On the stabilization of persistently excited linear systems

Yacine Chitour, Mario Sigalotti

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Yacine Chitour† Mario Sigalotti‡

Abstract

We consider control systems of the type $\dot{x} = Ax + \alpha(t)bu$, where $u \in \mathbf{R}$, (A, b) is a controllable pair and α is an unknown time-varying signal with values in $[0, 1]$ satisfying a persistent excitation condition i.e., $\int_t^{t+T} \alpha(s)ds \geq \mu$ for every $t \geq 0$, with $0 < \mu \leq T$ independent on t. We prove that such a system is stabilizable with a linear feedback depending only on the pair (T, μ) if the eigenvalues of A have non-positive real part. We also show that stabilizability does not hold for arbitrary matrices A. Moreover, the question of whether the system can be stabilized or not with an arbitrarily large rate of convergence gives rise to a bifurcation phenomenon in dependence of the parameter μ/T.

1 Introduction

The present paper is a continuation of [9], where the study of control linear systems subject to scalar persistently excited PE-signals was initiated. The general form of such systems is given by

$$\dot{x} = Ax + \alpha(t)Bu\,, \tag{1}$$

where $x \in \mathbf{R}^n$, $u \in \mathbf{R}^m$ and the function α is a *scalar* PE-signal, i.e., α takes values in $[0, 1]$ and there exist two positive constants μ, T such that, for every $t \geq 0$,

$$\int_t^{t+T} \alpha(s)ds \geq \mu. \tag{2}$$

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Given two positive real numbers μ and T with $\mu \leq T$, we use $\mathcal{G}(T,\mu)$ to denote the class of all PE signals verifying (2).

In (1), the PE-signal α can be seen as an input perturbation modeling the fact that the instants where the control u acts on the system are not exactly known. If α only takes the values 0 and 1, then (1) actually switches between the uncontrolled system $\dot{x} = Ax$ and the controlled one $\dot{x} = Ax + Bu$. In that context, the PE condition (2) is designed to guarantee some action on the system. (For a more detailed discussion on the interpretation of persistently excited systems and on the related literature, see [9].)

Our main concern will be the global asymptotic stabilization of system (1) with a constant linear feedback $u = -Kx$ where the gain matrix K is required to be the same *for all* signals in the considered class $\mathcal{G}(T,\mu)$ i.e., K depends only on A, b, T, μ and not on a specific element of $\mathcal{G}(T,\mu)$. We refer to such a gain matrix K as a (T,μ)-stabilizer. It is clear that (A, B) must be stabilizable for hoping that a (T,μ)-stabilizer exists and we will suppose that throughout the paper. Moreover, the stabilizability analysis can be reduced to the controllability subspace and thus to the case where (A, B) is controllable.

The questions studied in this paper find their origin in a problem stemming from identification and adaptive control (cf. [3]). Such a problem deals with the linear system $\dot{x} = -P(t)u$, where the matrix $P(\cdot)$ is symmetric non-negative definite and plays the role of α. If $P \equiv I$, then $u^* = x$ trivially stabilizes the system exponentially. But what if $P(t)$ is only semi-positive definite for all t? Under which conditions on P does $u^* = x$ still stabilize the system? The answer for this particular case, can be found in the seminal paper [13] which asserts that, if $x \in \mathbf{R}^n$ and $P \geq 0$ is bounded and has bounded derivative, it is *necessary and sufficient*, for the global exponential stability of $\dot{x} = -P(t)x$, that P is also *persistently exciting*, *i.e.*, that there exist $\mu, T > 0$ such that

$$\int_t^{t+T} \xi^T P(s)\xi ds \geq \mu, \tag{3}$$

for all unitary vectors $\xi \in \mathbf{R}^n$ and all $t \geq 0$. Therefore, as regards the stabilization of (1), the notion of *persistent excitation* seems to be a reasonable additional assumption on the signals α.

Let us recall the main results of [9]. We first addressed the issue of controllability of (1), uniformly with respect to $\alpha \in \mathcal{G}(T,\mu)$. We proved that, if the pair (A, B) is controllable, then (1) is (completely) controllable in time t if and only if $t > T - \mu$. We next focused on the existence of (T,μ)-stabilizers. We first treated the case where A is neutrally stable and we showed that in this case the gain $K = B^T$ is a (T,μ)-stabilizer for system (1) (see also [3]). Note that in the neutrally stable case K does not depend on T and μ. We next turned to the case where A is not stable. In such a situation, even in the one-dimensional case, a stabilizer K cannot be chosen independently of T and μ. In [9], we considered the first nontrivial unstable case, namely the double integrator $\dot{x} = J_2x + \alpha b_0 u$, where

J_2 denotes the 2×2 Jordan block, the control is scalar and $b_0 = (0, 1)^T$. We showed that, for every pair (T, μ), there exists a (T, μ)-stabilizer for $\dot{x} = J_2 x + \alpha b_0 u$, $\alpha \in \mathcal{G}(T, \mu)$.

In this paper, we restrict ourselves to the single-input case

$$\dot{x} = Ax + \alpha(t) bu, \qquad u \in \mathbf{R}, \qquad \alpha \in \mathcal{G}(T, \mu), \tag{4}$$

and we provide two sets of results. The first one concerns the stabilizability of (4). Given two arbitrary constants μ and T with $0 < \mu \leq T$, we prove the existence of a (T, μ)-stabilizer for (4) when the eigenvalues of A have non-positive real part. The second set of results concerns the possibility of obtaining an arbitrary rate of convergence once stabilization is achieved. We essentially focus on the two-dimensional case and we point out an interesting phenomenon: there exists $\rho_* \in (0, 1)$ so that, for every controllable two-dimensional pair (A, b), every $T > 0$ and every $\mu \in (0, \rho_* T)$, the maximal rate of convergence of (4) is finite. Here maximality is evaluated with respect to all possible (T, μ)-stabilizers. As a consequence, we prove the existence of matrices A (e.g., $J_2 + \lambda \mathrm{Id}_2$ with λ large enough) such that for every $T > 0$ and every $\mu \in (0, \rho_* T)$, the PE system (4) does not admit (T, μ)-stabilizers. The latter result is rather surprising when one compares it with the following two facts: let $\rho \in (0, 1]$; (i) given a sequence $(\alpha_n)_{n \in \mathbf{N}}$ with $\alpha_n \in \mathcal{G}(T_n, \rho T_n)$ and $\lim_{n \to +\infty} T_n = 0$, all its weak-$\star$ limit points $\alpha_\star$ take values in $[\rho, 1]$ (see Lemma 2.5) and (ii) the two-dimensional switched system $\dot{x} = J_2 x + \alpha_\star b_0 u$ can be stabilized, uniformly with respect to $\alpha_\star \in L^\infty(\mathbf{R}_{\geq 0}, [\rho, 1])$, with an arbitrary rate of convergence. The weak-$\star$ convergence considered in (i) is the natural one in this context since it renders the input-output mapping continuous.

Let us briefly comment on the technics used in this paper. First of all, it is clear that the notion of *common Lyapunov function*, rather powerful in the realm of switched systems, cannot be of (direct) help here since, at the differential level, one can evolve with an unstable dynamics $\dot{x} = Ax$, when $\alpha = 0$ takes the value zero. More refined tools as multiple and non-monotone Lyapunov functions (see, e.g., [1, 2, 7, 10, 14, 16]) do not seem well-adapted to persistently excited systems, at least for what concerns the proof of their stability. It seems to us that one must rather perform a trajectory analysis, on a time interval of length at least equal to T, in order to achieve any information which is uniform with respect to $\alpha \in \mathcal{G}(T, \mu)$. This viewpoint is more similar to the geometric approach to switched systems behind the results in [4, 5, 6]. As a second consideration, notice that point (i) described above, which is systematically used in the paper, presents formal similarities with the technique of *averaging* but is rather different from it, since no periodicity nor constant-average assumption is made here. Moreover, for a given persistently excited system, T is fixed and thus it does not tend to zero.

The paper is organized as follows. In Section 2 we introduce the notations of the paper, the basic definitions and some useful technical lemmas. We gather in Section 3 the stabilizability results for matrices whose spectrum has non-positive real part. Finally, the analysis of the maximal rates of convergence and divergence is the object of Section 4. Since many of our results give rise to further challenging questions, we propose in Section 5 several conjectures and open problems.

2 Notations and definitions

Let $\mathbf{N}$ denote the set of positive integers. Given n and m belonging to $\mathbf{N}$, we use $0_{n\times m}$ to denote the $n \times m$ matrix made of zeroes, $M_n(\mathbf{R})$ the set of real-valued $n \times n$ matrices, and Id_n the $n \times n$ identity matrix. We also write 0_n for $0_{n\times 1}$, $\sigma(A)$ for the spectrum of a matrix $A \in M_n(\mathbf{R})$, and $\Re(\lambda)$ (respectively, $\Im(\lambda)$) for the real (respectively, imaginary) part of a complex number λ.

Definition 2.1 (PE signal and (T, μ)-signal) *Let μ and T be positive constants with $\mu \le T$. A* (T, μ)-signal *is a measurable function $\alpha : \mathbf{R}_{\ge 0} \to [0, 1]$ satisfying*

$$\int_t^{t+T} \alpha(s)ds \ge \mu\,, \quad \forall t \in \mathbf{R}_{\ge 0}\,. \tag{5}$$

We use $\mathcal{G}(T, \mu)$ to denote the set of all (T, μ)-signals. A PE signal *is a measurable function $\alpha : \mathbf{R}_{\ge 0} \to [0, 1]$ such that there exist T, μ for which α is a (T, μ)-signal.*

Definition 2.2 (PE system) *Given two positive constants μ and T with $\mu \le T$ and a controllable pair $(A, b) \in M_n(\mathbf{R}) \times \mathbf{R}^n$, we define the* PE system *associated to T, μ, A, and b as the family of linear control systems given by*

$$\dot{x} = Ax + \alpha ub, \quad \alpha \in \mathcal{G}(T, \mu). \tag{6}$$

Given a PE system (6), we address the following problem. We want to stabilize (6) *uniformly* with respect to every (T, μ)-signal α, i.e., we want to find a vector $K \in \mathbf{R}^n$ which makes the origin of

$$\dot{x} = (A - \alpha(t)bK^T)x \tag{7}$$

globally asymptotically stable, with K depending only on A, b, T and μ.

More precisely, referring to $x(\cdot\,; t_0, x_0, K, \alpha)$ as the solution of (7) with initial condition $x(t_0; t_0, x_0, K, \alpha) = x_0$, we introduce the following definition.

Definition 2.3 ((T, μ)-stabilizer) *Let μ and T be positive constants with $\mu \le T$. The gain K is said to be a* (T, μ)-stabilizer *for (6) if (7) is globally asymptotically stable, uniformly with every (T, μ)-signal α. Since (7) is linear in x, this is equivalent to say that (7) is exponentially stable, uniformly with respect to $\alpha \in \mathcal{G}(T, \mu)$, i.e., there exist $C, \gamma > 0$ such that every solution $x(\cdot\,; t_0, x_0, K, \alpha)$ of (7) satisfies*

$$\|x(t; t_0, x_0, K, \alpha)\| \le Ce^{-(t-t_0)\gamma}\|x_0\|, \quad \forall t \ge t_0.$$

The next two lemmas collect some properties of PE signals.

Lemma 2.4 *1. If $\alpha(\cdot)$ is a (T,μ)-signal, then, for every $t_0 \geq 0$, the same is true for $\alpha(t_0+\cdot)$.*

2. If $0 < \rho' < \rho$ and $T > 0$ then $\mathcal{G}(T,\rho T) \subset \mathcal{G}(T,\rho' T)$.

3. For $\eta \in (0,\mu)$, $\mathcal{G}(T,\mu) \subset \mathcal{G}(T+\eta,\mu) \cap \mathcal{G}(T-\eta,\mu-\eta)$.

4. If $T \geq \tau > 0$ and $\rho > 0$, then $\mathcal{G}(\tau,\rho\tau) \subset \mathcal{G}(T,(\rho/2)T)$.

5. For every $0 < \rho' < \rho$ there exists $M > 0$ such that for every $T \geq M\tau > 0$ one has $\mathcal{G}(\tau,\rho\tau) \subset \mathcal{G}(T,\rho' T)$.

Proof. We only provide an argument for points 4 and 5. Fix $t \geq 0$, $T \geq \tau$, $\rho > 0$ and $\alpha \in \mathcal{G}(\tau,\rho\tau)$. Let l be the integer part of T/τ. Since $l \geq \max(1, T/\tau - 1)$, then $\int_t^{t+T} \alpha(s)ds \geq l\rho\tau \geq \max(\tau, T-\tau)\rho \geq T\rho/2$. For $\rho' \in (0,\rho)$ and T/τ large enough, one has $\max(\tau, T-\tau) \geq (\rho'/\rho)T$ and so $\int_t^{t+T} \alpha(s)ds \geq \rho' T$. ■

Let

$$b_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad A_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Recall that an element f of $L^\infty(\mathbf{R}_{\geq 0},[0,1])$ is the weak-$\star$ limit of a sequence $(f_k)_{k\in\mathbf{N}}$ of elements of $L^\infty(\mathbf{R}_{\geq 0},[0,1])$ if, for every $g \in L^1(\mathbf{R}_{\geq 0},\mathbf{R})$,

$$\int_0^\infty f(s)g(s)ds = \lim_{k\to\infty} \int_0^\infty f_k(s)g(s)ds. \tag{8}$$

It is well known that $L^\infty(\mathbf{R}_{\geq 0},[0,1])$ endowed with the weak-$\star$ topology is compact (see, for instance, [8]). Hence, each $\mathcal{G}(T,\mu)$ is weak-$\star$ compact. Unless specified, limit points of sequences of PE signals are to be understood as limits of subsequences with respect to the weak-$\star$ topology of $L^\infty(\mathbf{R}_{\geq 0},[0,1])$.

Lemma 2.5 *Let $(\alpha^{(n)})_{n\in\mathbf{N}}$ and $(\nu_n)_{n\in\mathbf{N}}$ be, respectively, a sequence of (T,μ)-signals and an increasing sequence of positive real numbers such that $\lim_{n\to\infty} \nu_n = \infty$.*

1. Define α_n as the $(T/\nu_n, \mu/\nu_n)$-signal given by $\alpha_n(t) = \alpha^{(n)}(\nu_n t)$ for $t \geq 0$. If $\alpha_\star$ is a limit point of the sequence $(\alpha_n)_{n\in\mathbf{N}}$, then $\alpha_\star$ takes values in $[\mu/T, 1]$ almost everywhere.

2. Let $j_0 \in \{0,1\}$ and $h \in \mathbf{N}$. Let ω_j, $j = j_0,\dots,h$, be real numbers with $\omega_j = 0$ if and only if $j = 0$ and $\{\pm\omega_j\} \neq \{\pm\omega_l\}$ for $j \neq l$. For every $t \geq 0$, let

$$v(t) = \begin{pmatrix} 1 \\ e^{\omega_1 A_0 t} b_0 \\ \vdots \\ e^{\omega_h A_0 t} b_0 \end{pmatrix} \quad \text{if } j_0 = 0 \quad \text{or} \quad v(t) = \begin{pmatrix} e^{\omega_1 A_0 t} b_0 \\ \vdots \\ e^{\omega_h A_0 t} b_0 \end{pmatrix} \quad \text{if } j_0 = 1.$$

For every signal α and every $t \geq 0$, define

$$\alpha^C(t) = \alpha(t)v(t)v(t)^T. \tag{9}$$

Then α^C is a time-dependent non-negative symmetric $(2h+1-j_0)\times(2h+1-j_0)$ matrix with $\alpha^C \leq \mathrm{Id}_{2h+1-j_0}$ and there exists $\xi > 0$ only depending on T, μ and $\omega_{j_0}, \ldots, \omega_h$ such that, for every $t \geq 0$,

$$\int_t^{t+T} \alpha^C(\tau)d\tau \geq \xi\, \mathrm{Id}_{2h+1-j_0}. \tag{10}$$

Define, moreover, $\alpha_n^C(t) = (\alpha^{(n)})^C(\nu_n t)$ for every $t \geq 0$ and every $n \in \mathbf{N}$. If $\alpha_\star^C$ is a limit point of the sequence $(\alpha_n^C)_{n\in\mathbf{N}}$ for the weak-$\star$ topology of $L^\infty(\mathbf{R}_{\geq 0}, M_{2h+1-j_0}(\mathbf{R}))$, then $\alpha_\star^C \geq (\xi/T)\mathrm{Id}_{2h+1-j_0}$ almost everywhere.

Proof. Let us first prove point 1. Let $\alpha_\star$ be the weak-$\star$ limit of some sequence $(\alpha_{n_k})_{k\geq 1}$. For every interval J contained in $\mathbf{R}_{\geq 0}$ of finite length $|J| > 0$, apply (8) by taking as g the characteristic function of J. Since each α_{n_k} is a $(T/\nu_{n_k}, \mu/\nu_{n_k})$-signal, it follows that

$$\frac{1}{|J|}\int_J \alpha_\star(s)ds = \lim_{k\to\infty}\frac{1}{|J|}\int_J \alpha_{n_k}(s)ds \geq \liminf_{k\to\infty}\frac{\mu}{|J|\nu_{n_k}}\mathcal{I}\left(\frac{|J|\nu_{n_k}}{T}\right) = \frac{\mu}{T},$$

where $\mathcal{I}(\cdot)$ denotes the integer part. Since $\alpha_\star$ is measurable and bounded, almost every $t > 0$ is a Lebesgue point for $\alpha_\star$, *i.e.*, the limit

$$\lim_{\varepsilon\to 0+}\frac{1}{2\varepsilon}\int_{t-\varepsilon}^{t+\varepsilon}\alpha_\star(s)ds$$

exists and is equal to $\alpha_\star(t)$ (see, for instance, [15]). We conclude that, as claimed, $\alpha_\star \geq \mu/T$ almost everywhere.

For the first part of point 2 fix $t \geq 0$ and notice that the map

$$\alpha \mapsto \int_t^{t+T}\alpha^C(s)ds$$

is continuous with respect to the weak-$\star$ topology and takes values in the set of non-negative symmetric matrices.

We claim that all such matrices are positive definite. Assume by contradiction that there exist $\alpha \in \mathcal{G}(T,\mu)$ and $x_0 \in \mathbf{R}^{2h+1-j_0}\setminus\{0_{2h+1-j_0}\}$ such that $\int_t^{t+T} x_0^T\alpha^C(s)x_0 ds = 0$. Then, for almost every $s \in [t, t+T]$, we would have $\alpha(s)x_0^T v(s) = 0$. Since $\alpha(s) \neq 0$ for s in a set of positive measure, we deduce that the real-analytic function $x_0^T v(\cdot)$ is identically equal to zero. Let $A_0^C = \mathrm{diag}(1, \omega_1 A_0, \ldots, \omega_h A_0)$ if $j_0 = 0$ or $A_0^C = \mathrm{diag}(\omega_1 A_0, \ldots, \omega_h A_0)$ if $j_0 = 1$. Then $x_0^T(A_0^C)^j v(0) = 0$ for every non-negative integer j. The contradiction is reached, since $(A_0^C, v(0))$ is a controllable pair and $x_0 \neq 0_{2h+1-j_0}$.

Then, by weak-$\star$ compactness of $\mathcal{G}(T,\mu)$, we deduce the existence of $\xi > 0$ independent of α such that (10) holds true. The independence of ξ with respect to t follows from the shift-invariance of $\mathcal{G}(T,\mu)$ pointed out in Lemma 2.4.

The second part of point 2 follows from the same argument used to prove point 1, noticing that, for every $t \geq 0$,

$$\int_t^{t+\frac{T}{\nu_n}} \alpha_n^C(\tau)d\tau \geq \frac{\xi}{\nu_n}\mathrm{Id}_{2h+1-j_0}.$$

■

3 Spectra with non-positive real part

We consider below the problem of whether a controllable pair (A, b) gives rise to a PE system that can be (T,μ)-stabilized for every choice of μ and T. We will see in Section 4 that this cannot be done in general. The scope of this section is to study the case in which each eigenvalue of A has non-positive real part.

The first step is to consider the special case of the n-integrator. Let $J_n \in M_n(\mathbf{R})$ be defined as

$$J_n = \begin{pmatrix} 0 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & & & \ddots & \ddots & \vdots \\ 0 & & \cdots & & 0 & 1 \\ 0 & & \cdots & \cdots & & 0 \end{pmatrix}.$$

Theorem 3.1 *Let $A = J_n$ and $b = (0,\dots,0,1)^T \in \mathbf{R}^n$. Then, for every T,μ with $T \geq \mu > 0$ there exists a (T,μ)-stabilizer for (6).*

Proof. In the special case of the n-integrator system (7) becomes

$$\left\{ \begin{array}{rcl} \dot{x}_j &=& x_{j+1}, \quad \text{for } j = 1,\dots,n-1, \\ \dot{x}_n &=& -\alpha(t)(k_1x_1 + \cdots + k_nx_n)\,, \end{array} \right. \tag{11}$$

where $K = (k_1,\dots,k_n)^T$.

For every $\nu > 0$, define

$$D_{n,\nu} = \mathrm{diag}(\nu^{n-1},\dots,\nu,1). \tag{12}$$

As done in [9] in the case $n = 2$, one easily checks that, in accordance with

$$\nu D_{n,\nu}^{-1} J_n D_{n,\nu} = J_n, \quad D_{n,\nu}b = b, \tag{13}$$

the time-space transformation

$$x_\nu(t) = D_{n,\nu}^{-1} x(\nu t)\,, \quad \forall t \geq \frac{t_0}{\nu}, \tag{14}$$

of the trajectory $x(\cdot) = x(\cdot\,; t_0, x_0, K, \alpha)$ satisfies

$$\frac{d}{dt} x_\nu(t) \;\; = \;\; J_n x_\nu(t) - \alpha(\nu t)\nu b K^T D_{n,\nu} x_\nu(t),$$

that is,

$$x_\nu(\cdot) = x(\cdot\,; t_0/\nu, D_{n,\nu}^{-1} x_0, \nu D_{n,\nu} K, \alpha(\nu\,\cdot)). \tag{15}$$

As a consequence, (11) admits a (T,μ)-stabilizer if and only if it admits a $(T/\nu, \mu/\nu)$-stabilizer. More precisely, K is a (T,μ)-stabilizer if and only if $\nu D_{n,\nu} K$ is a $(T/\nu, \mu/\nu)$-stabilizer.

Let us introduce, for every gain K, the switched system

$$\begin{cases} \dot{x}_j & = & x_{j+1}, \quad \text{for } j = 1, \dots, n-1, \\ \dot{x}_n & = & -\alpha_\star(t)(k_1 x_1 + \cdots + k_n x_n), \end{cases} \qquad \alpha_\star \in L^\infty(\mathbf{R}_{\geq 0}, [\mu/T, 1]). \tag{16}$$

Recall that (16) is said to be *globally uniformly exponentially stable* as a switched system if the origin is globally exponentially stable, uniformly with respect to $\alpha_\star \in L^\infty(\mathbf{R}_{\geq 0}, [\mu/T, 1])$, for the dynamics of (16). (For this and other notions of stability of switched systems see, for instance, [12].) For every K such that $k_1 \neq 0$, define $X_1 = k_1 x_1 + \cdots + k_n x_n$, $X_2 = k_1 x_2 + \cdots + k_{n-1} x_n$, $\dots$, $X_n = k_1 x_n$. The global uniform exponential stability of (16) is clearly equivalent to that of

$$\dot{X}_j = X_{j+1} - \alpha_\star \bar{k}_j X_1, \qquad j = 1, \dots, n, \quad \alpha_\star(t) \in [\mu/T, 1], \tag{17}$$

where $\bar{k}_j = k_{n+1-j}$ and, by convention, $X_{n+1} = 0_n$.

It has been proven in Gauthier and Kupka [11, Lemma 4.0] (where the result is attributed to W.P. Dayawansa), that there exist $\overline{K} \in \mathbf{R}^n$, a scalar $\gamma > 0$ and a symmetric positive definite $n \times n$ matrix S such that

$$(J_n - \bar{\alpha}\overline{K}(1,0,\dots,0))^T S + S(J_n - \bar{\alpha}\overline{K}(1,0,\dots,0)) \leq -\gamma \mathrm{Id}_n, \tag{18}$$

for every (constant) $\bar{\alpha} \in [\mu/T, 1]$.

Hence, there exist a gain $K \in \mathbf{R}^n$ such that (16) is globally uniformly exponentially stable and a positive definite matrix S' such that the quadratic Lyapunov function $V(x) = x^T S' x$ decreases uniformly on all trajectories of (16). In particular, there exists a time τ such that every trajectory of (16) starting in $B_2^V = \{x \in \mathbf{R}^n \mid V(x) \leq 2\}$ at time 0 lies in $B_1^V = \{x \in \mathbf{R}^n \mid V(x) \leq 1\}$ for every time larger than τ.

We claim that, for some $\nu > 0$, every trajectory of $\dot{x} = (A - \alpha_\nu(t) b K^T)x$ with initial condition in B_2^V and corresponding to a $(T/\nu, \mu/\nu)$-signal α_ν stays in B_1^V for every time larger than 2τ. (In particular, by homogeneity, K is a $(T/\nu, \mu/\nu)$-stabilizer and thus $\nu^{-1}D_{n,\nu}^{-1}K$ is a (T, μ)-stabilizer.) Assume, by contradiction, that for every $l \in \mathbf{N}$ there exist $x_{0,l} \in B_2^V$, $t_l \in [2\tau, 4\tau]$ and $\alpha_l \in \mathcal{G}(T/l, \mu/l)$ such that

$$x(t_l; 0, x_{0,l}, K, \alpha_l) \notin B_1^V \quad \text{for every } l \in \mathbf{N}. \tag{19}$$

By compactness of $B_2^V \times [2\tau, 4\tau]$ and by weak-$\star$ compactness of $L^\infty(\mathbf{R}_{\geq 0}, [0, 1])$, we can assume that, up to extracting a subsequence, $x_{0,l} \to x_{0,\star} \in B_2^V$, $t_l \to t_\star \in [2\tau, 4\tau]$ and α_l converges weakly-$\star$ to $\alpha_\star \in L^\infty(\mathbf{R}_{\geq 0}, [0, 1])$ as l goes to infinity. Then $x(t_l; 0, x_{0,l}, K, \alpha_l)$ converges, as l goes to infinity, to $x(t_\star; 0, x_\star, K, \alpha_\star)$ (see [9, Appendix] for details). Since $\alpha_\star \geq \mu/T$ almost everywhere (point 1 of Lemma 2.5), then $\alpha_\star$ can be taken as an admissible signal in (16).

By homogeneity of the linear system (16) and because $t_\star \geq 2\tau$, we have that

$$V(x(t_\star; 0, x_\star, K, \alpha_\star)) \leq 1/2.$$

Therefore, for l large enough $x(t_l; 0, x_{0,l}, K, \alpha_l) \in B_1^V$ contradicting (19). ■

Let us now turn the general case where the spectrum of A has non-positive real part. The main technical difficulties in order to adapt the proof of Theorem 3.1 come from the fact that A may have several Jordan blocks of different sizes.

Theorem 3.2 *Let $(A, b) \in M_n(\mathbf{R}) \times \mathbf{R}^n$ be a controllable pair and assume that the eigenvalues of A have non-positive real part. Then, for every T, μ with $T \geq \mu > 0$ there exists a (T, μ)-stabilizer for (6).*

Proof. Fix a controllable pair $(A, b) \in M_n(\mathbf{R}) \times \mathbf{R}^n$. Up to a linear change of variable, A and b can be written as

$$A = \begin{pmatrix} A_1 & A_3 \\ 0_{(n-n') \times n'} & A_2 \end{pmatrix}, \quad b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix},$$

where $n' \in \{0, \ldots, n\}$, $A_1 \in M_{n'}(\mathbf{R})$ is Hurwitz and all the eigenvalues of $A_2 \in M_{n-n'}(\mathbf{R})$ have zero real part. From the controllability assumption, we deduce that (A_2, b_2) is controllable. Setting $x = (x_1^T, x_2^T)^T$ according to the above decomposition, system (1) can be written as

$$\dot{x}_1 = A_1 x_1 + A_3 x_2 + \alpha(t) b_1 u, \tag{20}$$

$$\dot{x}_2 = A_2 x_2 + \alpha(t) b_2 u. \tag{21}$$

If there exists a (T, μ)-stabilizer K_2 for (21), then

$$K = \begin{pmatrix} 0_{n'} \\ K_2 \end{pmatrix}$$

is a (T,μ) stabilizer for (1). It is therefore enough to prove the theorem under the extra hypothesis that all eigenvalues of A lie on the imaginary axis.

Denote the distinct eigenvalues of A by $\pm i\omega_j$, $j\in\{j_0,j_0+1,\dots,h\}$, where $j_0=1$ if $0\notin\sigma(A)$ and $j_0=0$ with $\omega_0=0$ otherwise. For every $j\in\{0,\dots,h\}$, let r_j be the multiplicity of $i\omega_j$, with the convention that $r_0=0$ if $0\notin\sigma(A)$.

Assume that A is decomposed in Jordan blocks. Since (A,b) is controllable, then A has a unique (complex) Jordan block associated with each $\{i\omega_j,-i\omega_j\}$, $j_0\le j\le h$. (Otherwise, the rank of the matrix $(A-i\omega_j\mathrm{Id}_n\mid b)$ would be strictly smaller than n, contradicting the Hautus test for controllability.) Therefore, for every $j=1,\dots,h$, the Jordan block associated to $i\omega_j$ is $\omega_j A^{(j)}+J^C_{r_j}$, where $A^{(j)}=\mathrm{diag}(A_0,\dots,A_0)\in M_{2r_j}(\mathbf{R})$ and $J^C_{r_j}\in M_{2r_j}(\mathbf{R})$ is defined as

$$J^C_{r_j}=\begin{pmatrix} 0_{2\times2} & \mathrm{Id}_2 & 0_{2\times2} & \cdots & \cdots & 0_{2\times2}\\ 0_{2\times2} & 0_{2\times2} & \mathrm{Id}_2 & 0_{2\times2} & \cdots & 0_{2\times2}\\ \vdots & & \ddots & \ddots\quad\ddots & \ddots & \vdots\\ & & & & & 0_{2\times2}\\ 0_{2\times2} & & \cdots & 0_{2\times2} & 0_{2\times2} & \mathrm{Id}_2\\ 0_{2\times2} & & \cdots & \cdots & 0_{2\times2} & 0_{2\times2}\end{pmatrix},$$

that is, in terms of the Kronecker product, $J^C_{r_j}=J_{r_j}\otimes\mathrm{Id}_2$.

All controllable linear control systems associated with a pair (A,b) that have in common the eigenvalues of A, counted according to their multiplicity, are state-equivalent, since they can be transformed by a linear transformation of coordinates into the same system under companion form. We exploit such an equivalence to deduce that, up to a linear transformation of coordinates, (1) can be written as

$$\left\{\begin{array}{rcl}\dot x_0 &=& J_{r_0}x_0+\alpha b^0 u,\\ \dot x_j &=& (\omega_j A^{(j)}+J^C_{r_j})x_j+\alpha b^j u,\quad \text{for } j=1,\dots,h,\end{array}\right. \tag{22}$$

where b^0 and b^j are respectively the vectors of $\mathbf{R}^{r_0}$ and $\mathbf{R}^{2r_j}$ with all coordinates equal to zero except the last one that is equal to one. Here $x_0\in\mathbf{R}^{r_0}$ and $x_j\in\mathbf{R}^{2r_j}$ for $j=1,\dots,h$

Write the feedback law as $u=-K^Tx=-K_0^Tx_0-\sum_{l=1}^h K_l^T x_l$ with $K_0\in\mathbf{R}^{r_0}$ and $K_j\in\mathbf{R}^{2r_j}$ for every $1\le j\le h$.

For every $\nu>0$ consider the following change of time-space variables: let

$$\begin{array}{rcl} y_0(t) &=& D^{-1}_{r_0,\nu}x_0(\nu t),\\ y_j(t) &=& (D^C_{r_j,\nu})^{-1}e^{-\nu t A^{(j)}}x_j(\nu t),\quad \text{for } 1\le j\le h,\end{array}$$

where $D_{r_0,\nu}$ is defined as in (12) and

$$D^C_{r_j,\nu}=D_{r_j,\nu}\otimes\mathrm{Id}_2\in M_{2r_j}(\mathbf{R}).$$

In accordance with

$$\nu(D^C_{r_j,\nu})^{-1}J^C_{r_j}D^C_{r_j,\nu} = J^C_{r_j}, \quad D^C_{r_j,\nu}b^j = b^j,$$

we end up with the following linear time-varying system

$$\left\{\begin{array}{lcl} \dot y_0 & = & J_{r_0}y_0 - \alpha_\nu(t)b^0\big(K^T_{0,\nu}y_0 + \sum_{l=1}^h K^T_{l,\nu}e^{\nu t\omega_l A^{(l)}}y_l\big), \\ \dot y_j & = & J^C_{r_j}y_j - \alpha_\nu(t)b^{j,\nu}(t)\big(K^T_{0,\nu}y_0 + \sum_{l=1}^h K^T_{l,\nu}e^{\nu t\omega_l A^{(l)}}y_l\big), \quad \text{for } j = 1,\dots,h, \end{array}\right. \tag{23}$$

where $K_{0,\nu} = \nu D_{r_0,\nu}K_0$, $K_{j,\nu} = \nu D^C_{r_j,\nu}K_j$ and $b^{j,\nu}(t) = e^{-\nu t\omega_j A^{(j)}}b^j$ for $j = 1,\dots,h$. Given $\nu > 0$, (7) admits a (T,μ)-stabilizer if and only if (23) admits a $(T/\nu,\mu/\nu)$-stabilizer.

For each $l = 1,\dots,h$, assume that K_l^T is of the form $(0,k^l_1,\dots,0,k^l_{r_l})$, that is,

$$K_l^T = \mathcal{K}_l \otimes (0,1), \quad \mathcal{K}_l = (k^l_1,\dots,k^l_{r_l}).$$

For uniformity of notations, we also write $\mathcal{K}_0 = K_0^T$.

Let $(\alpha_\nu)_{\nu>0}$ be a family of signals satisfying $\alpha_\nu \in \mathcal{G}(T/\nu,\mu/\nu)$ for every $\nu > 0$. Consider a sequence $(\nu_n)_{\in\mathbf{N}}$ going to infinity as $n\to\infty$ such that the matrix-valued curve $\alpha^C_{\nu_n}(\cdot)$, defined as in (9), has a weak-$\star$ limit as $n\to\infty$ in $L^\infty(\mathbf{R}_{\geq 0}, M_{2h+1-j_0}(\mathbf{R}))$. Denote the weak-$\star$ limit by $C_\star$. It follows form point 2 of Lemma 2.5 that $C_\star(t)$ is symmetric and

$$C_\star(t) \geq \xi \mathrm{Id}_{2h+1-j_0},$$

for almost every $t\geq 0$, for some positive scalar ξ only depending on T,μ and $\sigma(A)$.

Define the 2×2 time-dependent matrices C_{jl}, $1\leq j,l\leq h$, the 1×2 time-dependent matrices C_{0j}, $1\leq j\leq h$, and the scalar time-dependent signal C_{00} by the relation

$$C_\star = (C_{jl})_{j_0\leq j,l\leq h}.$$

Consider, for every $n\in\mathbf{N}$, system (23) with $\nu = \nu_n$ and $K_\nu = K$. All coefficients of the sequence of systems obtained in this way are weakly-$\star$ convergent as n goes to infinity. The limit system is

$$\left\{\begin{array}{lcl} \dot y_0 & = & J_{r_0}y_0 - b^0\big(C_{00}\mathcal{K}_0y_0 + \sum_{l=1}^h C_{0l}(\mathcal{K}_l\otimes \mathrm{Id}_2)y_l\big), \\ \dot y_j & = & J^C_{r_j}y_j - (b^j\otimes \mathrm{Id}_2)\left(C^T_{0j}\mathcal{K}_0y_0 + \sum_{l=1}^h C_{jl}(\mathcal{K}_l\otimes\mathrm{Id}_2)y_l\right), \quad \text{for } j = 1,\dots,h. \end{array}\right. \tag{24}$$

We consider (24) as a switched system depending on K in which the admissible switching laws are all the time-varying matrix-valued coefficients C_{jl} obtained from the limit procedure described above.

In the sequel, we only treat the case where 0 is not an eigenvalue of A. The general case presents no extra mathematical difficulties and can be treated similarly. Then system (24) takes the form

$$\dot y_j = J^C_{r_j}y_j - (b^j\otimes\mathrm{Id}_2)\sum_{l=1}^h C_{jl}(\mathcal{K}_l\otimes\mathrm{Id}_2)y_l, \quad \text{for } j = 1,\dots,h. \tag{25}$$

We also assume that the multiplicities $r_1, \dots, r_h$ of the eigenvalues of A form a non-increasing sequence.

Let us impose a further restriction on the structure of the feedback K. Assume that there exist $\bar{k}_1, \dots, \bar{k}_{r_1} \in \mathbf{R}$, each of them different from zero, such that

$$k^l_\xi = \bar{k}_{r_l+1-\xi}, \qquad \text{for } 1 \le l \le h \text{ and } 1 \le \xi \le r_l.$$

We find it useful to provide an equivalent representation of system (25) in a higher dimensional vector space, introducing some redundant variables. In order to do so, for $l \in \{1, \dots, r_1\}$, associate to $y = (y_1, \dots, y_h)$ the $2h$-vector

$$Y_l = \begin{pmatrix} (\mathcal{K}_1 \otimes \mathrm{Id}_2)(J^C_{r_1})^{l-1} y_1 \\ \vdots \\ (\mathcal{K}_h \otimes \mathrm{Id}_2)(J^C_{r_h})^{l-1} y_h \end{pmatrix}.$$

Notice that the last $2h - 2m_l$ coordinates of Y_l are equal to zero, where m_l denotes the number of Jordan blocks of A of size not smaller than l, that is,

$$m_l = \#\{j \mid 1 \le j \le h,\ r_j \ge l\}.$$

For $l \in \{1, \dots, r_1\}$, let p_l be the orthogonal projection of $\mathbf{R}^{2h}$ onto $\mathbf{R}^{2m_l} \times \{0_{2h-2m_l}\}$, i.e.,

$$p_l = \mathrm{diag}(\mathrm{Id}_{2m_l}, 0_{(2h-2m_l)\times(2h-2m_l)}).$$

By construction we have $p_1 = \mathrm{Id}_{2r_1}$ and $p_l Y_j = Y_j$ for $1 \le l \le j \le r_1$.

Notice that the map $(y_1, \dots, y_h) \mapsto (Y_1, \dots, Y_{r_1})$ is a bijection between $\mathbf{R}^n$ and the subspace $E^h_{m_1,\dots,m_{r_1}}$ of $\mathbf{R}^{2hr_1}$ defined by

$$E^h_{m_1,\dots,m_{r_1}} = \{(Y_1, \dots, Y_{r_1}) \mid Y_l \in \mathbf{R}^{2h} \text{ and } p_l Y_l = Y_l \text{ for } l = 1, \dots, r_1\}.$$

Indeed, the matrix corresponding to the transformation is upper triangular, with the $\bar{k}_l$'s as elements of the diagonal, if one considers the following choice of coordinates on $E^h_{m_1,\dots,m_{r_1}}$: take the first two coordinates of the first copy of $\mathbf{R}^{2h}$, then the first two of its second copy and so on until the r_1^{th} copy; then take the third and fourth coordinates of the first copy of $\mathbf{R}^{2h}$ and repeat the procedure until its r_2^{th} copy; and so on, until the last two coordinates of the r_h^{th} copy of $\mathbf{R}^{2h}$.

If y is a solution of system (25), then $Y = (Y_1, \dots, Y_{r_1})$ is a trajectory in $E^h_{m_1,\dots,m_{r_1}}$ satisfying the system of equations

$$\dot{Y}_l = Y_{l+1} - \bar{k}_l p_l C_\star Y_1, \quad \text{for } l = 1, \dots, r_1, \tag{26}$$

where, by convention, $Y_{r_1+1} = 0_{2h}$.

We prove in the following proposition that there exist $\bar{k}_1, \dots, \bar{k}_{r_1} \neq 0$ such that system (26), restricted to $E^h_{m_1,\dots,m_{r_1}}$, is exponentially stable uniformly with respect to all time-dependent measurable symmetric matrices $C_\star$ satisfying $\xi \mathrm{Id}_{2h} \le C_\star(t) \le \mathrm{Id}_{2h}$ almost everywhere.

Proposition 3.3 *For every $h, r_1 \in \mathbf{N}$, for every non-increasing sequence of non-negative numbers $m_1, \dots, m_{r_1}$ such that $m_1 \leq h$ and for every $\xi > 0$, there exist $\lambda, \bar{k}_1, \dots, \bar{k}_{r_1} > 0$ and a symmetric positive definite $2hr_1 \times 2hr_1$ matrix S such that, for every $C_\star \in L^\infty(\mathbf{R}_{\geq 0}, M_{2h}(\mathbf{R}))$, if $C_\star(t)$ is symmetric and satisfies $\xi \mathrm{Id}_{2h} \leq C_\star(t) \leq \mathrm{Id}_{2h}$ almost everywhere, then any solution $Y : \mathbf{R}_{\geq 0} \to E^h_{m_1,\dots,m_{r_1}}$ of (26) satisfies for almost every $t \geq 0$ the inequality*

$$\frac{d}{dt}\left(Y(t)^T S Y(t)\right) \leq -\lambda \|Y(t)\|^2.$$

Proof. The proof is similar to that of [11, Lemma 4.0] and goes by induction on r_1.

We start the argument for $r_1 = 1$, with $h \in \mathbf{N}$, $0 \leq m_1 \leq h$ and $\xi > 0$ arbitrary. In that case the system reduces to

$$\dot{Y}_1 = -\bar{k}_1 p_1 C_\star Y_1,$$

with $Y_1 \in E^h_{m_1} = \mathbf{R}^{2m_1} \times \{0_{2h-2m_1}\}$. The conclusion follows by taking $\bar{k}_1 = 1$ and $S = \mathrm{Id}_{2h}$.

Let r_1 be a positive integer. Assume that the proposition holds true for every positive integer $j \leq r_1$ and for every $h \in \mathbf{N}$, $0 \leq m_1 \leq \cdots \leq m_{r_1} \leq h$ and $\xi > 0$. Consider system (26) where l runs between 1 and $r_1 + 1$. Set $Y = (Y_2^T, \dots, Y_{r_1+1}^T)^T$. Note that if $(Y_1^T, \dots, Y_{r_1+1}^T)^T \in E^h_{m_1,\dots,m_{r_1+1}}$, then $Y \in E^h_{m_2,\dots,m_{r_1+1}}$. The dynamics of (Y_1, Y) are given by

$$\begin{cases} \dot{Y}_1 = -\bar{k}_1 C_\star Y_1 + \Pi_1 Y, \\ \dot{Y} = -\overline{K} C_\star Y_1 + \mathcal{J} Y, \end{cases}$$

where

$$\begin{aligned} \Pi_1 &= (\mathrm{Id}_{2h}, 0_{2h \times 2h(r_1-1)}), \\ \overline{K} &= \begin{pmatrix} \bar{k}_2 p_2 \\ \vdots \\ \bar{k}_{r_1+1} p_{r_1+1} \end{pmatrix}, \\ \mathcal{J} &= J_{r_1} \otimes \mathrm{Id}_{2h}. \end{aligned}$$

Define the linear change of variables (Z_1, Z) given by

$$Z_1 = Y_1, \qquad Z = Y + \Omega Y_1,$$

where

$$\Omega = \begin{pmatrix} \eta_2 p_2 \\ \vdots \\ \eta_{r_1+1} p_{r_1+1} \end{pmatrix}$$

and the η_l's are scalar constants to be chosen later. Note that Z belongs to $E^h_{m_2,\dots,m_{r_1+1}}$ if Y does. The dynamics of (Z_1, Z) is given by

$$\begin{cases} \dot{Z}_1 = (-\bar{k}_1 C_\star + \Pi_1\Omega)Z_1 + \Pi_1 Z, \\ \dot{Z} = -\big((\overline{K} + \bar{k}_1\Omega)C_\star + (\mathcal{J} + \Omega\Pi_1)\Omega\big)Z_1 + (\mathcal{J} + \Omega\Pi_1)Z. \end{cases} \tag{27}$$

Let us apply the induction hypothesis to the system

$$\dot{Z} = (\mathcal{J} + \Omega\Pi_1)Z, \tag{28}$$

which is well defined on $E^h_{m_2,\dots,m_{r_1+1}}$ and has the same structure as system (26). (Here $\mathbf{C}_\star \equiv \mathrm{Id}_{2h}$ and therefore one can take as ξ any positive constant smaller than one.) We deduce the existence of $\lambda > 0$, $\eta_l < 0$, $2 \le l \le r_1 + 1$ and a symmetric positive definite matrix S such that $\dot{V}(t) \le -\lambda \|Z(t)\|^2$ where $V(t) = Z(t)^T S Z(t)$ and $Z(t)$ is any trajectory of (28) in $E^h_{m_2,\dots,m_{r_1+1}}$. Therefore,

$$\big[(\mathcal{J} + \Omega\Pi_1)^T S + S(\mathcal{J} + \Omega\Pi_1)\big]\big|_{E^h_{m_2,\dots,m_{r_1+1}}} \le -\lambda\,\mathrm{Id}_{E^h_{m_2,\dots,m_{r_1+1}}}.$$

Since Ω is fixed, for every $\bar{k}_1 > 0$ there exists a unique $\overline{K}(\bar{k}_1)$ such that $\overline{K}(\bar{k}_1) + \bar{k}_1\Omega = 0_{2r_1h\times 2h}$. Assume that $\overline{K} = \overline{K}(\bar{k}_1)$ and notice that the corresponding $\bar{k}_2, \dots, \bar{k}_{r_1+1}$ are positive.

Choose $S' = (1/2)\mathrm{diag}(\mathrm{Id}_{2h}, S)$ and define the corresponding Lyapunov function $W(Z_1, Z) = \|Z_1\|^2/2 + Z^T S Z/2$. If (Z_1, Z) is a trajectory of (27), then

$$\begin{aligned} \frac{d}{dt}W(Z_1, Z) &= -Z_1^T((\bar{k}_1 C_\star - \Pi_1\Omega)Z_1 - \Pi_1 Z) - Z^T S((\mathcal{J} + \Omega\Pi_1)\Omega Z_1 - (\mathcal{J} + \Omega\Pi_1)Z) \\ &\le Z_1^T(-\bar{k}_1 C_\star + \Pi_1\Omega)Z_1 - \lambda\|Z\|^2 + (\|\Pi_1\| + \|S(\mathcal{J} + \Omega\Pi_1)\Omega\|)\|Z_1\|\|Z\| \\ &\le (-\bar{k}_1\xi + \delta_1)\|Z_1\|^2 - \lambda\|Z\|^2 + \delta_2\|Z_1\|\|Z\|, \end{aligned}$$

where the constants $\delta_1, \delta_2 > 0$ do not depend on $\bar{k}_1$. Since

$$\|Z_1\|\|Z\| \le \varepsilon^2\|Z_1\|^2 + \frac{\|Z\|^2}{\varepsilon^2}$$

for every $\varepsilon > 0$, then

$$\frac{d}{dt}W(Z_1, Z) \le \left(-\bar{k}_1\xi + \delta_1 + \frac{\delta_2}{\varepsilon^2}\right)\|Z_1\|^2 + (-\lambda + \varepsilon^2\delta_2)\|Z\|^2.$$

Choosing ε^2 small enough in order to have $-\lambda + \varepsilon^2\delta_2 \le -\lambda/2$ and $\bar{k}_1$ large enough, we have

$$\frac{d}{dt}W(Z_1, Z) \le -\frac{\lambda}{2}(\|Z_1\|^2 + \|Z\|^2).$$

The proof is concluded, since (Z_1, Z) and (Y_1, Y) are equivalent systems of coordinates on the space $E^h_{m_1,\dots,m_{r_1+1}}$. ■

The proof of Theorem 3.2 is completed by applying the same contradiction argument as in the proof of Theorem 3.1. ■

4 Maximal rates of exponential convergence and divergence

Let $(A, b) \in M_n(\mathbf{R}) \times \mathbf{R}^n$ be a controllable pair, K belong to $\mathbf{R}^n$ and T, μ be positive constants such that $T \geq \mu$. For $\alpha \in \mathcal{G}(T, \mu)$ let $\lambda^+(\alpha, K)$ and $\lambda^-(\alpha, K)$ be, respectively, the maximal and minimal Lyapunov exponents associated with $\dot{x} = (A - \alpha b K^T)x$, i.e.,

$$\lambda^+(\alpha, K) = \sup_{\|x_0\|=1} \limsup_{t\to+\infty} \frac{\log(\|x(t; 0, x_0, K, \alpha)\|)}{t}, \quad \lambda^-(\alpha, K) = \inf_{\|x_0\|=1} \liminf_{t\to+\infty} \frac{\log(\|x(t; 0, x_0, K, \alpha)\|)}{t}.$$

The *rate of convergence* (respectively, the *rate of divergence*) associated with the family of systems $\dot{x} = (A - \alpha b K^T)x$, $\alpha \in \mathcal{G}(T, \mu)$, is defined as

$$\mathrm{rc}(A, b, T, \mu, K) = -\sup_{\alpha\in\mathcal{G}(T,\mu)} \lambda^+(\alpha, K) \quad \left(\text{respectively, } \mathrm{rd}(A, b, T, \mu, K) = \inf_{\alpha\in\mathcal{G}(T,\mu)} \lambda^-(\alpha, K)\right). \tag{29}$$

Notice that

$$\mathrm{rc}(A, b, T, \mu, K) \leq \min_{\bar{\alpha}\in[\mu/T,1]} \min\{-\Re(\sigma(A - \bar{\alpha} b K^T))\}, \tag{30}$$

and

$$\mathrm{rd}(A, b, T, \mu, K) \leq \min_{\bar{\alpha}\in[\mu/T,1]} \min\{\Re(\sigma(A - \bar{\alpha} b K^T))\}.$$

Moreover, since a linear change of coordinates $x' = Px$ does not affect Lyapunov exponents, then

$$\mathrm{rc}(A, b, T, \mu, K) = \mathrm{rc}(PAP^{-1}, Pb, T, \mu, (P^{-1})^T K), \tag{31}$$

and

$$\mathrm{rd}(A, b, T, \mu, K) = \mathrm{rd}(PAP^{-1}, Pb, T, \mu, (P^{-1})^T K). \tag{32}$$

Define the maximal rate of convergence associated with the PE system $\dot{x} = Ax + \alpha bu$, $\alpha \in \mathcal{G}(T, \mu)$, as

$$\mathrm{RC}(A, T, \mu) = \sup_{K\in\mathbf{R}^n} \mathrm{rc}(A, b, T, \mu, K), \tag{33}$$

and similarly, the maximal rate of divergence as

$$\mathrm{RD}(A, T, \mu) = \sup_{K\in\mathbf{R}^n} \mathrm{rd}(A, b, T, \mu, K). \tag{34}$$

Notice that neither $\mathrm{RC}(A, T, \mu)$ nor $\mathrm{RD}(A, T, \mu)$ depend on b, as it follows from (31) and (32).

Remark 4.1 Let us collect some properties of RC and RD that follow directly from their definition. First of all, one has

$$\mathrm{RC}(A + \lambda \mathrm{Id}_n, T, \mu) = \mathrm{RC}(A, T, \mu) - \lambda, \quad \mathrm{RD}(A + \lambda \mathrm{Id}_n, T, \mu) = \mathrm{RD}(A, T, \mu) + \lambda. \tag{35}$$

Then, by time-rescaling,

$$\mathrm{RC}(A, T, \rho T) = \mathrm{RC}(A/T, 1, \rho), \quad \mathrm{RD}(A, T, \rho T) = \mathrm{RD}(A/T, 1, \rho). \tag{36}$$

Notice moreover that, thanks to (13), both $\mathrm{RC}(J_n, T, \rho T)$ and $\mathrm{RD}(J_n, T, \rho T)$ only depend on ρ and thus are equal to $\mathrm{RC}(J_n, 1, \rho)$ and $\mathrm{RD}(J_n, 1, \rho)$, respectively. Finally, because of point 2 in Lemma 2.4, RC and RD are monotone with respect to their third argument.

Remark 4.2 Given a controllable pair (A, b) and a class $\mathcal{G}(T, \mu)$ of PE signals, whether or not RC and RD are both infinite can be understood as whether or not a pole-shifting type property holds true for the PE control system $\dot{x} = Ax + \alpha bu$, $\alpha \in \mathcal{G}(T, \mu)$.

The study of the pole-shifting type property for two-dimensional PE systems actually reduces to that of their maximal rates of convergence as a consequence of the following property.

Proposition 4.3 *Consider the two-dimensional PE systems $\dot{x} = Ax + \alpha bu$, $\alpha \in \mathcal{G}(T, \mu)$, with (A, b) controllable. Then $\mathrm{RC}(A, T, \mu) = +\infty$ if and only if $\mathrm{RD}(A, T, \mu) = +\infty$.*

Proof. According to (31), (32) and (35), it is enough to prove the result for (A, b) in companion form and with $\mathrm{Tr}(A) = 0$. Let then

$$A = \begin{pmatrix} 0 & 1 \\ a & 0 \end{pmatrix} \qquad b = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{37}$$

with $a \in \mathbf{R}$.

Assume that $\mathrm{RC}(A, T, \mu) = +\infty$. By definition, for every $C > 0$ there exists $K \in \mathbf{R}^2$ such that $\mathrm{rc}(A, b, T, \mu, k) > C$. Therefore, by definition of rc,

$$\limsup_{t \to +\infty} \frac{\log(\|x(t; 0, x_0, K, \alpha)\|)}{t} < -C, \qquad \forall \alpha \in \mathcal{G}(T, \mu), \forall \|x_0\| = 1. \tag{38}$$

Moreover, due to (30), for C large enough we can assume that k_1, k_2 and k_1/k_2 are large positive numbers.

Let $K_- = (k_1, -k_2)$. We claim that if C is large enough then $\mathrm{RD}(A, b, T, \mu, K_-) \geq C$. Assume by contradiction that there exists $\bar{\alpha} \in \mathcal{G}(T, \mu)$ such that $\lambda^-(\bar{\alpha}, K_-) < C$. Then there exists $\bar{x} \in \mathbf{R}^2$ of norm one and an increasing sequence $(t_n)_{n \in \mathbf{N}}$ of positive times going to infinity such that

$$\frac{\log(\|x(t_n; 0, \bar{x}, K_-, \bar{\alpha})\|)}{t_n} < C, \quad \forall \in \mathbf{N}.$$

Notice that for every $t \in [0, t_n]$,

$$x(t; 0, \bar{x}, K_-, \bar{\alpha}(\cdot)) = \mathrm{diag}(1, -1) x(t_n - t; 0, x_n, K, \bar{\alpha}(t_n - \cdot)),$$

where $x_n = \mathrm{diag}(1, -1) x(t_n; 0, \bar{x}, K_-, \bar{\alpha})$.

Therefore, by homogeneity,

$$\frac{\log\left(\left\| x\left(t_n; 0, \frac{x_n}{\|x_n\|}, K, \bar{\alpha}(t_n - \cdot)\right)\right\|\right)}{t_n} = -\frac{\log(\|x_n\|)}{t_n} = -\frac{\log(\|x(t_n; 0, \bar{x}, K_-, \bar{\alpha})\|)}{t_n} > -C. \tag{39}$$

This would contradict (38) if, for some positive integer n, $x_n/\|x_n\| = \bar{x}$ and the signal obtained by repeating $\bar{\alpha}|_{[0,t_n)}$ by periodicity over $\mathbf{R}_{\geq 0}$ belonged to $\mathcal{G}(T,\mu)$. Indeed, in such a case,

$$\frac{\log\left(\|x\left(kt_n;0,\bar{x},K,\tilde{\alpha}(\cdot)\right)\|\right)}{kt_n} > -C \tag{40}$$

for every $k \geq 1$, where $\tilde{\alpha} \in \mathcal{G}(T,\mu)$ denotes the signal obtained by repeating $\bar{\alpha}|_{[0,t_n)}(t_n - \cdot)$ by periodicity over $\mathbf{R}_{\geq 0}$.

In order to recover the periodic case, we are going to extend $\bar{\alpha}$ backwards in time over an interval $[-2\mu - \tau_n, 0)$ as follows. First set $A_1^- = A - bK_-^T$. We take $\bar{\alpha} = 1$ on the intervals $[-\mu, 0)$ and $[-2\mu - \tau_n, -\mu - \tau - n)$ and we extend $\bar{\alpha}$ on $[-\mu - \tau_n, -\mu)$ in such a way that the trajectory corresponding to $\bar{\alpha}|_{[-\mu-\tau_n,-\mu)}$ and to the gain K_- connects the half-line $\mathbf{R}_{\geq 0}x_n^+$ to $\bar{x}^-$, where $x_n^+ = \exp(\mu A_1^-)\mathrm{diag}(1,-1)x_n$ and $\bar{x}^- = \exp(-\mu A_1^-)\bar{x}$. We show below that this can be done fulfilling the PE condition and with τ_n upper bounded by a constant independent of n. Hence, the signal obtained extending $\bar{\alpha}_{[-2\mu-\tau_n,t_n]}$ by periodicity belongs to $\mathcal{G}(T,\mu)$ and we have

$$\begin{aligned} x\left(t_n + 2\mu + \tau_n; 0, x_n, K, \bar{\alpha}(t_n + 2\mu + \tau_n - \cdot)\right) \quad &\in \quad \mathbf{R}_{\geq 0}x_n \\ \log\left(\left\| x\left(t_n + 2\mu + \tau_n; 0, \frac{x_n}{\|x_n\|}, K, \bar{\alpha}(t_n + 2\mu + \tau_n - \cdot)\right)\right\|\right) \quad &= \quad \log\left(\|\tilde{x}\|\right) - \log(\|x(t_n; 0, \bar{x}, K_-, \bar{\alpha})\|), \end{aligned}$$

where $\tilde{x} = x(\tau_n + 2\mu; 0, \mathrm{diag}(1,-1)\bar{x}, K, \bar{\alpha}|_{[-2\mu-\tau_n,0]}(-\cdot))$. Note that $\log(\|\tilde{x}\|)$ can be lower bounded independently of n, because of the uniform boundedness of τ_n. Therefore,

$$\frac{\log\left(\|x\left(t_n + 2\mu + \tau_n; 0, x_n, K, \bar{\alpha}(t_n + 2\mu + \tau_n - \cdot)\right)\|\right)}{t_n + 2\mu + \tau_n} \quad > \quad \frac{\log\left(\|\tilde{x}\|\right)}{t_n + 2\mu + \tau_n} - \frac{Ct_n}{t_n + 2\mu + \tau_n}$$

is larger than $-C$ for n large enough and we can conclude as in (40).

We are left to prove that the control system on the unit circle whose admissible velocities are the projections of the linear vector fields $x \mapsto (A - \xi bK_-^T)x$, $\xi \in [0,1]$, is completely controllable in finite time by controls $\xi = \xi(t)$ satisfying the PE condition. Notice that the equilibria of the projection of a linear vector field $x \mapsto A'x$ on the unit circle are given by the eigenvalues of A'. All other trajectories are heteroclinic connections between the equilibria, unless the eigenvalues of A' are non-real, in which case the phase portrait is given by a single periodic trajectory.

Denote by θ a point on the unit circle, identified with $\mathbf{R}/2\pi\mathbf{Z}$. Then, the above mentioned control system on the unit circle can be written

$$\dot{\theta} = a\cos^2(\theta) - \sin^2(\theta) + \xi\cos(\theta)\left(k_2\sin(\theta) - k_1\cos(\theta)\right), \quad \xi \in [0,1]. \tag{41}$$

We prove the controllability of (41) by exhibiting a trajectory $\bar{\theta}$ of (41) corresponding to a PE control $\bar{\xi}$, starting at some $\theta_0 \in \mathbf{R}/2\pi\mathbf{Z}$, making a complete turn and going back in finite time to θ_0.

The PE condition will be verified by checking that the control $\bar{\xi} = 0$ is applied for a total time that is smaller than $T - \mu$. Define the angle $\theta_K \in (0, \pi/2)$ by

$$\tan\left(\theta_K\right) = 2\frac{k_2}{k_1}.$$

Notice that the eigenvectors of A_1^- are proportional to the vectors $(2, k_2 \pm \sqrt{k_2^2 - 4(k_1 - a)})$. Therefore, assuming that k_1 is larger than a, the angle between any real eigenvector of A_1^- and the vertical axis is smaller than θ_K.

Take $\theta_0 = \pi/2$ and apply $\bar{\xi} = 0$ until $\bar{\theta}$ reaches $\pi/2 - \theta_K$. Since k_2/k_1 is small and θ_K is of the same order as k_2/k_1, then we can assume that $a\cos^2(\theta) - \sin^2(\theta) < -1/2$ for $\theta \in [\pi/2 - \theta_K, \pi/2]$. Therefore, the time needed to go from $\pi/2$ to $\pi/2 - \theta_K$ can be assumed to be smaller than $(T - \mu)/2$. When the trajectory $\bar{\theta}$ reaches $\pi/2 - \theta_K$, switch to $\bar{\xi} = 1$ and apply it until $\bar{\theta}$ reaches (in finite time) $-\pi/2$. This is possible since either the eigenvectors of A_1^- are non-real or they are contained in the cone

$$\{(r\cos\theta, r\sin\theta) \mid r > 0,\ \theta \in (\pi/2 - \theta_K + m\pi, \pi/2 + m\pi),\ m \in \mathbf{Z}\}.$$

In both cases the dynamics of (41) with $\xi = 1$ describe a non-singular clockwise rotation on the arc of the unit circle corresponding to $[\pi/2, \pi/2 - \theta_K]$. The trajectory is completed, by homogeneity, taking $\bar{\xi} = 0$ until $\bar{\theta}$ reaches $-\pi/2 - \theta_K$ and finally $\bar{\xi} = 1$ until $\bar{\theta}$ reaches $-3\pi/2 = \pi/2 \pmod{2\pi}$. As required, the sum of the lengths of the intervals on which $\bar{\xi} = 0$ does not exceed $T - \mu$.

This concludes the proof that $\mathrm{RC}(A, T, \mu) = +\infty$ implies $\mathrm{RD}(A, T, \mu) = +\infty$. The converse can be proven by a perfectly analogous argument. ∎

4.1 Arbitrary rates of convergence and divergence for ρ large enough

This section aims at proving that for ρ large enough a persistently excited system can be either stabilized with an arbitrarily large rate of exponential convergence or destabilized with an arbitrarily large rate of exponential divergence. This will be done by adapting the classical high-gain technique.

Proposition 4.4 *Let n be a positive integer. There exists $\rho^* \in (0, 1)$ such that for every controllable pair $(A, b) \in M_n(\mathbf{R}) \times \mathbf{R}^n$, every $T > 0$ and every $\rho \in (\rho^*, 1]$ one has* $\mathrm{RC}(A, T, \rho T) = \mathrm{RD}(A, T, \rho T) = +\infty$.

Proof. Fix $T > 0$ and let $(A, b) \in M_n(\mathbf{R}) \times \mathbf{R}^n$ be a controllable pair in companion form. According to (35), it is enough to establish the result with the extra hypothesis that $\mathrm{Tr}(A) = 0$. We therefore assume in the sequel that $b = (0, \dots, 0, 1)^T$, $A = J_n + bK_A^T$ and $K_A^T b = 0$.

We first prove the stabilization result. Fix $K \in \mathbf{R}^n$ such that $J_n - bK^T$ is Hurwitz. Let P be the unique positive definite $n \times n$ matrix that solves the Lyapunov equation

$$(J_n - bK^T)^T P + P(J_n - bK^T) = -\mathrm{Id}_n.$$

Define $V(x) = x^T P x$. Then, for every $\alpha \in L^\infty(\mathbf{R}, [0, 1])$ and every solution of $\dot{x} = (J_n - \alpha bK^T)x$, one has

$$\frac{d}{dt} V(x(t)) \le -C_1 V(x(t)) + C_2(1 - \alpha(t))V(x(t)),$$

with C_1, C_2 two positive constants only depending on K. Choose $\rho \in (0,1)$ and assume that α is a $(T, T\rho)$-signal. Then, for every $t \geq 0$,

$$V(x(t+T)) \leq V(x(t)) \exp(-T(C_1 - C_2(1-\rho))).$$

Therefore, if $\rho > 1 - (C_1/2C_2)$ then $\mathrm{RC}(J_n, T, T\rho) \geq C_1/2 > 0$. For every $\gamma > 0$, set $K_\gamma = \gamma D_\gamma K$ (where, as in the previous section, $D_\gamma = \mathrm{diag}(\gamma^{n-1}, \dots, \gamma, 1)$). Recall that J_n and D_γ satisfy (13). Take a solution of $\dot{x} = (A - \alpha b K_\gamma^T)x$ with $\alpha \in \mathcal{G}(T, \rho T)$. Set $z(\cdot) = D_\gamma x(\cdot)$ and notice that for every $\gamma > 1$

$$\frac{d}{dt} V(z(t)) \leq \gamma(-C_1 + C_2(1-\alpha(t)) + C_A/\gamma^2) V(z(t)),$$

where C_A only depends on K_A and P. Then clearly $\mathrm{RC}(A, T, T\rho) \geq \gamma C_1/3$ for $\rho > 1 - (C_1/2C_2)$ and γ large enough. Thus, $\mathrm{RC}(A, T, T\rho) = +\infty$ and one can choose $\rho^* \geq 1 - (C_1/2C_2)$.

The destabilization result can be obtained by a similar argument based on the Lyapunov equation

$$(J_n - bL^T)^T Q + Q(J_n - bL^T) = \mathrm{Id}_n,$$

verified for some $L \in \mathbf{R}^n$ and some symmetric positive definite matrix Q. ■

4.2 Finite maximal rate of convergence for ρ small enough

In this section we restrict our attention to the case $n = 2$.

Proposition 4.5 *There exists $\rho_* \in (0,1)$ such that for every controllable pair $(A, b) \in M_2(\mathbf{R}) \times \mathbf{R}^2$, every $T > 0$ and every $\rho \in (0, \rho_*)$ one has* $\mathrm{RC}(A, T, \rho T) < +\infty$.

Proof. Thanks to Remark 4.1, it suffices to show that there exists $\rho_* \in (0,1)$ such that, for every controllable pair $(A, b) \in M_2(\mathbf{R}) \times \mathbf{R}^2$ with $\mathrm{Tr}(A) = 0$, one has $\mathrm{RC}(A, 1, \rho_*) < +\infty$.

As in (37), take (A, b) in companion form, ie,

$$A = J_2 + aH, \quad b = (0,1)^T,$$

with $a \in \mathbf{R}$ and $H = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.

For $\theta \in [-\pi, \pi)$ set $e_\theta = (\sin\theta, \cos\theta)^T$ and define $y_0 = (-1,\ 0)^T$. Every gain can be written as

$$K_{\theta,\gamma} = \gamma D_\gamma e_\theta,$$

with $\gamma \geq 0$ and $\theta \in [-\pi, \pi)$.

Moreover, if $A - bK^T$ is Hurwitz with $K = \gamma D_\gamma e_\theta$ then the sum and the product of its two eigenvalues are, respectively, $\gamma \cos\theta > 0$ and $\gamma^2 \sin\theta - a > 0$. In particular, $\theta \in (-\pi/2, \pi/2)$ and

$\gamma^2 \sin\theta > a$. If $\theta \in (-\pi/2, 0]$ with $A - bK^T$ Hurwitz, then $|a - \sin\theta\gamma^2| \le |a| = -a$ and therefore the convergence rate of $A - bK^T$ is upper bounded by a constant only depending on a.

Let $\Omega_0 = (0, \pi/2) \times (0, \infty)$. We show in the following the existence of $\rho > 0$ and $\Omega = \{(\theta, \gamma) \mid 0 < \theta < \pi/2,\ 0 < \gamma < \gamma(\theta)\} \subset \Omega_0$ such that

$$\text{if } (\theta, \gamma) \in \Omega_0 \text{ and } K_{\theta,\gamma} \text{ is a } (1, \rho)\text{-stabilizer of } \dot{x} = Ax + \alpha bu, \text{ then } (\theta, \gamma) \in \Omega, \tag{42}$$

and

$$\sup_{(\theta,\gamma)\in\Omega} \min\{-\Re(\sigma(A - bK_{\theta,\gamma}^T))\} < +\infty, \tag{43}$$

and the conclusion then follows from (30).

Fix $\theta \in (0, \pi/2)$. In order to find, for γ large enough, $\alpha \in \mathcal{G}(1, \rho)$ and $x_0 \in \mathbf{R}^2$ such that the trajectory of

$$\dot{x} = Ax - \alpha bK_{\theta,\gamma} x, \quad x(0) = x_0,$$

is unbounded, we apply the transformation $y_\gamma(\cdot) = D_\gamma x(\cdot/\gamma)$: the problem is now to find, for γ large enough, $\alpha \in \mathcal{G}(\gamma, \rho\gamma)$ and an unbounded trajectory of

$$\dot{y} = \left(J_2 + \frac{a}{\gamma^2} H\right) y - \alpha b e_\theta y. \tag{44}$$

Due to the homogeneity of the system, the latter fact reduces to determine τ large enough and $\alpha \in \mathcal{G}(\tau, 2\rho\tau)$ such that the solution $y(\cdot\,; 0, y_0, e_\theta, \alpha)$ of (44) satisfies $y(\tau; 0, y_0, \alpha) = -\xi y_0$ with $\xi > 1$. Indeed, for every $\gamma > \tau$ the extension of $\alpha|_{[0,\tau)}$ by periodicity is a $(\gamma, \rho\gamma)$-signal (see point 4 in Lemma 2.4) and the sequence $\|y(m\tau; 0, y_0, \alpha)\| = \xi^m$ goes to infinity as m goes to infinity.

Set

$$M_\theta = J_2 - be_\theta^T, \qquad N_{a,\theta,\gamma} = J_2 + \frac{a}{\gamma^2} H - be_\theta^T.$$

Consider $h > 0$ small to be fixed later. We distinguish two cases depending on whether $\theta \in (0, h)$ or not.

The case $\theta \in [h, \pi/2)$.

We construct a PE signal α as follows: starting at y_0 take $\alpha = 1$ until the trajectory $y(\cdot\,; 0, y_0, e_\theta, \alpha)$ of (44) reaches, at time T_1, the switching line $\sin(\theta)x + \cos(\theta)y = 0$. In order to ensure that the switching line is reached in finite time and, moreover, that T_1 is lower and upper bounded by two positive constants only depending on h (and not on $\theta \in [h, \pi/2)$), it suffices to choose $\gamma > \Gamma_1(a, h) > 0$ with $\Gamma_1(a, h)$ only depending on a and h. (Indeed, the bounds hold for all matrices in a neighborhood of $\{M_\theta \mid \theta \in [h, \pi/2)\}$ and it suffices to ensure that $N_{a,\theta,\gamma}$ belongs to such neighborhood.)

From $y(T_1; 0, y_0, e_\theta, \alpha)$ set $\alpha = 0$ until the first coordinate of $y(\cdot\,; 0, y_0, e_\theta, \alpha)$ takes, at time $T_1 + T_2$, the value 1. Finally, take $\alpha = 1$ until the second coordinate of $y(\cdot\,; 0, y_0, e_\theta, \alpha)$ reaches, at time $T_1 + T_2 + T_3$, the value 0. (See Figure 1.)

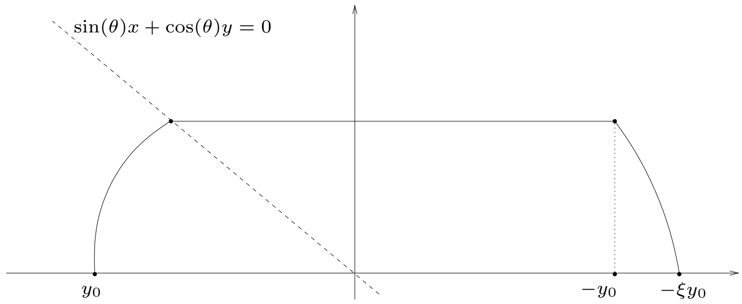

Figure 1: The trajectory $y(\cdot\,; 0, y_0, e_\theta, \alpha)$ when $\theta \in [h, \pi/2)$

Analogously to what happens for T_1, the values T_2 and T_3 admit lower and upper positive bounds only depending on h.

Define $\tau = T_1 + T_2 + T_3$ and notice that it admits an upper bound $\mathcal{T}_1(h)$ only depending on h. Finally, $\frac{T_1+T_3}{T_1+T_2+T_3}$ admits a lower bound ρ_1 only depending on h. The construction of the required $(\tau, \rho_1\tau)$-signal is achieved and we set

$$\gamma(\theta) \equiv \max(\Gamma_1(a, h), \mathcal{T}_1(h)). \tag{45}$$

The case $\theta \in (0, h)$.

Notice that the condition for $N_{a,\theta,\gamma}$ to be Hurwitz is that $\gamma^2 > |a|/\sin\theta$. Choose $\gamma > \Gamma_2(a, \theta) = M\sqrt{|a|}/\sin\theta$ with M large (to be fixed later independently of all parameters). In particular, for M large enough and $h_0 > 0$ small enough (independent of all parameters), for every $\theta \in (0, h_0)$ and every $\gamma > \Gamma_2(a, \theta)$ the matrix $N_{a,\theta,\gamma}$ has two real eigenvalues, denoted by $\mu_+(a, \theta, \gamma) > \mu_-(a, \theta, \gamma)$ and

$$-2 < \mu_-(a, \theta, \gamma) < -1/2, \quad -2\sin\theta < \mu_+(a, \theta, \gamma) < -\sin\theta/2. \tag{46}$$

From now on we assume $h \in (0, h_0)$.

Similarly to what has been done above, we construct a PE signal α as follows: starting at y_0 take $\alpha = 1$ in (44) for a time $T_1 = \bar{\rho}M/|\mu_+(a, \theta, \gamma)|$ with $\bar{\rho} \in (0, 1)$ to be fixed later. Set $y_1 = y(T_1; 0, y_0, e_\theta, \alpha)$.

From y_1 set $\alpha = 0$ for a time $T_2 = M/|\mu_+(a, \theta, \gamma)|$ and denote by y_2 the point $y(T_1+T_2; 0, y_0, e_\theta, \alpha)$. Finally, take $\alpha = 1$ until the second coordinate of $y(\cdot\,; 0, y_0, e_\theta, \alpha)$ assumes, at time $T_1 + T_2 + T_3$, the value 0. (See Figure 2.)

We next show that there exist $\bar{\rho}$ and M independent of θ and a such that T_3 is well defined and $y(T_1 + T_2 + T_3; 0, y_0, e_\theta, \alpha) = -\xi y_0$ with $\xi > 1$.

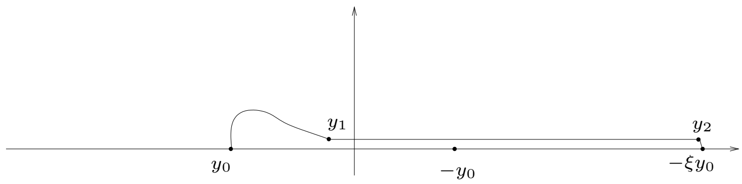

Figure 2: The trajectory $y(\cdot\,;0,y_0,e_\theta,\alpha)$ when $\theta\in(0,h)$

A simple computation yields

$$\begin{aligned} y_1 &= \frac{1}{\mu_-(a,\theta,\gamma)-\mu_+(a,\theta,\gamma)}\begin{pmatrix} e^{\mu_-(a,\theta,\gamma)T_1}\mu_+(a,\theta,\gamma)-e^{\mu_+(a,\theta,\gamma)T_1}\mu_-(a,\theta,\gamma) \\ \mu_-(a,\theta,\gamma)\mu_+(a,\theta,\gamma)(e^{\mu_-(a,\theta,\gamma)T_1}-e^{\mu_+(a,\theta,\gamma)T_1}) \end{pmatrix} \\ &= e^{-\bar\rho M}\begin{pmatrix} -1 \\ \mu_+(a,\theta,\gamma) \end{pmatrix}+O(\theta^2), \end{aligned}$$

with $\|O(\theta^2)\|\leq C\theta^2$ and C only depending on M and $\bar\rho$. (Similarly, in the sequel the symbol $O(\theta)$ stands for a function of θ upper bounded by $C\theta$ with C only depending on M and $\bar\rho$.)

In addition, one also gets that the first coordinate of y_2 is equal to

$$\begin{cases} e^{-M\bar\rho}(M-1)+O(\theta) & \text{if } a=0, \\ e^{-M\bar\rho}\left(M\frac{\mu_+(a,\theta,\gamma)}{\sin\theta}\sinh\left(\frac{\sin\theta}{\mu_+(a,\theta,\gamma)}\right)-\cosh\left(\frac{\sin\theta}{\mu_+(a,\theta,\gamma)}\right)\right)+O(\theta) & \text{if } a>0, \\ e^{-M\bar\rho}\left(M\frac{\mu_+(a,\theta,\gamma)}{\sin\theta}\sin\left(\frac{\sin\theta}{\mu_+(a,\theta,\gamma)}\right)-\cos\left(\frac{\sin\theta}{\mu_+(a,\theta,\gamma)}\right)\right)+O(\theta) & \text{if } a<0. \end{cases}$$

Using (46) one deduces that the first coordinate of y_2 is larger than

$$\begin{cases} e^{-M\bar\rho}(M/2\sinh(1/2)-\cosh(2))+O(\theta) & \text{if } a>0, \\ e^{-M\bar\rho}(M/2\sin(1/2)-\cos(2))+O(\theta) & \text{if } a<0. \end{cases}$$

Then in all three cases the first coordinate of y_2 becomes larger than

$$e^{-M\bar\rho}(MC_0-C_1+O(\theta)),$$

and one also gets that the second coordinate of y_2 can always be lower bounded by

$$\sin\theta e^{-M\bar\rho}(C_1-C_0/M+O(\theta)),$$

with $C_0>0$ and $C_1>0$ independent of all the parameters.

Fix M large and $\bar\rho\in(0,1)$ such that

$$e^{-M\bar\rho}(MC_0-C_1)\geq 2,\quad e^{-M\bar\rho}(C_1-C_0/M)\geq C_1/2.$$

Finally, by eventually reducing h in order to make each $O(\theta)$ uniformly small, one can ensure that the first coordinate of y_2 remains larger than 1 and that its second coordinate is positive.

Similar computations to the ones provided above show that it is possible to further ensure that $T_3 \leq 2T_1$.

Define $\tau = T_1 + T_2 + T_3$. Then $M/(2\sin\theta) < \tau < 8M/\sin\theta = \mathcal{T}_2(\theta)$. Choose now

$$\gamma(\theta) = M(8 + \sqrt{a})/\sin\theta \geq \max(\mathcal{T}_2(\theta), \Gamma_2(a,\theta)). \tag{47}$$

By construction, $\alpha \in \mathcal{G}(\tau, \bar{\rho}\tau)$. To conclude the proof it is enough to check condition (43) on

$$\Omega_* = \{(\theta,\gamma) \mid 0 < \theta < h,\ 0 < \gamma < \gamma(\theta)\}.$$

For $(\theta,\gamma) \in \Omega_*$ define

$$A^{\mathrm{stab}}_{\theta,\gamma} = A - bK^T_{\gamma,\theta} = \begin{pmatrix} 0 & 1 \\ a - \gamma^2\sin\theta & -\gamma\cos\theta \end{pmatrix}.$$

Then

$$0 < \det(A^{\mathrm{stab}}_{\theta,\gamma}) \leq C_0|\mathrm{Tr}(A^{\mathrm{stab}}_{\theta,\gamma})| + |a|,$$

with $C_0 = 2M(8 + \sqrt{|a|})$, implying (43). ∎

The following corollary is a direct consequence of Remark 4.1 and Proposition 4.5.

Corollary 4.6 *Take ρ_* as in the statement of Proposition 4.5. For every controllable pair $(A,b) \in M_2(\mathbf{R}) \times \mathbf{R}^2$, every $T > 0$ and every $\rho < \rho_*$, if $\lambda > 0$ is large enough, then $(A + \lambda\mathrm{Id}_2, b)$ is not $(T,\rho T)$-stabilizable. Moreover, if $0 < \rho < \rho_*$ and $\lambda > \mathrm{RC}(J_2, 1, \rho)$, then $(J_2 + \lambda\mathrm{Id}_2, b_0)$ is not $(T,\rho T)$-stabilizable for every $T > 0$.*

The above corollary establishes the existence of non-stabilizable PE systems if the ratio $\rho = \mu/T > 0$ is small enough and regardless of T. This is rather intriguing when one recalls, on the one hand, that any weak-$\star$ limit point $\alpha_\star$ of a sequence (α_n), with $\alpha_n \in \mathcal{G}(T_n, \rho T_n)$ and $\lim_{n\to+\infty} T_n = 0$, takes values in $[\rho, 1]$ (see point 1 of Lemma 2.5) and, on the other hand, that the switched system $\dot{x} = J_2 x + \alpha_\star(t) b_0 u$, $\alpha_\star(t) \in [\rho, 1]$, can be uniformly stabilized with an arbitrary rate of convergence by taking the feedback law $u_\gamma = -\gamma D_\gamma K x$, where $\gamma > 0$ is arbitrarily large and K is provided by [11, Lemma 4.0].

Remark 4.7 One possible interpretation of Proposition 4.5 goes as follows. Consider the destabilizing signals built in the argument of the proposition back in the original time-scale, i.e., as $(1,\rho)$-signals. These signals take only the values $0, 1$ over time intervals of length proportional to $1/\gamma$. Therefore, the fundamental solution associated to $\dot{x} = (A - \alpha b_0 K_{\gamma,\theta})x$ is a power of the product $A_1 A_2 A_3$, where $A_1 = \exp(T_1(A - b_0 K_{\gamma,\theta})/\gamma)$, $A_2 = \exp(T_2 A/\gamma)$ and $A_3 = \exp(T_3(A - b_0 K_{\gamma,\theta})/\gamma)$. The stabilizing effect of $A - b_0 K_{\gamma,\theta}$ is countered by the overshoot phenomenon occurring when the exponential of $A - b_0 K_{\gamma,\theta}$ is taken only over small intervals of time. If γ is large enough, such overshoot eventually destabilizes $\dot{x} = (A - \alpha b_0 K_{\gamma,\theta})x$.

4.3 Further discussion on the maximal rate of convergence

Let $(A,b) \in M(n,\mathbf{R}) \times \mathbf{R}^n$ be a controllable pair. Define

$$\rho(A,T) = \inf\{\rho \in (0,1] \mid \mathrm{RC}(A,T,T\rho) = +\infty\}. \tag{48}$$

Notice that $\rho(A,T)$ is equal to $\rho(A/T,1)$ and does not depend on $\mathrm{Tr}(A)$ (see Remark 4.1).

Proposition 4.4 implies that $\rho(A,T) \leq \rho^*$ for some $\rho^* \in (0,1)$ only depending on n. In the case $n = 2$, moreover Proposition 4.5 establishes a uniform lower bound $\rho(A,T) \geq \rho_* > 0$.

The following lemma collects some further properties of the function $T \mapsto \rho(A,T)$.

Lemma 4.8 *Let $(A,b) \in M_n(\mathbf{R}) \times \mathbf{R}^n$ be a controllable pair. Then (i) $T \mapsto \rho(A,T)$ is locally Lipschitz on $(0,+\infty)$; (ii) there exist $\lim_{T\to+\infty} \rho(A,T) = \sup_{T>0} \rho(A,T)$ and $\lim_{T\to 0^+} \rho(A,T) = \inf_{T>0} \rho(A,T)$.*

Proof. In order to prove (i), notice that point 3 in Lemma 2.4 implies that if $\mathrm{RC}(A,T,\rho T) < +\infty$ then for every $\eta \in (0,\rho T)$,

$$\mathrm{RC}\left(A, T+\eta, \frac{\rho T}{T+\eta}(T+\eta)\right) \quad < \quad +\infty, \tag{49}$$

$$\mathrm{RC}\left(A, T-\eta, \frac{\rho T-\eta}{T-\eta}(T-\eta)\right) \quad < \quad +\infty. \tag{50}$$

From (49) we deduce that for every $\eta \in (0, \rho(A,T)T)$,

$$\rho(A,T+\eta) \geq \frac{\rho(A,T)T}{T+\eta}, \tag{51}$$

and thus

$$\rho(A,T) - \rho(A,T+\eta) \leq \eta/T.$$

Similarly, (50) implies that, for every $\eta \in (0, \rho(A,T)T)$,

$$\rho(A,T-\eta) \geq \frac{\rho(A,T)T-\eta}{T-\eta}.$$

Therefore, one has

$$\rho(A,T) \geq \frac{\rho(A,T+\eta)(T+\eta)-\eta}{T} \tag{52}$$

for every η satisfying $0 < \eta < \rho(A,T+\eta)(T+\eta))$ and in particular for every $\eta \in (0, \rho(A,T)T)$ (see (51)). We obtain from (52) that $\rho(A,T+\eta) - \rho(A,T) \leq \eta/T$ and we conclude that

$$|\rho(A,T+\eta) - \rho(A,T)| \leq \frac{\eta}{T}$$

for every $\eta \in (0, \rho(A,T)T)$.

As for point (ii), it suffices to deduce from point 5 in Lemma 2.4 that if $0 < \rho' < \rho < 1$ then there exists $M > 0$ such that whenever $\mathrm{RC}(A,T,\rho T) = +\infty$ one has $\mathrm{RC}(A,\gamma,\rho'\gamma) = +\infty$ for every $\gamma > 0$ such that $\gamma/T > M$. ■

Remark 4.9 In the case $A = J_n$ equality (15) implies that the function $T \mapsto \rho(J_n, T)$ is constant. When $n = 2$ its constant value is positive, due to Proposition 4.5.

5 Open problems

We conclude the paper by providing some questions that arose from our investigation of single-input persistently excited linear systems.

Open problem 1 Does Proposition 4.3 still hold true in dimension bigger than two? Notice that the proof provided here essentially relies on the controllability of (41) in finite time.

Open problem 2 Consider the constant ρ_n^* defined as the upper lower bound for all the ρ^*'s satisfying the statement of Proposition 4.4 (n fixed). What can be said on the dependance of ρ_n^* on n as $n \to \infty$?

Open problem 3 We conjecture that Proposition 4.5 holds true in dimension $n > 2$. Note however that the proof given in the 2D case cannot be easily extended to the case in which $n > 2$. Indeed, our strategy is based on a complete parameterization of the candidate feedbacks for stabilization and on the explicit construction of a destabilizing signal α for every value of the parameter θ, which takes values in the one-dimensional sphere. In the general case, the parameter would belong to an $(n-1)$-dimensional manifold and an explicit construction, if possible, would be more intricate.

Open problem 4 It is a challenging question to determine whether the function $T \mapsto \rho(A, T)$ (defined in (48)) is constant for a general matrix A. If this is true, one may wonder whether its constant value depends on A. Otherwise, a natural question would be to understand the dependence of $\lim_{T\to 0^+} \rho(A, T)$ and $\lim_{T\to +\infty} \rho(A, T)$ on the matrix A.

Open problem 5 Proposition 4.5 states that, for $n = 2$ and μ/T small, the PE control system $\dot{x} = Ax + \alpha bu$, $\alpha \in \mathcal{G}(T, \mu)$, does not have the pole-shifting property (see Remark 4.2). It makes therefore sense to investigate additional conditions to impose on the PE signals (periodicity, positive dwell-time, uniform bounds on the derivative of the PE signal, etc) so that the pole-shifting property holds true for these restricted classes of PE signals, regardless of the ratio μ/T. First of all, the subclass of periodic PE signals must be excluded, since the destabilizing inputs constructed in Proposition 4.5 are periodic. It is also clear that, for the subclass of $\mathcal{G}(T, \mu)$ given by all signals with a positive dwell time $t_d > 0$, one gets arbitrary rate of convergence (or divergence) with a linear constant feedback, for every choice of T, μ, t_d. Here follows our conjecture.

Given $T, M > 0$ and $\rho \in (0, 1]$, let $\mathcal{D}(T, \rho, M)$ be the subset of $\mathcal{G}(T, \rho T)$ whose signals are globally Lipschitz over $[0, +\infty)$ with Lipschitz constant bounded by M. Then, given a controllable pair (A,b), we conjecture that it is possible to stabilize (respectively, destabilize) by a linear feedback the system $\dot{x} = Ax + \alpha bu$, $\alpha \in \mathcal{D}(T,\rho,M)$, with an arbitrarily large rate of convergence (respectively, divergence), i.e., we conjecture that for every $C > 0$ there exist two gains K_1 and K_2 such that for every $\alpha \in \mathcal{D}(T,\rho,M)$ the maximal Lyapunov exponent of $\dot{x} = (A - \alpha b K_1^T)x$ is smaller than $-C$ and the the minimal Lyapunov exponent of $\dot{x} = (A - \alpha b K_2^T)x$ is larger than C.

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