕ is actually just a simple adaptation of β so as to take the embedding into account. In contrast, $\phi^{-1} = \psi$ is not just an adaptation of β^{-1}. Indeed ψ is significantly simpler, it operates on the embedding in a local way, whereas β^{-1} requires some non-local manipulations on the edge-poset.

3.2. From N-avoiding plane bipolar posets to N-avoiding transversal structures. The next step is to complete an N-avoiding plane bipolar poset into a transversal structure. The procedure, called ϕ', is very similar to ϕ. Precisely, given a plane bipolar poset P (N-avoiding or not) the associated transversal structure $X = \phi'(P)$ is defined as follows, see Figure 3(c)-(d):

- *Create the outer quadrangle.* Insert one vertex, denoted W, in the left special face of P and one vertex, denoted E, in the right special face of P. Connect W and E to the source and sink of P,

denoted respectively by S and N, thus creating an outer quadrangle with vertices (W, N, E, S) in clockwise order. The bipolar poset augmented from P by insertion of the quadrangle —edges of the quadrangle oriented from S toward N— is denoted $\widetilde{P}$.

- *Insert the blue edges.* For each inner face f of $\widetilde{P}$, insert (in a planar way) blue edges inside f so as to create the following adjacencies, see Figure 4(c): left lateral vertices of f are connected to the bottomright lateral vertex of f, and right lateral vertices of f are connected to the topleft lateral vertex of f. The inserted blue edges inside f are directed from the left lateral vertices to the right lateral vertices of f. In other words, f is triangulated by transversal blue edges in the unique way avoiding blue N-patterns inside f.

LEMMA 7. *Let P be a plane bipolar poset. Then $\phi'(P)$ is a transversal structure whose red bipolar poset is P and whose blue bipolar poset is N-avoiding.*

PROOF. Clearly $X := \phi'(P)$ is a transversal structure (conditions (T1) and (T2) are satisfied) whose red bipolar poset is P. The blue bipolar poset is N-avoiding due to the two following observations: i) two blue edges with the same origin are inside the same face of P, and two blue edges with the same end are inside the same face of P, hence any blue N-pattern must be inside a face of P; ii) there is no blue N-pattern inside a face of P. $\square$

Call ψ' the mapping that associates to a transversal structure its red bipolar poset.

PROPOSITION 8. *For $n \geq 1$ and $i \geq 0$, the mapping ϕ' is a bijection between N-avoiding plane bipolar posets with n non-special vertices and i inner faces, and N-avoiding transversal structures with n inner vertices and $n + i + 1$ red edges. The inverse mapping of ϕ' is ψ'.*

PROOF. According to Lemma 7, $\psi'(\phi'(P)) = P$ for any plane bipolar poset P, so ψ' is the left inverse of ϕ'. Let us show that ψ' is also the right inverse of ϕ' when the mappings are restricted to N-avoiding structures. Let X be an N-avoiding transversal structure, with $X_\mathrm{r} := \psi'(X)$ its red bipolar poset and X_b its blue bipolar poset. When applying ϕ' to X_r, each inner face f of X_r is triangulated by transversal blue edges going from a left lateral vertex to a right lateral vertex of f. It is easily checked that the only such N-avoiding configuration is the one where all edges are incident either to the topleft or to the bottomright lateral vertex of f. In other words, the blue edges inside f are placed according to the insertion process of ϕ', hence $\phi'(P) = X$. Finally, the parameter-correspondence is due to the Euler relation, which ensures that a plane bipolar poset with n non-special vertices and i inner faces has $n + i + 1$ edges. $\square$

To conclude, we have described a bijection $\Phi = \phi' \circ \phi$ between plane bipolar orientations with $n \geq 1$ edges and $i + 2$ vertices and N-avoiding transversal structures with n inner vertices and $n + i + 1$ red edges. The bijection Φ operates in two steps ϕ and ϕ', the intermediate combinatorial structures being the N-avoiding plane bipolar posets.

3.3. Counting Baxter families. Plane bipolar orientations with n edges and i non-special vertices are known to be counted by the coefficients

$$(2) \qquad \Theta_{n,i} := \frac{2}{n(n+1)^2} \binom{n+1}{i} \binom{n+1}{i+1} \binom{n+1}{i+2}.$$

The formula has first been proved by Rodney Baxter [**Bax01**], guessing the answer from a recurrence satisfied by the counting sequence; a direct computation based on the "obstinate Kernel method" is due to Bousquet-Mélou [**BM02**]. A bijective proof has been found recently [**FPS09**]. Summing over i the coefficients $\Theta_{n,i}$, the number of plane bipolar orientations with n edges is

$$\theta_n := \sum_i \frac{2}{n(n+1)^2} \binom{n+1}{i} \binom{n+1}{i+1} \binom{n+1}{i+2}.$$

The numbers θ_n, called the *Baxter numbers* (after another Baxter, Glen Baxter, whose name is given to a family of permutations counted by θ_n), surface recurrently in the enumeration of combinatorial structures, see [**FFNO08**] for a recent survey on "Baxter families". Our bijections with N-avoiding structures, as described in Section 3, bring new Baxter families to the surface:

PROPOSITION 9. *For $n \geq 1$ and $i \geq 0$, the number $\Theta_{n,i}$ counts:*

- *N-avoiding plane bipolar posets with n non-special vertices and i inner faces,*
- *N-avoiding transversal structures with n inner vertices and $n + i + 1$ red edges.*

Taking into account the parameter n only, these structures are counted by the Baxter number θ_n.

4. Bijection between non-separable maps and irreducible triangulations

4.1. Non-separable maps as specific plane bipolar orientations. For rooted maps, plane bipolar orientations are always assumed to have the root edge going from the source to the sink. As already mentioned in Section 2.2, bipolar orientations are naturally associated with the property of non-separability; a rooted map is non-separable if and only if it admits a plane bipolar orientation. Even more is true, namely, each rooted non-separable map can be endowed with a specific plane bipolar orientation in a canonical way. Given a plane bipolar orientation, we define a *left-oriented piece*, shortly a LOP, as a 4-tuple (v_1, v_2, f_1, f_2) made of two distinct vertices v_1, v_2 and two distinct faces f_1, f_2 of O such that the following conditions are satisfied, see Figure 5(a):

- the vertex v_1 is the sink of f_2 and is a left lateral vertex of f_1,
- the vertex v_2 is a right lateral vertex of f_2 and is the source of f_1.

Note that v_1 and v_2 must be non-special vertices and f_1 and f_2 must be inner faces.

FACT 10 ([**dFOdMR95, OdM94**]). *A rooted map is non-separable if and only if it admits a bipolar orientations. A rooted non-separable map M has a unique plane bipolar orientation with no LOP, called the* minimal plane bipolar orientation *of M.*

The terminology is due to the fact that the set of plane bipolar orientations of a fixed rooted non-separable map is a distributive lattice, the minimal element of the lattice being the bipolar orientation with no LOP, see [**OdM94**]. The set of plane bipolar orientations with no LOP is denoted $\mathcal{O}_{\min}$.

LEMMA 11 (link with plane bipolar posets). *Let O be a plane bipolar orientation, and let $P = \phi(O)$ be the associated N-avoiding plane bipolar poset. Then O has no LOP if and only if P has no LOP.*

PROOF. Assume that O has a LOP (v_1, v_2, f_1, f_2). Let w_1 be the vertex of P inserted in the bottomright edge of f_2 and w_2 the vertex of P in the topleft edge of f_1. Let f_1' be the face of P corresponding to v_2 and f_2' the face of P corresponding to v_1. Then (w_1, w_2, f_1', f_2') is easily checked to be a LOP of P, as shown in Figure 5(b).

Conversely, assume that P has a LOP $(w_1, w_2, f_1' = f_{v_1}, f_2' = f_{v_2})$, with v_1 and v_2 the vertices of O associated respectively to the two faces f_2' and f_1' of P. Let f_1 be the face of O containing the ingoing edges of w_1, and let f_2 be the face of O containing the outgoing edges of w_2. Let us first check that f_1 and f_2 can not be equal. Assume $f_1 = f_2 = f$, then w_1 is in the right lateral path and w_2 is in the left lateral path of f; since $P = \phi(O)$ and since there is a path from w_2 to w_1 (due to the LOP of P), the vertices w_2 and w_1 must be adjacent, so the LOP of P is in fact reduced to an N-pattern, a contradiction. Thus f_1 and f_2 are distinct. From the definition of ψ and the fact that w_2 and w_1 are not adjacent, it is easily checked that (v_1, v_2, f_1, f_2) is a LOP of $O = \psi(P)$, see Figure 5(b). □

Let $\mathcal{P}_{\min}$ be the set of plane bipolar posets with no LOP. Note that an N-pattern—with e its central edge—in a plane bipolar poset induces a LOP (v_1, v_2, f_1, f_2) where v_1 is the end of e, v_2 is the origin of e, f_1 is the face on the right of e, and f_2 is the face on the left of e. Hence $\mathcal{P}_{\min}$ is also the set of N-avoiding plane bipolar posets with no LOP, which by Lemma 11 is in bijection with $\mathcal{O}_{\min}$. Thus we obtain

$$(3) \qquad\qquad\qquad \mathcal{O}_{\min} \simeq_\phi \mathcal{P}_{\min}.$$

4.2. Irreducible triangulations as specific transversal structures. As we recall here from [**Fus09**], one can endow an irreducible triangulation with a specific transversal structure in a canonical way. Given a transversal structure X, an *alternating 4-cycle* of X is a cycle of four edges that alternate in color, that is, the cycle has two opposite red edges and two opposite blue edges. Given v a vertex of R, let e and e' be the two edges of R incident to v, such that e' follows e in a counterclockwise walk around R; e is called the *left-edge* of v and e is called the *right-edge* of v. A boundary incidence of R is the incidence of an edge e inside R with a vertex v on R; it is called a *left incidence* if e has the color of the left-edge of v and a *right incidence* if e has the color of the right-edge of v.
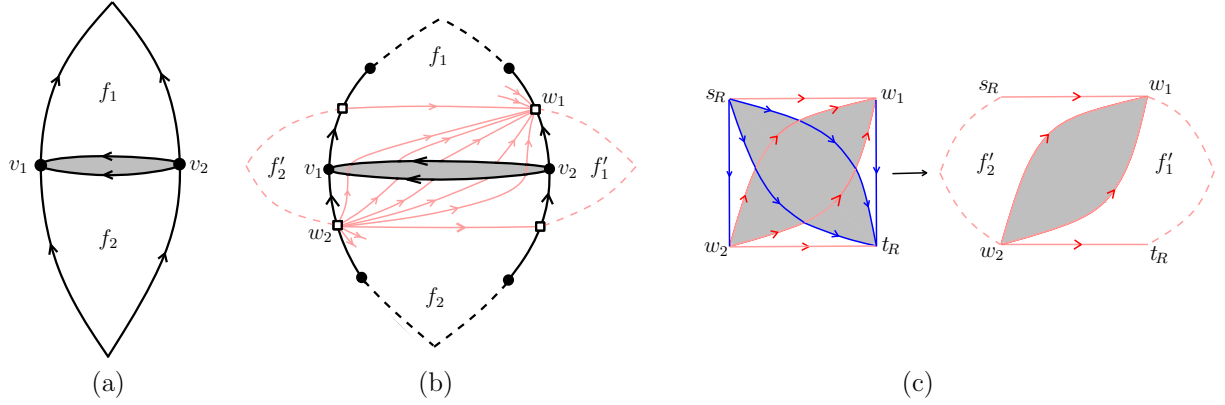
FIGURE 5. (a) A left-oriented piece (LOP) of a plane bipolar orientation. (b) Each LOP in a plane bipolar orientation O yields a LOP in $P = \phi(O)$. (c) Each right alternating 4-cycle in an N-avoiding transversal structure X yields a LOP in the red bipolar poset of X.

As proved in [**Fus09**], R is of two possible types: either all boundary incidences for R incidences are left incidences, in which case R is called a *left alternating 4-cycle*, or all boundary incidences of R are right incidences, in which case R is called a *right alternating 4-cycle*.

Let us give a slightly different formulation. By Condition (T1), the cycle R is made of two oriented paths of length 2 with the same origin s_R and the same end t_R, which are called the *source* and the *sink* of R. Let X_R be X restricted to R and its interior, forgetting the colors and directions of the edges of R. Then, if R has at least one interior vertex, X_R is a transversal structure. If s_R is source of red edges, R is a left alternating 4-cycle; if s_R is source of blue edges, R is a right alternating 4-cycle. In the degenerated case where R has no vertex inside—thus it has a unique edge e inside— consider the triple formed by e and the two edges of R of the same color as e. Then either the triple forms an V-pattern, in which case R is a left alternating 4-cycle, or the triple forms an N-pattern, in which case R is a right alternating 4-cycle.

FACT 12 ([**Fus09**]). *A triangulation of the 4-gon is irreducible if and only if it admits a transversal structure. An irreducible triangulation admits a unique transversal structure with no right alternating 4-cycle, called its* minimal *transversal structure.*

Again the terminology refers to the fact that the set of transversal structures of an irreducible triangulation is a distributive lattice whose minimal element is the transversal structure with no right alternating 4-cycle, see [**Fus09**].

LEMMA 13 (link with plane bipolar posets). *Let X be an N-avoiding transversal structure, and let X_r be the red bipolar poset of X. Then X has no right alternating 4-cycle if and only if X_r has no LOP.*

PROOF. Assume that X has a right alternating 4-cycle R, the vertices of R in clockwise order being denoted (s_R, w_1, t_R, w_2), with s_R the source and t_R the sink of R. Let f_1' be the face of X_r containing the blue edge (w_1, t_R), and let f_2' be the face of X_r containing the blue edge (s_R, w_2). As discussed at the end of the definition of right alternating 4-cycles, a right alternating 4-cycle with no vertex inside yields an N-pattern. Hence, since X is N-avoiding, R has at least one vertex inside, so the faces f_1' and f_2' are distinct. It is easily checked that (w_1, w_2, f_1', f_2') is a LOP, as shown in Figure 5(c). Conversely assume that X_r has a LOP (w_1, w_2, f_1', f_2'). Let s_R be the topleft lateral vertex of f_2', and let t_R be the bottomright lateral vertex of f_1'. Since X is N-avoiding, X is obtained from its red bipolar poset as $X = \phi'(X_r)$. By definition of ϕ' (see Figure 4(c)), in X there is a blue edge from s_R to w_2 and a blue edge from w_1 to t_R. Hence, the 4-cycle $R = (s_R, w_1, t_R, w_2)$ is an alternating 4-cycle. As there are red edges inside R incident to w_1 and w_2, s_R is source of blue edges, so R is a right alternating 4-cycle. $\qquad\square$

Let $\mathcal{T}_{\min}$ be the set of transversal structures with no right alternating 4-cycle (minimal transversal structures). Note that an N-pattern in a transversal structure yields a right alternating 4-cycle: the cycle delimited by the faces on both sides of the central edge of the N-pattern. Hence $\mathcal{T}_{\min}$ is also the set of

N-avoiding transversal structures with no right alternating 4-cycle, which by Lemma 13 is in bijection with $\mathcal{P}_{\min}$. Thus we obtain:

$$\mathcal{P}_{\min} \simeq_{\phi'} \mathcal{T}_{\min}. \tag{4}$$

4.3. The bijection. To conclude, we have

$$\mathcal{O}_{\min} \simeq_{\phi} \mathcal{P}_{\min} \simeq_{\phi'} \mathcal{T}_{\min}.$$

Hence $\Phi = \phi' \circ \phi$ specializes into a bijection between minimal plane bipolar orientations and minimal transversal structures. Since minimal plane bipolar orientations identify to (rooted) non-separable maps and minimal transversal structures identify to (rooted) irreducible triangulations, we obtain a bijection, called F_1, between these two map families. Precisely, given a rooted non-separable map M with at least 2 edges $T := F_1(M)$ is obtained as follows:

- endow M with its minimal plane bipolar orientation O, the root edge being deleted,
- compute the transversal structure X associated to O: $X := \Phi(O)$,
- return T as the irreducible triangulation underlying X, rooted at the edge going from N to W,

and $M = F_1^{-1}(T)$ is obtained as follows:

- endow T with its minimal transversal structure,
- compute the plane bipolar orientation O associated to X: $O := \psi(\psi'(X))$,
- return the rooted non-separable map M underlying O (M receives an additional root edge going from the source to the sink).

THEOREM 14. *For $n \geq 2$, the mapping F_1 is a bijection between rooted non-separable maps with n edges and irreducible triangulations with $n + 3$ vertices.*

PROOF. The bijection results from $\mathcal{O}_{\min} \simeq \mathcal{T}_{\min}$, as discussed above. The parameter-correspondence is inherited from the parameter-correspondences of ϕ and ϕ'. □

4.4. Counting non-separable maps. Brown and Tutte [**BT64**] used the recursive approach introduced by Tutte to show that the number of rooted non-separable maps with $n + 1$ edges and $i + 2$ vertices is

$$\Lambda_{n,i} = \frac{(n+i)!(2n-i-1)!}{(i+1)!(n-i)!(2i+1)!(2n-2i-1)!}. \tag{5}$$

And the number of rooted non-separable with $n + 1$ edges is $\lambda_n = \sum_{i=0}^{n-1} \Lambda_{n,i}$, which simplifies to

$$\lambda_n = \frac{2(3n)!}{(n+1)!(2n+1)!}. \tag{6}$$

Bijective proofs have been given later on, the first one being a correspondence with ternary trees bearing labels so as to satisfy a positivity condition [**JS98**]. A more direct construction has been described by Schaeffer in his PhD [**Sch98**], as a 4-to-$(2n + 2)$ correspondence between ternary trees with n nodes and rooted non-separable maps with $n + 1$ edges. The correspondence is based on local "closure" operations on a ternary tree, with the effect of closing one by one the faces of the associated map.

Our bijections relating maps via specific combinatorial structures, as described in Section 4, yields the following counting results:

PROPOSITION 15. *For $n \geq 1$ and $i \geq 0$, the number $\Lambda_{n,i}$ counts:*

- *rooted non-separable maps with $n + 2$ vertices, $i + 2$ faces, and whose minimal plane bipolar orientation (forgetting the root-edge) is a plane bipolar poset,*
- *rooted irreducible triangulations with n inner vertices and whose minimal transversal structure has $n + i + 1$ red edges.*

Taking into account the parameter n only, these map families are counted by λ_n.

REMARK. As for rooted non-separable maps, there exists a simple 4-to-$(2n + 2)$ correspondence between ternary trees with n nodes and rooted irreducible triangulations with n inner vertices [**Fus09**], again based on "closure" operations on the tree.

It turns out that our bijection F_1 puts in correspondence a non-separable map M and an irreducible triangulation T that have the same underlying (unrooted) ternary tree. This is no surprise, as the ternary

tree associated to M (T) is a spanning tree that arises from deleting specific edges in the minimal bipolar orientation of M (minimal transversal structure of T, resp.), whereas F_1 results from ϕ matching minimal plane bipolar orientations with minimal transversal structures. Hence, F_1 can be considered as a direct bijective construction on maps that has the same effect as computing the ternary tree associated to a non-separable map and then closing the tree into an irreducible triangulation.

5. Bijection between loopless maps and triangulations

5.1. Decomposing loopless maps into non-separable components. It is well known in graph theory that a connected graph G is decomposed into a collection of non-separable components called the *blocks*, which are the maximal non-separable subgraphs of G [**HP73, MT01**]. Conversely, the process of gluing non-separable graphs at common vertices in a tree-like fashion yields any connected graph in a unique way. This classical decomposition readily adapts to rooted loopless maps [**Tut63, GJ83**]. For our purpose it proves convenient to define the size $|M|$ of a rooted loopless map as its number of edges including the root. In this section we consider the vertex-map—made of a unique vertex and no edge—as a rooted loopless map. In contrast, non-separable maps are required to have at least one edge.

FACT 16. *The following process:*
- *take a rooted non-separable map C, and order the corners of C in a canonical way as $\alpha_1, \ldots, \alpha_{2|C|}$;*
- *take a collection $M_1, \ldots, M_{2|C|}$ of rooted loopless maps; for each $1 \leq i \leq 2|C|$, insert M_i in the corner α_i of C, merging the outer face of M_i with the face incident to α_i and the root vertex of M_i with the vertex incident to α_i,*

yields a rooted loopless map, and each rooted loopless map M is obtained in a unique way from this process.

The map C is called the core-map *of M, and the $(2|C|+1)$-tuple $\mathfrak{D}(M) = (C; M_1, \ldots, M_{2|C|})$ is called the* decomposition-tuple *of M. (Note that the size of M is the sum of the sizes of the maps in the decomposition-tuple.)*

5.2. Decomposing triangulations into irreducible components. Similarly as for loopless maps, there exists a classical decomposition of triangulations into components of higher connectivity: each triangulation on the topological sphere is obtained from a collection of 4-connected triangulations (which are triangulations where all 3-cycles are facial) glued at common triangles in a tree-like fashion. The same idea —decomposition at separating 3-cycles— works as well to decompose rooted triangulations of the 4-gon into rooted irreducible components [**Tut62**]. First let us introduce some terminology. Given a rooted triangulation T of the 4-gon, denote by N, E, S, W the outer vertices of T in clockwise order around the outer face, with N the root-vertex; then T is called *WE-diagonal* if W and E are adjacent, T is called *SN-diagonal* if S and N are adjacent, and T is called *non-diagonal* otherwise. Notice that the two diagonal cases are disjoint by planarity of T. The unique rooted irreducible triangulation that is *WE*-diagonal is the quadrangle (N, E, S, W) split by a diagonal (W, E); this map is called the *WE-link-map*. Similarly the unique rooted irreducible triangulation that is *SN*-diagonal is the quadrangle (N, E, S, W) split by a diagonal (S, N); this map is called the *SN-link-map*. For our purpose it is convenient to define the size of a triangulation or of a triangulation of the 4-gon T as $||T|| := (|V(T)| - 3)$. Notice that $||T||$ is the number of inner vertices if T is a triangulation and $2||T||$ is the number of inner faces if T is a triangulation of the 4-gon, by the Euler relation.

FACT 17. *The following process:*
- *take a rooted irreducible triangulation I, and order the inner faces of I in a canonical way as $f_1, \ldots, f_{2||I||}$;*
- *take a collection $T_1, \ldots, T_{2||I||}$ of rooted triangulations; for each $1 \leq i \leq 2||I||$, substitute f_i by T_i in a canonical way (e.g., the outer triangle of T_i fits with the contour of f_i and the root vertex of T_i fits with a distinguished vertex of f_i)*

yields a rooted triangulation of the 4-gon; and each rooted triangulation T of the 4-gon is obtained in a unique way from this process.

The map I is called the core-triangulation *of T; T is non-diagonal if I has at least one inner vertex, is WE-diagonal if I is the WE-link-map, and is SN-diagonal if I is the SN-link-map. The $(2||I|| + 1)$-tuple $\mathfrak{D}(T) = (I; T_1, \ldots, T_{2||I||})$ is called the* decomposition-tuple *of T. (Note that the size of T is the sum of the sizes of the maps in the decomposition-tuple.)*

As a corollary we can describe the decomposition-tuples corresponding to rooted triangulations. Indeed, in each size $k \geq 1$, there is a simple bijection between rooted triangulations and rooted triangulations of the 4-gon that are not WE-diagonal. Starting from a rooted triangulation T, delete the edge following the root in counterclockwise order around the outer face; the obtained map is a triangulation of the 4-gon, and W and E are not adjacent, since the unique edge connecting them in T has been deleted. Conversely, starting from a rooted triangulation $\widetilde{T}$ of the 4-gon that is not WE-diagonal, add an edge between E and W. Since W and E are not adjacent in $\widetilde{T}$, the obtained map T has no multiple edges (nor loops) and has clearly all faces of degree 3, so T is a triangulation.

COROLLARY 18. *In each size $k \geq 1$, rooted triangulations can be identified with rooted triangulations of the 4-gon that are not WE-diagonal, which themselves identify to decomposition-tuples whose core-triangulation is not the WE-link-map.*

5.3. The bijection. Our size-preserving bijection, called F_2, between rooted loopless maps and rooted triangulations is defined recursively from the bijection F_1. Recall that F_1 maps a non-separable map of size at least 2 (at least one non-root edge) to an irreducible triangulation of the same size (at least one inner vertex). It is convenient here to augment the correspondence F_1 into a bijection $\widetilde{F}_1$ with one object of size 1 on each side: the edge-map—made of two vertices connected by an edge—corresponds to the SN-link-map. We denote by $\mathcal{L}$ the family of (rooted) loopless maps including the vertex-map, by $\mathcal{L}_{\leq n}$ the set of loopless maps of size at most n, and by $\mathcal{L}_n$ the set of loopless maps of size n. Similarly, denote by $\mathcal{T}$ the family of (rooted) triangulations including the triangle, by $\mathcal{T}_{\leq n}$ the set of triangulations of size at most n, and by $\mathcal{T}_n$ the set of triangulations of size n. Finally, denote by $\mathcal{N}$ the family of (rooted) non-separable maps with at least one edge and by $\mathcal{I}$ the family of (rooted) irreducible triangulations with at least one inner vertex. Note that $\widetilde{F}_1$ is a size-preserving bijection between $\mathcal{N}$ and $\mathcal{I} \cup \{SN\text{-link-map}\}$.

The size-preserving bijection F_2 between $\mathcal{L}$ and $\mathcal{T}$ is specified recursively as follows. First, the vertex-map—the unique rooted loopless map M such that $|M| = 0$—is mapped by F_2 to the triangle-map—the unique rooted triangulation T such that $||T|| = 0$. Given $n \geq 0$, assume that F_2 is a well defined bijection from $\mathcal{L}_{\leq n}$ to $\mathcal{T}_{\leq n}$, that is size-preserving: $||F_2(M)|| = |M|$ for every $M \in \mathcal{L}_{\leq n}$.

Let us now extend the bijection F_2 to maps of size $n + 1$. By Fact 16, a tuple $(C; M_1, \ldots, M_{2|C|})$ corresponds to a loopless map in $\mathcal{L}_{n+1}$ if and only if C is in $\mathcal{N}$, the M_i's are in $\mathcal{L}$, and $|C| + \sum_i |M_i| = n+1$, in which case all the M_i's must be in $\mathcal{L}_{\leq n}$. Let $\mathcal{DL}_{n+1}$ be the set of such decomposition-tuples, note that $\mathcal{DL}_{n+1} \simeq \mathcal{L}_{n+1}$. Similarly, by Fact 17 and Corollary 18, a tuple $(I; T_1, \ldots, T_{2||I||})$ corresponds to a triangulation in $\mathcal{T}_{n+1}$ if and only if I is in $\mathcal{I} \cup \{SN - \text{link} - \text{map}\}$, the T_i's are in $\mathcal{T}$, and $||I|| + \sum_i ||T_i|| = n + 1$, in which case all the T_i's must be in $\mathcal{T}_{\leq n}$. Let $\mathcal{DT}_{n+1}$ be the set of such decomposition-tuples, note that $\mathcal{DT}_{n+1} \simeq \mathcal{T}_{n+1}$. Since $\widetilde{F}_1$ is a size-preserving bijection between $\mathcal{N}$ and $\mathcal{I} \cup \{SN\text{-link-map}\}$ and since, by induction, F_2 is a bijection between $\mathcal{L}_{\leq n}$ and $\mathcal{T}_{\leq n}$, we obtain a bijection $\mathcal{DL}_{n+1} \simeq \mathcal{DT}_{n+1}$: for $(C; M_1, \ldots, M_{2|C|}) \in \mathcal{DL}_{n+1}$, the associated tuple in $\mathcal{DT}_{n+1}$ is $(\widetilde{F}_1(C); F_2(M_1), \ldots, F_2(M_{2|C|}))$. Since $\mathcal{DL}_{n+1} \simeq \mathcal{L}_{n+1}$ and $\mathcal{DT}_{n+1} \simeq \mathcal{T}_{n+1}$, we conclude that $\mathcal{L}_{n+1}$ is in bijection with $\mathcal{T}_{n+1}$, and we call F_2 the bijection extended to $\mathcal{L}_{\leq n+1}$.

THEOREM 19. *For $n \geq 0$, the mapping F_2 is a bijection between rooted loopless maps with n edges and rooted triangulations with n inner vertices.*

REMARK. Let us make a few comments on this bijection. Denote by $\mathcal{L}_{n,k}$ the family of rooted loopless maps with n edges and root-vertex of degree k, and denote by $\mathcal{T}_{n,k}$ the family of rooted triangulations with n inner vertices and root-vertex of degree $k+2$. The equality $|\mathcal{L}_n| = |\mathcal{T}_n|$ is known for long [**Tut62, Tut73**], as both $|\mathcal{L}_n|$ and $|\mathcal{T}_n|$ are equal to

$$(7) \qquad\qquad a_n := \frac{2(4n+1)!}{(n+1)!(3n+2)!}.$$

A first bijective proof of $|\mathcal{L}_n| = |\mathcal{T}_n|$, more precisely of $|\mathcal{L}_{n,k}| = |\mathcal{T}_{n,k}|$, has been found by Wormald [**Wor80**], based on the observation that the generating trees (induced by root-edge-deletion) of $\mathcal{L} = \cup_{n,k}\mathcal{L}_{n,k}$ and of $\mathcal{T} = \cup_{n,k}\mathcal{T}_{n,k}$ are isomorphic. The bijection is thus recursive; a rooted loopless map is first completely decomposed to find out the place it occupies in the generating tree of $\mathcal{L}$, then the associated rooted triangulation is the one occupying the same place in the generating tree of $\mathcal{T}$.

Our bijection has the original feature of being a mixing of a recursive construction (the parallel recursive decompositions into non-separable/irreducible components) and a direct construction (the mapping F_2 to match non-separable/irreducible components). The advantage is that, in practice, there should be fewer levels of recursion in our bijection than in Wormald's one. Indeed, as shown in [**BFSS01**], a typical loopless map is made of a giant non-separable component —of linear size— having small connected components attached in each corner.

Let us also mention that direct bijective proofs of $|\mathcal{T}_n| = a_n$ have been found recently [**PS06, FPS08**], again based on closure operations on trees. However, no bijection with trees is known yet to explain $|\mathcal{L}_n| = a_n$. It would be interesting to find out such a construction for loopless maps, and hopefully derive from it a direct (nonrecursive) bijection between $\mathcal{L}_n$ and $\mathcal{T}_n$.

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