Directionally Convex Ordering of Random Measures, Shot Noise Fields and Some Applications to Wireless Communications

Bartlomiej Blaszczyszyn, D. Yogeshwaran

DIRECTIONALLY CONVEX ORDERING OF RANDOM MEASURES, SHOT NOISE FIELDS AND SOME APPLICATIONS TO WIRELESS COMMUNICATIONS

BARTŁOMIEJ BŁASZCZYSZYN,* *INRIA/ENS and Math. Inst. University of Wrocław*

D. YOGESHWARAN,** *INRIA/ENS*

Abstract

Directionally convex (dcx) ordering is a tool for comparison of dependence structure of random vectors that also takes into account the variability of the marginal distributions. When extended to random fields it concerns comparison of all finite dimensional distributions. Viewing locally finite measures as non-negative fields of measure-values indexed by the bounded Borel subsets of the space, in this paper we formulate and study the dcx ordering of random measures on locally compact spaces. We show that the dcx order is preserved under some of the natural operations considered on random measures and point processes, such as deterministic displacement of points, independent superposition and thinning as well as independent, identically distributed marking. Further operations such as position dependent marking and displacement of points are shown to preserve the order on Cox point processes. We also examine the impact of dcx order on the second moment properties, in particular on clustering and on Palm distributions. Comparisons of Ripley's functions, pair correlation functions as well as examples seem to indicate that point processes higher in dcx order cluster more.

As the main result, we show that non-negative integral shot-noise fields with respect to dcx ordered random measures inherit this ordering from the measures. Numerous applications of this result are shown, in particular to comparison of various Cox processes and some performance measures of wireless networks, in both of which shot-noise fields appear as key ingredients. We also mention a few pertinent open questions.

Keywords: stochastic ordering, directional convexity, random measures, random fields, point processes, wireless networks

2000 Mathematics Subject Classification: Primary 60E15

Secondary 60G60, 60G57, 60G55

* Postal address: ENS DI TREC, 45 rue d'ulm, 75230 Paris, FRANCE.

** Postal address: ENS DI TREC, 45 rue d'ulm, 75230 Paris, FRANCE.

1. Introduction

Point processes (p.p.) have been at the centre of various studies in stochastic geometry, both theoretical and applied. Most of the work involving quantitative analysis of p.p. have dealt with Poisson p.p.. One of the main reasons being that characteristics of Poisson p.p. are amenable to computations and yield nice closed form expressions in many cases. Computations have been difficult in great many cases, even for Cox (doubly stochastic Poisson) p.p..

Comparison of point processes To improve upon this situation, qualitative, comparative studies of p.p. have emerged as useful tools. The first method of comparison of p.p. has been coupling or stochastic domination (see [18,20,32]). In our terminology, these are known as strong ordering of p.p.. When two p.p. can be coupled, one turns out to be a subset of the other. This ordering is very useful for obtaining various bounds and proving limit theorems. However, using it one cannot compare two different p.p. with same mean measures. An obvious example is an homogeneous Poisson p.p. and a stationary Cox p.p. with the same intensity. The question arises of what ordering is suitable for such p.p.? This is an important question since it is expected that by comparing p.p. of the same intensity one should achieve a tighter bound than by coupling. For some more details on strong ordering of p.p. and need for other orders, see remarks in [29, Section 5.4 and Section 7.4.2].

From convex to *dcx* order Two random variables X and Y with the same mean $\mathrm{E}(X) = \mathrm{E}(Y)$ can be compared by how "spread out" their distributions are. This *statistical variability* (in statistical ensemble) is captured to a limited extent by the variance, but more fully by convex ordering, under which X is less than Y if and only if for all convex f, $\mathrm{E}(f(A)) \leq \mathrm{E}(f(B))$. In multi-dimensions, besides different statistical variability of marginal distributions, two random vectors can exhibit different *dependence properties* of their coordinates. The most evident example here is comparison of the vector composed of several copies of one random variable to a vector composed of independent copies sampled from the same distribution. A useful tool for comparison of the dependence structure of random vectors with fixed marginals is the *supermodular order*. The *dcx* order is another integral order (generated by a class of *dcx* functions in the same manner as convex functions generate the convex order) that can be seen as a generalization of the supermodular one, which in addition takes into account the variability of the marginals (cf [29, Section 3.12]). It can be naturally extended to random fields by comparison of all finite dimensional distributions.

The *dcx* order of random measures In this paper we make an obvious further extension that consists in *dcx* ordering of locally finite measures (to which belong p.p.) viewed as non-negative fields of measure-values on all bounded subsets of the space. We show that the *dcx* order is preserved under some of the natural operations considered on random measures and point processes, such as independent superposition and thinning. Also, we examine the impact of *dcx* order on the second moment properties, in particular on clustering, and Palm distributions.

Integral shot-noise fields Many interesting characteristics of random measures, both in the theory and in applications have the form of integrals of some non-negative kernels. We call them *integral shot-noise fields.* For example, many classes of Cox p.p., with the most general being Lévy based Cox p.p. (cf. [14]), have stochastic intensity fields, which are shot-noise fields. They are also key ingredients of the recently proposed, so-called "physical" models for wireless networks, as we will explain in what follows (see also [1, 8, 11]). It is thus particularly appealing to study the shot-noise fields generated by *dcx* ordered random measures.

Since integrals are linear operators on the space of measures, and knowing that a linear function of a vector is trivially *dcx*, it is naturally to expect that the integral shot-noise fields with respect to *dcx* ordered random measures will inherit this ordering from the measures. However, this property cannot be concluded immediately from the finite dimensional *dcx* ordering of measures. The formal proof of this fact that is the main result of this paper involves some arguments from the theory of integration combined with the closure property of *dcx* order under joint weak convergence and convergence in mean.

Ordering in queueing theory and wireless communications The theory of stochastic ordering provides elegant and efficient tools for comparison of random objects and is now being used in many fields. In particular in queueing theory context, in [33], Ross made a conjecture that replacing a stationary Poisson arrival process in a single server queue by a stationary Cox p.p. with the same intensity should increase the average customer delay. There have been many variations of these conjectures which are now known as Ross-type conjectures. They triggered the interest in comparison of queues with similar inputs ([6,25,31]). The notion of a *dcx* function was partially developed and used in conjunction with the proving of Ross-type conjectures ([21,22, 34]). Much earlier to these works, a comparative study of queues motivated by neuron-firing models can be found in [16]. Also comparison of variances of point processes and fibre processes was studied in [36] and hence it can be considered as a forerunner to our article. The applicability of these results has generated sufficient interest in the theory of stochastic ordering as can be seen from the diverse results in the book of Müller and Stoyan ([29]). As most works on ordering of p.p. were motivated by applications to queueing theory, results were primarily focused on one-dimensional point processes. An attempt to rectify the lack of work in higher dimensions was made in [24], where comparison results for shot-noise fields of spatial stationary Cox p.p. were given. The results of [24] are the starting point of our investigation.

Our interest in ordering of point processes, and in particular in the shot-noise fields they generate, has roots in the analysis of wireless communications, where these objects are primarily used to model the so called *interference* that is the total power received from many emitters scattered in the plane or space and sharing the common Hertzian medium. According to a new emerging methodology, the interference-aware stochastic geometry modeling of wireless communications provides a way of defining and computing macroscopic properties of large wireless networks by some averaging over all potential random patterns for node locations in an infinite plane and radio channel characteristics, in the same way as queuing theory provides averaged response times or congestion over all potential arrival patterns within a given parametric class. These macroscopic properties will allow one to characterize the key dependencies of the network performance characteristics in function of a relatively small number of parameters.

In the above context, Poisson distribution of emitters/receiver/users is often too simplistic. Statistics show that the real patterns of users exhibits more clustering effects ("hots spots") than observed in an homogeneous Poisson point processes. On the other hand, good packet-collision-avoidance mechanisms scheme should create some "repulsion" in the pattern of nodes allowed to access simultaneously to the channel. This rises questions about the analysis of non-Poisson models, which could be to some extent tackled on the ground of the theory of stochastic ordering. Interestingly, we shall show that there are certain performance characteristics in wireless networks that *improve* with more variability in the input process.

The remaining part of the article is organized as follows. In the next section, we will present the main definitions and state the main results concerning *dcx* ordering of the integral shot-noise fields. Section 3 will explore the various consequences of ordering of random measures. The proofs of the main results are given in Section 4. Examples illustrating the use and application of the theorems shall be presented in Section 5. Section 6 will sketch some of the possible applications of results in the context of wireless communications. Finally, we conclude with some remarks and questions in Section 7. There is an Appendix (Section 8) containing some properties of stochastic orders and their extensions that are used in the paper.

2. Definitions and the Main Result

The order $\leq$ on $\mathbb{R}^n$ shall denote the component-wise partial order, i.e., $(x_1, \ldots, x_n) \leq (y_1, \ldots, y_n)$ if $x_i \leq y_i$ for every i.

Definition 2.1.
- We say that a function $f : \mathbb{R}^d \to R$ is *directionally convex* (*dcx*) if for every $x, y, p, q \in \mathbb{R}^d$ such that $p \leq x, y \leq q$ and $x + y = p + q$,

$$f(x) + f(y) \leq f(p) + f(q).$$

- Function f is said to be *directionally concave* (*dcv*) if the inequality in the above equation is reversed.
- Function f is said *directionally linear* (*dl*) if it is *dcx* and *dcv*.

Function $f = (f_1, \ldots, f_n) : \mathbb{R}^d \to \mathbb{R}^n$ is said to be *dcx*(*dcv*) if each of its component f_i is *dcx*(*dcv*). Also, we shall abbreviate *increasing* and *dcx* by *idcx* and *decreasing* and *dcx* by *ddcx*. Similar abbreviations shall be used for *dcv* functions. Moreover, we abbreviate *non-negative* and *idcx* by $idcx^+$.

In the following, let $\mathfrak{F}$ denote some class of functions from $\mathbb{R}^d$ to $\mathbb{R}$. The dimension d is assumed to be clear from the context. Unless mentioned, when we state $\mathrm{E}(f(X))$ for $f \in \mathfrak{F}$ and X a random vector, we assume that the expectation exists, i.e., for each random vector X we consider the sub-class of $\mathfrak{F}$ for which the expectations exist with respect to (w.r.t) X.

Definition 2.2.
- Suppose X and Y are real-valued random vectors of the same dimension. Then X is said to be *less than* Y *in* $\mathfrak{F}$ *order* if $\mathrm{E}(f(X)) \leq \mathrm{E}(f(Y))$ for all $f \in \mathfrak{F}$ (for which both expectations are finite). We shall denote it as $X \leq_{\mathfrak{F}} Y$.
- Suppose $\{X(s)\}_{s \in S}$ and $\{Y(s)\}_{s \in S}$ are real-valued random fields, where S is an arbitrary index set. We say that $\{X(s)\} \leq_{\mathfrak{F}} \{Y(s)\}$ if for every $n \geq 1$ and $s_1, \ldots, s_n \in S$, $(X(s_1), \ldots, X(s_n)) \leq_{\mathfrak{F}} (Y(s_1), \ldots, Y(s_n))$.

In the remaining part of the paper, we will mainly consider $\mathfrak{F}$ to be the class of *dcx*, *idcx* and *idcv* functions; the negation of these functions give rise to *dcv*, *ddcv* and *ddcx* orders respectively. If $\mathfrak{F}$ is the class of increasing functions, we shall replace $\mathfrak{F}$ by *st* (strong) in the above definitions. These are standard notations used in literature.

As concerns random measures, we shall work in the set-up of [17]. Let $\mathbb{E}$ be a locally compact, second countable Hausdorff (LCSC) space. Such spaces are polish, i.e., complete and separable metric space. Let $\mathtt{B}(\mathbb{E})$ be the Borel σ-algebra and $\mathtt{B}_b(\mathbb{E})$ be the σ-ring of *bounded, Borel subsets* (bBs). Let $\mathbb{M} = \mathbb{M}(\mathbb{E})$ be the space of non-negative Radon measures on $\mathbb{E}$. The Borel σ-algebra $\mathcal{M}$ is generated by the mappings $\mu \mapsto \mu(B)$ for all B bBs. A random measure Λ is a mapping from a probability space

$(\Omega, \mathcal{F}, \mathrm{P})$ to $(\mathbb{M}, \mathcal{M})$. We shall call a random measure Φ a p.p. if $\Phi \in \bar{\mathbb{N}}$, the subset of counting measures in $\mathbb{M}$. Further, we shall say a p.p. Φ is simple if a.s. $\Phi(\{x\}) \leq 1$ for all $x \in \mathbb{E}$. Throughout, we shall use Λ for an arbitrary random measure and Φ for a p.p.. A random measure Λ can be viewed as a random field $\{\Lambda(B)\}_{B \in \mathrm{B}_b(\mathbb{E})}$. With this viewpoint and the previously introduced notion of ordering of random fields, we define ordering of random measures.

Definition 2.3. Suppose $\Lambda_1(\cdot)$ and $\Lambda_2(\cdot)$ are random measures on $\mathbb{E}$. We say that $\Lambda_1(\cdot) \leq_{dcx} \Lambda_2(\cdot)$ if for any $I_1, \ldots, I_n$ bBs in $\mathbb{E}$,

$$(\Lambda_1(I_1), \ldots, \Lambda_1(I_n)) \leq_{dcx} (\Lambda_2(I_1), \ldots, \Lambda_2(I_n)). \tag{1}$$

The definition is similar for other orders, i.e., when $\mathfrak{F}$ is the class of *idcx/idcv/ddcx/ddcv/st* functions.

Definition 2.4. Let S be any set and $\mathbb{E}$ a LCSC space. Given a random measure Λ on $\mathbb{E}$ and a measurable (in the first variable alone) *response function* $h(x,y) : \mathbb{E} \times S \to \bar{\mathbb{R}}^+$ where $\bar{\mathbb{R}}^+$ denotes the completion of positive real-line with infinity, the *(integral) shot-noise field* is defined as

$$V_\Lambda(y) = \int_{\mathbb{E}} h(x,y)\Lambda(dx). \tag{2}$$

With this brief introduction, we are ready to state our key result that will be proved in Section 4.1.

Theorem 2.1. *1. If $\Lambda_1 \leq_{idcx\,(resp.\,idcv)} \Lambda_2$, then $\{V_{\Lambda_1}(y)\}_{y \in S} \leq_{idcx\,(resp.\,idcv)} \{V_{\Lambda_2}(y)\}_{y \in S}$.*

2. Let $\mathrm{E}(V_{\Lambda_i}(y)) < \infty$, for all $y \in S$, $i = 1, 2$. If $\Lambda_1 \leq_{dcx} \Lambda_2$, then $\{V_{\Lambda_1}(y)\}_{y \in S} \leq_{dcx} \{V_{\Lambda_2}(y)\}_{y \in S}$.

The first part of the above theorem for the one-dimensional marginals of bounded shot-noise fields generated by lower semi-continuous response functions is proved in [24] for the special case of spatial stationary Cox p.p.. It is conspicuous that we have generalized the earlier result to a great extent. This more general result will be used in many places in this paper, in particular to prove ordering of independently, identically marked p.p. (Proposition 3.2), Ripley's functions (Proposition 3.4), Palm measures (Proposition 3.5), independently marked Cox processes (Proposition 3.7), extremal shot-noise fields (Proposition 4.1). Apart form these results, Sections 5 and 6 shall amply demonstrate examples and applications that shall need Theorem 2.1.

3. Ordering of Random Measures and Point Processes

We shall now give a sufficient condition for random measures to be ordered, namely that the condition (1) in Definition 2.3 needs to be verified only for disjoint bBs. The necessity is trivial. This is a much easier condition and will be used many times in the remaining part of the paper.

Proposition 3.1. *Suppose* $\Lambda_1(\cdot)$ *and* $\Lambda_2(\cdot)$ *are two random measures on* $\mathbb{E}$. *Then* $\Lambda_1(\cdot) \leq_{dcx} \Lambda_2(\cdot)$ *if and only if condition (1) holds for all* mutually disjoint *bBs. The same results holds true for idcx and idcv order.*

Proof. We need to prove the 'if' part alone. We shall prove for *dcx* order and the same argument is valid for f being *idcx* or *idcv*. Let condition (1) be satisfied for all mutually disjoint bBs. Let $f : \mathbb{R}_+^n \to R$ be *dcx* function and $B_1, \ldots, B_n$ be bBs. We can choose mutually disjoint bBs $A_1, \ldots, A_m$ such that $B_i = \cup_{j \in J_i} A_j$ for all i. Hence $\Lambda(B_i) = \sum_{j \in J_i} \Lambda(A_j)$. Now define $g : \mathbb{R}_+^m \to \mathbb{R}_+^n$ as $g(x_1, \ldots, x_m) = (\sum_{j \in J_1} x_j, \ldots, \sum_{j \in J_n} x_j)$. Then g is *idl* and so $f \circ g$ is *dcx*. Moreover, $f(\Lambda(B_1), \ldots, \Lambda(B_n)) = f \circ g(\Lambda(A_1), \ldots, \Lambda(A_m))$ and thus the result for *dcx* follows. □

3.1. Simple Operations Preserving Order

Point processes are special cases of random measures and as such will be subject to the considered ordering. It is known that each p.p. Φ on a LCSC space $\mathbb{E}$ can be represented as a countable sum $\Phi = \sum_i \varepsilon_{X_i}$ of Dirac measures ($\varepsilon_x(A) = 1$ if $x \in A$ and 0 otherwise) in such a way that X_i are random elements in $\mathbb{E}$. We shall now show that all the three orders $dcx, idcx, idcv$ preserve some simple operations on random measures and p.p., as deterministic mapping, independent identically distributed (i.i.d.) thinning and independent superposition.

Let $\phi : \mathbb{E} \to \mathbb{E}'$ be a measurable mapping to some LCSC space $\mathbb{E}'$. By the image of a (random) measure Λ by ϕ we understand $\Lambda'(\cdot) = \Lambda(\phi^{-1}(\cdot))$. Note that the image of a p.p. Φ by ϕ consists in deterministic displacement of all its points by ϕ.

Let $\Phi = \sum_i \varepsilon_{x_i}$. By *i.i.d. marking* of Φ, with marks in some LCSC space $\mathbb{E}'$, we understand a p.p. on the product space $\mathbb{E} \times \mathbb{E}'$, with the usual product Borel σ-algebra, defined by $\tilde{\Phi} = \sum_i \varepsilon_{(x_i, Z_i)}$, where $\{Z_i\}$ are i.i.d. random variables (r.v.), so called marks, on $\mathbb{E}'$. By *i.i.d. thinning* of Φ, we understand $\overline{\Phi} = \sum_i Z_i \varepsilon_{x_i}$, where Z_i are i.i.d. 0-1 Bernoulli random variables r.v.. The probability $\mathrm{P}\{Z = 1\}$ is called the *retention probability. Superposition* of p.p. is understood as addition of (counting) measures. Measures on Cartesian products of LCSC spaces are always considered with their corresponding product Borel σ-algebras.

Proposition 3.2. *Suppose* $\Lambda_i, i = 1, 2$ *are random measures and* $\Phi_i, i = 1, 2$ *are p.p..*

Assume that $\Lambda_1 \leq_{dcx\,(resp.\,idcx;\,idcv)} \Lambda_2$ *and* $\Phi_1 \leq_{dcx\,(resp.\,idcx;\,idcv)} \Phi_2$.

1. *Let* Λ'_i *be the image of* Λ_i, $i = 1, 2$, *by some mapping* $\phi : \mathbb{E} \to \mathbb{E}'$. *Then* $\Lambda'_1 \leq_{dcx\,(resp.\,idcx;\,idcv)} \Lambda'_2$. *As a special case, the same holds true for the displacement of points of* Φ_i*'s by* ϕ.

2. *Let* $\Phi_i, i = 1, 2$, *be simple p.p. and* $\tilde{\Phi}_i, i = 1, 2$, *be the corresponding i.i.d. marked p.p. with the same distribution of marks. Then* $\tilde{\Phi}_1 \leq_{dcx\,(resp.\,idcx\,;idcv)} \tilde{\Phi}_2$.

3. *Then* $\overline{\Phi}_i$ *be i.i.d. thinning of* Φ_i, $i = 1, 2$, *with the same retention probability. Then* $\overline{\Phi}_1 \leq_{dcx\,(resp.\,idcx\,;idcv)} \overline{\Phi}_2$.

4. *Let* Λ'_1 *and* Λ'_2 *be two random measures such that* $\Lambda'_1 \leq_{dcx\,(resp.\,idcx;\,idcv)} \Lambda'_2$. *Assume that* Λ'_i*'s are independent of* Λ_i*'s. Then* $\Lambda_1 + \Lambda'_1 \leq_{dcx\,(resp.\,idcx;\,idcv)} \Lambda_2 + \Lambda'_2$, *where* $+$ *is understood as the addition of measures.*

5. *Suppose the random measures are on the product space* $\mathbb{E} \times \mathbb{E}'$. *Then* $\Lambda_1(\mathbb{E} \times \cdot) \leq_{dcx\,(resp.\,idcx;\,idcv)} \Lambda_2(\mathbb{E} \times \cdot)$, *provided the respective projections are Radon measures.*

Proof. 1. The result follows immediately from the Definition 2.3.

2. We shall prove $\tilde{\Phi}_1 \leq_{dcx} \tilde{\Phi}_2$ and the proof for the other orders is similar. Since $\mathbb{E}$ is a LCSC space, there exists a *null-array of partitions* $\{B_{n,j} \subset \mathbb{E}\}_{n\geq 1, j\geq 1}$, i.e., $\{B_{n,j}\}_{j\geq 1}$ form a finite partition of $\mathbb{E}$ for every n and $\max_{j\geq 1}\{|B_{n,j}|\} \to 0$ as $n \to \infty$ where $|\cdot|$ denotes the diameter in any fixed metric (see [17, page 11]). For every $x \in \mathbb{E}$, let $j(n,x)$ be the unique index such that $x \in B_{n,j(n,x)}$. Let $\overline{Z} = \{Z_{n,j}\}_{n\geq 1, j\geq 1}$ be a family of $\mathbb{E}'$-valued i.i.d. random variables with distribution $F(\cdot)$. Define marked p.p. $\tilde{\Phi}^n_i = \sum_{X_k \in \Phi_i} \varepsilon_{(X_k, Z_{n,j(n,X_k)})}$ for $i = 1, 2$. We shall now verify that the sequences $\tilde{\Phi}^n_i$'s satisfy the assumption of Lemma 8.2 with limits $\tilde{\Phi}_i$'s respectively.

   Firstly let $B_1, \ldots, B_m \subset \mathbb{E} \times \mathbb{E}'$ be bBs and $g : \mathbb{R}^m \to \mathbb{R}$ be a continuous bounded function. Since B_i's are bounded and Φ_i's are simple, given $\Phi_i, i = 1, 2$, there exists a.s. $N(\Phi_i) \in \mathbb{N}$ such that for $n \geq N(\Phi_i)$, the indices $j(n, X_k) \neq j(n, X_l)$ for $X_k \neq X_l$, $X_k, X_l \in \Phi_i \cap (B_1 \cup \ldots \cup B_m)$. Hence for $n \geq N(\Phi_i)$, $\mathrm{E}(g(\tilde{\Phi}^n_i(B_1), \ldots, \tilde{\Phi}^n_i(B_m))|\Phi_i) = \mathrm{E}(g(\tilde{\Phi}_i(B_1), \ldots, \tilde{\Phi}_i(B_m))|\Phi_i)$ and in consequence $\mathrm{E}(g(\tilde{\Phi}^n_i(B_1), \ldots, \tilde{\Phi}^n_i(B_m))|\Phi_i) \to \mathrm{E}(g(\tilde{\Phi}_i(B_1), \ldots, \tilde{\Phi}_i(B_m))|\Phi_i)$ a.s. as $n \to \infty$. Since g is bounded, by dominated convergence theorem we have that $\mathrm{E}(g(\tilde{\Phi}^n_i(B_1), \ldots, \tilde{\Phi}^n_i(B_m))) \to \mathrm{E}(g(\tilde{\Phi}_i(B_1), \ldots, \tilde{\Phi}_i(B_m)))$. Thus $(\tilde{\Phi}^n_i(B_1), \ldots, \tilde{\Phi}^n_i(B_m)) \xrightarrow{\mathrm{D}} (\tilde{\Phi}^n_i(B_1), \ldots, \tilde{\Phi}^n_i(B_m))$. Secondly it is easy to check that for $B_1 = B' \times B''$, we have $\mathrm{E}(\tilde{\Phi}^n_i(B_1)) = \mathrm{E}(\Phi_i(B'))F(B'') = \mathrm{E}(\tilde{\Phi}_i(B_1))$ and hence by an appropriate approximation $\mathrm{E}(\tilde{\Phi}^n_i(B_1)) = \mathrm{E}(\tilde{\Phi}_i(B_1))$ for any bBs B_1.

Finally for any bBs $B \subset \mathbb{E}\times\mathbb{E}'$ and any realization $\overline{Z} = \overline{z} = \{z_{n,j}\}_{n\geq 1, j\geq 1}$, define $V_i^{\overline{z}}(B) := \int_{\mathbb{E}} \mathbf{1}[(x, z_{n,j(n,x)}) \in B]\Phi_i(dx)$. Since $z_{n,j(n,\cdot)}$ is a piecewise constant function, $1[(x, z_{n,j(n,x)}) \in B]$ is a measurable function in x and so $V_i^{\overline{z}}$'s are integral shot-noise fields (as per Definition 2.4) indexed by bBs of $\mathbb{E}\times\mathbb{E}'$. Thus from Theorem 2.1, we have that for any *dcx* function f,

$$
\begin{aligned}
&\mathrm{E}(f(\tilde{\Phi}_1^n(B_1),\ldots,\tilde{\Phi}_1^n(B_m))|\overline{Z}=\overline{z}) = \mathrm{E}(f(V_1^{\overline{z}}(B_1),\ldots,V_1^{\overline{z}}(B_m))) \\
&\quad\leq \ \mathrm{E}(f(V_2^{\overline{z}}(B_1),\ldots,V_2^{\overline{z}}(B_m))) = \mathrm{E}(f(\tilde{\Phi}_2^n(B_1),\ldots,\tilde{\Phi}_2^n(B_m))|\overline{Z}=\overline{z})
\end{aligned}
$$

Now, taking further expectations we get $(\tilde{\Phi}_1^n(B_1),\ldots,\tilde{\Phi}_1^n(B_m)) \leq_{dcx} (\tilde{\Phi}_2^n(B_1),\ldots,\tilde{\Phi}_2^n(B_m))$. Since the approximation satisfies the assumption of Lemma 8.2, the proof follows.

3. We need to prove $\mathrm{E}(f(\overline{\Phi}_1(A_1),\ldots,\overline{\Phi}_1(A_n))) \leq \mathrm{E}(f(\overline{\Phi}_2(A_1),\ldots,\overline{\Phi}_1(A_n)))$ for *dcx* (resp. *idc*; *idcv*) function f and mutually disjoint A_k, $k=1,\ldots,n$; cf. Proposition 3.1. Note that given $\Phi(A_k) = n_k$, we have $\overline{\Phi}(A_k) = \sum_{i=1}^{n_k} Z_i^k$, where Z_i^k are i.i.d. copies of the Bernoulli thinning variable. Thus the result follows from the first statement of Lemma 8.3.

4. Using the following fact from [29]: $X \leq_{dcx\,(\text{resp.}\,idcx;\,idcv)} Y$ implies $X+Z \leq_{dcx\,(\text{resp.}\,idcx;\,idcv)} Y+Z$ provided Z is independent of X and Y one can easily show that $\Lambda_1 + \Lambda_1' \leq_{dcx\,(\text{resp.}\,idcx;\,idcv)} \Lambda_2 + \Lambda_1'$ assuming Λ_1' independent of Λ_2. The same argument shows that $\Lambda_2 + \Lambda_1' \leq_{dcx\,(\text{resp.}\,idcx;\,idcv)} \Lambda_2 + \Lambda_2'$. The result follows by the transitivity of the order.

5. This result follows easily from Lemma 8.2 using an increasing approximation of $\mathbb{E}$ by bBs.

□

3.2. Impact on Higher Order Properties

We will state now some results involving ordering of moments of random measures and draw some conclusions concerning the so called second order properties. These latter ones make it possible to characterize the clustering in p.p..

By the nth power of random measure Λ, we understand a random measure Λ^k on the product space $\mathbb{E}^k$ given by $\Lambda^k(A_1 \times \ldots \times A_k) = \prod_{j=1}^k \Lambda(A_j)$. Its expectation, $\alpha^k(\cdot) = \mathrm{E}(\Lambda^k(\cdot))$ is called the kth moment measure. The first moment measure $\alpha(\cdot) = \alpha^1(\cdot)$ is called the mean measure.

Proposition 3.3. *Consider random measures $\Lambda_1 \leq_{idcx} \Lambda_2$. Then $\Lambda_1^k \leq_{idcx} \Lambda_2^k$ and $\alpha_1^k(\cdot) \leq \alpha_2^k(\cdot)$. Moreover, if $\Lambda_1 \leq_{dcx} \Lambda_2$ then $\alpha_1(\cdot) = \alpha_2(\cdot)$.*

Proof. By the standard arguments, one can approximate any bBs set C_i, $i = 1, \dots, n$ in $\mathbb{E}^k$ by increasing unions of rectangles. By Lemma 8.2 and using a similar argument about composition of a *idcx* and *idl* function as in the proof of Proposition 3.1, to prove the first statement, it is enough to show the respective inequality for *idcx* function $f : \mathbb{R}^n \to \mathbb{R}$ taken of the values of the moment measures on n rectangles in $\mathbb{E}^k$. In this context, consider $g : \mathbb{R}^m \to \mathbb{R}$ given by

$$g(y_1, \dots, y_m) = f\Big(\prod_{j \in J_1} y_j, \dots, \prod_{j \in J_n} y_j \Big) ,$$

where $J_1, \dots, J_n$ are k-element subsets of the set $\{1, \dots, m\}$. Note for non-negative arguments that if f is *idcx* then g is *idcx*.

The second statement follows easily from the first one by the fact that $f(x) = x$ is *idcx*. For the first moment (mean measure) note that both $f(x) = x$ and $f(x) = -x$ are *dcx*. $\square$

We shall explore now the relation between *dcx* ordering and clustering of points in a p.p. One of the most popular functions for the analysis of this effect is the *Ripley's K function* $K(r)$ (reduced second moment function); see [35]. Assume that Φ is a stationary p.p. on $\mathbb{R}^d$ with finite intensity $\lambda = \alpha(B)$, where B is a bBs of Lebesgue measure 1. Then

$$K(r) = \frac{1}{\lambda |G|} \, \mathrm{E}\Big(\sum_{X_i \in \Phi \cap G} (\Phi(B_{X_i}(r)) - 1) \Big) ,$$

where $B_x(r)$ is the ball centered at x of radius r and $|G|$ denotes the Lebesgue measure of a bBs G; due to stationarity, the definition does not depend on the choice of G.

Proposition 3.4. *Consider two stationary p.p.* Φ_i, $i = 1, 2$, *with same finite intensity and denote by* $K_i(r)$ *their Ripley's K functions. If* $\Phi_1 \le_{dcx} \Phi_2$ *then* $K_1(\cdot) \le K_2(\cdot)$.

Proof. Denote $I_i = \mathrm{E}\Big(\sum_{X_j \in \Phi_i \cap G} (\Phi_i(B_{X_j}(r)) - 1) \Big)$, $i = 1, 2$. By the equality of mean measures (Proposition 3.3), it is enough to prove that $I_1 \le I_2$. Note that I_i can be written as the value of some shot noise evaluated with respect to Φ_i^2, the second product of the p.p..

$$I_i = \sum_{X_j, X_k \in \Phi_i} \mathbf{1}[X_j \in G] \mathbf{1}[0 < |X_k - X_j| \le r] ,$$

where $\mathbf{1}[\cdot]$ denotes the indicator function. Thus, the result follows from Proposition 3.3 and Theorem 2.1.

Another useful characteristic is the *pair correlation function* defined on $\mathbb{R}^2$ as $g(x, y) = \frac{\rho_2(x,y)}{\rho_1(x)\rho_1(y)}$, where ρ_k is the *k th product intensity*, equal (outside the diagonals) to the density of the kth moment measure α^k with respect to the Lebesgue measure.

We avoid discussion on questions such as existence etc. The following result follows from Proposition 3.3.

Corollary 3.1. *Consider p.p. such that $\Phi_1 \leq_{dcx} \Phi_2$. Then their respective pair correlation functions satisfy $g_1(x,y) \leq g_2(x,y)$ almost everywhere with respect to the Lebesgue measure.*

3.3. Impact on Palm Measures

For the following definitions and results regarding Palm distributions of random measures see [17, Section 10].

Definition 3.1. For a fixed measurable f such that $0 < \mathrm{E}(\int_{\mathbb{E}} f(x)\Lambda(dx)) < \infty$, the f-mixed Palm version of Λ, denoted by $\Lambda_f \in \mathbb{M}$, is defined as having the distribution

$$\mathrm{P}(\Lambda_f \in M) = \frac{\mathrm{E}(\int_{\mathbb{E}} f(x)\Lambda(dx)\mathbf{1}[\Lambda \in M])}{\mathrm{E}(\int_{\mathbb{E}} f(x)\Lambda(dx))}, \quad M \in \mathcal{M}.$$

In case Λ (say on the Euclidean space $\mathbb{E} = \mathbb{R}^d$) has a density $\{\lambda(x)\}_{x\in\mathbb{R}^d}$, we define for each $x \in \mathbb{R}^d$ the Palm version Λ_x of Λ by the formula

$$\mathrm{P}(\Lambda_x \in M) = \frac{\mathrm{E}(\lambda(x)\mathbf{1}[\Lambda \in M])}{\mathrm{E}(\lambda(x))}, \quad M \in \mathcal{M}.$$

Palm versions Λ_x can be defined for a general random measure via some Radon-Nikodym derivatives. However, we shall state our result for Λ_x as defined above as well as for mixed Palm versions Λ_f in order to avoid the arbitrariness related to the non-uniqueness of Radon-Nikodym derivatives.

Proposition 3.5. *Suppose $\Lambda_i, i = 1,2$ are random measures.*

1. *If $\Lambda_1 \leq_{dcx} \Lambda_2$ then $(\Lambda_1)_f \leq_{idcx} (\Lambda_2)_f$ for any non-negative measurable function f such that $0 < \int_{\mathbb{E}} f(x)\alpha(dx) < \infty$, where α is the (common) mean measure of Λ_i, $i = 1,2$.*

2. *Suppose that Λ_i has locally finite mean measure and almost surely (a.s.) locally Riemann integrable density λ_i, $i = 1,2$. If $\{\lambda_1(x)\} \leq_{dcx} \{\lambda_2(x)\}$, then $\Lambda_1 \leq_{dcx} \Lambda_2$ and for every $x \in \mathbb{R}^d$, $(\Lambda_1)_x \leq_{idcx} (\Lambda_2)_x$.*

Proof. 1. Denote $I_i = \int_{\mathbb{E}} f(x)\Lambda_i(dx)$, $i = 1,2$. By Proposition 3.3, $\Lambda_1 \leq_{dcx} \Lambda_2$ implies that the mean measures are equal and thus $\mathrm{E}(I_1) = \mathrm{E}(I_2)$. It remains to prove

$$\mathrm{E}(g(\Lambda_1(B_1),\ldots,\Lambda_1(B_n))I_1) \leq \mathrm{E}(g(\Lambda_2(B_1),\ldots,\Lambda_2(B_n))I_2)$$

for *idcx* function g. This follows from Theorem 2.1 and the fact that $h(x_0, x) = x_0 g(x) : \mathbb{R}^{n+1} \to \mathbb{R}$ is *idcx*, for non-negative argument x_0.

2. The first part follows immediately from the second statement of Lemma 8.4. For the second part, use the same argument about $h(x_0, x) = x_0 g(x)$ as above.

$\square$

Remark 3.1. Compared to earlier results where *dcx* ordering led to *dcx* ordering, one might tend to believe that the loss here (as *dcx* implies *idcx* only) is more technical. However the following illustrates that it is natural to expect so: consider a Poisson p.p. Φ and its (deterministic) intensity measure $\alpha(\cdot)$ (i.e., its mean measure $\alpha(\cdot) = \mathrm{E}(\Phi(\cdot))$. Using the complete independence property of the Poisson p.p. and the fact that each *dcx* function is component-wise convex, one can show that for disjoint bBs $A_1, \ldots, A_n$ and any *dcx* function f, $f(\alpha(A_1), \ldots, \alpha(A_n)) \leq \mathrm{E}(f(\Phi(A_1), \ldots, \Phi(A_n))$. Thus $\alpha \leq_{dcx} \Phi$. It is easy to see that $\alpha_f(\cdot) = \alpha(\cdot)$ (mixed Palm version of a deterministic measure is equal to the original measure). Take $f(x) = 1[x \in A]$ for some bBs A. Then $\mathrm{E}(\Phi_f(A)) = \mathrm{E}((\Phi(A))^2)/\alpha(A) = \alpha(A) + 1$ since $\Phi(A)$ is a Poisson r.v.. Thus $\alpha_f(A) < \mathrm{E}(\Phi_f(A))$ disproving $\alpha_f(A) \leq_{dcx} \Phi_f(A)$. Another counterexample involving Poisson-Poisson cluster p.p. will be given in Remark 5.2.

3.4. Cox Point Processes

We will consider now Cox p.p. (see e.g. [35, III 5.2]), known also as doubly stochastic Poisson p.p., which constitute a rich class often used to model patterns which exhibit more clustering than in Poisson p.p..

Recall that a *Cox*(Λ) *p.p.* Φ_Λ on $\mathbb{E}$ generated by the random *intensity measure* $\Lambda(\cdot)$ on $\mathbb{E}$ is defined as having the property that Φ_Λ conditioned on $\Lambda(\cdot)$ is a Poisson p.p. with intensity $\Lambda(\cdot)$. Note that Cox p.p. may be seen as a result of an operation transforming some random (intensity) measure into a point (Cox) p.p..

One can easily show that this operation preserves our orders.

Proposition 3.6. *Consider two ordered random measures* $\Lambda_1 \leq_{dcx\,(resp.\,idcx;\,idcv)} \Lambda_2$. *Then* $\Phi_{\Lambda_1} \leq_{dcx\,(resp.\,idcx;\,idcv)} \Phi_{\Lambda_2}$.

Proof. Taking a *dcx* (resp. *idcx*; *idcv*) function ϕ, assuming (by Proposition 3.1) mutually disjoint bBs A_k, $k = 1, \ldots, n$, using the definition of Cox p.p. and the second statement of the Lemma 8.3 one shows for $i = 1, 2$ that that the conditional expectation

$$\mathrm{E}(\phi(\Phi_{\Lambda_i}(A_1), \ldots, \Phi_{\Lambda_i}(A_n))|\Lambda_i)$$

given the intensity measure Λ_i is a *dcx* (resp. *idcx*; *idcv*) function of $(\Lambda_i(A_1), \ldots, \Lambda_i(A_n))$. The result follows thus from the assumption of the measures Λ_i being *dcx* ordered. $\square$

We will show now using Theorem 2.1 that $dcx, idcx, idcv$ ordering of Cox intensity measures is preserved by independent (not necessarily identically distributed) marking and thinning, as well as independent displacement of points of the p.p..

By *independent marking* of p.p. Φ on $\mathbb{E}$ with *marks* on some LCSC space $\mathbb{E}'$, we understand a p.p. $\tilde{\Phi} = \sum_i \varepsilon_{(x_i,Z_i)}$ such that given $\Phi = \sum_i \varepsilon_{x_i}$, Z_i are independent random elements in $\mathbb{E}'$, with distribution $\mathrm{P}\{Z_i \in \cdot | \Phi = \sum_i \varepsilon_{x_i}\} = F_{x_i}(\cdot)$ given by some probability (mark) kernel $F_x(\cdot)$ from $\mathbb{E}$ to $\mathbb{E}'$. The fact that $F_x(\cdot)$ may depend on x (in contrast to i.i.d. marking) is sometimes emphasized by calling $\tilde{\Phi}$ a "position dependent" marking. Independent thinning can be seen as the projection on $\mathbb{E}$ of the subset $\tilde{\Phi}(\cdot, \{1\})$ of the independently marked p.p. $\tilde{\Phi}$ where the marks $Z_i \in \{0,1\} = \mathbb{E}'$, are independent Bernoulli thinning variables $Z_i = Z_i(x)$, whose distributions may be dependent on x_i. Similarly, the projection of an independently marked p.p. $\tilde{\Phi} = \sum_i \varepsilon_{(x_i,Z_i)}$ on the space of marks $\mathbb{E}'$; i.e., $\tilde{\Phi}(\mathbb{E} \times \cdot) = \sum_i \varepsilon_{Z_i}$ can be seen as independent displacement of points of Φ to the space $\mathbb{E}'$. Special examples are i.i.d. shifts of points in the Euclidean space, when $Z_i = x_i + Y_i$, where Y_i are i.i.d.

Proposition 3.7. *Suppose $\Phi_i, i = 1,2$, are two Cox(Λ_i) p.p.. Assume that their intensity measures are ordered $\Lambda_1 \leq_{dcx\,(resp.\,idcx;\,idcv)} \Lambda_2$. Let $\tilde{\Phi}_i, i = 1,2$ be the corresponding independently marked p.p. with the same mark kernel $F_x(\cdot)$. Then $\tilde{\Phi}_1 \leq_{dcx\,(resp.\,idcx;\,idcv)} \tilde{\Phi}_2$.*

From the above Proposition, the following corollary follows immediately by the last statement of Proposition 3.2.

Corollary 3.2. *Independent thinning and displacement of points preserves dcx (resp. idcx; idcv) order of the intensities of Cox p.p..*

Proof. (*Prop. 3.7*) Let Φ_i be Cox(Λ_i) $i = 1,2$ respectively. Assume $\Lambda_1 \leq_{dcx(idcx,idcv)} \Lambda_2$. It is known that independent marking of Cox(Λ_i) p.p. is a Cox$(\tilde{\Lambda}_i)$ p.p. with intensity measure $\tilde{\Lambda}_i$ on $\mathbb{E} \times \mathbb{E}'$ given by $\tilde{\Lambda}_i(\cdot) = \int_{\mathbb{E}} \int_{\mathbb{E}'} \mathbf{1}[(x,y) \in \cdot\,] F_x(dy) \Lambda_i(dx)$; cf. [35, Secs 4.2 and 5.2]. Let S be the family of bBs in $\mathbb{E} \times \mathbb{E}'$; for $x \in \mathbb{E}$ and bBs $C \subset \mathbb{E} \times \mathbb{E}'$ consider $h(x,C) = \int_{\mathbb{E}'} \mathbf{1}[(x,y) \in C]\, F_x(dy)$. Then the integral shot noise $V_{\Lambda_i}(C) = \int_{\mathbb{E}} h(x,C)\, \Lambda_i(dx)$ satisfies $V_{\Lambda_i}(C) = \tilde{\Lambda}_i(C)$ for all bBs C. Thus, by Theorem 2.1 $\tilde{\Lambda}_1 \leq_{dxc\,(\mathrm{resp.}\,idcx;\,idxv)} \tilde{\Lambda}_2$ and the result follows from Proposition 3.6. $\square$

If $\Lambda(\cdot) \in \mathbb{M}(\mathbb{R}^d)$ a.s has a density $\{\lambda(x)\}_{x \in \mathbb{R}^d}$ with respect to Lebesgue measure then the density is referred to as the intensity field of the Cox p.p., which will be called in this case Cox(λ) p.p. and denoted by Φ_λ.

It is known that Cox p.p. is over-dispersed with respect to the Poisson p.p., i.e., $\mathrm{Var}(\Phi_1(B)) \leq \mathrm{Var}(\Phi_2(B))$ where Φ_1, Φ_2 are, respectively, Poisson and Cox p.p. with the same mean measure. Hence, it is clear that a Cox p.p. can only be greater in *dcx* order than a Poisson p.p. with the same mean measure. Indeed, in Section 5 we will show several examples when this stronger result holds, namely Cox p.p. that are *dcx* ordered (larger) with respect to the corresponding Poisson p.p., as well as Cox p.p. *dcx* ordered with respect to each other.

3.5. Alternative Definition of *dcx* Order

We viewed a random measure as a random field and have defined ordering from this viewpoint. Alternatively, one can consider a random measure as an element of the space of Radon measures $\mathbb{M}$ and define ordering between two $\mathbb{M}$-valued random elements. This can be done once we define what is a *dcx* function on $\mathbb{M}$. The *dcx* order can be defined on more general spaces; [22] extends the notion of *dcx* ordering to lattice ordered Abelian semigroups with some compatibility conditions between the lattice structure and the Abelian structure ($LOAS^+$). The space $\mathbb{M}$ can be equipped with the following lattice and algebraic structure. Consider the following partial order: for $\mu, \nu \in \mathbb{M}$, we say $\mu \leq \nu$ if $\mu(B) \leq \nu(B)$ for all bBs B in $\mathbb{E}$ and addition $(\mu + \nu)(B) = \mu(B) + \nu(B)$. Under this definition, the space $\mathbb{M}$ forms a $LOAS^+$ as required by [22]. Then one can define a directionally convex function on $\mathbb{M}$ as in Definition 2.1. Call it a dcx^1 function. This gives rise to dcx^1 order of random measures analogously to the first part of the Definition 2.2.

Now we have two reasonable definitions of ordering of random measures. It is easy to see that dcx^1 ordering implies *dcx* ordering. In light of Example 5.1.7 of [29], existence of a counterexample to the converse looks plausible, though we failed in our attempts to construct one. However, the result of [3] proves that *convex* ordering of real valued stochastic process $\{X_n\}_{n\in\mathbb{N}}$ implies *continuous, convex* ordering of the corresponding elements of the infinite-dimensional Euclidean spaces $\mathbb{R}^{\mathbb{N}}$. This suggests that *dcx* of random measures may imply a dcx^{1*} order induced by some subclass of dcx^1 functionals of random measures, which are regular in some sense. Leaving this general question as an open problem, we remark only that the integral shot-noise fields studied in the next section can be seen as some particular class of functionals of random measures, which are dcx^1 (in fact linear on $\mathbb{M}$) and regular enough for their means to satisfy the required inequality provided the random measures are *dcx* ordered. It is natural thus to have them in the suggested dcx^{1*} class.

Recall also that for *strong* order of p.p. there is the full equivalence between these two definitions, and both imply the possibility of a coupling of the ordered p.p. such that the smaller one is a.s. a subset of the greater one; cf [32].

4. Ordering of Shot-Noise Fields

In this section we will prove Theorem 2.1 concerning *dcx* ordering of integral shot-noise fields, which is the main result of this paper. We will also consider the so called extremal shot-noise fields.

4.1. Integral Shot-Noise Fields

Usually shot-noise fields are defined for p.p. as the following sum (thus sometimes called additive shot-noise fields) $V_\Phi(y) = \sum_{X_n \in \Phi} h(X_n, y)$ where $\Phi = \sum_n \varepsilon_{X_n}$ and h is a non-negative response function. In definition 2.4 we have made a significant but natural generalization of this definition. It is pretty clear as to why we call this generalization integral shot-noise field. The extension to unbounded response functions is not just a mathematical generalization alone. It shall provide us a simple proof of ordering for extremal-shot-noise fields for p.p..

Now, we shall prove Theorem 2.1. The proof is inspired by [24].

Proof. (*Theorem 2.1*) We shall prove the second statement first. The necessary modifications for the proof of the first statement shall be indicated later on.

2. We need to show that $(V^1(y_1),\dots,V^1(y_m)) \leq_{dcx} (V^2(y_1),\dots,V^2(y_m))$ for $y_i \in S, 1 \leq i \leq m$ and $V^j(\cdot) = V_{\Lambda_j}(\cdot)$, $j = 1,2$. The proof relies on the construction of two sequences of random vectors $(V_k^j(y_1),\dots,V_k^j(y_m))$, $k = 1,2\dots$, $j = 1,2$ satisfying the assumptions of Lemma 8.2.

   Choose an increasing sequence of compact sets K_k, $k \geq 1$ in $\mathbb{E}$, such that $K_k \nearrow \mathbb{E}$. Since h is measurable in its first argument, we know that there exists a sequence of simple functions $h_k(\cdot, y_i), k \in \mathbb{N}$ such that as $k \to \infty$, $h_k(\cdot, y_i) \uparrow h(\cdot, y_i)$ for $1 \leq i \leq m$. They can be written down explicitly as follows:

$$\begin{aligned} h_k(\cdot, y_i) &= \gamma_k \mathbf{1}[\{x \in K_k : h(x, y_i) = \infty\}] \\ &\quad + \sum_{n=1}^{\gamma_k} \frac{n-1}{2^k} \mathbf{1}[\{x \in K_k : \frac{n-1}{2^k} \leq h(x, y_i) < \frac{n}{2^k}\}](\cdot) \end{aligned}$$

   for $1 \leq i \leq m$, where $\gamma_k = k2^k$. Put $I^i_{kn} = \{x \in K_k : \frac{n-1}{2^k} \leq h(x, y_i) < \frac{n}{2^k}\}$ and $I^i_{k\infty} = \{x \in K_k : h(x, y_i) = \infty\}$ for $1 \leq i \leq m$ and $1 \leq n \leq \gamma_k$. Note that all I^i_{kn} $n = 1,\dots,\infty$ are bBs and the sequence of random vectors we are looking for is

$$V_k^j(y_i) = \int_{\mathbb{E}} h_k(x, y_i) \Lambda_j(dx) = \gamma_k \Lambda_j(I^i_{k\infty}) + \sum_{n=1}^{\gamma_k} \frac{n-1}{2^k} \Lambda_j(I^i_{kn}),$$

   for $j = 1,2$. By the definition of integral, it is clear that for $j = 1,2$ as $k \to \infty$, $(V_k^j(y_1),\dots,V_k^j(y_m)) \uparrow (V^j(y_1),\dots,V^j(y_m))$ a.s. and hence in distribution.

By monotone convergence theorem, the expectations, which are finite by the assumption, also converge. What remains to prove is that for each $k \in \mathbb{N}$, the vectors are *dcx* ordered.

Fix $k \in \mathbb{N}$. Now observe that for $j = 1, 2$, $i = 1, \ldots, m$, $V_k^j(y_i)$ are increasing linear functions of the vectors $(\Lambda_j(I_{kn}^i) : n = 1, \ldots, \gamma_k, \infty)$, $j = 1, 2$. The latter are *dcx* ordered by the assumptions. And since composition of *dcx* with increasing linear functions is *dcx*, it follows that $(V_k^1(y_1), \ldots, V_k^1(y_m)) \leq_{dcx} (V_k^2(y_1), \ldots, V_k^2(y_m))$.

1. For vectors $(V_k^j(y_1), \ldots, V_k^j(y_m))$, $k = 1, 2 \ldots$, $j = 1, 2$ defined as above, $f(V_k^j(y_1), \ldots, V_k^j(y_m)) \uparrow f(V^j(y_1), \ldots, V^j(y_m))$ a.s. for f *idcx* (resp. *idcv*) and hence
$\mathrm{E}(f(V_k^j(y_1), \ldots, V_k^j(y_m))) \uparrow \mathrm{E}(f(V^j(y_1), \ldots, V^j(y_m)))$, $j = 1, 2$. The proof is complete by noting that $\mathrm{E}\, f(V_k^1(y_1), \ldots, V_k^1(y_m)) \leq \mathrm{E}\,(f(V_k^2(y_1), \ldots, V_k^2(y_m))$ for all $k \geq 1$ and f *idcx* (resp. *idcv*).

$\square$

4.2. Extremal Shot-Noise Fields

We recall now the definition of the extremal shot-noise, first introduced in [13].

Definition 4.1. Let S be any set and $\mathbb{E}$ a LCSC space. Given a p.p. Φ on $\mathbb{E}$ and a measurable (in the first variable alone) response function $h(x, y) : \mathbb{E} \times S \to \mathbb{R}$, the extremal shot-noise field is defined as

$$U_\Phi(y) = \sup_{X_i \in \Phi} \{h(X_i, y)\}. \tag{3}$$

In order to state our result for extremal shot-noise fields, we shall use the *lower orthant* (*lo*) order.

Definition 4.2. Let X and Y be random $\mathbb{R}^d$ vectors. We say $X \leq_{lo} Y$ if $\mathrm{P}(X \leq t) \geq \mathrm{P}(Y \leq t)$ for every $t \in \mathbb{R}^d$.

On the real line, this is the same as strong order (i.e., when $\mathfrak{F}$ consists of increasing functions) but in higher dimensions it is different. Obviously *st* order implies *lo* order and examples of random vectors which are ordered in *lo* but not in *st* are known; see ([29]). Thus, it is clear that the following proposition is a generalization of the corresponding one-dimensional result in [24] where the proof method was similar to the proof of the ordering of integral shot-noise fields. We shall give a much simpler proof using the already proved result.

Proposition 4.1. *Let* $\Phi_1 \leq_{idcv} \Phi_2$. *Then* $\{U_{\Phi_1}(y)\}_{y\in S} \leq_{lo} \{U_{\Phi_2}(y)\}_{y\in S}$.

Proof. The probability distribution function of the extremal shot-noise can be expressed by the Laplace transform of some corresponding (additive) one as follows. Let $\{x_1,\ldots,x_m\} \subset S$ and $(a_1,\ldots,a_m) \in R^m$. Then

$$\begin{aligned} \mathrm{P}(U(y_i) \leq a_i, 1 \leq i \leq m) &= \mathrm{E}(\prod_i \mathbf{1}[\sup_n\{h(X_n,y_i) \leq a_i\}]) \\ &= \mathrm{E}(\prod_i \prod_n \mathbf{1}[h(X_n,y_i) \leq a_i]) \\ &= \mathrm{E}(\prod_i \prod_n e^{\log \mathbf{1}[h(X_n,y_i)\leq a_i]}) \\ &= \mathrm{E}(\prod_i e^{-\sum_n -\log \mathbf{1}[h(X_n,y_i)\leq a_i]}) \\ &= \mathrm{E}(e^{-\sum_i \hat{U}(y_i)}) \end{aligned}$$

where $\hat{U}(y_i) = \sum_n -\log \mathbf{1}[h(X_n,y_i) \leq a_i]$ is an additive shot-noise with response function taking values in $[0,\infty]$. The response function is clearly non-negative and measurable. The function $f(x_1,\ldots,x_m) = e^{-\sum_i x_i}$ is a ddcx function on $(-\infty,\infty]$. The result follows by the first statement of Theorem 2.1. $\square$

The extremal shot-noise field can be used to define the Boolean model. Given a (generic) random closed set (RACS; see [35, Ch. 6]) G, let $h((x,G),y) = \mathbf{1}[y \in x+G]$.

Definition 4.3. By a *Boolean model* with the p.p. of germs Φ and the typical grain G we call the random set $C(\Phi,G) = \{y : U_{\tilde{\Phi}}(y) > 0\}$ where $\tilde{\Phi} = \sum_i \varepsilon_{(X_i,G_i)}$ is i.i.d. marking of Φ with the mark distribution equal to this of G.

We shall call G a fixed grain if there exists a closed set B such that $G = B$ a.s.. We shall demonstrate in Section 6.1 as to how one can obtain comparison results for the Boolean model using the results of this section.

5. Examples of *dcx* Ordered Measures and Point Processes

In this section, we shall provide some examples of *dcx* ordered measures and p.p. on the Euclidean space $\mathbb{E} = \mathbb{R}^d$. The examples are intended to be illustrative and not encyclopaedic. The purpose of the examples is to show that there are *dcx* ordered p.p. as well as demonstrate some methods to prove that two p.p. are *dcx* ordered. Many of the examples seem to indicate that p.p. higher in *dcx* order cluster more, at least for Cox p.p..

5.1. Ising-Poisson Cluster Point Processes

Let $\{\lambda(s)\}_{s\in\mathbb{R}^d}$ be a stationary random intensity field. Define a new field, which is random but constant in space $\{\lambda_m(s) = \lambda(0)\}$ and deterministic constant field $\{\lambda_h(s) = \mathrm{E}(\lambda(0))\}$. $\mathrm{Cox}(\lambda_m)$ is known as mixed Poisson p.p. and $\mathrm{Cox}(\lambda_h)$ is just the well-known homogeneous Poisson p.p.. Denote the random intensity measures of the Cox, mixed and homogeneous Poisson p.p., by Λ, Λ_m and Λ_h respectively (i.e., $\Lambda(dx) = \lambda(x)\,dx$, etc.) It is proved in [24] that $\Lambda \leq_{dcx} \Lambda_m$ and when $\{\lambda(s)\}$ is a conditionally increasing field, $\Lambda_h \leq_{dcx} \Lambda$. Recall that a random field $\{X(s)\}$ is a *conditionally increasing field* if for any k and $s_1,\ldots,s_k \in \mathbb{R}^d$ the expectation $\mathrm{E}(f(X(s_1))|X(s_j) = a_j\ \forall\ 2 \leq j \leq k)$ is increasing in a_j for all increasing f. However, no example of a conditionally increasing field was given in [24]. Now we construct one.

Consider the d-dimensional lattice $\mathbb{Z}^d$. Let $\{X(z)\}_{z\in\mathbb{Z}^d}$ be i.i.d. random variables taking values in $\{+1,-1\}$. Call $\{X(z)\}$ a (random) configuration of spins. In order to obtain a stationary field consider a random shift of the origin of $\mathbb{Z}^d$ to U with uniform distribution on $[0,1]^d$ (U independent of $\{X(z)\}$). Let the lattice shifted by U be denoted by $\mathbb{Z}^d_*$. Pick two numbers $\mu_2 \leq \mu_1$. For $s \in \mathbb{R}^d$, define $\lambda(s) = \mu_1\mathbf{1}[X(\dot{s}) = 1] + \mu_2 1[X(\dot{s}) = -1]$ where $\dot{s}$ represents the unique "lower left" point in $\mathbb{Z}^d_*$ nearest to s. The intensity field is clearly stationary. We shall now show that $\{\lambda(s)\}$ is conditionally increasing. Note that

$$f(\lambda(s)) = 1[x(\dot{s}) = 1](f(\mu_1) - f(\mu_2)) + f(\mu_2) \tag{4}$$

From Theorem 1.2.15 of [29], it is sufficient to show the conditional increasing property conditioned on U, the random origin of the lattice $\mathbb{Z}^d_*$. Hence it is enough for the Ising model to possess the following property:

$$\begin{aligned} &\mathrm{P}(X(z_1) = 1|X(z_2) = -1, X(z_j) = a_j, j = 3,\ldots,k) \\ &\quad\leq\ \mathrm{P}(X(z_1) = 1|X(z_2) = 1, X(z_j) = a_j, j = 3,\ldots,k), \end{aligned}$$

where $a_i \in \{+1,-1\}$ and $z_i \in \mathbb{Z}^d, i = 1,\ldots,k$. This follows easily from the fact that the spins are i.i.d.

We call the Cox p.p. generated by the above conditionally increasing field $\{\lambda(s)\}$ the *Ising-Poisson cluster* p.p. By the arguments presented in [24], it is *dcx* larger than the homogeneous Poisson p.p. with the same intensity. Note that intuitively the Ising-Poisson cluster p.p. "clusters" its points more than a homogeneous Poisson p.p. In what follows, we will see more examples of cluster (Cox) p.p. which are *dcx* larger than the corresponding homogeneous Poisson p.p..

5.2. Lévy Based Cox Point Processes (LCPs)

This class of p.p. is being introduced in [14]. One can find many examples of LCPs in the above mentioned paper. In simple terms, a LCP is a p.p. whose intensity field is an integral shot-noise field of a Lévy basis. A random measure $L \in \mathbb{M}(\mathbb{R}^d)$ is said to be a non-negative Lévy basis if

- for any sequence $\{A_n\}$ of disjoint, bBs of $\mathbb{R}^d$, $L(A_n)$ are independent random variables (*complete independence*) and $L(\bigcup A_n) = \sum L(A_n)$ a.s. provided $\cup A_n$ is also a bBs of $\mathbb{R}^d$.
- for every bBs A of $\mathbb{R}^d$, $L(A)$ is infinitely divisible.

We shall consider only non-negative Lévy bases, even though there exist signed Lévy bases too (see [14]). Hence, we shall omit the reference to non-negativity in future.

A Cox p.p. Φ is said to be a LCP, if its intensity field is of the form

$$\lambda(y) = \int_{\mathbb{R}^d} k(x,y) L(dx),$$

where L is a Lévy basis and the kernel k is a non-negative function such that $k(x,y)$ is a.s. integrable with respect to L and $k(.,y)$ is integrable with respect to Lebesgue measure. In [14] the response function k and the Lévy basis L is chosen such that $\int_B \lambda(y)\, dy < \infty$ a.s. for all bBs B, for which a sufficient condition is $\int_B \mathrm{E}(\lambda(y))\, dy < \infty$. In our considerations, in order to be able to use Lemma 8.4, we will require that $\lambda(y)$ is a.s. locally Riemann integrable.

Remark 5.1. Note that a sufficient condition for this is that $\lambda(y)$ is a.s. continuous, for which, in turn, it is enough to assume that k is continuous in its second argument and that for all $x \in \mathbb{R}^d$, there exist $B_x(\epsilon_x)$, $\epsilon_x > 0$ such that $\int_{\mathbb{R}^d} \sup_{z \in B_x(\epsilon_x)} k(z,y)\, \alpha(dx)$ $< \infty$ for all y, where $\alpha(B) = \mathrm{E}(L(B))$, the mean measures of the Lévy bases; (cf [1]).

Lemma 5.1. *Let L_1 and L_2 be Lévy bases with mean measure α_i. Let $\Phi_i, i = 1,2$ be LCPs with Lévy bases $L_i, i = 1,2$ respectively.*

1. *$L_1 \le_{dcx\,(resp.\,idcx;\,idcv)} L_2$ if and only if $L_1(A) \le_{cx\,(resp.\,icx;\,icv)} L_2(A)$ for all bBs A of $\mathbb{R}^d$, where cx, icx, icv stands, respectively for convex, increasing convex and increasing concave.*

2. *If $L_1 \le_{dcx\,(resp.\,idcx;\,idcv)} L_2$, then $\Phi_1 \le_{dcx\,(resp.\,idcx;\,idcv)} \Phi_2$ provided the intensity fields $\lambda_i(y)$ of LCP Φ_i is a.s. locally Riemann integrable with these integrals, in case of dcx, having finite means.*

3. *$\alpha_i \le_{dcx} L_i$.*

Proof. The first part is due to Proposition 3.1 and the complete independence property of Lévy bases. As for the second part, it is a simple consequence of Theorem 2.1, Lemma 8.4 and Proposition 3.6. The third part follows from complete independence and Jensen's inequality. □

We shall now give some examples of *dcx* ordered Lévy basis.

Example 5.1. Let $\{x_i\}$ be a locally finite deterministic configuration of points in $\mathbb{R}^d$. Let $\{X_i^j\}_{i\geq 1}, j = 1,2$ be i.i.d sequence of infinite divisible random variables such that $X_1^1 \leq_{cx} X_1^2$. (For example, X_1^1 can be sum of two independent exponential r.v. with mean 1/2 and X_1^2be an exponential r.v. with mean 1.) Define the Lévy bases as follows:

$$L_j(A) = \sum_{x_i \in A} X_i^j,$$

where A is a bBs of $\mathbb{R}^d$ and $j = 1,2$. By Lemma 5.1 and the fact that $X_1^1 \leq_{cx} X_1^2$ it follows that $L_1 \leq_{dcx} L_2$.

Example 5.2. Let $\tilde{\Phi} = \sum_i \varepsilon_{(x_i,Z_i)}$ be an homogeneous Poisson p.p. on $\mathbb{R}^d$ independently marked by random variables $\{Z_i\}$ with mean λ_0. Consider two random measures $\Lambda_1 = \sum_{(x_i,Z_i)\in\tilde{\Phi}} \lambda_0 \varepsilon_{x_i}$ and $\Lambda_2 = \sum_{(x_i,Z_i)\in\tilde{\Phi}} Z_i \varepsilon_{x_i}$. Note that L_i, $i = 1,2$ are Levy basis. By Lemma 5.1 and the fact that $\lambda_0 \leq_{cx} Z_i$, conditioning on the number of points and using the same argument as in the proof of the second statement of Proposition 3.2 one can prove that $\Lambda_1 \leq_{dcx} \Lambda_2$.

5.3. Poisson-Poisson Cluster Point Processes

By Poisson-Poisson cluster p.p., we understand a LCP with the Levy basis being a Poisson p.p. This class deserves a separate mention due to the generality of the ordering results that are possible. For rest of the section, assume that $h(x)$ is a non-negative measurable function such that $\int_{\mathbb{R}^d} h(x)dx = \lambda_0 < \infty$.

We shall now give an example of a parametric family of *dcx* ordered Poisson-Poisson cluster p.p.. Fix $\lambda > 0$. Let $\Phi_c, c > 0$ be a family of homogeneous Poisson p.p. on $\mathbb{R}^d$ of intensity $c\lambda$. Let a non-negative function $h : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ be given and consider a family of shot noise fields $\lambda_c(y) = \int_{\mathbb{R}^d} (h(x,y)/c)\ \Phi_c(dx)$, which are assumed a.s. locally Riemann integrable with $\int_B \mathrm{E}(\lambda_c(y))\, dy < \infty$ for bBs B.

Proposition 5.1. *The family of shot-noise fields $\{\lambda_c(y)\}_{y\in\mathbb{R}^d}$ is decreasing in dcx, i.e., for $0 < c_1 \leq c_2$ we have $\{\lambda_{c_2}(y)\} \leq_{dcx} \{\lambda_{c_1}(y)\}$. Consequently Cox($\lambda_{c_2}$)$\leq_{dcx}$ Cox(λ_{c_1}).*

Proof. Note that $\{\lambda_c(x)\}$ can be seen as a shot-noise field generated by the response function h and the Levy basis $L_c = (1/c)\Phi_{\lambda c}$. By Lemma 5.1 and Theorem 2.1, it is enough to prove that $L_{c_2}(A) \leq_{cx} L_{c_1}(A)$ for A bBs and $c_2 > c_1 > 0$.

Since, $X \leq_{cx} Y$ implies that $aX \leq_{cx} aY$ for all scalars $a > 0$, it suffices to prove that $L_{ca}(A) \leq_{cx} L_a(A)$ for A bBs and $c > 1, a > 0$. This essentially boils down to proving that $N_{ca} \leq_{cx} cN_a,\ c > 1, a > 0$, where N_a stands for a Poisson r.v. with mean a. Let $\{X_i^n\}_{1\leq i\leq n}$ and $\{Y_i^n\}_{1\leq i\leq n},\ n \geq 1$ be i.i.d. sequences of Bernoulli r.v's with probability of success ca/n and a/n, respectively, with $n \geq ca$. Let $X^n = \sum_{i=1}^n X_i^n$ and $Y^n = \sum_{i=1}^n cY_i^n$. It is well known that X^n, Y^n converge weakly to N_{ca}, N_a respectively, as $n \to \infty$. As convex order preserves weak convergence, we need to only prove that $X^n \leq_{cx} cY^n$. By the independence of summands, it is enough to prove that $X_i^n \leq_{cx} cY_i^n$, which we shall do in what follows. Let f be a convex and differentiable function. Define $g(c) := \mathrm{E}\, f(X_i^n) - \mathrm{E}\, f(cY_i^n) = \frac{a}{n}\{c(f(1) - f(0)) - f(c) + f(0)\}$. Note that $g(1) = 0$. Hence, our proof is complete if we show that g is decreasing in $c > 1$. Indeed,

$$\begin{aligned} g'(c) &= \frac{a}{n}\{(f(1) - f(0)) - f'(c)\} \\ &= \frac{a}{n}\{f'(b) - f'(c)\} \leq 0, \quad (b < c) \end{aligned}$$

where $b \in (0,1)$ by mean-value theorem and f' is increasing due to convexity. □

Poisson-Poisson cluster p.p. can be also *dcx* compared to a homogeneous Poisson p.p.. Let Φ and Φ' be homogeneous Poisson p.p. with intensities $\lambda < \infty$ and $\lambda \times \lambda_0$ respectively. Define $\mu(y) = \sum_{X_i\in\Phi} h(X_i - y)$. Let Φ'' be $\mathrm{Cox}(\mu(x))$.

Proposition 5.2. *Let Φ, Φ', $\{\mu(y)\}$ be as above. Assume that $\mu(y)$ is a.s. locally Riemann integrable and $\mathrm{E}(\mu(y)) = \mathrm{E}(\mu(0)) < \infty$. Then $\Phi' \leq_{dcx} \Phi''$.*

Proof. By the last statement of Lemma 5.1 we have $\lambda\, dx \leq_{dcx} \Phi(dx)$. Note that $\lambda \times \lambda_0 = \int_{\mathbb{R}^d} h(x - y)\,\lambda dx$ and thus by the second statement of Theorem 2.1 (note the assumption $\mathrm{E}(\mu(y)) < \infty$) $\{\lambda \times \lambda_0\} \leq_{dcx} \{\mu(y)\}$, where the *dcx* smaller field is a deterministic, constant. The result follows now from the second statement of Lemma 8.4 by assumption that $\mu(y)$ is a.s. Riemann integrable and observing that $\mathrm{E}(\int_A \mu(y)\, dy) = \mathrm{E}(\mu(0)) \int_A dy < \infty$ for all bBs A. □

Remark 5.2. Consider Poisson p.p. Φ' and $\mathrm{Cox}(\mu)$ as in Proposition 5.2. It is known that the Palm version (given a point at the origin) of Φ' can be constructed taking $\Phi' + \varepsilon_0$. By [27, Proposition 2], analogously, Palm version of $\mathrm{Cox}(\mu)$ can be taken as $\mathrm{Cox}(\mu) + \varepsilon_0 + \Phi''$, where Φ'' is an independent of $\mathrm{Cox}(\mu)$ Poisson p.p. with intensity $h(y - \xi)$ where ξ is sampled from the distribution $h(dx)/\int h(y)dy$. This shows that one cannot expect *dcx* ordering of the Palm versions of Φ' and $\mathrm{Cox}(\mu)$.

5.4. Log Cox Point Processes

This class of p.p. are defined by the logarithm of their intensity fields.

An extension of LCP studied in [14] is Log-Lévy driven Cox process (LLCPs). Under the notation of the previous subsection, a p.p. Φ is said to be a LLCP if its intensity field is given by

$$\lambda(y) = \exp\left(\int_{\mathbb{R}^d} k(x,y)L(dx)\right).$$

[14] allows for negative kernels and signed Lévy measures but they do not fit into our framework. Suppose that $L_1 \leq_{idcx} L_2$, then $\Phi_1 \leq_{idcx} \Phi_2$ where $\Phi_i, i = 1,2$ are the respective LLCPs of $L_i, i = 1,2$ with kernel $k(.,.)$. These are simple consequences of Theorem 2.1 and the exponential function being icx.

Another class is the Log-Gaussian Cox process (LGCPs)(see [26]). A p.p. Φ is said to be a LGCP if its intensity field is $\lambda(y) = \exp\{X(y)\}$ where $\{X(y)\}$ is a Gaussian random field. Suppose $\{X_i(y)\}, i = 1,2$ are two Gaussian random fields, then $\{X_1(y)\} \leq_{idcx} \{X_2(y)\}$ if and only if $\mathrm{E}(X_1(y)) \leq \mathrm{E}(X_2(y))$ for all $y \in \mathbb{R}^d$ and $\mathrm{cov}(X_1(y_1), X_1(y_2)) \leq \mathrm{cov}(X_2(y_1), X_2(y_2))$ for all $y_1, y_2 \in \mathbb{R}^d$. From the composition rules of $idcx$ order, it is clear that $idcx$ ordering of Gaussian random fields implies $idcx$ ordering of the corresponding LGCPs. An example of parametric dcx ordered Gaussian random field is given in [24, Sec 4].

5.5. Generalized Shot Noise Cox Processes (GNSCPs)

This class of Cox p.p. was first introduced and its various statistics were studied in [28]. In simple terms, these are Cox p.p. whose random intensity field is a shot-noise field of a p.p. We say a Cox p.p. is GNSCP if the random intensity field $\{\lambda(y)\}_{y \in \mathbb{R}^d}$ driving the Cox p.p. is of the following form : $\lambda(y) = \sum_j \gamma_j k_{b_j}(c_j, y)$ where $(c_j, b_j, \gamma_j) \in \Phi$, a p.p. on $\mathbb{R}^d \times (0,\infty) \times (0,\infty)$. Also we impose the following condition on the kernel k : $k_{b_j}(c_j, y) = \frac{k_1(c_j/b_j, y/b_j)}{b_j^d}$ where $k_1(c_j, .)$ is a density with respect to the Lebesgue measure on $\mathbb{R}^d$. We shall denote the GNSCP driven by Φ as Φ^G. This class includes various known p.p. such as Neyman-Scott p.p., Thomas p.p., Matérn Cluster p.p. among others. The case when b_j's are constants and $\{(c_j, \gamma_j)\}$ is a Poisson p.p. is called as Shot Noise Cox process (See [27]). Shot Noise Cox process are also LCPs. Suppose two p.p. $\Phi_1 \leq_{dcx\,(\text{resp.}\, idcx;\, idcv)} \Phi_2$, then from Theorem 2.1, we infer that $\Phi_1^G \leq_{dcx\,(\text{resp.}\, idcx;\, idcv)} \Phi_2^G$.

5.6. Ginibre-Radii Like Point Process

Let $\{\Phi_i\}_{i \geq 0}$ be an i.i.d. family of p.p. on $\mathbb{R}^+$. So, the points of each p.p. Φ_i can be sequenced based on their distance from the origin. Let Φ be the p.p. formed *by picking the ith point of* Φ_i for $i \geq 1$. We shall from now on abbreviate $\Phi([0,b])$ by $\Phi(b)$

for $b > 0$ and similarly for other p.p. used. Note the following representation for $\Phi(b)$ and $\Phi_0(b)$:

$$\Phi(b) = \sum_{k\geq 1} \mathbf{1}[\Phi_k(b) \geq k] \ ; \ \Phi_0(b) = \sum_{k\geq 1} \mathbf{1}[\Phi_0(b) \geq k].$$

Let

$$\Phi^m(b) = \sum_{k\geq 1}^{m} \mathbf{1}[\Phi_k(b) \geq k] \ ; \ \Phi_0^m(b) = \sum_{k\geq 1}^{m} \mathbf{1}[\Phi_0(b) \geq k].$$

By Lorentz's inequality (see [29, Th. 3.9.8]), it follows that $(\Phi_1(b), \ldots, \Phi_m(b)) \leq_{sm} (\Phi_0(b), \ldots, \Phi_0(b))$, where sm stands for *supermodular* (see [29, § 3.9]). Define the $f : \mathbb{N}^m \to \mathbb{R}$ as follows : $f(n_1, \ldots, n_m) = \sum_{k\geq 1} \mathbf{1}[n_k \geq k]$. It is easy to verify that both f and $-f$ are sm and $f(n \wedge m) \leq f(n), f(m) \leq f(n \vee m)$. In consequence $g \circ f$ is sm provided g is cx and $\mathrm{E}(g(\Phi^m(b))) = \mathrm{E}(g \circ f(\Phi_1(b), \ldots, \Phi_m(b))) \leq \mathrm{E}(g \circ f(\Phi_0(b), \ldots, \Phi_0(b))) = \mathrm{E}(g(\Phi_0^m(b)))$. Hence $\Phi^m(b) \leq_{cx} \Phi_0^m(b)$ and using Lemma 8.2, we get that $\Phi(b) \leq_{cx} \Phi_0(b)$. To complete the proof $\Phi \leq_{dcx} \Phi_0$, one would require a multi-variate generalization of Lorentz's inequality which we have been unable to prove.

We shall now explain the reasons for considering the above p.p. Φ. If we assume that Φ_i above are Poisson, then Φ is know to be a representation of the p.p. of the squared radii $|\Phi_G|^2 = \{|X_n|^2 \ : \ X_n \in \Phi_G\}$ of the *Ginibre process* Φ_G (see [4,19]). It has been observed in simulations that this *determinental* p.p. exhibits less clustering than the homogeneous Poisson p.p. Our result can be seen as a first step towards a formal statement of this property.

6. Applications to Wireless Communication Networks

From the point of view of applications of our main result, what remains is examples of interesting dcx functions. In what follows, we will provide such functions arising in the context of wireless networks. In many of the models we have assumed ordered point processes with i.i.d. marks. However due to Propnosition 3.7, the results hold for independently marked Cox p.p. provided the respective intensity measures are ordered.

6.1. Coverage Process with Independent Grains

The Boolean model $C(\Phi, G)$ defined earlier (see Definition 4.3) is the main object of analysis in the theory of Coverage processes (see [12]). The percolation properties of the Boolean model has been studied in [23] while the connectivity properties of the Boolean model has been studied in [30]. For $\tilde{\Phi}$ as in the Definition 4.3 of the Boolean model, denote by $V(y) = \sum_{(X_i,G_i)\in\tilde{\Phi}} \mathbf{1}[y \in X_i + G_i]$ the number of grains covering

$y \in \mathbb{R}^d$. Denote by $\psi(s_1, \ldots, s_n)$ the *joint probability generating functional (p.g.f) of the number of grains* covering locations $y_1, \ldots, y_n \in \mathbb{R}^d$ $\psi(s_1, \ldots, s_n) = \mathrm{E}\Big(\prod_{j=1}^n s_j^{V(y_j)}\Big)$, $s_j \geq 0$, $j = 1, \ldots, n$. Note that the function $g(v_1, \ldots, v_n) = \prod_j^n s_j^{v_j}$ is *idcx* when $s_j \geq 1$ for all $j = 1, \ldots, n$ and is *ddcx* when $0 \leq s_j \leq 1$ for all j.

Thus the following result follows immediately from Theorem 2.1, Proposition 3.2 and Proposition 3.7.

Corollary 6.1. *Let Φ_i, $i = 1, 2$ be a simple p.p. (of germs) on $\mathbb{R}^d$. Consider the corresponding Boolean models with the typical grain G and, as above, denote the respective coverage number fields by $\{V_i(y)\}$ and and their p.g.f by ψ_i. If $\Phi_1 \leq_{dcx\,(resp.\,idcx;\,idcv)} \Phi_2$ then $\{V_1(y)\} \leq_{dcx\,(resp.\,idcx;\,idcv)} \{V_2(y)\}$, with the result for dcx holding provided $\mathrm{E}(V_i(y)) < \infty$ for all y. In particular, if $\Phi_1 \leq_{idcx} \Phi_2$ then $\mathrm{E}(V_1(y)^\beta) \leq \mathrm{E}(V_2(y)^\beta)$ for all $\beta \geq 1$. If $\Phi_1 \leq_{idcx\,(resp.\,ddcx)} \Phi_2$ then $\psi_1(s_1, \ldots, s_n) \leq \psi_2(s_1, \ldots, s_n)$ for $s_j \geq 1$ (resp. $s_j \leq 1$) $j = 1, \ldots, n$.*

Note that $1 - \psi(0, \ldots, 0)$ represents the expected *coverage measure*, i.e.,the probability whether the locations $y_1, \ldots, y_n$ are covered by at least one grain. In [12, Section 3.8] it is shown that expected one-point coverage (or volume fraction in case of stationary p.p.) for a stationary Cox p.p. and some clustered p.p. is lower than that of a stationary, homogeneous Poisson p.p..

Coverage processes arise in various applications. In particular, in wireless communications the points of the p.p. (germs) usually represent locations of antennas and their grains the respective communication regions. In this context $V(y)$ is the number of antennas covering the point y and the coverage measure is the indicator that at least one of them is able to reach y. The application of the Boolean model to the modeling of wireless communications dates back to the article of Gilbert [10] in 1961.

6.2. Random Geometric Graphs (RGGs)

This class of graphs has increasingly found applications in spatial networks. For a detailed study of these graphs, see [30]. A random geometric graph is defined as a graph with Φ as the vertex set and the edge-set $E = \{\{X_i, X_j\} : |X_i - X_j| \leq r\}$. Clearly this is related to the Boolean model defined in the previous subsection. One of the objects of interest in a RGG is the typical degree. Under the notation of the previous subsection, the typical degree ($deg(\Phi, G)$) for a RGG formed by a stationary p.p. Φ and grain distribution G is $deg(\Phi, G) = \frac{1}{\lambda|A|} \sum_{X_i, X_j \in \Phi} \mathbf{1}[X_i \in A]\mathbf{1}[X_i \neq X_j]\mathbf{1}[(X_i + G_i) \cap (X_j + G_j) \neq \emptyset]$, where A is a bBs. If $G = B_0(r), r > 0$, then $\mathrm{E}(deg(\Phi, G)) = K(r)$ is the *Ripley's K function* defined in Section 3.2. The following result follows easily from Theorem 2.1, Proposition 3.3 and Proposition 3.7.

Corollary 6.2. *Suppose that simple p.p. $\Phi_1 \leq_{dcx} \Phi_2$, then $deg(\Phi_1, G) \leq_{idcx} deg(\Phi_2, G)$.*

6.3. Interference in Wireless Communications

The Boolean model is not sufficient for analyzing wireless networks as it ignores the fact that in radio communications signal received from one particular transmitter is jammed by the signals received from the other transmitters. According to information theory as well as existing technology, the quality of a given radio communication link is determined by the so called *signal to interference and noise ratio* (SINR) at the receiver of this link. a mathematical point of view, the interference in the above considerations is just the sum of the powers of the signals received from all transmitters (perhaps except own transmitter(s)). It is then the shot-noise field of received powers that plays important role in determining the connectivity and the capacity of the network in a broad sense. The foundations of the theory of SINR coverage processes are quite recent (see [1, 2, 8, 11]). In what follows, we shall study the impact of structure of the p.p. of interferers on given radio links.

Consider a set of n emitters $\{x_i\}$ and n receivers $\{y_i\}$. Suppose that the signal received by y_i from x_k is S_{ki}. These $\{S_{ik}\}$ are assumed to be independent. The assumption of independence is due to the phenomenon of *fading.* Let the set of additional interferers be modeled by a i.i.d. marked p.p. $\tilde{\Phi} = \varepsilon_{(X_j,(Z_j^1,\ldots,Z_j^n))}$, independent of $\{S_{ik}\}$, where Z_j^n is the power received by the receiver y_i from the interferer located at X_j. Denote the *background noise* random variable by W.

We say that the signal from x_i is successfully received by y_i if $S_{ii}/(W + I_i + V_i) > T$ where $I_i = \sum_{k\neq i} S_{ki}$ and $V_i = \sum_j Z_j^i$ is the interference received at y_i from the set of other emitters $\{x_k : k \neq i\}$ and interferers in $\tilde{\Phi}$, respectively, and $T > 0$ is some (assume constant) required *SINR threshold.* If we denote by p, the probability of successful reception of signals from each x_i to y_i, then

$$
\begin{aligned}
p &= \mathrm{P}(S_{ii} > (W + I_i + V_i)T \quad \forall i = 1,\ldots,n) \\
&= \mathrm{E}(\prod_i \overline{F}_{ii}(T(W + I_i + V_i)))\,, \qquad (5)
\end{aligned}
$$

where $\overline{F}_{ii}(s) = \mathrm{P}(S_{ii} \geq s)$ and the second equality is due to independence. Given $\{I_i : i = 1,\ldots,n\}$ and W, the expression under expectation in (5) can be viewed as a function of the value of the shot-noise vector $(V_1,\ldots,V_n)$ evaluated with respect to $\tilde{\Phi}$. Theorem 2.1 and Proposition 3.7 implies the following result concerning the impact of the structure of the set of interferers on p.

Corollary 6.3. *Consider emitters $\{x_i\}$, receivers $\{y\}_i$, powers $\{S_{ki}\}$ as above. Let $\tilde{\Phi}_u$, $u = 1,2$ be two simple marked p.p. of interferers. Denote by p_u, $u = 1,2$ the probability of successful reception given by (5) in the model with the set of interferers $\tilde{\Phi}_u$. Assume the product of tail distribution functions of the received powers $\prod_{i=1}^n \overline{F}_{ii}(s_i)$*

be dcx. If $\Phi_1 \leq_{ddcx} \Phi_2$ then $p_1 \leq p_2$.

It is quite natural to assume *ddcx* $\prod_{i=1}^n \overline{F}_{ii}(s_i)$. For example the constant emitted power P, omni-directional path-loss function $l(r)$ and *Rayleigh fading* in the radio channel implies $S_{ki} = PH_{ki}/l(|x_k - y_i|)$, where $|H_{ki}|$ are i.i.d. exponential random variables with mean 1. In this case $\prod_{i=1}^n \overline{F}_{ii}(s_i)$ is ddcx. (*)

7. Conclusions and Open Questions

To the best of our knowledge, this is the first study of *dcx* ordering of random measures and p.p.. We have defined the *dcx* order and characterized it by finite dimensional distributions of the measure values on disjoint bBs of the space. As the main result, we have proved that the integrals of some non-negative kernels with respect to *dcx* ordered random measures inherit this ordering from the measures. This was shown to be a very useful tool in study of many particular characteristics of random measures and in the construction and analysis of stochastic models.

In this paper, we have also left several open questions. Here we briefly summarize them.

- Our *dcx* order is defined via finite dimensional distributions of random measures. This makes the verification of *dcx* order more easy but requires additional work when studying functionals, which cannot be explicitly expressed in terms of the values of the measure on some finite collection of bBs as, e.g., an integral of the measure. Considering a dcx^{1*} order on the space of measures could facilitate the former task. However, the precise regularity conditions of the dcx^{1*} functional on the space of measures which would guarantee the equivalence between these two approaches are not known (cf Section 3.5).
- Comparisons of Ripley's functions (see Proposition 3.4) and pair correlation functions (Corollary 3.1) seem to indicate that the higher in *dcx* order processes cluster their points more. We have shown examples of p.p., which are larger than Poisson one, namely Cox p.p., which indeed exhibit more clustering than in Poisson p.p.. It would be interesting to show examples of p.p. which are *dcx* smaller than Poisson one, and which exhibit less clustering than it. Matérn "hard

(*) Recently in [9], under the assumption of Rayleigh fading, direct analytical methods have been used to compare the probability of successful reception in Poisson p.p. and a class of Poisson-Poisson cluster p.p. known as Neyman-Scott p.p. for both stationary and Palm versions. These results relay on explicit expressions for this probability known in the considered cases. Further, it is shown that for a certain choice of parameters, Palm version of the Poisson-Poisson cluster p.p. has a worser probability of successful reception than the Poisson p.p.. In our terminology, it simply means that the corresponding Palm versions aren't *ddcx* ordered as the connectivity probability is a *ddcx* function (Eqn. 5) of the integral shot-noise fields of the corresponding Palm versions. This strengthens Remark 5.2 by showing that *idcx* ordering of Palm versions is the best one can obtain in full generality.

core" p.p. and Ginibre p.p. are some natural candidates for this.

- We have studied *dcx* order that takes into account the dependence structure and the variability of the marginals or random measures. It seems plausible to study in a similar manner other orders such as convex, component-wise convex order etc. Note however that the supermodular order does not seem to be a reasonable one in the context of random measures. The reason is that it allows to compare only measures with the same finite dimensional distributions, and thus a Poisson p.p. can only be (trivially) compared in this order to itself. Indeed, Poisson finite dimensional distributions imply total independence property and thus uniquely characterize Poisson p.p. (cf [7, Lemma 2.3.I]).

8. Appendix

In order to make the paper more self-contained, we shall recall now some basic results on stochastic orders used in the main stream of the paper. The following two lemmas can be found in [29, Chapter 3].

Lemma 8.1. *1. A twice differentiable function f is directionally convex if and only if $\frac{\partial^2}{\partial x_i \partial x_j} f(x) \geq 0$, for all $x, 1 \leq i, j \leq n$.*

2. The stochastic order relation $\leq_{dcx}$ is generated by infinitely differentiable dcx functions.

Due to the above lemma, at some places we only prove that two random vectors are ordered with respect to twice differentiable *dcx* functions and conclude that they are *dcx* ordered.

We denote by $\xrightarrow{\mathrm{D}}$ convergence in distribution (weak convergence).

Lemma 8.2. *Let $(X^{(k)} : k = 1, \ldots)$ and $(Y^{(k)} : k = 1, \ldots)$ be sequences of random vectors. Suppose $X^{(k)} \leq_{dcx} Y^{(k)}$ for all $k \in \mathbb{N}$. If $X^{(k)} \xrightarrow{\mathrm{D}} X$ and $Y^{(k)} \xrightarrow{\mathrm{D}} Y$ and if moreover $\mathrm{E}(X^{(k)}) \to \mathrm{E}(X)$ and $\mathrm{E}(Y^{(k)}) \to \mathrm{E}(Y)$, then $X \leq_{dcx} Y$.*

The following result is from Lemmas 2.17 and 2.18 of [21].

Lemma 8.3. *1. For $i = 1, \ldots, m$ let $(S_j^i : j = 1, \ldots)$ be independent sequences of i.i.d. non-negative random variables. Suppose f is dcx (resp. idcx; idcv), then $g(n_1, \ldots, n_m) = \mathrm{E}(f(\sum_{j=1}^{n_1} S_j^1, \ldots, \sum_{j=1}^{n_m} S_j^m))$ is also dcx (resp. idcx; idcv).*

2. Let $N_i, i = 1, \ldots, k$ denote k mutually independent Poisson r.v. where the mean of N_i is λ_i. If $\phi : \mathbb{N}^k \to \mathbb{R}$ is dcx (resp. idcx; idcv), then $g(\lambda_1, \ldots, \lambda_k) = \mathrm{E}(\phi(N_1, \ldots, N_k))$ is also dcx (resp. idcx; idcv).

The first part of the following lemma is an easy extension of the one-dimensional version in [21]. The second part, which we prove in what follows, is a further extension of it.

Lemma 8.4. *Suppose $\{X(s)\}_{s\in\mathbb{R}^d}$ and $\{Y(s)\}_{s\in\mathbb{R}^d}$ are two non-negative real-valued and a.s. locally Riemann integrable random fields. For some $n \geq 1$ and disjoint bBs $I_1,\dots,I_n$ denote $J_X^i = \int_{I_i} X(s)ds$, $J_Y^i = \int_{I_i} Y(s)ds$.*

1. *If $\{X(s)\} \leq_{idcx\,(resp.\,idcv)} \{Y(s)\}$, then $(J_X^1,\dots,J_X^n) \leq_{idcx\,(resp.\,idcv)} (J_Y^1,\dots,J_Y^n)$. for any n and for any $I_1,\dots,I_n$ disjoint bBs.*
2. *Suppose further that $\mathrm{E}(\int_A X(x)dx) < \infty$ for all bBs A in $\mathbb{R}^d$ and similarly for $\{Y(x)\}$. If $\{X(x)\} \leq_{dcx} (dcv)\{Y(x)\}$, then $(J_X^1,\dots,J_X^n) \leq_{dcx(dcv)} (J_Y^1,\dots,J_Y^n)$.*

Proof. (2) We shall prove for $d = 1$ and as can be seen from the proof, the generalization is fairly straightforward.

We need to prove that $(\int_{I_1} X(s)ds,\dots,\int_{I_n} X(s)ds) \leq_{dcx} (\int_{I_1} Y(s)ds,\dots,\int_{I_n} Y(s)ds)$, for $I_i, i = 1,\dots,n$ disjoint bBs. We shall give an approximation satisfying the assumptions of Lemma 8.2. Let $I_i = [a_i, b_i]; a_i, b_i \in \mathbb{R}, i = 1,\dots,n$. Let $\{(t^i_{mj})_{1\leq j\leq k_m}, i = 1,\dots,n\}$ be the sequences of mth nested partition of each interval. The middle Riemann sum can be given as follows : $X^m(I_i) = \sum_j X(t^i_{mj})(t^i_{m(j+1)} - t^i_{mj}), i = 1,\dots,n, k \in \mathbb{N}$ and similarly for $Y(x)$. These are the variables satisfying the approximation as in Lemma 8.2. As $X(s)$ is Riemann integrable,

$$(X^m(I_1),\dots,X^m(I_n)) \to (J_X^1,\dots,J_X^n)$$

a.s. and hence in distribution. It is also clear the middle Riemann sums of $X(\cdot)$ and $Y(\cdot)$ are ordered. What remains to prove is that $\mathrm{E}\,X^m(I_i) \to \mathrm{E}\,J_X^i$. In the last term, by Fubini, we can interchange the expectation and integral and hence it suffices to prove $\mathrm{E}\,X^m(I_i) \to \int_{I_i} \mathrm{E}\,X(s)ds$. Our assumption implies that this is true. $\square$

Acknowledgement

DY was supported by a grant from EADS, France. DY also wishes to thank Prof. Fred Huffer for sharing the technical report.

References

[1] Baccelli, F. and Błaszczyszyn, B. (2001). On a coverage process ranging from the Boolean model to the Poisson-Voronoi tessellation with applications to wireless communications. *Adv. Appl. Probab.* **33**, 293–323.

[2] Baccelli, F., Błaszczyszyn, B. and Mühlethaler, P. (2006). An aloha Protocol for multihop mobile wireless networks. *IEEE Trans. Inf. Theory* **52**, 421–436.

[3] BASSAN, B. AND SCARSINI, M. (1991). Convex orderings for stochastic processes. *Comment. Math. Univ. Carolin.* **32**, 115–118.

[4] BEN HOUGH, J., KRISHNAPUR, M., PERES,Y. AND VIRAG,B. (2006). Determinantal processes and Independence. *Probability Surveys* **3,** 206–229.

[5] BILLINGSLEY, P. (1979). *Probability and Measure*, Wiley, New York.

[6] CHANG, C.-S., CHAO, X.L. AND PINEDO, M. (1991) Monotonicity results for queues with doubly stochastic Poisson arrivals:Ross conjecture. *Adv. Appl. Probab.* **23**, 210–228.

[7] DALEY, D.J. AND VERE-JONES, D. (1988). *An Introduction to the Theory of Point Processes*, Springer Verlag, New York.

[8] DOUSSE, O., FRANCESCHETTI, M., MACRIS, N., MEESTER, R. AND THIRAN, P. (2006). Percolation in the signal to interference ratio graph. *J. Appl. Probab.* **43**, 552–562.

[9] GANTI, R. AND HAENGGI, M. (2008). Interference and Outage in Clustered Wireless Ad Hoc Networks. *arXiv:0706.2434v1*, `http://arxiv.org/abs/0706.2434`

[10] GILBERT, E. N. (1961) Random plane networks. *SIAM J.* **9**, 533–543.

[11] GUPTA, P. AND KUMAR, P. R. (2000) The capacity of wireless networks. *IEEE Trans. Inf. Theory* **42(2)**, 388–404.

[12] HALL, P.G. (1988). *Introduction to the Theory of Coverage processes*, Wiley, New York.

[13] HEINRICH, L. AND MOLCHANOV, I.S. (1994). Some limit theorems for extremal and union shot-noise processes. *Mathematische Nachrichten* **168,** 139-159.

[14] HELLMUND, G., PROKEŠOVÁ, M. AND VEDEL JENSEN, E.B. (2008). Lévy based Cox point processes. *Adv. Appl. Probab.* **40**, 603–629.

[15] HUFFER, F. (1984). Inequalities for $M/G/\infty$ queues and related shot noise processes *J. Appl. Probab.* **24**, 978–989.

[16] HUFFER, F. (1984). Inequalities for $M/G/\infty$ queues and related shot noise processes. *Technical report No.351.* Dept. of Statistics, Stanford University, California.

[17] KALLENBERG, O. (1983). *Random Measures*, Academic Press, London.

[18] KAMAE, T., KRENGEL, U. AND O'BRIEN, G.L. (1977). Stochastic Inequalities on partially ordered spaces. *Ann. Probab.* **5**, 899–912.

[19] KOSTLAN, E. (1992). On the spectra of Gaussian matrices. *Linear Algebra Appl.* **162/164**, 385–388.

[20] LINDVALL, T. (1992) *Lectures on Coupling Method*, John Wiley, London.

[21] MEESTER, L.E. AND SHANTHIKUMAR, J.G. (1993). Regularity of Stochastic processes - A Theory based on directional convexity. *Probab. Eng Inform. Sc.* **7**, 343–360.

[22] MEESTER L.E. AND SHANTHIKUMAR, J.G. (1999). Stochastic Convexity on a general space. *Math. Operat. Res.* **24**, 472–494.

[23] Meester, R. and Roy, R. (1996). *Continuum Percolation*, Cambridge University Press.

[24] Miyoshi, N. (2004). A Note on bounds and monotonicity of spatial stationary cox noise processes. *Probab. Eng Inform. Sc.* **18**, 561–571.

[25] Miyoshi, N. and Rolski, T. (2004) Ross type conjectures on Monotonicity of queues *Australian & New Zealand Journal of Statistics.* **46**, 121–131.

[26] Møller, J., Syversveen, A.R. and Waagepetersen, R.P. (1998). Log Gaussian Cox processes *Scandinavian Journal of Statistics.* **25**, 451-482.

[27] Møller, J. (2003). Shot Noise Cox processes. *Adv. Appl. Probab.* **35**, 614–640.

[28] Møller, J. and Torrisi, G. L. (2005). Generalized Shot Noise Cox processes. *Adv. Appl. Probab.* **37**, 48–74.

[29] Müller, A. and Stoyan, D. (2002). *Comparison Methods for Stochastic Models and Risks* , John Wiley & Sons, New York.

[30] Penrose, M.D. (2003). *Random Geometric Graphs*, Clarendon Press, Oxford.

[31] Rolski, T. (1989). Upper bounds for single server queues with doubly stochastic Poisson arrivals. *Math. Operat. Res.* **11**, 442–450.

[32] Rolski, T. and Szekli, R. (1991). Stochastic ordering and thinning of point processes. *Stoch. Proc. Appl.* **37**, 299–312.

[33] Ross, S.M. (1978) Average delay in queues with non-stationary Poisson arrivals. *J. Appl. Probab.* **15**, 602–609.

[34] Shaked, M. and Shanthikumar, J.G. (1990). Parametric stochastic convexity and concavity of stochastic processes. *Annals of the Institute of Statistical Mathematics.* **42**, 509–531.

[35] Stoyan, D., Kendall, W. and Mecke, J. (1995). *Stochastic Geometry and its Applications*, John Wiley & Sons, Chichester.

[36] Stoyan, D. (1983). Inequalities and bounds for variances of point processes and fibre processes. *Math. Operationsforsch. Statist. Ser. Statist.* **14**, 409–419.