Results on hypergraph planarity

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Abstract. Using the notion of planarity and drawing for hypergraphs introduced respectively by Johnson and Pollak [9] and Mäkinen [14], we show in this paper that any hypergraph having less than nine hyperedges is vertex-planar and can be drawn in the edge standard and in the subset standard without edge crossing.

Key words: hypergraphs, planarity, vertex-planarity, drawing in the edge standard, drawing in the subset standard, Euler diagrams

1 Introduction

Hypergraphs can be viewed as generalizations of graphs: a hypergraph is an ordered pair $(V, \mathcal{E})$ where V is a set of vertices and $\mathcal{E}$ is a set of hyperedges, each hyperedge being a subset of V.
Drawings of hypergraphs are used in various contexts as, for example, vlsi design [1], databases [6, 4, 3] and information visualization [8]. Obtaining "good looking" layouts of hypergraphs becomes an important point in information visualization where readable cartographies may constitute an useful help to access huge sets of documents. In this context, we want to create Euler-like diagrams to represent the interconnections of a collection of semantic fields (cf. [19, 17] for a more detailed description of our purpose). As the number of hyperedges corresponds to the number of semantic fields, we need to study the existence of planar hypergraphs layouts with respect to the number of hyperedges.
In this paper, we study the following problem:
what is the maximum number of hyperedges a hypergraph can contain to ensure that it has a planar representation ?
After having described in section 1.1 the existing graphical representations for hypergraphs, we show in section 1.2 the equivalence between the notions of vertex-planarity and planar drawing in edge standard for hypergraphs. Finally, we prove in section 2 that any hypergraph having less than nine hyperedges is vertex planar.

1.1 Graphical representation of hypergraphs

Several graphical representations have been introduced for hypergraphs (see figure 1 for an example):

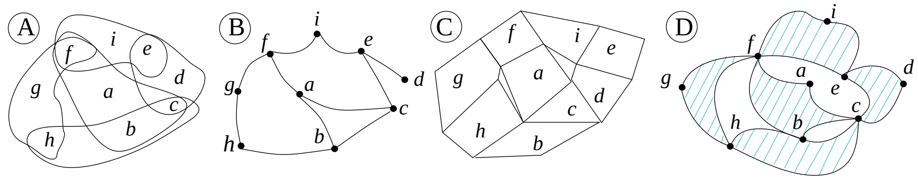

Fig. 1. The hypergraph $H = (V, \mathcal{E})$ with $V = \{a, b, c, d, e, f, g, h, i\}$ and $\mathcal{E} = \{\{a, b, c, f\}, \{c, d, e\}, \{e, f, i\}, \{f, g, h\}, \{b, c, h\}\}$ is drawn in the subset standard (A) and in the edge standard (B). A *vertex-based Venn diagram* representing H is drawn in (C). The hypergraph H is Zykov-planar and its representation is drawn in (D).

- Zykov, as described in [9], considers that a hypergraph is planar when it can be represented by the faces of a planar map, the vertices belonging to the boundaries of the faces (cf. [7] for a study on the minimal non planar hypergraphs using Zykov's definition of planarity).
- Johnson and Pollak [9] introduce two notions of planarity for hypergraphs, based on dual generalizations of Venn diagrams: the edge-planarity and the vertex-planarity. They show that the general problem of determining whether a hypergraph is (vertex- / edge-) planar is NP-complete (other complexity results related to vertex-planarity and hypergraphs can be found in [1]). Let us recall the definition of vertex-planarity [9]:
  "Given a hypergraph $H = (V, \mathcal{E})$, a *vertex-based Venn diagram representing* H consists of a planar graph G, an embedding M of G into the plane, and a one-to-one map from the set V of vertices of H to the set of faces of M, such that for each hyperedge $e \in \mathcal{E}$, the union of the faces corresponding to vertices in e comprises a region of the plane whose interior is connected. A hypergraph is said *vertex-planar* if there is a *vertex-based Venn diagram* that represents it."
  As noticed in [15, 12, 16, 5], the name *Venn* corresponds to diagrams containing 2^n regions when H has n hyperedges. The vertex-based Venn diagrams defined are similar to the *extended Euler diagrams* introduced in [18, 19].
- Mäkinen [14] introduces two types of drawings for hypergraphs: for "the edge standard", the vertices belonging to a same hyperedge are connected together (in [10] an implementation of a edge standard drawing method is described); for the "subset standard", hyperedges are represented by planar regions bounded by curves (see [2] for a description of a drawing system based on this definition).

In fact, the notion of Zykov-planarity and vertex-planarity can be considered as a generalization of the planarity's notion for graphs because we have [9]: a graph G is planar in the ordinary sense if and only if it is vertex-planar (resp. Zykov-planar) when viewed as a hypergraph. Then, as the graph $K_{3,3}$ is the non planar graph having the smallest number of edges, which is equal to nine, we know that there is at least one non vertex-planar hypergraph having nine hyperedges (we use the notations of [13] for $K_{3,3}$ and K_5 and Kuratowski's theorem [11]).
Thus, as we want to know the lower bound n on the number of hyperedges such that the following assertion is true:

"all the hypergraphs having at most n hyperedges are vertex-planar".

We already know that n is lower than nine.
We show in this paper that n is equal to eight, i.e. that all the hypergraphs having at most

eight hyperedges are vertex-planar. This is made by a constructive proof in section 2. Let us first of all compare the notions of vertex-planarity and planar drawing in the edge standard for hypergraphs.

1.2 Vertex-planarity and drawing in the edge standard

Let $H = (V, \mathcal{E})$ be a hypergraph, with the set of vertices V and the set of hyperedges $\mathcal{E}$. Following Mäkinen, we define an equivalence relation r on V by:
vrv' if and only if for each edge e in $\mathcal{E}$, $v \in e$ if and only if $v' \in e$.
Then the *condensation* of H is the hypergraph $H' = (V', \mathcal{E}')$ in which V' contains a vertex v' for each equivalence class of V w.r.t. r and $\mathcal{E}'$ has an edge e' with vertex $v' \in V'$ if and only if the corresponding edge e in $\mathcal{E}$ contains v' (cf. figure 2).

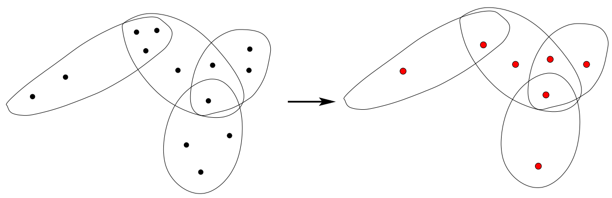

Fig. 2. A hypergraph and its condensation.

One can notice that the condensation of H is vertex-planar if and only if H is vertex-planar. Then, we will work with the condensation of H instead of hypergraphs H in the rest of the paper.

Remark 1. According to Mäkinen's definition, a drawing of H in the edge standard is strictly constrained by the fact that a hyperedge is represented by a path of edges connecting all its vertices together. Then, when H is vertex-planar, the adjacency graph of the regions forming *the vertex-based Venn diagram representing* H is a drawing of H in the edge standard. Conversely, when H has a planar drawing in the edge standard, a vertex based Venn diagram representing H can be build (cf. figure 3 (B) and (C) for an example). Thus, the two notions: "H is vertex-planar" and "H has a planar drawing in edge standard" are equivalent.

More formally, we say that a graph $G = (V, E)$ is *a representation of* $H = (V, \mathcal{E})$ *in the edge standard* when it satisfies: $\forall e \in \mathcal{E}$, the subgraph of G induced by e is connected. By extension, a graph $G' = (V', E')$, where $V' \subset V$, is *a representation of* H *restricted to* V' *in the edge standard* if $\forall e \in \mathcal{E}$, the subgraph of G induced by $e \cap V'$ is connected. Then we can say that "*H has a planar drawing in edge standard*" or "*H is vertex-planar*" when there exists a planar graph $G = (V, E)$ which is *a representation of* $H = (V, \mathcal{E})$ *in the edge standard*.

2 The constructive proof

To show that a hypergraph $H = (V, \mathcal{E})$ having less than nine hyperedges is vertex planar, we will show how to build a planar representation of H in the edge standard.
Let $H = (V, \mathcal{E})$, v be a vertex of V and W a subset of V. We note: $e(v)$ the subset of $\mathcal{E}$

formed by the hyperedges containing v; $n_e(v)$ the number of hyperedges of $\mathcal{E}$ containing v; $e(W)$ the set of hyperedges of $\mathcal{E}$ containing at least a vertex of W; $e_u(v, W)$ the subset of $e(v)$ for which v is the unique vertex in W, and $e_u(W) = \bigcup_{v \in W} e_u(v, W)$.

Given $H = (V, \mathcal{E})$, the condensation of a hypergraph having $n < 9$ hyperedges, to show how to build a planar graph representation of H in the edge standard, we proceed as follows:

1. we first choose a subset V_0 of V such that:
   (i) any hyperedge of $\mathcal{E}$ has a vertex in V_0
   (ii) V_0 has a minimal number of vertices
   (iii) V_0 is maximal w.r.t. the relation $\succeq$: $W \succeq W'$ if the ordered list $(n_e(v))_{v \in W}$ is greater than the ordered list $(n_e(v'))_{v' \in W'}$ for the lexicographic order.
   (cf. example 1 for an illustration of those properties)
2. a planar *representation of H restricted to V_0 in the edge standard*, $G_0 = (V_0, E_0)$, is built.
   The properties of V_0 and the planarity of G_0 are studied in section 2.1.
3. $V \backslash V_0$ is partitioned into k sets of vertices V_1,...,V_k such that $v \in V_i$ when i is the minimum number of edges to be inserted in $G_0 = (V_0, E_0)$ to obtain a representation of H restricted to $V_0 \cup \{v\}$ in the edge standard.
   The construction of the partition of $V \backslash V_0$ is described in section 2.2.
4. a planar *representation of H in the edge standard* is built by inserting progressively the vertices of V_k,V_{k-1}, ..., V_1 in G_0.
   This is the subject of section 2.3.

2.1 Properties of V_0

In the following, $H = (V, \mathcal{E})$ is the condensation of a hypergraph having less than nine hyperedges and V_0 is a subset of V satisfying (i),(ii) and (iii).
As V_0 has a minimal number of vertices, we have:

Lemma 1. *$e_u(V_0)$ contains at least $card(V_0)$ elements. and $card(V_0) \leq card(\mathcal{E})$.*

Proposition 1. *When $card(\mathcal{E}) < 9$, there is a planar* representation of H restricted to V_0 in the edge standard*;*

Proof. The complete graph $K_{card(V_0)}$ on V_0 is a representation of H restricted to V_0 in the edge standard. Let $G_0 = (V_0, E_0)$ be a subgraph of $K_{card(V_0)}$, with minimal number of edges, and being a representation of H restricted to V_0 in the edge standard. We have the following cases to consider:
$-$ $\boldsymbol{card(V_0) \leq 4}$. $G_0 = (V_0, E_0)$ and is planar because we need at least 5 vertices to build a non planar graph.
$-$ $\boldsymbol{card(V_0) \geq 5}$. By definition, we have $card(e_u(V_0)) \geq 5$ then at most 3 hyperedges have to be represented by a path in G_0. As G_0 is minimal in number of edges among the representations of H restricted to V_0 in the edge standard, it cannot contains any subdivision of K_5. When $card(V_0) \geq 6$, only 2 hyperedges have to be represented by a path in G_0, thus G_0 cannot contain a subdivision of $K_{3,3}$. $\square$

In the rest of the paper, $G_0 = (V_0, E_0)$ denotes a planar representation of H restricted to V_0 in the edge standard.

2.2 The partition of $V \backslash V_0$

Once a subset V_0 of V satisfying (i), (ii) and (iii) is chosen, a partition of $V \backslash V_0$ is made, classifying the vertices v of $V \backslash V_0$ with respect to the number of edges necessary to extend V_0 to $V_0 \cup \{v\}$ while keeping the property of being a representation of H restricted to $V_0 \cup \{v\}$ in the edge standard. More precisely:
V is partitioned into $k+1$ sets of vertices V_0,...,V_k. Each V_i, with $i > 0$ is such that $v \in V_i$ if and only if i is the minimum number of edges to be inserted in $G_0 = (V_0, E_0)$ to obtain a representation of H restricted to $V_0 \cup \{v\}$ in the edge standard. These edges connect v and vertices of V_0. In the following, $W_i(v) = \{w_1, ..., w_t\}$ denotes a minimum subset of V_0 such that its vertices can be connected to v to form a representation of H restricted to $V_0 \cup \{v\}$ in the edge standard. We have: $card(W_i(v)) = i$.

Example 1 *Consider the two hypergraphs (cf. figure 3):*
• *For the hypergraph of figure 1, we have* $\mathcal{E} = \{e_1, ..., e_5\}$ *with* $e_1 = \{a, b, c, f\}$, $e_2 = \{c, d, e\}$, $e_3 = \{e, f, i\}$, $e_4 = \{f, g, h\}$ *and* $e_5 = \{b, c, h\}$. *Then* $e(a) = \{e_1\}$, $e(b) = \{e_1, e_5\}$, $e(c) = \{e_1, e_2, e_5\}$, $e(d) = \{e_2\}$, $e(e) = \{e_2, e_3\}$, $e(f) = \{e_1, e_3, e_4\}$, $e(g) = \{e_4\}$ *and* $e(h) = \{e_4, e_5\}$. *The set of vertices* V *is partitioned into three sets:* $V_0 = \{c, f\}$, $V_1 = \{a, b, d, g, i\}$ *and* $V_2 = \{e, h\}$ *(case A of figure 3).*
• *Suppose* $H = (V, \mathcal{E})$ *with* $V = \{a, b, c, d, e, f, g, h\}$ *and* $\mathcal{E} = \{e_1, ..., e_8\}$ *with* $e_1 = \{a, f, i\}$, $e_2 = \{b, g\}$, $e_3 = \{c, h\}$, $e_4 = \{d, i\}$, $e_5 = \{a, b, e\}$, $e_6 = \{b, c, f, h\}$, $e_7 = \{c, d, e, g\}$ *and* $e_8 = \{a, g\}$. *We have* $e(a) = \{e_1, e_5, e_8\}$, $e(b) = \{e_2, e_5, e_6\}$, $e(c) = \{e_3, e_6, e_7\}$, $e(d) = \{e_4, e_7\}$, $e(e) = \{e_7, e_5\}$, $e(f) = \{e_1, e_6\}$, $e(g) = \{e_2, e_5, e_8\}$, $e(h) = \{e_3, e_6\}$ *and* $e(i) = \{e_1, e_4\}$. *The sets of vertices* $W = \{a, b, c, d\}$ *and* $W' = \{a, g, c, i\}$ *both satisfy (i), (ii) and (iii): the ordered list of the* $(n_e(v))_{v \in W}$ *and the ordered list of the* $(n_e(v'))_{v' \in W'}$ *are both equal to* $(3, 3, 3, 2)$. *The set* $W'' = \{a, g, h, i\}$ *satisfies (i) and (ii) but not (iii) because the ordered list of the* $(n_e(v''))_{v'' \in W''}$ *is equal to* $(3, 3, 2, 2)$ *which is lower than the others for the lexicographic order. Let us take* $V_0 = \{a, b, c, d\}$. *Then* $V_1 = \{h\}$, $V_2 = \{e, f, i\}$ *and* $V_3 = \{g\}$. *Two different subsets of* V_0 *can be connected to* f *with a minimal number of vertices:* $W_2(f)$ *can be equal to* $\{a, b\}$ *or to* $\{a, c\}$.

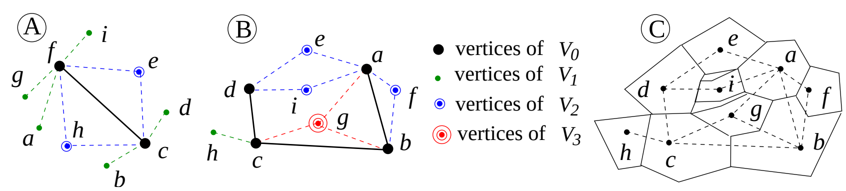

Fig. 3. The graph G_0 and the partition of V into $V_0, ..., V_k$ for the two hypergraphs of example 1. The dashed lines connect vertices of V_i, $i > 0$, with vertices of V_0 according to $\mathcal{E}$. The solid lines represent the graph G_0. in (A), V_0 contains two vertices and G_0 contains one edge connecting them. In (B), three edges connect the four vertices of V_0. In (C), a vertex-based Venn diagram representing the second hypergraph is drawn.

Because of the minimality of V_0 and the definition of the V_i, we have:

Lemma 2. *(1) if* v *is a vertex of* V_i, $i > 0$, $e(v)$ *contains at least* i *hyperedges of* $e_u(W_i(v))$.

(2) V_i is empty when $i > card(V_0)$
(3) V_i is empty when $i > card(\mathcal{E}) - card(V_0)$

2.3 Extension of G_0

Remark 2. Suppose that G' is a planar representation of H restricted to $V' = V_0 \cup W$ in the edge standard, where W is a subset of $V \backslash V_0$.

- a vertex v of V_1 is inserted in G' by adding an edge to obtain a graph G'', a representation of H restricted to $V' \cup \{v\}$ in the edge standard. As v is connected to only one vertex and G' is planar, G'' is also planar. Thus any insertion of a vertex of V_1 can be made without breaking the planarity.

Without loss of generality, we will suppose in the rest of the paper that any vertex of V is included in at least two hyperedges of $\mathcal{E}$. As a consequence, vertices of V_0 are included in at least two hyperedges of $\mathcal{E}$.

- a vertex v of of V_2 is inserted in G' by adding two edges joining v with two vertices w_1 and w_2 of V_0. If (w_1, w_2) is an edge of G', this insertion leads to a planar representation of H restricted to $V' \cup \{v\}$ in the edge standard. In the other cases, we must show that the edge (w_1, w_2) does not break the planarity while being inserted into G'.

Considering the cardinalities of $\mathcal{E}$ and of V_0, we have the following cases to consider to show that the insertion of vertices of $V \setminus V_0$ leads to a planar representation of H in the edge standard:

$card(V_0)$	$V \setminus V_0$					
	$card(\mathcal{E}) = 8$	$card(\mathcal{E}) = 7$	$card(\mathcal{E}) = 6$	$card(\mathcal{E}) = 5$	$card(\mathcal{E}) = 4$	$card(\mathcal{E}) = 3$
1	V_1	V_1	V_1	V_1	V_1	V_1
2	$V_1 \cup V_2$	$V_1 \cup V_2$	$V_1 \cup V_2$	$V_1 \cup V_2$	$V_1 \cup V_2$	V_1
3	$\boldsymbol{V_1 \cup ... V_3}$	$\boldsymbol{V_1 \cup ... V_3}$	$\boldsymbol{V_1 \cup .. V_3}$	$V_1 \cup V_2$	V_1	$\emptyset$
4	$\boldsymbol{V_1 \cup ... V_4}$	$\boldsymbol{V_1 \cup ... V_3}$	$V_1 \cup V_2$	V_1	$\emptyset$	$\emptyset$
5	$\boldsymbol{V_1 \cup ... V_3}$	$V_1 \cup V_2$	V_1	$\emptyset$	$\emptyset$	$\emptyset$
6	$V_1 \cup V_2$	V_1	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
7	V_1	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$

Remark 3. Only the cases written in bold letters have to be examined precisely. The other cases are solved as follows:

- When $card(V_0) = 2$, all the vertices v of V_2 are inserted in G_0 by adding two edges joining v with the two vertices of V_0 without breaking the planarity of the resulting graph.
- When $card(\mathcal{E}) - card(V_0) = 2$, two hyperedges e' and e'' of $\mathcal{E}$ contain several vertices of V_0 and $card(V_0)$ hyperedges of $\mathcal{E}$ contain only one vertex of V_0. Then G_0 is composed of at most two paths connecting the vertices of V_0 belonging to e' (resp. e''). As V_0 has a minimal number of vertices, a vertex of V_2 cannot belong to more than one hyperedge of $e_u(V_0)$ and then it must belong to either e' or e''. Thus the insertions of vertices of V_2 correspond to insertions of edges between vertices of e' and vertices of e'' in G_0. These insertions can be made without breaking the planarity of G_0 (cf. figure 4).

Let us now study how vertices of V_3 can be inserted in G_0. Using the fact that $card(\mathcal{E}) < 9$ and that a vertex of V_i is included in at least i hyperedges of $\mathcal{E}$, we have:

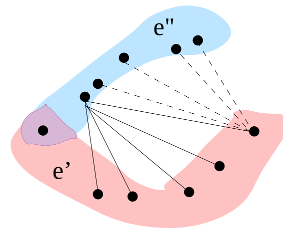

Fig. 4. When $card(\mathcal{E}) - card(V_0) = 2$, edges corresponding to insertion of vertices of V_2 included in e' (resp. e'') are drawn in dashed (resp. solid) lines.

Lemma 3. *V_3 contains at most two vertices v and v' which are not both included in a same hyperedge of $\mathcal{E}$.*

Remark 4. When two vertices v and v' of V_3 are both linked to $W_3(v) = \{w_1, w_2, w_3\}$ and when the face defined by $W_3(v)$ do not contain any edge in G_0, if there exists i in $\{1, 2, 3\}$ with $e(v) \cap e_u(w_i, W_3(v)) = e(v') \cap e_u(w_i, W_3(v))$ then these vertices can be inserted inside the face defined by $W_3(v)$ without edge crossing (cf figure 5 (A)).
- v is inserted inside the face (w_1, w_2, w_3) by adding three edges.
- by adding the edges (v, v'), (w_2, v') and (w_3, v'), we obtain a planar representation of H restricted to $V_0 \cup \{v, v'\}$ in the edge standard.

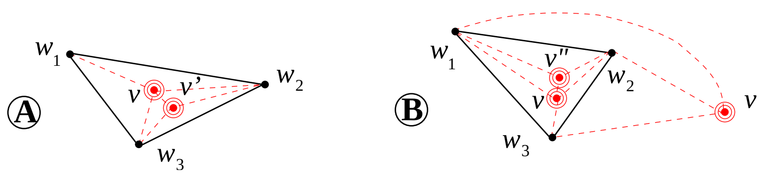

Fig. 5. The insertion is made by adding the dashed edges. A: insertion of v and v' inside the face (w_1, w_2, w_3). B: when $V_0 = \{w_1, w_2, w_3\}$, the insertion of v, v' and v'' can be made either inside or outside the face (w_1, w_2, w_3).

Proposition 2. *When $card(V_0) = 3$ and $card(\mathcal{E}) < 9$, the insertions of vertices of $V \backslash V_0$ in G_0 lead to a planar representation of H in the edge standard.*

Proof. Vertices of V_3 are inserted as follows (cf figure 5 (B)):
- a vertex v of V_3 is inserted inside the face (w_1, w_2, w_3) by adding three edges.
- if a vertex v' of V_3, is such that for any $i = 1, 2, 3$, $e(v) \cap e_u(w_i, W_3(v)) \neq e(v') \cap e_u(w_i, W_3(v))$, then v' is inserted outside the face defined by V_0 by adding three edges.
- if v has been inserted inside (w_1, w_2, w_3) and v' inserted outside (w_1, w_2, w_3), then, as V_0 has a minimum number of vertices and $card(\mathcal{E}) < 9$, any other vertex v'' of V_3 must satisfy : there exists w_i in V_0 such that either $e_u(w_i, V_0) \cap e(v) = e_u(w_i, V_0) \cap e(v'')$ or $e_u(w_i, V_0) \cap e(v') = e_u(w_i, V_0) \cap e(v'')$. Then v" is inserted as described in remark 4.
We proceed similarly for the other vertices of V_3 and obtain a planar representation of H restricted to $V_0 \cup V_3$ in the edge standard.
Vertices of V_2 are inserted by adding paths along the edges of the triangle defined by the three vertices of V_0. The resulting graph is a planar representation of H in the edge standard. $\square$

Now, to prove that any hypergraph having less than 9 hyperedges has a planar representation in the edge standard, we have the following cases to consider:

- $card(V_0) = 4$ and $card(\mathcal{E}) = 7$ or 8.
- $card(V_0) = 5$ and $card(\mathcal{E}) = 8$.

Proposition 3. *When $card(V_0) = 4$ and $card(\mathcal{E}) < 9$, the insertions of vertices of $V \backslash V_0$ in G_0 lead to a planar representation of H in the edge standard.*

Proof. Let us consider the two cases separately:
-A- When $\boldsymbol{card(\mathcal{E}) = 7}$, we first prove that V_4 is empty and that the vertices of V_3 can always be inserted inside the faces defined by $W_3(v)$. Then, as K_4 is planar, the resulting graph will be a planar representation of H in the edge standard.
V_4 is empty: suppose that v is a vertex of V_4, then v belongs to at least four hyperedges of $e_u(V_0)$. As V_0 satisfies (iii), V_0 contains a vertex v_0 which belongs to at least four hyperedges. There is only one hyperedge of $\mathcal{E}$ which contains both v and v_0. Then the set $\{v, v_0\}$ satisfies (i) and contains only two vertices which contradicts the hypothesis on V_0.
If V_3 contains two vertices v and v' associated to $W_3(v) = \{w_1, w_2, w_3\}$ and such that for any $i = 1, 2, 3$, $e(v) \cap e_u(w_i, W_3(v)) \neq e(v') \cap e_u(w_i, W_3(v))$, then at most one hyperedge e of $\mathcal{E}$ does not contain v or v'. Let v_0 be a vertex of V_0 belonging to e. then the set $\{v, v', v_0\}$ satisfies (i) and contains only three vertices which contradicts the hypothesis on V_0.
-B- When $\boldsymbol{card(\mathcal{E}) = 8}$, V_4 may not be empty.
-B-1. Suppose $\boldsymbol{V_4}$ **contains a vertex** $\boldsymbol{v_4}$.
We will show that the insertion of v_4 in G_0 leads to a planar representation of H restricted to $V_0 \cup \{v_4\}$ in the edge standard and that the insertion of vertices of $V \backslash (V_0 \cup \{v_4\})$ can be made by adding at most two edges to this graph.
When v_4 is a vertex of V_4, four hyperedges of $e_u(V_0)$ contains v. Suppose that there is a vertex v_0 in V_0 such that $card(e_u(v_0, V_0)) = 1$. If v_0 belongs to 4 hyperedges, then there is v_0' in V_0 such that $\{v_4, v_0, v_0'\}$ satisfies (i) and contains only three vertices which contradicts the hypothesis on V_0.
Otherwise, $V' = \{v_4\} \cup V_0 \backslash \{v_0\}$ satisfies (i), (ii) and is such that $V' \succeq V_0$ which contradicts the hypothesis on V_0.
Thus when V_4 contains a vertex v_4, $card(e_u(v, V_0)) = 2$ for any vertex v of V_0 and G_0 does not contain any edge. v_4 is inserted in G_0 by adding four edges and the resulting graph is planar.
Let v be a vertex of $V \backslash (V_0 \cup \{v_4\})$. Suppose that there are two hyperedges e and e' of $\mathcal{E}$ which contain v but not v_4. Let v_1 and v_2 be two vertices of V_0 included neither in e nor in e'. Then $V' = \{v_1, v_2, v, v_4\}$ satisfies (i), (ii) and is such that $V' \succeq V_0$ which contradicts the hypothesis on V_0.
Thus the vertices of $V \backslash (V_0 \cup \{v_4\})$ can be inserted by adding two edges connecting v_4 and a vertex of V_0 which leads to a planar graph.

-B-2. Suppose $\boldsymbol{V_4}$ **is empty**.

Let us examine the insertion of vertices of V_3:

• If all the vertices of V_3 can be inserted inside the faces defined by K_4 without edge crossing then, as K_4 is planar, the resulting graph will be planar.

• Otherwise, suppose that the two vertices v and v' of V_3 are both associated to

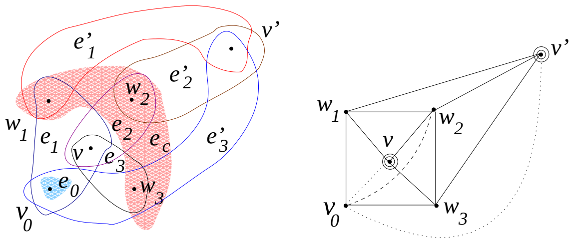

Fig. 6. When $card(V_0) = 4$, $card(\mathcal{E}) = 8$ and vertices v and v' are both associated to $\{w_1, w_2, w_3\}$. The path v_0, w_2 is replaced by one of the two paths v_0, v' or v_0, v.

$W_3(v) = \{w_1, w_2, w_3\}$ and are such that $\forall i = 1, ..., 3$, $e_u(w_i, W_3(v)) \cap e(v) \neq e_u(w_i, W_3(v)) \cap e(v')$. Then, six hyperedges of $\mathcal{E}$ contain either v or v'. Thus using the fact that $card(\mathcal{E}) < 9$, that v and v' belong to V_3 and the hypothesis satisfied by V_0, we have (cf. figure 6 for an example):

(a) $\forall i = 1, 2, 3$, $e_u(w_i, W_3(v)) = \{e_i, e'_i\}$, with v included in e_i and v' in e'_i, and there is a hyperedge e_c in $\mathcal{E}$ which contains the three vertices w_1, w_2 and w_3.

(b) the vertex v_0 of $V_0 \backslash W_3(v)$ has only one hyperedge in $e_u(v_0, V_0)$; v_0 is not included in e_c and cannot be included in both e_i and e'_i for $i = 1, 2, 3$.

We insert v inside the face defined by $W_3(v)$ and v' outside this face. This leads to a planar graph G'. Then one of the three edges joining v_0 with the w_i, for example the edge (w_2, v_0) cannot be drawn without edge crossing in G', as in figure 6. Thus to insert the other vertices of V_3 and the vertices of V_2 in G' without edge crossing, we replace the paths joining v_0 to w_2 by either a path joining v_0 to v or a path joining v_0 to v' (this can always be done because e_2 and e'_2 cannot both contain v_0). These insertions lead to a planar representation of H in the edge standard. □

The following remark will be used in the proof of proposition 4.

Remark 5. Let v be a vertex of $V \backslash V_0$ included in $e_1 \cap e_2$ where $\{e_1\} = e_u(v_1, V_0)$ $\{e_2\} = e_u(v_2, V_0)$ with v_1 and v_2 two distinct vertices of V_0. Then we have:

1. $\mathcal{E}$ contains a hyperedge $e_{1,2}$ of such that $e_{1,2} \cap V_0 = \{v_1, v_2\}$, otherwise V_0 could be replaced by $(V_0 \backslash \{v_1, v_2\}) \cup \{v\}$.
2. Consequently, the edge (v_1, v_2) belongs to G_0. Then, if v belongs to V_2, it can be inserted in G_0 without breaking the planarity.
3. Moreover, when v belongs to V_3 and $card(V_0) = 5$, we have:
   (a) v_1 and v_2 must be included in more than two hyperedges of $\mathcal{E}$, otherwise V_0 would not be maximal w.r.t. the relation $\succeq$.
   (b) Consequently, if v is included in a hyperedge e of $\mathcal{E} \backslash e_u(V_0)$, then $\mathcal{E} \backslash e_u(V_0)$ must contain at least three hyperedges.

Because of (a), at least two hyperedges of $\mathcal{E}\backslash e_u(V_0)$ are used for v_1 and v_2 and they are distinct from e.

(c) if v is included in a hyperedge of $e_u(v_0, V_0)$ with $v_0 \in V_0\backslash\{v_1, v_2\}$, then v_0 must be such that $card(e_u(v_0, V_0)) > 1$.
Otherwise, using point 1 of this remark, six hyperedges are necessary for v_1, v_2 and v_0 (e_i for each $e_u(v_i, V_0)$ and $e_{i,j}$ for each couple (v_i, v_j)). Then when $card(V_0) = 5$, three other hyperedges are needed for the two vertices of $V_0\backslash\{v_0, v_1, v_2\}$, which is impossible when $card(\mathcal{E}) < 9$.

Proposition 4. *When $card(V_0) = 5$ and $card(\mathcal{E}) = 8$ the insertions of vertices of $V\backslash V_0$ in G_0 lead to a planar representation of H in the edge standard.*

Proof. We have: $V\backslash V_0 = V_1 \cup V_2 \cup V_3$. We will prove that the insertion of vertices of $V\backslash V_0$ leads to a planar representation of H in the edge standard considering the number N of hyperedges of $\mathcal{E}\backslash e_u(V_0)$. Because of the cardinality of $\mathcal{E}$ and V_0, we have $3 \geq N > 0$.

- case A- Only one hyperedge e of $\mathcal{E}$ contains several vertices of V_0.
There are at least one vertex v_0 such that $card(e_u(v_0, V_0)) > 1$, and three vertices of V_0, v_1, v_2 and v_3, such that $card(e_u(v_i, V_0)) = 1$ and v_i belongs to e for $i = 1, 2, 3$.

- Suppose v_0 is the unique vertex of V_0 satisfying $card(e_u(v_0, V_0)) > 1$. Then V_3 is empty, otherwise this would contradict the fact that V_0 has a minimum number of vertices (cf. figure 7(A)).

- Suppose V_0 contains another vertex v_0' such that $card(e_u(v_0', V_0)) > 1$. Then, to satisfy the minimality condition of V_0, the vertices of V_3 must be included in e, in one hyperedge of $e_u(v_0, V_0)$ and in one hyperedge of $e_u(v_0', V_0)$. These vertices can be inserted inside the face defined by (v_0, v_0', v_1) as done in remark 4 (cf. figure 7(B)).

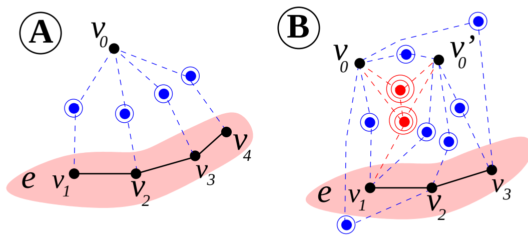

Fig. 7. When $card(V_0) = 5$, $card(\mathcal{E}) = 8$ and only one hyperedge e of $\mathcal{E}$ contains several vertices of V_0. The paths corresponding to potential insertions of elements of V_2 and V_3 are drawn in dashed lines. In this figure, the positions of the vertices of V_2 and V_3 w.r.t. e are not significant.

Consider now the subgraph G_0' of G_0 restricted to the set of vertices $\{v_1, v_2, v_3\}$. As G_0 is minimal in number of edges and e is the unique hyperedge containing v_1, v_2 and v_3, G_0' does not contain a K_3. Using remark 5, a vertex v of V_2 cannot be such that $W_2(v) = \{v_i, v_j\}$ with $i, j \in \{1, 2, 3\}$. As the elements of V_3, if they exist, are inserted inside the face defined by (v_0, v_0', v_1) , the insertion of vertices of $V\backslash V_0$ in G_0' cannot create a subdivision of K_3. Thus, the insertions of vertices of $V\backslash V_0$ in G_0 do not create a subdivision of K_5, as illustrated in figure 7 (A) et (B).

- case B - Two hyperedges e and e' of $\mathcal{E}$ contain several vertices of V_0.
Then exactly one vertex v_0 of V_0 is such that $e_u(v_0, V_0) = \{e_0, e'_0\}$.
Let us examine the vertices of V_3.

- If a vertex v of V_3 is included in the hyperedge e_1 s.t. $\{e_1\} = e_u(v_1, V_0)$. As V_0 must be maximal w.r.t. the relation $\succeq$, v_1 must be included in e and in e'. Then as v belongs to V_3, v must be included in e_2, $\{e_2\} = e_u(v_2, V_0)$ with $v_2 \neq v_1$ and in one hyperedge of $e_u(v_0, V_0)$, because of point c of remark 5.3. Thus we have the following configuration: $e(v_0) = \{e_0, e'_0\}$ or $e(v_0) = \{e_0, e'_0, e'\}$, $e(v_i) = \{e_i, e, e'\}$ for $i = 1, 2$ and $e(v_i) = \{e_i, e'\}$ for $i = 3, 4$, and v is inserted inside the face $(v_0 v_1 v_2)$, as illustrated in figure 8 (A).

- Otherwise, V_3 may contain vertices that must be included in e, in e' and in one of the hyperedges of $e_u(v_0, V_0)$. They can be inserted in a unique face, following remark 4 and we have the configuration of figure 8 (B).

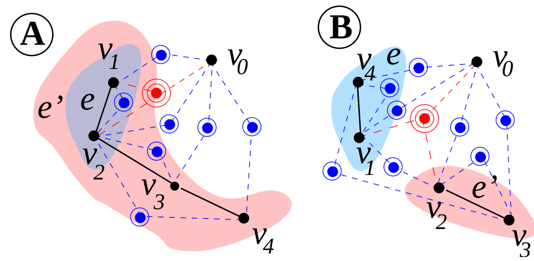

Fig. 8. When $card(V_0) = 5$, $card(\mathcal{E}) = 8$ and two hyperedges, e and e', of $\mathcal{E}$ contains several vertices of V_0. The paths corresponding to potential insertions of elements of V_2 and V_3 are drawn in dashed lines. In this fi gure, the positions of the vertices of V_2 and V_3 w.r.t. e and e' are not signifi cant.

Consider now the subgraph G'_0 of G_0 restricted to $V_0 \backslash \{v_0\}$. As G_0 is minimal in number of edges and there are only two hyperedges e and e' containing several vertices of $V_0 \backslash \{v_0\}$, G'_0 does not contain a K_4.
As noticed in remark 5 .2, a vertex of V_2 belonging to two hyperedges of $e_u(V_0) \backslash e_u(v_0, V_0)$ can be inserted in G_0 along an existing edge of G_0. The other vertices of V_2 associated to vertices of $V_0 \backslash \{v_0\}$ belong to at least one hyperedge of $\{e, e'\}$ and can be inserted as described in figure 4 without creating a subdivision of K_4. Thus the insertion of the vertices of $V \backslash V_0$ can be done without creating a subdivision of K_5 (as illustrated in figure 8 (A) and (B)). Please, notice that to simplify the drawing, multiple insertions between two vertices have been represented only once in those figures.
- case C- Three hyperedges e, e' and e'' of $\mathcal{E}$ contain several vertices of V_0.
As $card(V_0) = 5$, we can remark that:

- any vertex of V_0 belongs to exactly one hyperedge of $e_u(V_0)$. Then using point c of remark 5.3, a vertex v of V_3 cannot contain three hyperedges of $e_u(V_0)$.

- there is at least one vertex of V_0 included in at least two hyperedges of $\{e, e', e''\}$. Then v must be included in at least one hyperedge e_0 with $\{e_0\} = e_u(v_0, V_0)$.
As V_0 is maximal w.r.t. the relation $\succeq$, v_0 must be included in two hyperedges e and e' of $\mathcal{E} \backslash e_u(V_0)$. Then v necessarily belongs to e'', e_0 and e_1 with $e \cap V_0 = \{v_0, v_1\}$. Thus V_0 must be such that: $e(v_0) = \{e_0, e, e'\}$, $e(v_1) = \{e_1, e, e'\}$, $\{e_2, e''\} \in e(v_2)$ and

$e \cap V_0 = \{v_0, v_1\}$.
Suppose that e'' contains exactly two vertices v_2 and v_3 of V_0 distinct from v_0 and v_1. V_3 could contain a second vertex v' included in e_2, e_3 and e but, in this case, V_0 would not be minimal because $e(\{v, v', v_4\}) = e(V_0)$.
Consequently, V_3 contains at most one vertex v that is inserted inside the face $(v_0v_1v_2)$ (cf. figure 9(A)).

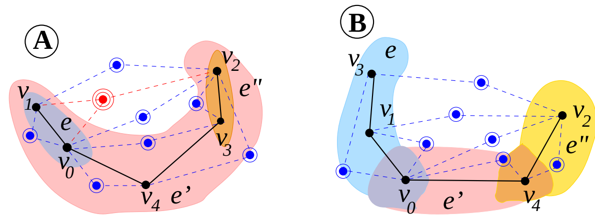

Fig. 9. When $card(V_0) = 5$, $card(\mathcal{E}) = 8$ and three hyperedges e, e' and e'' of $\mathcal{E}$ contains several vertices of V_0. The paths corresponding to potential insertions of elements of V_2 and V_3 are drawn in dashed lines. In this figure, the positions of the vertices of V_2 and V_3 w.r.t. e, e' and e'' are not significant.

Let us now examine the vertices of V_2. As previously, we suppose that v_0 and v_2 are vertices of V_0 respectively included in $\{e, e'\}$ and in e''. A vertex v of V_2 is:

- either included in two hyperedges of $e_u(V_0)$. Following remark 5.2, v is inserted along an existing edge of G_0.

- either included in only one hyperedge of $e_u(V_0)$. Suppose that v is included in $e_u(v_1, V_0)$. Then v will be inserted by adding two edges between either v_1 and v_0 or between v_1 and v_2.

- or included only in hyperedges of $\mathcal{E} \backslash e_u(V_0)$. Then v will be inserted by adding two edges between v_2 and v_0.

The paths created by these insertions join v_0 or v_2 with the other vertices of V_0 or are located along existing edges of G_0, as shown in figure 9 (A) and (B). Then, the resulting graph does not contain a subdivision of a K_5. Finally, we obtain a planar representation of H in the edge standard. □

Then using remark 3, proposition 2, 3 and 4, we have:

Theorem 1. *Any hypergraph having at most eight hyperedges is vertex-planar and has a planar representation in the edge standard.*

3 Conclusion

We have shown by a constructive proof that any hypergraph having less than nine hyperedges is vertex-planar and has a planar drawing in the edge standard.

The vertex-based Venn diagram representing a hypergraph H is an extended Euler diagram (cf. [19]). The only difference with a planar drawing of H in the subset standard is that a hyperedge is represented by several closed curves: one of the curves is the external contour and the others are the internal contours. The internal contours are included in the planar region bounded by the external contour and represent holes in this region. The planar region defined by this set of curves is connected. Then, by adding curves

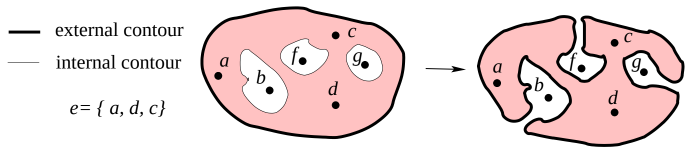

Fig. 10. The region corresponding to the hyperedge e in a vertex-based Venn diagram is transformed into a region bounded by a curve.

connecting the external curve and the internal curves and opening the internal curves as in figure 10, the vertex-based Venn diagram representing H can be easily transformed into a planar drawing of H in the subset standard. Thus our method can also be used to compute a planar drawing of any hypergraph having less than nine hyperedges in the subset standard (cf. figure 11 for an example).

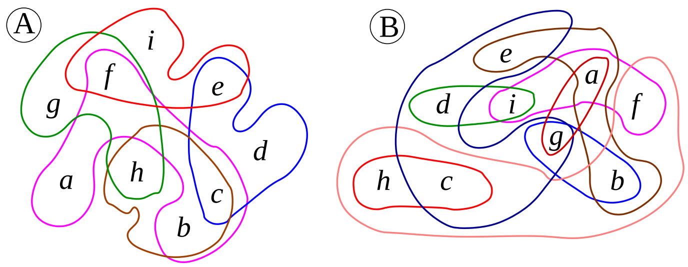

Fig. 11. A drawing in the subset standard of the two hypergraphs of example 1, according to the representation in the edge standard built with our method.

We are currently implementing a system computing a planar drawing in the edge standard, given a hypergraph having less than nine hyperedges. This work will be integrated in a graphical user interface for digital library access [17].

References

1. Circuit verification via hypergraph realization. *Simulation standard, Silvaco international*, 9(3):7–9, March 1998.
2. F. Bertault and P. Eades. Drawing hypergraphs in the subset standard. In *Graph Drawing 2000*, LNCS 1984, pages 164–169. Springer Verlag, 2000.
3. T. Catarci, M.F. Costabile, S. Levialdi, and C.Batini. Visual query systems for databases: a survey. *Journal of Visual Languages and Computing*, 8:215–260, 1997.
4. T. Catarci and L. Tarantino. A hypergraph-based framework for visual interaction with databases. *Journal of Visual Languages and Computing*, 6(2):135–166, 1995.
5. S. Chow and F. Ruskey. Drawing area proportional Venn and Euler diagrams. In *Graph Drawing 2003*, number 2912 Springer 2004 in Lecture Notes in Computer Science, pages 466–477, Perugia, Italy, September 2003.
6. M. Consens. *Creating and filtering structural data visualizations using Hygraph patterns*. Tech. Rep. CSRI– 302, Toronto, Canada, February 1994.

7. M. Debowsky. Results on planar hypergraphs and on cycle decompositions. Master's thesis, University of Vermont, Burlington, April 2002.
8. D. Harel. On visual formalisms. *Communications of the ACM*, 31(5):514–530, 1988.
9. D.S. Johnson and H. O. Pollak. Hypergraph planarity and the complexity of drawing Venn diagrams. *Journal of Graph Theory*, 11(3):309–325, 1987.
10. H. Klemetti, I. Lapinleimu, E. Mäkinen, and M. Sieranta. A programming project: trimming the spring algorithm for drawing hypergraphs. *SIGCSE Bulletin*, 27(3):34–38, 1995.
11. K. Kuratowski. Sur le problème des courbes gauches en topologie. *Fundamenta Mathematicae*, 15:271–283, 1930.
12. O. Lemon and I. Pratt. Spatial logic and the complexity of diagrammatic reasoning. *Machine Graphics and Vision (Special Issue on Diagrammatic Reasoning)*, 6(1):89–109, 1997.
13. A. Liebers. Planarizing graphs a survey and annotated bibliography. *Journal of Graph Algorithms and Applications*, 5(1):1–74, 2001.
14. E. Mäkinen. How to draw a hypergraph. *International Journal of Computer Mathematics*, 34:177–185, 1990.
15. F. Ruskey. A survey of Venn diagrams. *The electronic journal of combinatorics*, March 2001.
16. S-J. Shin and O. Lemon. Diagrams. In Edward N. Zalta, editor, *The Stanford Encyclopedia of Philosophy*. 2003. http://plato.stanford.edu/entries/diagrams/.
17. J. Thièvre, M.L. Viaud, and A. Verroust-Blondet. Using Euler diagrams in traditional library environment. In *Euler Diagrams 2004*, Brighton, U.K., September 2004.
18. A. Verroust and M.L. Viaud. Ensuring the drawability of extended Euler diagrams for up to 8 sets. Technical Report 4973, INRIA, Rocquencourt, France, October 2003.
19. A. Verroust and M.L. Viaud. Ensuring the drawability of extended Euler diagrams for up to 8 sets. In *Diagrammatic Representation and Inference, Diagrams 2004*, number 2980 in Lecture Notes in Artificial Intelligence, pages 128–141, Cambridge, U.K., March 2004.