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The main tool for stochastic calculus with respect to a multidimensional process B with small Hölder regularity index is rough path theory. Once B has been lifted to a rough path, a stochastic calculus – as well as solutions to stochastic differential equations driven by B – follow by standard arguments.

Although such a lift has been proved to exist by abstract arguments [19], a first general, explicit construction has been proposed in [27, 28] under the name of Fourier normal ordering.

The purpose of this short note is to convey the main ideas of the Fourier normal ordering method in the particular case of the iterated integrals of lowest order of fractional Brownian motion with arbitrary Hurst index.

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0 Introduction

The (two-sided) fractional Brownian motion $t \to B_t$, $t \in \mathbb{R}$ (fBm for short) with Hurst exponent α, $\alpha \in (0,1)$, defined as the centered Gaussian process with covariance

$$\mathbb{E}[B_s B_t] = \frac{1}{2}(|s|^{2\alpha} + |t|^{2\alpha} - |t-s|^{2\alpha}), \tag{0.1}$$

is a natural generalization in the class of Gaussian processes of the usual Brownian motion (which is the case $\alpha = \frac{1}{2}$), in the sense that it exhibits two fundamental properties shared with Brownian motion, namely, it has stationary increments, viz. $\mathbb{E}[(B_t - B_s)(B_u - B_v)] = \mathbb{E}[(B_{t+a} - B_{s+a})(B_{u+a} - B_{v+a})]$ for every $a, s, t, u, v \in \mathbb{R}$, and it is self-similar, viz.

$$\forall \lambda > 0, \quad (B_{\lambda t}, t \in \mathbb{R}) \stackrel{(law)}{=} (\lambda^\alpha B_t, t \in \mathbb{R}). \tag{0.2}$$

One may also define a d-dimensional vector Gaussian process (called: *d-dimensional fractional Brownian motion*) by setting $B_t = (B_t(1), \ldots, B_t(d))$ where $(B_t(i), t \in \mathbb{R})_{i=1,\ldots,d}$ are d independent (scalar) fractional Brownian motions.

Its theoretical interest lies in particular in the fact that it is (up to normalization) the only Gaussian process satisfying these two properties.

A standard application of Kolmogorov's theorem shows that fBm has a version with α^--Hölder continuous (i.e. κ-Hölder continuous for every $\kappa < \alpha$) paths. In particular, fBm with small Hurst parameter α is a natural, simple model for continuous but very irregular processes.

There has been a widespread interest during the past ten years in constructing a stochastic integration theory with respect to fBm and solving stochastic differential equations driven by fBm, see for instance [15, 9, 1, 22, 23]. The multi-dimensional case is very different from the one-dimensional case. When one tries to integrate for instance a stochastic differential equation driven by a two-dimensional fBm $B = (B(1), B(2))$ by using any kind of Picard iteration scheme, one encounters very soon the problem of defining the Lévy area of B which is the antisymmetric part of $\mathcal{A}_{ts} := \int_s^t dB_{t_1}(1) \int_s^{t_1} dB_{t_2}(2)$. This is the simplest occurrence of iterated integrals $\boldsymbol{B}^k_{ts}(i_1, \ldots, i_k) := \int_s^t dB_{t_1}(i_1) \ldots \int_s^{t_{k-1}} dB_{t_k}(i_k)$, $i_1, \ldots, i_k \leq d$ for d-dimensional fBm $B = (B(1), \ldots, B(d))$ which lie at the heart of the rough path theory due to T. Lyons, see [17, 18]. An alternative construction has been given in [10] under the name of 'algebraic rough path theory', which we now propose to describe briefly.

Assume $\Gamma_t = (\Gamma_t(1), \ldots, \Gamma_t(d))$ is some non-smooth d-dimensional path which is α-Hölder continuous. Integrals such as $\int f_1(\Gamma_t) d\Gamma_t(1) + \ldots + f_d(\Gamma_t) d\Gamma_t(d)$ do not make sense a priori because Γ is not differentiable (Young's integral [16] works for $\alpha > \frac{1}{2}$ but not beyond). In order to define the integration of a differential form along Γ, it is enough to define a *truncated multiplicative functional* or *geometric rough path* $(\mathbf{\Gamma}^1, \ldots, \mathbf{\Gamma}^{\lfloor 1/\alpha \rfloor})$ lying above Γ, $\lfloor 1/\alpha \rfloor$=entire part of $1/\alpha$, where $\mathbf{\Gamma}^1_{ts} = (\delta\Gamma)_{ts} := \Gamma_t - \Gamma_s$ is the *increment* of Γ between s and t, and each $\mathbf{\Gamma}^k = (\mathbf{\Gamma}^k(i_1, \ldots, i_k))_{1 \le i_1, \ldots, i_k \le d}$, $k \ge 2$ is a *substitute* for the iterated integrals $\int_s^t d\Gamma_{t_1}(i_1) \int_s^{t_1} d\Gamma_{t_2}(i_2) \ldots \int_s^{t_{k-1}} d\Gamma_{t_k}(i_k)$ with the following three properties:

(i) (*Hölder continuity) each component of* $\mathbf{\Gamma}^k$ *is* $k\alpha^-$*-Hölder continuous, that is to say,* $k\kappa$*-Hölder for every* $\kappa < \alpha$;

(ii) (multiplicative or Chen property) letting $\delta\mathbf{\Gamma}^k_{tus} := \mathbf{\Gamma}^k_{ts} - \mathbf{\Gamma}^k_{tu} - \mathbf{\Gamma}^k_{us}$*, one requires*

$$\delta\mathbf{\Gamma}^k_{tus}(i_1, \ldots, i_k) = \sum_{k_1+k_2=k} \mathbf{\Gamma}^{k_1}_{tu}(i_1, \ldots, i_{k_1}) \mathbf{\Gamma}^{k_2}_{us}(i_{k_1+1}, \ldots, i_k). \quad (0.3)$$

(iii) (geometric or shuffle property)

$$\mathbf{\Gamma}^{n_1}_{ts}(i_1, \ldots, i_{n_1}) \mathbf{\Gamma}^{n_2}_{ts}(j_1, \ldots, j_{n_2}) = \sum_{\boldsymbol{k} \in \mathrm{Sh}(\boldsymbol{i}, \boldsymbol{j})} \mathbf{\Gamma}^{n_1+n_2}_{ts}(k_1, \ldots, k_{n_1+n_2}) \quad (0.4)$$

where $\mathrm{Sh}(\boldsymbol{i}, \boldsymbol{j})$ *is the subset of permutations of* $i_1, \ldots, i_{n_1}, j_1, \ldots, j_{n_2}$ *which do not change the orderings of* $(i_1, \ldots, i_{n_1})$ *and* $(j_1, \ldots, j_{n_2})$.

Then there is a standard procedure which allows to define out of these data iterated integrals of any order and to solve differential equations driven by Γ.

The multiplicativity property (0.3) and the geometric property (0.4) are satisfied by smooth paths, as can be checked by direct computation. So the most natural way to construct such a multiplicative functional is to start from some smooth approximation Γ^ε, $\varepsilon \overset{>}{\to} 0$ of Γ such that each iterated integral $\mathbf{\Gamma}^{k,\varepsilon}_{ts}(i_1, \ldots, i_k)$, $k \le \lfloor 1/\alpha \rfloor$ converges in the $k\kappa$-Hölder norm for every $\kappa < \alpha$.

This general scheme has been applied to fBm in a paper by L. Coutin and Z. Qian [5] and later in a paper by the author [26], using different schemes of approximation of B by a family of Gaussian processes B^ε (living on the same probability space) with $\varepsilon \to 0$. In both cases, the variance of the Lévy area has been proved to diverge in the limit $\varepsilon \to 0$ when $\alpha \leq 1/4$.

Let us explain briefly our construction for the second-order iterated integral $\mathbf{B}^2_{ts}(i_1, i_2)$ (by abuse of language, we shall call this object a Lévy area, although the Lévy area is usually defined as the corresponding antisymmetrized quantity).

Let $\alpha \in (0, \frac{1}{2})$ (or even $\alpha \in (0, 1/4)$), and consider the natural iterated integral $\mathbf{B}^{2,\varepsilon}_{ts}(i_1, i_2)$ for some family of approximations B^ε of B. Assume $\mathbf{Z}^\varepsilon(i_1, i_2)$ is some a.s. $2\alpha^-$-Hölder random function (living on the same probability space as B^ε), antisymmetric in (i_1, i_2). Then

$$\mathcal{R}\mathbf{B}^{2,\varepsilon}_{ts}(i_1,i_2) := \mathbf{B}^{2,\varepsilon}_{ts}(i_1,i_2) + \delta\mathbf{Z}^\varepsilon_{ts}(i_1,i_2) = \mathbf{B}^{2,\varepsilon}_{ts}(i_1,i_2) + (\mathbf{Z}^\varepsilon_t(i_1,i_2) - \mathbf{Z}^\varepsilon_s(i_1,i_2)) \tag{0.5}$$

satisfies properties (i), (ii) and (iii). The multiplicativity property is preserved because δ acting on an increment vanishes [10],

$$\begin{aligned} \delta(\delta\mathbf{Z}^\varepsilon(i_1,i_2))_{tus} &= (\mathbf{Z}^\varepsilon_t(i_1,i_2) - \mathbf{Z}^\varepsilon_s(i_1,i_2)) - (\mathbf{Z}^\varepsilon_t(i_1,i_2) - \mathbf{Z}^\varepsilon_u(i_1,i_2)) \\ &- (\mathbf{Z}^\varepsilon_u(i_1,i_2) - \mathbf{Z}^\varepsilon_s(i_1,i_2)) = 0, \end{aligned} \tag{0.6}$$

and the geometric property is also preserved because $\mathbf{Z}^\varepsilon(i_1,i_2) + \mathbf{Z}^\varepsilon(i_2,i_1) = 0$.

The function $\mathbf{Z}^\varepsilon$ may be seen as a *counterterm*. Now (see section 1) $\mathbf{Z}^\varepsilon$ may be chosen so as to make the *regularized Lévy area* $\mathcal{R}\mathbf{B}^{2,\varepsilon}_{ts}(i_1,i_2)$ converge for every $\kappa < \alpha$ and $T > 0$ in $L^2(\Omega, \mathcal{C}^{2\kappa}([-T,T]))$ to a finite Lévy area for B, where $\mathcal{C}^{2\kappa}([-T,T])$ is the Banach space of 2κ-Hölder 1-increments on $[-T,T]$ in the sense of Gubinelli [10], equipped with the Hölder norm $||f||_{2\kappa,T} = \sup_{s,t\in[-T,T]} \frac{|f_{t,s}|}{|t-s|^{2\kappa}}$. More precisely, $\mathbf{Z}^\varepsilon$ may be chosen in the second chaos of B. One may prove bounds of the type

$$\mathbb{E}\mathcal{R}\mathbf{B}^{2,\varepsilon}_{ts}(i_1,i_2)^2 \leq C|t-s|^{4\alpha}, \quad \mathbb{E}\left(\mathcal{R}\mathbf{B}^{2,\varepsilon}_{ts}(i_1,i_2) - \mathcal{R}\mathbf{B}^{2,\eta}_{ts}(i_1,i_2)\right)^2 \leq C|\varepsilon-\eta|^{2\alpha} \tag{0.7}$$

from which (see [27], Proposition 1.5) the convergence in $L^2(\Omega, \mathcal{C}^{2\kappa}([-T,T]))$ may be proved by using standard arguments, in particular the Garsia-Rodemich-Rumsey lemma [8].

The first section concerns the construction of the counterterm $\mathbf{Z}^\varepsilon$ and the regularization of the Lévy area. The main tool is Fourier transform. Splitting iterated integrals into *increment/boundary terms* (see below) and reordering Fourier components by Fubini's theorem in such a way that *innermost integrals bear highest Fourier frequencies* ensures the proper Hölder regularity for every *boundary term*, hence the name of *Fourier normal ordering*. Then *increment terms* are regularized by introducing an adequate cut in the Fourier domain of integration. The multiplicative rule is preserved by doing this (see eq. (0.6)).

The sketchy generalization to higher-order iterated integrals in section 2 involves tree combinatorics linked with the *Hopf algebra* structure of the algebra of decorated rooted trees, as defined by A. Connes and D. Kreimer in a series of papers [2, 3, 4] devoted to a mathematical axiomatization of the BPHZ (Bogolioubov and coauthors) algorithm of renormalization of Feynman graphs in quantum field theory [12]. Our algorithm has nothing to do with BPHZ though, since it is based on a somewhat arbitrary but convenient regularization of *skeleton integrals* (higher-order generalizations of the increment term of the Lévy area), which are *tree iterated integrals* of a particular type. The proof of the multiplicative/Chen and geometric/shuffle property in [28] relies on Hopf algebra computations though. A BPHZ or dimensional renormalization scheme could be applied to skeleton integrals instead of a blunt regularization (work in progress). This algebraic approach has proved to be useful in a variety of contexts ranging from numerical analysis to quantum chromodynamics or the study of polylogarithms or multi-zeta functions, see for instance [14, 21, 29].

1 Definition of a Lévy area

Recall that fractional Brownian motion B may be defined via the harmonizable representation [24]

$$B_t = c_\alpha \int_{\mathbb{R}} |\xi|^{\frac{1}{2}-\alpha} \frac{e^{\mathrm{i}t\xi}-1}{\mathrm{i}\xi} \, W(d\xi) \tag{1.1}$$

where $(W_\xi, \xi \ge 0)$ is a complex Brownian motion extended to $\mathbb{R}$ by setting $W_{-\xi} = -\overline{W}_\xi$ $(\xi \ge 0)$, and $c_\alpha = \frac{1}{2}\sqrt{-\frac{\alpha}{\cos \pi\alpha \Gamma(-2\alpha)}}$.

We shall use the following approximation of B by a family of centered Gaussian processes $(B^\varepsilon, \varepsilon > 0)$ living in the first chaos of B.

Definition 1.1 (approximation B^ε) *Let, for $\varepsilon > 0$,*

$$B_t^\varepsilon = c_\alpha \int_{\mathbb{R}} e^{-\varepsilon|\xi|} |\xi|^{\frac{1}{2}-\alpha} \frac{e^{\mathrm{i}t\xi}-1}{\mathrm{i}\xi} \, W(d\xi). \tag{1.2}$$

The process B^ε is easily seen to have a.s. smooth paths. The infinitesimal covariance $\mathbb{E}(B^\varepsilon)'_s(B^\varepsilon)'_t$ may be computed explicitly using the Fourier transform [6]

$$\mathcal{F}K_\varepsilon^{',-}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} K_\varepsilon^{',-}(x) e^{-\mathrm{i}x\xi} dx = -\frac{\pi\alpha}{2\cos\pi\alpha\Gamma(-2\alpha)} e^{-2\varepsilon|\xi|} |\xi|^{1-2\alpha} \mathbf{1}_{|\xi|>0}, \tag{1.3}$$

where $K_\varepsilon^{',-}(s-t) := \frac{\alpha(1-2\alpha)}{2\cos\pi\alpha}(-\mathrm{i}(s-t)+2\eta)^{2\alpha-2}$. By taking the real part of these expressions, one finds that B^ε has the same law as the analytic approximation of B defined in [26], namely, $B^\varepsilon = \Gamma_{t+\mathrm{i}\varepsilon} + \Gamma_{t-\mathrm{i}\varepsilon} = 2\mathrm{Re}\ \Gamma_{t+\mathrm{i}\varepsilon}$, where Γ is the analytic fractional Brownian motion (see also [25]).

This second-order case is too simple to capture the combinatorial features of Fourier normal ordering. On the other hand, the proof of convergence for the regularized Lévy area (after substraction of the counterterm) is short, and the proof for tree integrals in the general case may be considered as a generalization.

Fourier normal ordering is the combination of (i) a number of equivalent splittings of the iterated integrals related by Fubini's theorem; (ii) a splitting of the Fourier domain of integration into a number of disjoint Fourier domains; (iii) an appropriate choice of splitting of the iterated integrals on each Fourier domain.

Let us first write down the two equivalent splittings in the case of the Lévy area. The components $B(1), B(2)$ are assumed to be constructed from i.i.d. Brownian motions $W(1)$, $W(2)$ via the above harmonizable representation. Note that $\mathbf{B}^{2,\varepsilon}_{ts}(j,j) = \frac{1}{2}(B^\varepsilon_t(j) - B^\varepsilon_s(j))^2$, $j = 1,2$ requires no regularization, hence the only problematic second-order iterated integrals are the mixed integrals $\mathbf{B}^{2,\varepsilon}(i,j)$, $i \neq j$, say, $i=1, j=2$.

Lemma 1.2 *Let* $\mathbf{B}^{2,\varepsilon}_{ts}(1,2) = \int_s^t dB^\varepsilon_{u_1}(1) \int_s^{u_1} dB^\varepsilon_{u_2}(2)$. *Then:*

$$\mathbf{B}^{2,\varepsilon}_{ts}(1,2) = c_\alpha^2 \int_{-\infty}^{+\infty} dW_{\xi_1}(1) \int_{-\infty}^{+\infty} dW_{\xi_2}(2) e^{-\varepsilon(|\xi_1|+|\xi_2|)} |\xi_1\xi_2|^{\frac{1}{2}-\alpha} I_{ts}(\xi_1,\xi_2) \tag{1.4}$$

where

$$I_{ts}(\xi_1,\xi_2) := \int_s^t e^{\mathrm{i}u_1\xi_1} du_1 \int_s^{u_1} e^{\mathrm{i}u_2\xi_2} du_2. \tag{1.5}$$

The proof is straightforward.

There are two apparently equivalent ways of splitting the integral $I_{ts}(\xi_1,\xi_2)$ into an **increment term**, $(\delta J)_{ts} = J_t - J_s$, and a **boundary term** denoted by the symbol ∂:

(i) either writing $I_{ts}(\xi_1,\xi_2)$ as $(G_t^+ - G_s^+)(\xi_1,\xi_2) + I_{ts}^+(\xi_1,\xi_2)(\partial)$, where (provided $\xi_1+\xi_2 \neq 0$)

$$G_u^+(\xi_1,\xi_2) = \frac{e^{\mathrm{i}u(\xi_1+\xi_2)}}{[\mathrm{i}(\xi_1+\xi_2)][\mathrm{i}\xi_2]}, \qquad I_{ts}^+(\xi_1,\xi_2)(\partial) = -\frac{e^{\mathrm{i}s\xi_2}}{\mathrm{i}\xi_2} \cdot \frac{e^{\mathrm{i}t\xi_1} - e^{\mathrm{i}s\xi_1}}{\mathrm{i}\xi_1}; \tag{1.6}$$

(ii) or (using Fubini's theorem)

$$\begin{aligned} I_{ts}(\xi_1,\xi_2) &= \int_s^t e^{\mathrm{i}u_2\xi_2} du_2 \int_{u_2}^t e^{\mathrm{i}u_1\xi_1} du_1 \\ &= (G_t^- - G_s^-)(\xi_2,\xi_1) + I_{ts}^-(\xi_2,\xi_1)(\partial), \end{aligned} \tag{1.7}$$

where

$$G_u^-(\xi_2,\xi_1) = -\frac{e^{\mathrm{i}u(\xi_1+\xi_2)}}{[\mathrm{i}(\xi_1+\xi_2)][\mathrm{i}\xi_1]}, \qquad I_{ts}^-(\xi_2,\xi_1)(\partial) = \frac{e^{\mathrm{i}t\xi_1}}{\mathrm{i}\xi_1} \cdot \frac{e^{\mathrm{i}t\xi_2} - e^{\mathrm{i}s\xi_2}}{\mathrm{i}\xi_2}. \tag{1.8}$$

In either case, the inner integral $\int_x^u e^{\mathrm{i}u'\xi} du'$, $(u,x,\xi) = (u_1,s,\xi_2)$ or (u_2,t,ξ_1) has been formally decomposed as $\int^u e^{\mathrm{i}u'\xi} du' - \int^x e^{\mathrm{i}u'\xi} du' = \frac{e^{\mathrm{i}u\xi}}{\mathrm{i}\xi} - \frac{e^{\mathrm{i}x\xi}}{\mathrm{i}\xi}$, which introduces an apparent infra-red divergence, since the single terms $\frac{e^{\mathrm{i}u\xi}}{\mathrm{i}\xi}$, $\frac{e^{\mathrm{i}x\xi}}{\mathrm{i}\xi}$ diverge when $\xi \to 0$. [1] Boundary terms come from the contribution of $\frac{e^{\mathrm{i}x\xi}}{\mathrm{i}\xi}$, $x = s$ or t, and are *not* increments.

The idea of Fourier normal ordering is that **innermost integrals should bear highest Fourier frequencies** in order to get correct Hölder estimates *separately* for the increment and the boundary term. Namely, using for instance the decomposition eq. (1.6) for arbitrary values of ξ_1,ξ_2 yields a boundary term $-B_s^\varepsilon(2)(B_t^\varepsilon(1) - B_s^\varepsilon(1))$ which is obviously only α^--Hölder, and not $2\alpha^-$-Hölder.

We shall say that a function of two arguments $(t,s) \to \mathbf{X}_{ts}^\varepsilon$ $(\varepsilon > 0)$ in the second chaos of B is *uniformly* $2\alpha^-$*-Hölder in* ε if

(i) $\mathbb{E}(\mathbf{X}_{ts}^\varepsilon)^2 \le C|t-s|^{4\alpha}$ for a constant C which is independent of ε, and furthermore

(ii) one has the following *rate of convergence* $\mathbb{E}\left(\mathbf{X}_{ts}^\varepsilon - \mathbf{X}_{ts}^\eta\right)^2 \le C|\varepsilon-\eta|^{2\alpha}$ when $\varepsilon,\eta \to 0$.

[1] Formally $\int^u e^{\mathrm{i}u\xi} du = \frac{e^{\mathrm{i}u\xi}}{\mathrm{i}\xi} = \int_{\pm\mathrm{i}\infty}^u e^{\mathrm{i}u\xi} du$ depending on the sign of ξ.

This implies (by the arguments given in the Introduction) that $\mathbf{X}^{\varepsilon}_{ts}$ converges in $L^2(\Omega;\mathcal{C}^{2\kappa}([-T,T]))$ for every $T>0$ and $\kappa<\alpha$.

Lemma 1.3 *(i) (**boundary term**) For every $\alpha\in(0,\frac{1}{2})$,*

$$\mathbf{B}^{2,\varepsilon,+}_{ts}(1,2)(\partial):=c_\alpha^2\int\int_{|\xi_1|\le|\xi_2|}dW_{\xi_1}(1)dW_{\xi_2}(2)e^{-\varepsilon(|\xi_1|+|\xi_2|)}|\xi_1\xi_2|^{\frac{1}{2}-\alpha}I^+_{ts}(\xi_1,\xi_2)(\partial) \tag{1.9}$$

and

$$\mathbf{B}^{2,\varepsilon,-}_{ts}(1,2)(\partial):=c_\alpha^2\int\int_{|\xi_2|\le|\xi_1|}dW_{\xi_1}(1)dW_{\xi_2}(2)e^{-\varepsilon(|\xi_1|+|\xi_2|)}|\xi_1\xi_2|^{\frac{1}{2}-\alpha}I^-_{ts}(\xi_2,\xi_1)(\partial) \tag{1.10}$$

are $2\alpha^-$-Hölder uniformly in ε.

*(ii) (**increment term**) For every $\alpha\in(1/4,1/2)$, the functions*

$$\delta G^{2,\varepsilon,+}_{ts}(1,2):=c_\alpha^2\int\int_{|\xi_1|\le|\xi_2|}e^{-\varepsilon(|\xi_1|+|\xi_2|)}|\xi_1\xi_2|^{\frac{1}{2}-\alpha}\left(G^+_t(\xi_1,\xi_2)-G^+_s(\xi_1,\xi_2)\right) \tag{1.11}$$

and

$$\delta G^{2,\varepsilon,-}_{ts}(1,2):=c_\alpha^2\int\int_{|\xi_2|\le|\xi_1|}e^{-\varepsilon(|\xi_1|+|\xi_2|)}|\xi_1\xi_2|^{\frac{1}{2}-\alpha}\left(G^-_t(\xi_2,\xi_1)-G^-_s(\xi_2,\xi_1)\right) \tag{1.12}$$

satisfy the first estimates (i) $\mathbb{E}|\delta G^{2,\varepsilon,\pm}_{ts}(1,2)|^2\le C|t-s|^{4\alpha}$.

Remarks.

1. Only the *increment* $\delta G^{2,\varepsilon,\pm}(1,2)$ makes sense: $G^{2,\varepsilon,\pm}(1,2)$ defined as the integral of $G^{\pm}_t(\xi_1,\xi_2)$ is infra-red divergent (see also remark after Lemma 1.4 below).

2. Even if $\alpha>1/4$, only the *sum of the two increment terms* $\delta G^{2,\varepsilon,+}_{ts}(1,2)+\delta G^{2,\varepsilon,-}_{ts}(1,2)$ satisfies the above rate of convergence estimate (ii). This is due to the spurious singularity on the diagonal $\xi_1=-\xi_2$, not to an ultra-violet divergence when $|\xi_1|,|\xi_2|\to\infty$. We skip the proof which is not needed.

Proof. Let us first remark that the symmetry $(W(1)\leftrightarrow W(2), s\leftrightarrow t)$ exchanges $G^+_t(\xi_1,\xi_2)-G^+_s(\xi_1,\xi_2)$ with $G^-_t(\xi_2,\xi_1)-G^-_s(\xi_2,\xi_1)$, and $I^+_{ts}(\xi_1,\xi_2)(\partial)$ with $I^-_{ts}(\xi_2,\xi_1)(\partial)$. Hence it is enough to prove Hölderianity for $\mathbf{B}^{2,\varepsilon,+}(1,2)(\partial)$ and $\delta G^{2,\varepsilon,+}(1,2)$.

We shall use a number of times the following elementary lemma, inspired by arguments of J.-P. Kahane concerning the regularity of random Fourier series [13]:

Lemma 1.4 *(i) Let $F(u)=\int_{\mathbb{R}} dW_\xi a(\xi)e^{\mathrm{i}u\xi}$, where $|a(\xi)|^2\le C|\xi|^{-1-2\beta}$ for some $0<\beta<1$: then, for every $u_1,u_2\in\mathbb{R}$,*

$$\mathbb{E}|F(u_1)-F(u_2)|^2\le C'|u_1-u_2|^{2\beta}. \tag{1.13}$$

(ii) Let $\tilde F(\varepsilon)=\int_{\mathbb{R}} dW_\xi a(\xi)e^{-\varepsilon|\xi|}$ $(\varepsilon>0)$, where $|a(\xi)|^2\le C|\xi|^{-1-2\beta}$ for some $0<\beta<1$: then, for every $\varepsilon_1,\varepsilon_2\in\mathbb{R}_+$,

$$\mathbb{E}|\tilde F(\varepsilon_1)-\tilde F(\varepsilon_2)|^2\le C'|\varepsilon_1-\varepsilon_2|^{2\beta}. \tag{1.14}$$

Proof. Bound $|e^{\mathrm{i}u_1\xi}-e^{\mathrm{i}u_2\xi}|$ by $|u_1-u_2||\xi|$ for $|\xi|\le\frac{1}{|u_1-u_2|}$ and by 2 otherwise, and similarly for $|e^{-\varepsilon_1|\xi|}-e^{-\varepsilon_2|\xi|}|$. Note the variance integral is infra-red convergent near $\xi=0$. □

Remark: Unless $|a(\xi)|^2$ is L^1_{loc} near $\xi=0$, only the *increments* $F(u_1)-F(u_2)$, $\tilde F(\varepsilon_1)-\tilde F(\varepsilon_2)$ are well-defined.

(i) Apply Lemma 1.4 (i) to $F_s(u)=\int_{\mathbb{R}} dW_{\xi_1}(1)a(\xi_1)e^{\mathrm{i}u\xi_1}$ with

$$a(\xi_1)=e^{-\varepsilon|\xi_1|}|\xi_1|^{-\frac12-\alpha}\int_{|\xi_2|\ge|\xi_1|} dW_{\xi_2}(2)e^{-\varepsilon|\xi_2|}e^{\mathrm{i}s\xi_2}|\xi_2|^{-\frac12-\alpha}; \tag{1.15}$$

since Var $a(\xi_1)\le C|\xi_1|^{-1-4\alpha}$, one gets the uniform Hölderianity estimates (i) $\mathbb{E}(\mathbf{B}^{2,\varepsilon,+}_{ts}(1,2)(\partial))^2\le C|t-s|^{4\alpha}$ for $\mathbf{B}^{2,\varepsilon,+}_{ts}(1,2)(\partial)$. As for the rate of convergence (ii) (see above Lemma 1.3), one rewrites $e^{-\varepsilon(|\xi_1|+|\xi_2|)}-e^{-\eta(|\xi_1|+|\xi_2|)}$ as $(e^{-\varepsilon|\xi_1|}-e^{-\eta|\xi_1|})e^{-\varepsilon|\xi_2|}+e^{-\eta|\xi_1|}(e^{-\varepsilon|\xi_2|}-e^{-\eta|\xi_2|})$. The first term may be bounded as F_s by applying Lemma 1.4 (ii). For the second term, Lemma 1.4 (ii) should be applied to $\tilde F_s(\varepsilon)=\int_{|\xi_2|\ge|\xi_1|} dW_{\xi_2}(2)e^{-\varepsilon|\xi_2|}e^{\mathrm{i}s\xi_2}|\xi_2|^{-\frac12-\alpha}$, which yields an exponent 2α instead of 4α.

(ii) Apply Lemma 1.4 (i) to $F(u)=\int_{\mathbb{R}} dW_\xi a(\xi)e^{\mathrm{i}u\xi}$ with (setting $\xi=\xi_1+\xi_2$)

$$a(\xi)=\frac{1}{\mathrm{i}\xi}\int_{|\xi_2|\ge|\xi-\xi_2|} dW_{\xi_2}|\xi-\xi_2|^{\frac12-\alpha}\frac{|\xi_2|^{\frac12-\alpha}}{\mathrm{i}\xi_2}e^{-\varepsilon(|\xi_2|+|\xi-\xi_2|)}. \tag{1.16}$$

Setting ε directly to 0 yields (assuming for instance $\xi>0$)

$$\text{Var } a(\xi)\le\frac{1}{\xi^2}\int_{\xi/2}^{+\infty}|\xi_2|^{-1-2\alpha}|\xi-\xi_2|^{1-2\alpha}d\xi_2, \tag{1.17}$$

which converges if and only if $\alpha > 1/4$, in which case

$$\text{Var } a(\xi) \le |\xi|^{-1-4\alpha} \int_{\frac{1}{2}}^{+\infty} |u|^{-1-2\alpha} |1-u|^{1-2\alpha} du = C|\xi|^{-1-4\alpha}. \quad (1.18)$$

□

All together one has proved (up to the proof for the rate of convergence for $\delta G^{2,\varepsilon,\pm}$) that

$$\mathbf{B}^{2,\varepsilon}_{ts} = (\mathbf{B}^{2,\varepsilon,+}_{ts}(1,2)(\partial) + \delta G^{2,\varepsilon,+}_{ts}(1,2)) + (\mathbf{B}^{2,\varepsilon,-}_{ts}(1,2)(\partial) + \delta G^{2,\varepsilon,-}_{ts}(1,2)) \quad (1.19)$$

is $2\alpha^-$-Hölder uniformly in ε, *provided* $\alpha > 1/4$.

So what should one do when $\alpha \le 1/4$?

Definition 1.5 (cut Fourier domain) *Let, for some constant* $C_{reg} \in (0,1)$*,*

$$\mathbb{R}^2_{reg} := \{(\xi_1, \xi_2) \mid |\xi_1| \le |\xi_2|, |\xi_1 + \xi_2| > C_{reg}|\xi_2|\}. \quad (1.20)$$

The condition $|\xi_1 + \xi_2| > C_{reg}|\xi_2|$ excludes a conical region along the singular line $\xi_1 = -\xi_2$.

Definition 1.6 (regularized Lévy area $\mathcal{R}\mathbf{B}^{2,\varepsilon}_{ts}$) *Let*

$$\mathcal{R}\mathbf{B}^{2,\varepsilon}_{ts} := (\mathbf{B}^{2,\varepsilon,+}_{ts}(1,2)(\partial) + \delta\mathcal{R}G^{2,\varepsilon,+}_{ts}(1,2)) + (\mathbf{B}^{2,\varepsilon,-}_{ts}(1,2)(\partial) + \delta\mathcal{R}G^{2,\varepsilon,-}_{ts}(1,2)) \quad (1.21)$$

where the following **regularized increment term** *has been introduced,*

$$\mathcal{R}G^{2,\varepsilon,+}_t(1,2) := c^2_\alpha \int\int_{(\xi_1,\xi_2)\in\mathbb{R}^2_{reg}} e^{-\varepsilon(|\xi_1|+|\xi_2|)} |\xi_1\xi_2|^{\frac{1}{2}-\alpha} G^+_t(\xi_1,\xi_2) \quad (1.22)$$

and

$$\mathcal{R}G^{2,\varepsilon,-}_t(1,2) := c^2_\alpha \int\int_{(\xi_2,\xi_1)\in\mathbb{R}^2_{reg}} e^{-\varepsilon(|\xi_1|+|\xi_2|)} |\xi_1\xi_2|^{\frac{1}{2}-\alpha} G^-_t(\xi_2,\xi_1). \quad (1.23)$$

The regularized Lévy area satisfies the multiplicative property (ii) of the Introduction because the corresponding counterterm

$$\delta\mathbf{Z}^{2,\varepsilon}(1,2) := \big(\delta\mathcal{R}G^{2,\varepsilon,+}(1,2) - \delta G^{2,\varepsilon,+}(1,2)\big) + \big(\delta\mathcal{R}G^{2,\varepsilon,-}(1,2) - \delta G^{2,\varepsilon,-}(1,2)\big) \quad (1.24)$$

(given by an integral on the conical Fourier domain $\mathbb{R}^2 \setminus \mathbb{R}^2_{reg}$ along the diagonal $\xi_1 = -\xi_2$) is an increment (see Introduction). It satisfies the geometric property (iii) because $\delta\mathbf{Z}^{2,\varepsilon}(1,2)$ is antisymmetric in $1 \leftrightarrow 2$, which follows in turn from the symmetry exhibited at the beginning of the proof of Lemma 1.3.

Theorem 1.1 *For every $\alpha \in (0,1/2)$, $\mathcal{R}\mathbf{B}^{2,\varepsilon}_{ts}(1,2)$ is $2\alpha^-$-Hölder uniformly in ε.*

Proof. Let us first prove the Hölder estimates (i) $\mathbb{E}|\mathcal{R}\mathbf{B}^{2,\varepsilon}_{ts}(1,2)|^2 \le C|t-s|^{4\alpha}$. Similarly to the proof of point (ii) in Lemma 1.3, we apply Lemma 1.4 (i) to $F_{reg}(u) = \int_{\mathbb{R}} dW_\xi a_{reg}(\xi) e^{iu\xi}$ with (setting $\xi = \xi_1 + \xi_2$)

$$a_{reg}(\xi) = \frac{1}{\mathrm{i}\xi} \int_D dW_{\xi_2} |\xi - \xi_2|^{\frac{1}{2}-\alpha} \frac{|\xi_2|^{\frac{1}{2}-\alpha}}{\mathrm{i}\xi_2} e^{-\varepsilon(|\xi_2|+|\xi-\xi_2|)}, \tag{1.25}$$

where $D = \{\xi_2 \in \mathbb{R} \mid |\xi_2| \ge |\xi - \xi_2|, |\xi| > C_{reg}|\xi_2|\}$. In particular, $|\xi - \xi_2| < \frac{1}{C_{reg}}|\xi|$, which implies $|\xi_2| \le C|\xi|$, so (assuming for instance $\xi > 0$)

$$\begin{aligned} \mathrm{Var}\ a_{reg}(\xi) &\le \frac{1}{\xi^2} \int_{\xi/2}^{C\xi} |\xi_2|^{-1-2\alpha} |\xi - \xi_2|^{1-2\alpha} d\xi_2 \\ &= |\xi|^{-1-4\alpha} \int_{1/2}^{C} |u|^{1-2\alpha} |1-u|^{-1-2\alpha}\, du = C'|\xi|^{-1-4\alpha} < \infty. \end{aligned} \tag{1.26}$$

For the rate of convergence, we rewrite $e^{-\varepsilon(|\xi_1|+|\xi_2|)} - e^{-\eta(|\xi_1|+|\xi_2|)}$ as the sum of two terms as in the proof of Lemma 1.3 (i). The *second* term may be studied using Lemma 1.4 (ii) by considering the coefficient of $dW_{\xi_2}(2)$; since $\frac{|\xi|}{2} \le |\xi_2| < \frac{|\xi|}{C_{reg}}$, one gets the same estimates as for the proof of the Hölder estimates (i), and hence a rate of convergence with exponent 4α. The *first* term involves the function $\tilde{F}(\varepsilon) = \int dW_{\xi_1} \tilde{a}_\eta(\xi_1) e^{iu\xi_1} e^{-\varepsilon|\xi_1|}$, with

$$\tilde{a}_\eta(\xi_1) = |\xi_1|^{\frac{1}{2}-\alpha} \int_{|\xi_2| \ge |\xi_1|} dW_{\xi_2} \frac{|\xi_2|^{\frac{1}{2}-\alpha}}{[\mathrm{i}(\xi_1+\xi_2)][\mathrm{i}\xi_2]} e^{-\eta|\xi_2|}. \tag{1.27}$$

The estimates $\mathbb{E}|\tilde{a}_\eta(\xi_1)|^2 \le C|\xi|^{-1-4\alpha}$ is easy to get using the fact that $|\xi_1 + \xi_2| > C_{reg}|\xi_2| \ge C_{reg}|\xi_1|$, which gives once again the same rate of convergence.

$\square$

Remark. Since the boundary terms $\mathbf{B}^{2,\varepsilon,\pm}_{ts}(1,2)(\partial)$ are uniformly $2\alpha^-$-Hölder in ε, one could simply have set $\mathcal{R}G^{2,\varepsilon,\pm}_{ts}(1,2)(\partial) \equiv 0$ (which looks somewhat drastic). Our point of view is to try and regularize as little as possible.

2 Iterated integrals of higher order: a sketchy overview

Combinatorics of Fourier normal ordering become non trivial starting from iterated integrals of third order; we shall concentrate on this case in this section, although some notions will be presented for the case of general iterated integrals. Detailed proofs should be found in [27].

Recall the two decompositions of $I_{ts}(\xi_1,\xi_2)$ into $\delta G^+_{ts}(\xi_1,\xi_2)+I^+_{ts}(\xi_1,\xi_2)(\partial)$ and $\delta G^-_{ts}(\xi_2,\xi_1)+I^-_{ts}(\xi_2,\xi_1)(\partial)$. As already noted, the symmetry $\xi_1\leftrightarrow\xi_2$, $s\leftrightarrow t$ maps G^+ to G^- and $I^+(\partial)$ to $I^-(\partial)$, hence terms with indices $\pm$ may be treated on an equal footing. This is no more the case for the integrals involved in $\mathbf{B}^{3,\varepsilon}_{ts}$. Namely, Fubini's theorem implies for instance (considering three among the six permutations of $\{1,2,3\}$, including the trivial one)

$$\int_s^t dB^\varepsilon_{u_1}(i_1)\int_s^{u_1} dB^\varepsilon_{u_2}(i_2)\int_s^{u_2} dB^\varepsilon_{u_3}(i_3) \tag{2.1}$$

$$= \int_s^t dB^\varepsilon_{u_2}(i_2)\int_{u_2}^{t} dB^\varepsilon_{u_1}(i_1)\int_s^{u_2} dB^\varepsilon_{u_3}(i_3) \tag{2.2}$$

$$= \int_s^t dB^\varepsilon_{u_2}(i_2)\int_s^{u_2} dB^\varepsilon_{u_3}(i_3)\int_{u_2}^{t} dB^\varepsilon_{u_1}(i_1) \tag{2.3}$$

Each of these may be represented as a finite sum of *tree iterated integrals* as we shall presently see. If $\mathbb{T}$ is a decorated rooted tree, i.e. a tree with a distinguished vertex v_1 called *root*, such that each vertex $v\in V(\mathbb{T})=\{\text{vertices of }\mathbb{T}\}$ wears a label $\ell(v)\in\{1,\ldots,d\}$, and $\Gamma=(\Gamma(1),\ldots,\Gamma(d))$ is a smooth d-dimensional path, then the integral of Γ along $\mathbb{T}$ is (denoting by v^-, $v\in V(\mathbb{T})\setminus\{v_0\}$ the unique ancestor of v, i.e. the unique vertex just below v)

$$[I_{\mathbb{T}}(\Gamma)]_{ts} := \int_s^t d\Gamma_{x_{v_1}}(\ell(v_1))\int_s^{x_{v_2^-}} d\Gamma_{x_{v_2}}(\ell(v_2))\ldots\int_s^{x_{v^-_{|V(\mathbb{T})|}}} d\Gamma_{x_{v_{|V(\mathbb{T})|}}}(\ell(v_{|V(\mathbb{T})|})) \tag{2.4}$$

where $(v_1,\ldots,v_{|V(\mathbb{T})|})$ is any ordering of $V(\mathbb{T})$ compatible with the tree partial ordering, i.e. such that $v^-<v$ for every $v\in V(\mathbb{T})\setminus\{v_1\}$ (in other words, such that the indices of the vertices decrease while going down the branches towards the root). The definition extends easily to *forests*, i.e. to (finite) disjoint unions of trees (which are seen as a commutative product of the trees), by multiplying the tree iterated integrals corresponding to each connected component of the forest, and then (by taking linear combinations) to the **algebra over $\mathbb{R}$ generated by decorated rooted trees**, $\mathcal{T}$, which is actually a **Hopf algebra** [2, 3].

Now replace in eq. (2.1,2.2,2.3) $\int_u^t$ by $\int_s^t - \int_s^u$ and $\int_u^{u'}$ by $\int_s^{u'} - \int_s^u$. Then eq. (2.1,2.2,2.3) may be represented resp. as $I_{\mathbb{T}_1}(B^\varepsilon)$, $I_{\mathbb{T}_{2,1}}(B^\varepsilon) - I_{\mathbb{T}_{2,2}}(B^\varepsilon)$, $I_{\mathbb{T}_{3,1}}(B^\varepsilon) - I_{\mathbb{T}_{3,2}}(B^\varepsilon)$ (see Fig. 1).

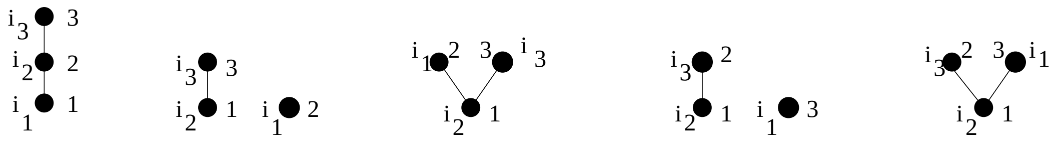

Figure 1: Example of iterated integrals. From left to right: $\mathbb{T}_1, \mathbb{T}_{2,1}, \mathbb{T}_{2,2}, \mathbb{T}_{3,1}, \mathbb{T}_{3,2}$.

More generally, to each permutation σ of $\{1,2,3\}$ corresponds a rewriting of $\mathbf{B}_{ts}^{3,\varepsilon}(i_1,i_2,i_3)$ as some finite sum,

$$\mathbf{B}_{ts}^{3,\varepsilon}(i_1,i_2,i_3) = \sum_j g(\sigma,j) I_{\mathbb{T}_j^\sigma}(B^\varepsilon), \tag{2.5}$$

where $g(\sigma,j) = \pm 1$ is a sign. Note that each $I_{\mathbb{T}_j^\sigma}(B^\varepsilon)$ equals more explicitly (after permutation of the names of the ξ-variables, $\xi_i \to \xi_{\sigma(i)}$)

$$[I_{\mathbb{T}_j^\sigma}(B^\varepsilon)]_{ts} = c_\alpha^3 \int_{\mathbb{R}} dW_{\xi_1}(i_{\sigma(1)}) \int_{\mathbb{R}} dW_{\xi_2}(i_{\sigma(2)}) \int_{\mathbb{R}} dW_{\xi_3}(i_{\sigma(3)}) |\xi_1\xi_2\xi_3|^{\frac{1}{2}-\alpha} \,. \\ e^{-\varepsilon(|\xi_1|+|\xi_2|+|\xi_3|)} \int_s^{x_{v_1^-}} e^{\mathrm{i}x_{v_1}\xi_1} dx_{v_1} \int_s^{x_{v_2^-}} e^{\mathrm{i}x_{v_2}\xi_2} dx_{v_2} \int_s^{x_{v_3^-}} e^{\mathrm{i}x_{v_3}\xi_3} dx_{v_3}, \tag{2.6}$$

where (v_1,v_2,v_3) is the natural ordering of $V(\mathbb{T}_j^\sigma)$ (compatible with the tree ordering) given by the position of the corresponding variable of integration inside the iterated integral *after* applying Fubini's theorem, see eq. (2.1,2.2,2.3) or Figure 1, and $x_{v_i^-} = t$ if v_i is a root. Note also that the labels on the trees are simply $\ell(v_j) = i_{\sigma(j)}$.

The idea of Fourier normal ordering consists in rewriting first $\mathbf{B}_{ts}^{3,\varepsilon}(i_1,i_2,i_3)$ as (letting Σ_3 be the group of permutations of $\{1,2,3\}$)

$$c_\alpha^3 \sum_{\sigma\in\Sigma_3} \sum_j g(\sigma,j) \int\int\int_{|\xi_1|\le|\xi_2|\le|\xi_3|} dW_{\xi_{v_1}}(i_{\sigma(1)}) dW_{\xi_{v_2}}(i_{\sigma(2)}) dW_{\xi_{v_3}}(i_{\sigma(3)}) \,. \\ .\, e^{-\varepsilon(|\xi_1|+|\xi_2|+|\xi_3|)} \int_s^{x_{v_1^-}} e^{\mathrm{i}x_{v_1}\xi_1} dx_{v_1} \int_s^{x_{v_2^-}} e^{\mathrm{i}x_{v_2}\xi_2} dx_{v_2} \int_s^{v_3^-} e^{\mathrm{i}x_{v_3}\xi_3} dx_{v_3}, \tag{2.7}$$

where $\{v_1, v_2, v_3\}$ are the vertices of $\mathbb{T}_j^{\sigma}$, so that innermost integrals bear highest Fourier indices.

The next task is to get rid of divergences by adding some counterterms (or in other words, by discarding the contribution to the integral of adequate Fourier subdomains). If one wants the multiplicative property to remain true, this must be done in compatibility with lower-order counterterms (i.e. with the definition of the regularized Lévy area at this stage). Hence one should have:

$$\mathbf{B}^{3,\varepsilon}_{ts}(i_1,i_2,i_3)-\mathbf{B}^{3,\varepsilon}_{tu}(i_1,i_2,i_3)-\mathbf{B}^{3,\varepsilon}_{us}(i_1,i_2,i_3) = \mathbf{B}^{1,\varepsilon}_{tu}(i_1)\mathbf{B}^{2,\varepsilon}_{us}(i_2,i_3)+\mathbf{B}^{2,\varepsilon}_{tu}(i_1,i_2)\mathbf{B}^{1,\varepsilon}_{us}(i_3). \tag{2.8}$$

This identity has been generalized to tree integrals (see [11]). Namely, letting $\mathbb{T}$ be a tree and $\boldsymbol{v}$ range over all *admissible cuts* of $\mathbb{T}$ (we shall use the notation: $\boldsymbol{v} \models V(\mathbb{T})$), i.e. over all non-empty subsets $\{v_1,\ldots,v_J\} \in V(\mathbb{T}) \setminus \{0\}$ (0=root of $\mathbb{T}$) such that no pair $\{v_i, v_j\} \subset \boldsymbol{v}$ is connected by going down or up the tree, then

$$[\delta I_{\mathbb{T}}(B^{\varepsilon})]_{tus} = \sum_{\boldsymbol{v}\models V(\mathbb{T})} [I_{L_{\boldsymbol{v}}\mathbb{T}}(B^{\varepsilon})]_{tu}[I_{R_{\boldsymbol{v}}\mathbb{T}}(B^{\varepsilon})]_{us} \tag{2.9}$$

where $R_{\boldsymbol{v}}\mathbb{T} \subset \mathbb{T}$ is the forest obtained as the union of branches lying above $v_1,\ldots,v_J$ (including the vertices $v_1,\ldots,v_J$), and $L_{\boldsymbol{v}}\mathbb{T} \subset \mathbb{T}$ is the subtree obtained after removing these branches (see example on Fig. 2). This identity called **tree multiplicative property** may be rephrased [28] by saying that $[\delta I(B^{\varepsilon})]_{tus}$ (viewed as a linear form on the Hopf algebra $\mathcal{T}$) is the *convolution* of $[I(B^{\varepsilon})]_{tu}$ and $[I(B^{\varepsilon})]_{us}$.

Figure 2: Admissible cuts of $\mathbb{T}_{2,2}$: $\boldsymbol{v} = \{3\}, \{2\}$ or $\{2,3\}$ is also in this particular case the set of vertices of $R_{\boldsymbol{v}}\mathbb{T}$.

Now (see footnote after Lemma 1.3), one wants to set formally $s = \pm \mathrm{i}\infty$ to generalize the increment/boundary decomposition of the previous section. This results in the following definition of *skeleton integrals*:

Definition 2.1 (skeleton integrals) *Let*

$$[\mathrm{Sk}I_{\mathbb{T}}(B^{\varepsilon})]_t = c_\alpha^3 \int_{\mathbb{R}} dW_{\xi_1}(i_{\sigma(1)}) \int_{\mathbb{R}} dW_{\xi_2}(i_{\sigma(2)}) \int_{\mathbb{R}} dW_{\xi_3}(i_{\sigma(3)}) |\xi_1\xi_2\xi_3|^{\frac{1}{2}-\alpha} \, . \\ e^{-\varepsilon(|\xi_1|+|\xi_2|+|\xi_3|)} \int^{x_{v_1^-}} e^{\mathrm{i}x_{v_1}\xi_1} dx_{v_1} \int^{x_{v_2^-}} e^{\mathrm{i}x_{v_2}\xi_2} dx_{v_2} \int^{x_{v_3^-}} e^{\mathrm{i}x_{v_3}\xi_3} dx_{v_3} \tag{2.10}$$

(see eq. (2.6)).

Formally $[\mathrm{Sk}I_{\mathbb{T}}(B^{\varepsilon})]_t = [I_{\mathbb{T}}(B^{\varepsilon})]_{t,\pm\mathrm{i}\infty}$. As was the case for $G^{2,\varepsilon,\pm}(1,2)$, only the *increment* $[\delta \mathrm{Sk}I_{\mathbb{T}}(B^{\varepsilon})]_{ts} = [\mathrm{Sk}I_{\mathbb{T}}(B^{\varepsilon})]_t - [\mathrm{Sk}I_{\mathbb{T}}(B^{\varepsilon})]_s$ makes sense. The tree multiplicative property yields then the **tree skeleton decomposition**

$$[I_{\mathbb{T}}(B^{\varepsilon})]_{tu} = [\mathrm{Sk}I_{\mathbb{T}}(B^{\varepsilon})]_t - [\mathrm{Sk}I_{\mathbb{T}}(B^{\varepsilon})]_u - \sum_{\boldsymbol{v} \models V(\mathbb{T})} [I_{L_{\boldsymbol{v}}\mathbb{T}}(B^{\varepsilon})]_{tu} [\mathrm{Sk}I_{R_{\boldsymbol{v}}\mathbb{T}}(B^{\varepsilon})]_u . \tag{2.11}$$

This is an inductive formula, which yields $I_{\mathbb{T}}(B^{\varepsilon})$ in terms of integrals or skeleton integrals of lower order. Once again, it may be interpreted as a convolution of characters of the Hopf algebra $\mathcal{T}$, as the convolution of $[\mathrm{Sk}I(B^{\varepsilon})]_t$ with the inverse of $[\mathrm{Sk}I(B^{\varepsilon})]_u$ (defined via the *antipode*) to be precise, see [28]. If e.g. $\mathbb{T}$ has 3 vertices, then each tree component of $R_{\boldsymbol{v}}\mathbb{T}$ has at most 2 vertices. A skeleton integral for a tree reduced to one vertex is simply

$$\int_{\mathbb{R}} dW_\xi e^{-\varepsilon|\xi|} |\xi|^{\frac{1}{2}-\alpha} \int^u e^{\mathrm{i}u\xi} du = \int_{\mathbb{R}} dW_\xi e^{-\varepsilon|\xi|} e^{\mathrm{i}u\xi} \frac{|\xi|^{\frac{1}{2}-\alpha}}{\mathrm{i}\xi} \tag{2.12}$$

(whose increments are those of B^{ε} itself) and requires no regularization, whereas for a tree with two vertices, one gets

$$\int_{\mathbb{R}} dW_{\xi_1}(1) \int_{\mathbb{R}} dW_{\xi_2}(2) e^{-\varepsilon(|\xi_1|+|\xi_2|)} |\xi_1\xi_2|^{\frac{1}{2}-\alpha} \int^u e^{\mathrm{i}x_{v_1}\xi_1} dx_{v_1} \int^{x_{v_1}} e^{\mathrm{i}x_{v_2}\xi_2} dx_{v_2} \\ = \int_{\mathbb{R}} dW_{\xi_1}(1) \int_{\mathbb{R}} dW_{\xi_2}(2) e^{-\varepsilon(|\xi_1|+|\xi_2|)} |\xi_1\xi_2|^{\frac{1}{2}-\alpha} G_u^{+}(\xi_1,\xi_2) \tag{2.13}$$

(which is divergent in the limit $\varepsilon \to 0$ when $\alpha < 1/4$, as proved in Lemma 1.3).

The reader may easily check that formula (2.11) gives precisely the increment/boundary decomposition of the previous section when $\mathbb{T}$ has two vertices.

In order to get convergent quantities in compatibility with the regularization of second-order integrals, one should (i) regularize the new skeleton integrals of third order $[\mathrm{Sk}I_{\mathbb{T}}(B^\varepsilon)]_t$; (ii) replace $[I_{L_v\mathbb{T}}(B^\varepsilon)]_{tu}$ and $[\mathrm{Sk}I_{R_v\mathbb{T}}(B^\varepsilon)]_u$ in the right-hand side of (2.11) by the corresponding regularized quantity of order (1 or) 2.

In other words, **regularization must be performed on each skeleton integral of order** ≥ 2.

Definition 2.2 (regularized skeleton integrals of order 3) *Let, for* $\mathbb{T} = \mathbb{T}_1, \mathbb{T}_{2,1}, \mathbb{T}_{2,2}, \mathbb{T}_{3,1}$ *or* $\mathbb{T}_{3,2}$ *corresponding to one of the three above permutations* $\sigma \subset \Sigma_3$*,*

$$[\mathcal{R}\mathrm{Sk}I_{\mathbb{T}}(B^\varepsilon)]_u = c_\alpha^3 \int\int\int_{(\xi_1,\xi_2,\xi_3)\in\mathbb{R}^{\mathbb{T}}_{reg}} dW_{\xi_1}(i_{\sigma(1)})dW_{\xi_2}(i_{\sigma(2)})dW_{\xi_3}(i_{\sigma(3)})$$
$$e^{-\varepsilon(|\xi_1|+|\xi_2|+|\xi_3|)}|\xi_1\xi_2\xi_3|^{\frac{1}{2}-\alpha}\int^{x_{v_1^-}} e^{\mathrm{i}x_{v_1}\xi_1}dx_{v_1}\int^{x_{v_2^-}} e^{\mathrm{i}x_{v_2}\xi_2}dx_{v_2}\int^{x_{v_3^-}} e^{\mathrm{i}x_{v_3}\xi_3}dx_{v_3}, \tag{2.14}$$

where $\mathbb{R}^{\mathbb{T}}_{reg} \subset \mathbb{R}^3$ *is the subdomain of integration defined by:*

(i) $\mathbb{R}^{\mathbb{T}_1}_{reg} = \{|\xi_1| \leq |\xi_2| \leq |\xi_3|;\quad |\xi_2+\xi_3|, |\xi_1+\xi_2+\xi_3| > C'_{reg}|\xi_3|\}$;

(ii) $\mathbb{R}^{\mathbb{T}_{2,1}}_{reg} = \{|\xi_1| \leq |\xi_2| \leq |\xi_3|;\quad |\xi_1+\xi_3| > C'_{reg}|\xi_3|\}$;

(iii) $\mathbb{R}^{\mathbb{T}_{2,2}}_{reg} = \{|\xi_1| \leq |\xi_2| \leq |\xi_3|;\quad |\xi_1+\xi_2+\xi_3| > C'_{reg}|\xi_3|\}$;

(iv) $\mathbb{R}^{\mathbb{T}_{3,1}}_{reg} = \{|\xi_1| \leq |\xi_2| \leq |\xi_3|;\quad |\xi_1+\xi_2| > C'_{reg}|\xi_2|\}$;

(v) $\mathbb{R}^{\mathbb{T}_{3,2}}_{reg} = \{|\xi_1| \leq |\xi_2| \leq |\xi_3|;\quad |\xi_1+\xi_2+\xi_3| > C'_{reg}|\xi_3|\}$

for some constant $C'_{reg} \in (0,1)$.

For the general definition of the cut Fourier domains of integrations, we refer the reader to [27]. Uniform Hölderianity with respect to ε has been defined before Lemma 1.3.

Theorem 2.1 *Assume* $\alpha < 1/3$. *Then the above regularized skeleton integrals are* $3\alpha^-$*-Hölder uniformly in* ε.

Proof (sketch). Let us just sketch the proof e.g. for $[\mathcal{R}\mathrm{Sk}I_{\mathbb{T}_1}(B^\varepsilon)]_u$, assuming $i_1 \neq i_2 \neq i_3$, say, $i_1 = 1$, $i_2 = 2$, $i_3 = 3$. The triple integral

$\int^{x_{v_1^-}} e^{\mathrm{i}x_{v_1}\xi_1} dx_{v_1} \int^{x_{v_2^-}} e^{\mathrm{i}x_{v_2}\xi_2} dx_{v_2} \int^{x_{v_3^-}} e^{\mathrm{i}x_{v_3}\xi_3} dx_{v_3}$ writes (by straightforward computation) $\frac{e^{\mathrm{i}t(\xi_1+\xi_2+\xi_3)}}{[\mathrm{i}\xi_3][\mathrm{i}(\xi_2+\xi_3)][\mathrm{i}(\xi_1+\xi_2+\xi_3)]}$. The idea is that the restriction of the domain of integration ensures that denominators are not *too* small. Apply Lemma 1.4 (i) to $F(u) = \int_{\mathbb{R}} dW_\xi a(\xi) e^{\mathrm{i}u\xi}$ with (setting $\xi = \xi_1 + \xi_2 + \xi_3$)

$$a(\xi) = \frac{1}{\mathrm{i}\xi} \int_{D_\xi} dW_{\xi_2}(2) dW_{\xi_3}(3) |\xi - \xi_2 - \xi_3|^{\frac{1}{2}-\alpha} \frac{|\xi_2|^{\frac{1}{2}-\alpha}}{\mathrm{i}(\xi_2+\xi_3)} \frac{|\xi_3|^{\frac{1}{2}-\alpha}}{\mathrm{i}\xi_3} \qquad (2.15)$$

on a domain D_ξ on which $\frac{|\xi|}{3} \le |\xi_3| < \frac{|\xi|}{C'_{reg}}$, $|\xi - \xi_2 - \xi_3|^{\frac{1}{2}-\alpha} \le |\xi_3|^{\frac{1}{2}-\alpha}$ and $\frac{|\xi_2|^{\frac{1}{2}-\alpha}}{|\xi_2+\xi_3|} \le \frac{1}{C'_{reg}} |\xi_3|^{-\frac{1}{2}-\alpha}$. Considering Var $a(\xi) \le \frac{C}{\xi^2} \int\int_{D_\xi} d\xi_2 d\xi_3 |\xi_3|^{-1-6\alpha}$, the integral over $|\xi_2| \le |\xi_3|$ contributes $O(|\xi_3|)$, while the integral over $\frac{|\xi|}{3} \le |\xi_3| < \frac{|\xi|}{C'_{reg}}$ leads to $\frac{C}{\xi^2} O(|\xi|^{1-6\alpha}) = O(|\xi|^{-1-6\alpha})$. Hence $\mathbb{E}\left([\delta \mathcal{R}\mathrm{Sk} I_{\mathbb{T}_1}(B^\varepsilon)]_{ts}\right)^2 \le C|t-s|^{6\alpha}$. By applying Lemma 1.4 (ii), a similar convergence rate may be proved. □

We may now finally define regularized iterated integrals of order 3 according to the above scheme.

Definition 2.3 (regularized integrals of order 3) *Let*

$$[\mathcal{R}I_{\mathbb{T}}(\mathbf{B}^\varepsilon)]_{tu}(i_1, i_2, i_3) = \sum_{\sigma \in \Sigma_3} \sum_j g(\sigma, j) [\mathcal{R}I_{\mathbb{T}_j^\sigma}(B^\varepsilon)]_{tu} \qquad (2.16)$$

where $g(\sigma, j)$ is as in eq. (2.5), and by definition

$$[\mathcal{R}I_{\mathbb{T}}(B^\varepsilon)]_{tu} = [\mathcal{R}\mathrm{Sk}I_{\mathbb{T}}(B^\varepsilon)]_t - [\mathcal{R}\mathrm{Sk}I_{\mathbb{T}}(B^\varepsilon)]_u - \sum_{\boldsymbol{v} \models V(\mathbb{T})} [\mathcal{R}I_{L_{\boldsymbol{v}}\mathbb{T}}(B^\varepsilon)]_{tu} [\mathcal{R}\mathrm{Sk}I_{R_{\boldsymbol{v}}\mathbb{T}}(B^\varepsilon)]_u. \qquad (2.17)$$

The definition mimicks the previous unregularized tree multiplicative property for skeleton integrals, eq. (2.11). All terms in it are convergent when $\varepsilon \to 0$. There remains only to prove that the multiplicative/Chen (ii) and geometric/shuffle property (iii) of the Introduction are preserved by the regularization. In principle (ii) should be true because eq. (2.17) is a multiplicative property in itself; it must only be shown that this multiplicative property implies the original multiplicative property (ii) of the Introduction after summing over all permutations σ, which is best done in a Hopf algebra language. The geometric/shuffle property is proved by showing that the regularized integration operator $\sum_\sigma \sum_j g(\sigma, j) \mathcal{R}\mathrm{Sk}I_{\mathbb{T}_j^\sigma}$ is a character of another Hopf algebra called *shuffle algebra*. Proofs should be found in [28].

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