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ABSTRACT. We study a discontinuous Galerkin method for the Stokes equations with a new stabilization of the viscous term. On the one hand, it allows us to recover, as the stabilization parameter tends towards infinity, some stable and well-known nonconforming approximations of the Stokes problem. On the other hand, we can easily define an *a posteriori* error indicator, based on the reconstruction of a locally conservative $H(div)$-tensor. An *a priori* error analysis is also carried out, yielding optimal convergence rates. Numerical tests illustrating the accuracy and the robustness of the scheme are presented.

1. INTRODUCTION

There is a large interest in discontinuous Galerkin finite element methods (dG) and the literature became quite impressive, that it would be impossible to mention here all the contributions. For a unified presentation of several existing dG methods for elliptic problems, we refer to [2].

Our approach for the Stokes equations follows the symmetric interior penalty method for elliptic problems of [1], which ensures well-posedness of the discrete formulation and optimal-order error estimates. It has been extended to the Stokes and Navier-Stokes equations by Girault, Rivière and Wheeler [17]. The velocity is looked for in $\boldsymbol{P}_k$ and the pressure belongs to P_{k-1}, with $1 \leq k \leq 3$. The employed stabilization term is a penalization of the jumps of the velocities across the edges, penalizing the nonconformity of the discrete solution. Instead, we propose to penalize the L^2-projection on P_{k-1} of the jumps, following a similar idea of Hansbo and Larson [19] for the elasticity problem. This approach presents two main advantages.

First, we prove that the solution of the dG formulation tends, as the stabilization parameter γ goes to infinity, towards the solution of the $\boldsymbol{P}_k \times P_{k-1}$ nonconforming approximation of the Stokes problem, which is known to be stable for $1 \leq k \leq 3$. Moreover, the inf-sup constant with respect to the energy norm of our method is independent of γ whereas that of [17] is $O(1/\sqrt{\gamma})$. Therefore, contrarily to [17], our method is robust for large stabilization parameters; this phenomenon is highlighted by numerical experiments. Note that we limit ourselves to the case $k \leq 3$, since for $k \geq 4$ it is known (cf. [27]) that $\boldsymbol{P}_k^{cont} \times P_{k-1}^{disc}$ is a stable pair of spaces for the Stokes problem, provided that special mesh constructions are avoided.

Second, our choice of the stabilization allows us to construct a locally conservative vector approximation in the Raviart-Thomas space RT_{k-1}, which is further used to define a simple *a posteriori* error estimator for the proposed numerical scheme. This feature is new as regards the dG approximations of the Stokes problem, though it has been previously mentioned by Kim [21] for the Laplace operator. Moreover, in the case $k = 1$ we show that the *a posteriori* dG error indicator tends, as $\gamma \to \infty$, towards the error indicator for the nonconforming Crouzeix-Raviart approximation of the Stokes equations. Recent works on the $H(div)$ flux reconstruction in view of *a posteriori* analysis include [23] for mixed finite element methods and [13] for dG approximation of elliptic equations; note that in the latter, the reconstructed flux belongs to a larger space than ours, that is RT_k.

We also discuss in this paper another variational formulation of the Stokes problem, written in terms of the strain rate tensor instead of the gradient of the velocity. The main advantage is the equivalence between its dG version and a three-fields formulation, allowing to recover the stress tensor in an obvious way. One may then be able to generalize it to non-Newtonian fluids or to impose other boundary conditions related to the normal stress. In order to retrieve a discrete Korn's type inequality for discontinuous velocities (cf. [6] or [24]), we consider an additional stabilization term in the case $k = 1$. The dG method thus obtained seems to be new, and can be easily generalized to more complex constitutive laws involving nonlinear and/or convective terms. Another velocity-pressure-stress formulations of the stationary Stokes equations is considered in [14], where an augmented mixed finite element method is studied. We show that the proposed discretizations are well-posed and yield optimal convergence rates, which are confirmed by numerical experiments.

Let us also cite a different approach for the discontinuous Galerkin approximation of the Stokes and Navier-Stokes equations, developed by Bassi and Rebay [4], [3] and Cockburn et al. [9], [8]. They all introduce the gradient of the velocity as a third variable but they discretize it by means of different numerical fluxes. The link between the stabilization of [4], the usual one (cf. [1]) and ours will be discussed in Section 3.

A possible extension of this work consists in considering the instationary Stokes equations and developing a space-time dG algorithm. In [28], such an algorithm is proposed for the time-dependent Oseen equations, by using anisotropic Sobolev spaces on the space-time domain. Their dG discretization of the viscous term is different from ours and uses the approach of [3], based on the introduction of an artificial flux. One could then apply the dG time discretization of [28] together with our space discretization. Two other extensions, to more general boundary conditions and to the Navier-Stokes equations, are briefly discussed in this article.

The outline of the paper is as follows. The model problem and three equivalent variational formulations are introduced in Section 2 and Section 3, respectively. Section 4 is devoted to the description of the numerical approximation, while in Section 5 we establish the well-posedness of the discrete problems. In Section 6, the relationship between our dG method and some well-known nonconforming approximations of the Stokes problem is discussed. Section 7 is devoted to the derivation of optimal *a priori* error bounds. Next, a simple *a posteriori* error indicator is introduced and its efficiency and reliability are proved in Section 8. Finally, some extensions are discussed in Section 9 and numerical tests are presented in the last section.

2. Notations and problem setting

We agree to write the vectors in bold letters and the second-order tensors in underlined letters, $\underline{\tau} = (\tau_{ij})_{1\le i,j\le 2}$; the product of two tensors will be denoted by :

$$\underline{\tau} : \underline{\sigma} = \tau_{ij}\sigma_{ij}.$$

For a given Hilbert space V, we put $\boldsymbol{V} = \{\mathbf{v} = (v_1, v_2);\ v_i \in V,\ 1 \le i \le 2\} = V \times V$, respectively $\underline{V} = \{\underline{\tau} = (\tau_{ij});\ \tau_{ij} \in V,\ 1 \le i, j \le 2\}$. We employ in this paper the summation convention of Einstein and we denote by the letter c any positive constant independent of the discretization parameter h. For any $k \in \mathbb{N}$, we denote by P_k the space of polynomials on $\mathbb{R}^2$ of total degree $\le k$.

In what follows, Ω denotes a Lipschitz domain of $\mathbb{R}^2$. We use classical notations : $H^m(\Omega)$ is the Sobolev space of order $m \in \mathbb{N}$, $\|\cdot\|_{m,\Omega}$ and $|\cdot|_{m,\Omega}$ are its usual Hilbert norm, respectively semi-norm.

We are interested in the stationary Stokes equations, which describe the steady flow of an incompressible Newtonian fluid at low Reynolds numbers. The governing equations are : the momentum conservation law

$$-\nabla \cdot (\underline{\tau} - p\underline{I}) = \boldsymbol{f},$$

the mass conservation law

$$\nabla \cdot \boldsymbol{u} = 0$$

and the constitutive law of a Newtonian fluid

$$\underline{\tau} = 2\mu \underline{D}(\boldsymbol{u}).$$

For the sake of simplicity, we consider a Dirichlet boundary condition

$$\boldsymbol{u} = \boldsymbol{g} \quad \text{on } \partial\Omega.$$

Here above, $\underline{\tau}$ denotes the viscous stress tensor, p the pressure, μ the fluid's viscosity (supposed to be constant) and $\underline{D}(\boldsymbol{u})$ the deformation tensor, given by

$$\underline{D}(\mathbf{u}) = \frac{1}{2}\left(\underline{\nabla}\boldsymbol{u} + (\underline{\nabla}\boldsymbol{u})^t\right)$$

with $\boldsymbol{u}$ the fluid's velocity. The data of the problem are $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$ and $\boldsymbol{g} \in \boldsymbol{H}^{1/2}(\partial\Omega)$.

3. Three equivalent variational formulations

In what follows, we shall consider three mixed variational formulations of the previous boundary value problem, in view of the numerical approximation. For this purpose, let $\boldsymbol{u}_g \in \boldsymbol{H}^1(\Omega)$ be a continuous lifting of the boundary data $\boldsymbol{g} \in \boldsymbol{H}^{1/2}(\partial\Omega)$, such that :

$$\boldsymbol{u}_g = \boldsymbol{g} \text{ on } \partial\Omega \quad \text{and} \quad \|\boldsymbol{u}_g\|_{1,\Omega} \leq c\,\|\boldsymbol{g}\|_{1/2,\partial\Omega}\,.$$

First, we consider the classical velocity-pressure formulation, obtained by substituting $\underline{\tau}$. The boundary value problem can then be written as follows :

$$\begin{cases} -\mu\Delta\boldsymbol{u} + \nabla p &= \boldsymbol{f} \quad \text{in } \Omega \\ \nabla\cdot\boldsymbol{u} &= 0 \quad \text{in } \Omega \\ \boldsymbol{u} &= \boldsymbol{g} \quad \text{on } \partial\Omega \end{cases}$$

and its variational formulation is :

$$(1) \qquad \begin{cases} (\boldsymbol{u},p) \in (\boldsymbol{u}_g + \boldsymbol{H}^1(\Omega)) \times L^2_0(\Omega) & & \\ a(\boldsymbol{u},v) + b(p,\boldsymbol{v}) & = l(\boldsymbol{v}) & \forall \boldsymbol{v} \in \boldsymbol{H}^1_0(\Omega) \\ b(q,\boldsymbol{u}) & = 0 & \forall q \in L^2_0(\Omega), \end{cases}$$

where the bilinear and the linear forms are defined by

$$\begin{aligned} a(\boldsymbol{u},\boldsymbol{v}) &= \mu\int_\Omega \underline{\nabla}\boldsymbol{u} : \underline{\nabla}\boldsymbol{v}dx, \\ b(p,\boldsymbol{v}) &= -\int_\Omega p\nabla\cdot\boldsymbol{v}dx, \\ l(\boldsymbol{v}) &= \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}dx. \end{aligned}$$

Second, we use that $\Delta\boldsymbol{u} = 2div\underline{D}(\boldsymbol{u})$ and we get the following formulation :

$$(2) \qquad \begin{cases} (\boldsymbol{u},p) \in (\boldsymbol{u}_g + \boldsymbol{H}^1(\Omega)) \times L^2_0(\Omega) & & \\ c(\boldsymbol{u},v) + b(p,\boldsymbol{v}) & = l(\boldsymbol{v}) & \forall \boldsymbol{v} \in \boldsymbol{H}^1_0(\Omega) \\ b(q,\boldsymbol{u}) & = 0 & \forall q \in L^2_0(\Omega) \end{cases}$$

with

$$c(\boldsymbol{u},\boldsymbol{v}) = 2\mu\int_\Omega \underline{D}(\boldsymbol{u}) : \underline{D}(\boldsymbol{v})dx.$$

We have used that the product between a symmetric and an anti-symmetric tensor vanishes, i.e.

$$\int_\Omega \underline{D}(\boldsymbol{u}) : \underline{\nabla}\boldsymbol{v}dx = \int_\Omega \underline{D}(\boldsymbol{u}) : \underline{D}(\boldsymbol{v})dx.$$

Finally, we consider a three-fields formulation in the unknowns $(\underline{\tau},\boldsymbol{u},p)$:

$$\begin{cases} -div\underline{\tau} + \nabla p &= \boldsymbol{f} \quad \text{in } \Omega \\ \nabla\cdot\boldsymbol{u} &= 0 \quad \text{in } \Omega \\ \underline{\tau} - 2\mu\underline{D}(\boldsymbol{u}) &= \underline{0} \quad \text{in } \Omega \\ \boldsymbol{u} &= \boldsymbol{g} \quad \text{on } \partial\Omega. \end{cases}$$

Its weak form is given by

$$
(3)\qquad \begin{cases} (\boldsymbol{u}, p, \underline{\tau}) \in (\boldsymbol{u}_g + \boldsymbol{H}^1(\Omega)) \times L^2_0(\Omega) \times \underline{X} & \\ \qquad\quad b(p, \boldsymbol{v}) + d(\underline{\tau}, \boldsymbol{v}) & = l(\boldsymbol{v}) \quad \forall \boldsymbol{v} \in \boldsymbol{H}^1_0(\Omega) \\ b(q, \boldsymbol{u}) & = 0 \qquad \forall q \in L^2_0(\Omega) \\ d(\underline{\theta}, \boldsymbol{u}) \qquad\quad - e(\underline{\theta}, \underline{\tau}) & = 0 \qquad \forall \underline{\theta} \in \underline{X}, \end{cases}
$$

where

$$
\begin{aligned} d(\underline{\tau}, \boldsymbol{v}) &= \int_\Omega \underline{\tau} : \underline{D}(\boldsymbol{v}) dx \\ e(\underline{\theta}, \underline{\tau}) &= \frac{1}{2\mu} \int_\Omega \underline{\theta} : \underline{\tau} dx \end{aligned}
$$

and where

$$
\underline{X} \quad = \quad \{\underline{\theta} = (\theta_{ij})_{1 \le i,j \le 2}; \theta_{ij} = \theta_{ji},\, \theta_{ij} \in L_2(\Omega),\ i, j = 1, 2\}\,.
$$

Note that the symmetry of the stress tensor is strongly imposed in the definition of the space $\underline{X}$.

The proofs of the well-posedness of each of these mixed formulations are well known. We refer for instance to [16] for the first two formulations, whereas the formulation (3) is equivalent to (2). Indeed, if $(\boldsymbol{u}, p)$ is the unique solution of (2) then $(\boldsymbol{u}, p, 2\mu\underline{D}(\boldsymbol{u}))$ satisfies (3), which shows the existence of a solution for (3); reciprocally, if $(\boldsymbol{u}, p, \underline{\tau})$ is a solution of (3) then $(\boldsymbol{u}, p)$ satisfies (2) and $\underline{\tau} = 2\mu\underline{D}(\boldsymbol{u})$, which ensures the uniqueness.

The three-fields formulation is useful in order to compute the flow of non-Newtonian fluids, which is our further goal. Indeed, when considering such fluids one cannot eliminate the stress tensor by means of the corresponding constitutive law, and hence one deals with formulations of at least three unknowns. Nevertheless, the numerical approximation which will be further introduced allows us to show the equivalence between the *discrete* versions of (3) and (2). This, in our opinion, justifies the interest of the two-fields formulation (2).

Another justification is related to boundary conditions of Neumann's type : in (1), one has to impose $(\underline{\nabla}\boldsymbol{u})\boldsymbol{n}$ on the boundary whereas in (2), one has to prescribe $\underline{D}(\boldsymbol{u})\boldsymbol{n}$ (i.e., $\underline{\tau}\boldsymbol{n}$ in the Newtonian case) which is more meaningful from a physical point of view.

4. Discretization by means of DG methods

4.1. **Preliminary notations and results.** From now on, we assume that Ω is a polygonal domain. Let $(\mathcal{T}_h)_{h>0}$ be a *regular* family of triangulations of Ω consisting of triangles : $\overline{\Omega} = \bigcup_{T \in \mathcal{T}_h} T$. We agree to denote by ε_h^{int} the set of internal edges of $\mathcal{T}_h$, by ε_h^{∂} the set of edges situated on the boundary $\partial\Omega$ and by ε_h the set of all edges of $\mathcal{T}_h$, $\varepsilon_h = \varepsilon_h^{int} \cup \varepsilon_h^{\partial}$. As usually, let h_T be the diameter of the triangle T and let $h = \max_{T \in \mathcal{T}_h} h_T$.

On every edge e belonging to ε_h^{int}, such that $\{e\} = \partial T_1 \cap \partial T_2$, we define once for all the unit normal $\boldsymbol{n}_e$ oriented from T_1 towards T_2. Then, for a given function φ, we define the average and the jump across the edge e as follows :

$$
\begin{aligned} [\varphi] &= \varphi_{/T_1} - \varphi_{/T_2}, \\ \{\varphi\} &= \frac{1}{2}\left(\varphi_{/T_1} + \varphi_{/T_2}\right). \end{aligned}
$$

If e is situated on the boundary $\partial\Omega$, we agree to take as $\boldsymbol{n}_e$ the outward unit normal $\boldsymbol{n}$; in this case, the jump and the average of φ are equal to the trace of φ on e.

We agree to denote the $L^2(e)$-orthogonal projection of a given function $\varphi \in L^2(e)$ on the polynomial space P_k ($k \in \mathbb{N}$) by $\pi_k\varphi$.

In what follows, we take $k = 1$, 2 or 3 and we introduce the finite dimensional spaces :

$$\begin{aligned} \boldsymbol{V}_h &= \left\{ \boldsymbol{v}_h \in \boldsymbol{L}^2(\Omega);\ (\boldsymbol{v}_h)_{/T} \in \boldsymbol{P}_k,\ \ \forall T \in \mathcal{T}_h \right\}, \\ Q_h &= \left\{ q_h \in L^2_0(\Omega);\ (q_h)_{/T} \in P_{k-1},\ \ \forall T \in \mathcal{T}_h \right\}, \\ \underline{X}_h &= \left\{ \underline{\theta}_h \in \underline{X};\ (\underline{\theta}_h)_{/T} \in \underline{P}_{k-1},\ \ \forall T \in \mathcal{T}_h \right\}. \end{aligned}$$

Let us next recall some approximation results for the spaces $\boldsymbol{V}_h$ and Q_h (see also [17]). For each $k = 1,\ 2,\ 3$, it is known that there exist two interpolation operators $i_h \in \mathcal{L}(L^2_0(\Omega); Q_h)$ and $\boldsymbol{I}_h \in \mathcal{L}(\boldsymbol{H}^1(\Omega); \boldsymbol{V}_h)$ such that, for any $T \in \mathcal{T}_h$ and any $e \in \varepsilon_h$, one has :

$$\int_T r(i_h q - q)dx \ = \ 0, \quad \forall r \in P_{k-1},\ \forall q \in L^2_0(\Omega) \tag{4}$$

and

$$\begin{aligned} \int_T r\nabla\cdot(\boldsymbol{I}_h\boldsymbol{v} - \boldsymbol{v})dx &= 0, \quad \forall r \in P_{k-1},\ \forall \boldsymbol{v} \in \boldsymbol{H}^1(\Omega) \\ \int_e \boldsymbol{r}\cdot[\boldsymbol{I}_h\boldsymbol{v}]ds &= 0, \quad \forall \boldsymbol{r} \in \boldsymbol{P}_{k-1},\ \forall \boldsymbol{v} \in \boldsymbol{H}^1_0(\Omega). \end{aligned} \tag{5}$$

Moreover, for $s \in [0, k]$ the following interpolation estimates hold :

$$\begin{aligned} &\forall q \in L^2_0(\Omega) \cap H^s(\Omega), && \|q - i_h q\|_{0,T} \le C h_T^s |q|_{s,T} \\ &\forall \boldsymbol{v} \in \boldsymbol{H}^{s+1}(\Omega), && |\boldsymbol{v} - \boldsymbol{I}_h\boldsymbol{v}|_{1,T} \le C h_T^s |\boldsymbol{v}|_{s+1,\Delta_T} \end{aligned} \tag{6}$$

where Δ_T is a suitable macro-element containing T. The case $k = 1$ follows from [11] (with $\Delta_T = T$), $k = 2$ from [15] and $k = 3$ from [10].

Let us now introduce a bilinear form on $(\boldsymbol{H}^1(\Omega) + \boldsymbol{V}_h) \times (\boldsymbol{H}^1(\Omega) + \boldsymbol{V}_h)$ representing our new stabilization term :

$$J(\boldsymbol{u}, \boldsymbol{v}) = \mu \sum_{e\in\varepsilon_h} \frac{1}{|e|} \int_e [\boldsymbol{\pi}_{k-1}\boldsymbol{u}]\cdot[\boldsymbol{\pi}_{k-1}\boldsymbol{v}]ds. \tag{7}$$

Note that $J(\boldsymbol{u}, \boldsymbol{v}) = 0$ for any $\boldsymbol{u} \in \boldsymbol{H}^1_0(\Omega)$.

Finally, let $|\cdot|_{1,h}$ denote the H^1- broken semi-norm, defined as follows :

$$|\ \boldsymbol{v}\ |_{1,h} = \left(\sum_{T\in\mathcal{T}_h} \|\nabla \boldsymbol{v}\|^2_{0,T} \right)^{1/2}, \quad \forall \boldsymbol{v} \in \boldsymbol{H}^1(\Omega) + \boldsymbol{V}_h$$

and let us also introduce the following semi-norm on $\boldsymbol{H}^1(\Omega) + \boldsymbol{V}_h$:

$$|||\boldsymbol{v}||| = \left(\mu|\boldsymbol{v}|^2_{1,h} + \gamma J(\boldsymbol{v}, \boldsymbol{v})\right)^{1/2}$$

where $\gamma > 0$ is a stabilization parameter. Then we can prove :

Lemma 4.1. *The application $\boldsymbol{v} \rightarrow|||\ \boldsymbol{v}\ |||$ is a norm on $\boldsymbol{V}_h$.*

Proof. Let $\boldsymbol{v} \in \boldsymbol{V}_h$ such that $|\boldsymbol{v}|_{1,h} = J(\boldsymbol{v}, \boldsymbol{v}) = 0$. On the one hand, it follows that $\boldsymbol{v}$ is piecewise constant on every triangle $T \in \mathcal{T}_h$ and on the other hand, one gets that $[\boldsymbol{\pi}_{k-1}\boldsymbol{v}] = 0$ on every edge $e \in \varepsilon_h$ ($k = 1,\ 2,\ 3$). The last assertion translates into :

$$\int_e [\boldsymbol{v}]\cdot\boldsymbol{r}ds = \int_e [\boldsymbol{\pi}_{k-1}\boldsymbol{v}]\cdot\boldsymbol{r}ds = 0, \quad \forall \boldsymbol{r} \in \boldsymbol{P}_{k-1} \text{ and } \forall e \in \varepsilon_h.$$

Hence, $\boldsymbol{v}$ is continuous, respectively zero at the k Gauss points of any edge $e \in \varepsilon_h^{int}$, respectively $e \in \varepsilon_h^{\partial}$. Together with the property $\boldsymbol{v}$ piecewise constant, it implies that $\boldsymbol{v} = \boldsymbol{0}$ on Ω, which concludes the proof.

4.2. **First two-fields formulation.** We consider the next discrete dG formulation of (1), where Nitsche's method (cf. [25]) is employed in order to treat the nonhomogeneous boundary condition :

$$
(8) \qquad \begin{cases} (\boldsymbol{u}_h, p_h) \in \boldsymbol{V}_h \times Q_h & \\ a_h(\boldsymbol{u}_h, \boldsymbol{v}_h) + b_h(p_h, \boldsymbol{v}_h) & = l_h(\boldsymbol{v}_h) \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h \\ b_h(q_h, \boldsymbol{u}_h) & = g_h(q_h) \quad \forall q_h \in Q_h. \end{cases}
$$

The bilinear, respectively linear forms are defined as follows :

$$
\begin{aligned}
a_h(\cdot,\cdot) &= A_0(\cdot,\cdot) + A_1(\cdot,\cdot) + \gamma J(\cdot,\cdot) \\
A_0(\boldsymbol{u}_h, \boldsymbol{v}_h) &= \mu \sum_{T \in \mathcal{T}_h} \int_T \underline{\nabla} \boldsymbol{u}_h : \underline{\nabla} \boldsymbol{v}_h dx \\
A_1(\boldsymbol{u}_h, \boldsymbol{v}_h) &= -\mu \sum_{e \in \varepsilon_h} \left(\int_e \left\{ \frac{\partial \boldsymbol{u}_h}{\partial \boldsymbol{n}_e} \right\} \cdot [\boldsymbol{v}_h] \, ds + \int_e \left\{ \frac{\partial \boldsymbol{v}_h}{\partial \boldsymbol{n}_e} \right\} \cdot [\boldsymbol{u}_h] \, ds \right) \\
b_h(q_h, \boldsymbol{v}_h) &= - \sum_{T \in \mathcal{T}_h} \int_T q_h \nabla \cdot \boldsymbol{v}_h dx + \sum_{e \in \varepsilon_h} \int_e \{q_h\} [\boldsymbol{v}_h \cdot \boldsymbol{n}_e] ds \\
l_h(\boldsymbol{v}_h) &= \sum_{T \in \mathcal{T}_h} \int_T \boldsymbol{f} \cdot \boldsymbol{v}_h dx - \mu \sum_{e \in \varepsilon_h^\partial} \int_e \frac{\partial \boldsymbol{v}_h}{\partial \boldsymbol{n}_e} \cdot \boldsymbol{g} ds + \mu \sum_{e \in \varepsilon_h^\partial} \frac{\gamma}{|e|} \int_e \boldsymbol{\pi}_{k-1} \boldsymbol{g} \cdot \pi_{k-1} \boldsymbol{v}_h ds \\
g_h(q_h) &= \sum_{e \in \varepsilon_h^\partial} \int_e q_h \boldsymbol{g} \cdot \boldsymbol{n}_e ds.
\end{aligned}
$$

The first part of $A_1(\cdot,\cdot)$ comes from the integration by part of $-\Delta \boldsymbol{u}$, whereas the last part is added in order to obtain a *symmetric* form $a_h(\cdot,\cdot)$.

In [17], the authors study a different discontinuous Galerkin method for the Stokes and Navier-Stokes equations, where the stabilization term is given by :

$$
J^*(\boldsymbol{u}_h, \boldsymbol{v}_h) = \mu \sum_{e \in \varepsilon_h} \frac{1}{\mid e \mid} \int_e [\boldsymbol{u}_h] \cdot [\boldsymbol{v}_h] ds.
$$

We shall prove in what follows that, contrarily to the dG method of [17], ours is robust with respect to the stabilization parameter, i.e. the solution of (8) tends, as γ tends to infinity, towards the unique solution of the Stokes problem discretized by $\boldsymbol{P}_k \times P_{k-1}$ nonconforming finite elements, for all $k = 1, 2, 3$. Moreover, we can compute the corresponding edge integrals by a lower degree Gauss formula than in [17]. Finally, our choice of the stabilization allows us to construct a locally conservative vector approximation in the Raviart-Thomas space, which is further used to define a simple *a posteriori* error indicator for the dG method.

Another stabilization term $\int_\Omega R([\boldsymbol{u}_h]) \cdot R([\boldsymbol{v}_h]) dx$ was proposed by Bassi and Rebay in [4], where R is a lifting of the jumps across the edges in $\boldsymbol{P}_k^{disc}$. In order to improve the computational efficiency and memory use, they replace the contributions from the global lifting operator R with a local lifting operator R_e, defined by

$$
\sum_{\bar{K} \supset e} \int_K R_e(\boldsymbol{w}) \cdot \boldsymbol{v}_h dx \quad = \quad \int_e \boldsymbol{w} \cdot \{\boldsymbol{v}_h\} \, ds, \quad \forall \boldsymbol{v}_h \in \boldsymbol{P}_k^{disc}.
$$

So finally the stabilization term is approximated by

$$
J^{\#}(\boldsymbol{u}_h, \boldsymbol{v}_h) = \sum_{e \in \varepsilon_h} \sum_{\bar{K} \supset e} \gamma_K \int_K R_e([\boldsymbol{u}_h]) \cdot R_e([\boldsymbol{v}_h]) dx.
$$

By means of a simple computation, one can notice that for $k=1$, one actually has that $R_e(\boldsymbol{w})_{/e} = \dfrac{|e|}{|K|}\left(\boldsymbol{\pi}_1\boldsymbol{w} + \frac{1}{2}\boldsymbol{\pi}_0\boldsymbol{w}\right)$ which gives :

$$\begin{aligned}\int_K R_e([\boldsymbol{u}_h])\cdot R_e([\boldsymbol{v}_h])dx &= \frac{1}{2}\int_e [\boldsymbol{u}_h]\cdot R_e([\boldsymbol{v}_h])ds \\ &= \frac{|e|}{2\,|K|}\int_e [\boldsymbol{u}_h]\cdot[\boldsymbol{v}_h]ds + \frac{|e|}{4\,|K|}\int_e [\boldsymbol{\pi}_0\boldsymbol{u}_h]\cdot[\boldsymbol{\pi}_0\boldsymbol{v}_h]ds.\end{aligned}$$

In conclusion, by choosing $\gamma_K = \gamma$ one gets on *regular* meshes that $J^{\#}(\cdot,\cdot) = J^*(\cdot,\cdot) + J(\cdot,\cdot)$. If we change the definition of the lifting operator and we look for it in $\boldsymbol{P}_{k-1}$ instead of $\boldsymbol{P}_k$, then it follows for $k=1$ that $R_e(\boldsymbol{w})_{/e} = \dfrac{|e|}{2\,|K|}\boldsymbol{\pi}_0\boldsymbol{w}$ and therefore $J^{\#}(\cdot,\cdot) = J(\cdot,\cdot)$, since

$$\int_K R_e([\boldsymbol{u}_h])\cdot R_e([\boldsymbol{v}_h])dx = \frac{|e|}{4\,|K|}\int_e [\boldsymbol{\pi}_0\boldsymbol{u}_h]\cdot[\boldsymbol{\pi}_0\boldsymbol{v}_h]ds.$$

Similar computations could be carried out for $k=2$ or 3, allowing to express $J^{\#}(\cdot,\cdot)$ in terms of $J^*(\cdot,\cdot)$ and $J(\cdot,\cdot)$.

4.3. **Second two-fields formulation.** We consider the following discrete version of (2) :

$$\left\{\begin{array}{lll}(\boldsymbol{U}_h, P_h)\in \boldsymbol{V}_h\times Q_h & & \\ c_h(\boldsymbol{U}_h,\boldsymbol{v}_h) + b_h(P_h,\boldsymbol{v}_h) & = f_h(\boldsymbol{v}_h) & \forall \boldsymbol{v}_h\in\boldsymbol{V}_h \\ b_h(q_h,\boldsymbol{U}_h) & = g_h(q_h) & \forall q_h\in Q_h\end{array}\right. \tag{9}$$

where :

$$\begin{aligned}c_h(\cdot,\cdot) &= C_0(\cdot,\cdot) + C_1(\cdot,\cdot) + \gamma J(\cdot,\cdot) + \gamma_1 J_1(\cdot,\cdot)\\ C_0(\boldsymbol{u}_h,\boldsymbol{v}_h) &= 2\mu\sum_{T\in\mathcal{T}_h}\int_T \underline{D}(\boldsymbol{u}_h):\underline{D}(\boldsymbol{v}_h)dx\\ C_1(\boldsymbol{u}_h,\boldsymbol{v}_h) &= -2\mu\sum_{e\in\varepsilon_h}\left(\int_e \{\underline{D}(\boldsymbol{u}_h)\boldsymbol{n}_e\}\cdot[\boldsymbol{v}_h]\,ds + \int_e \{\underline{D}(\boldsymbol{v}_h)\boldsymbol{n}_e\}\cdot[\boldsymbol{u}_h]\,ds\right)\\ J_1(\boldsymbol{u}_h,\boldsymbol{v}_h) &= \sum_{e\in\varepsilon_h^{int}}\frac{1}{|e|}\int_e [\boldsymbol{\pi}_1(\boldsymbol{u}_h\cdot\boldsymbol{n}_e)][\boldsymbol{\pi}_1(\boldsymbol{v}_h\cdot\boldsymbol{n}_e)]ds\\ f_h(\boldsymbol{v_h}) &= \sum_{T\in\mathcal{T}_h}\int_T \boldsymbol{f}\cdot\boldsymbol{v}_h dx - 2\mu\sum_{e\in\varepsilon_h^\partial}\int_e (\underline{D}(\boldsymbol{v}_h)\boldsymbol{n}_e)\cdot\boldsymbol{g}ds\\ &\quad +\mu\gamma\sum_{e\in\varepsilon_h^\partial}\frac{1}{|e|}\int_e \boldsymbol{\pi}_{k-1}\boldsymbol{g}\cdot\pi_{k-1}\boldsymbol{v}_h ds\end{aligned}$$

and where $\gamma > 0$ and $\gamma_1 > 0$ are now two stabilization parameters (which can be chosen independent of h). A second stabilization term $J_1(\cdot,\cdot)$ is added in order to retrieve the coercivity of the bilinear form $c_h(\cdot,\cdot)$, thanks to a discrete Korn's type inequality.

In view of the analysis of (9), we introduce the following semi-norm on $\boldsymbol{H}^1(\Omega) + \boldsymbol{V}_h$:

$$[[\boldsymbol{v}]] = \left(2\mu\sum_{T\in\mathcal{T}_h}\|\underline{D}(\boldsymbol{v})\|_{0,T}^2 + \gamma J(\boldsymbol{v},\boldsymbol{v}) + \gamma_1 J_1(\boldsymbol{v},\boldsymbol{v})\right)^{1/2}$$

and we show the next result :

Lemma 4.2. *The application $\boldsymbol{v}\to[[\boldsymbol{v}]]$ is a norm on $\boldsymbol{V}_h$, for all $k = 1,\ 2,\ 3$.*

Proof. Let $[[\boldsymbol{v}]] = 0$. Then $[\boldsymbol{\pi}_{k-1}\boldsymbol{v}] = \mathbf{0}$ on every edge $e \in \varepsilon_h$, $[\boldsymbol{\pi}_1\boldsymbol{v} \cdot \boldsymbol{n}_e] = 0$ on every internal edge $e \in \varepsilon_h^{int}$ and $\underline{D}(\boldsymbol{v}) = \underline{0}$ on every triangle $T \in \mathcal{T}_h$. The last relation implies that $\boldsymbol{v}$ is a rigid motion on every triangle $T \in \mathcal{T}_h$ (i.e. $\boldsymbol{v} = \boldsymbol{a}_T + b_T\boldsymbol{x}^\perp$ with $\boldsymbol{a}_T \in \mathbb{R}^2$, $b_T \in \mathbb{R}$ and $\boldsymbol{x}^\perp = (x_2, -x_1)$), so $\boldsymbol{v}$ is piecewise linear.

For $k = 2$ or 3, it follows that $\boldsymbol{\pi}_{k-1}\boldsymbol{v} = \boldsymbol{v}$ and therefore we can deduce similarly to the previous lemma that $\boldsymbol{v}$ is continuous across the edges and null on the boundary. So $\boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega)$ and the classical Korn inequality implies that $\boldsymbol{v} = \mathbf{0}$ on Ω.

For $k = 1$, we have that $\boldsymbol{v} \cdot \boldsymbol{n}_e$ is continuous across the internal edges but $\boldsymbol{v}$ is continuous only at the midpoints of the edges. Let us note (cf. [24]) that the tangential trace on a segment of a rigid motion is a constant (and not a fully linear function) : indeed, $(b_T\boldsymbol{x}^\perp) \cdot \boldsymbol{t}_e = b_T\boldsymbol{x} \cdot \boldsymbol{n}_e$ is constant along the edge e of equation $\boldsymbol{x} \cdot \boldsymbol{n}_e = const$. Then we can immediately deduce that $\boldsymbol{v} \cdot \boldsymbol{t}_e$ is also continuous across the internal edges, hence $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$. So now $\boldsymbol{v} = \boldsymbol{a} + b\boldsymbol{x}^\perp$ on Ω with $\boldsymbol{\pi}_0\boldsymbol{v} = \mathbf{0}$ on every boundary edge $e \in \varepsilon_h^\partial$, which implies $\boldsymbol{v} = \mathbf{0}$.

Remark. For $k = 2$ or 3, it is not necessary to add the new stabilization term $J_1(\cdot,\cdot)$ in order to enhance the coercivity of $c_h(\cdot,\cdot)$ since $KerJ \subset KerJ_1$. We only add it in order to obtain a unified presentation of the method for all k.

4.4. **Three-fields formulation.** We consider the following dG approximation of (3) :

$$\text{(10)} \qquad \begin{cases} (\boldsymbol{U}_h^{'}, P_h^{'}, \underline{\tau}_h^{'}) \in \boldsymbol{V}_h \times Q_h \times \underline{X}_h \\ k_h(\boldsymbol{U}_h^{'}, \boldsymbol{v}_h) + b_h(P_h^{'}, \boldsymbol{v}_h) + d_h(\underline{\tau}_h^{'}, \boldsymbol{v}_h) & = f_h(\boldsymbol{v}_h) & \forall \boldsymbol{v}_h \in \boldsymbol{V}_h \\ b_h(q_h, \boldsymbol{U}_h^{'}) & = g_h(q_h) & \forall q_h \in Q_h \\ d_h(\underline{\theta}_h, \boldsymbol{U}_h^{'}) \qquad\qquad - e(\underline{\theta}_h, \underline{\tau}_h^{'}) & = 0 & \forall \underline{\theta}_h \in \underline{X}_h, \end{cases}$$

where

$$\begin{aligned} k_h(\cdot,\cdot) &= C_1(\cdot,\cdot) + \gamma J(\cdot,\cdot) + \gamma_1 J_1(\cdot,\cdot) \\ d_h(\underline{\theta}_h, \boldsymbol{v}_h) &= \sum_{T \in \mathcal{T}_h} \int_T \underline{\theta}_h : \underline{D}(\boldsymbol{v}_h) dx. \end{aligned}$$

5. Well-posedness of the approximated problems

The aim of this section is to prove the well-posedness of the previous discrete formulations. For this purpose, we shall apply the Babŭska-Brezzi theorem for the two-fields formulations (8) and (9), and we shall directly prove the well-posedness of the three-fields formulation (10) by showing its equivalence with (9). Let us begin by checking the coercivity of the bilinear forms $a_h(\cdot,\cdot)$, respectively $c_h(\cdot,\cdot)$ on the discrete kernel $Ker_h b_h$, as well as the inf-sup condition for $b_h(\cdot,\cdot)$ with respect to the norms $|||\cdot|||$, respectively $[[\cdot]]$.

Lemma 5.1. *For γ large enough, there exists a constant $\alpha_1 > 0$ independent of h and μ such that*

$$\forall \boldsymbol{v} \in \boldsymbol{V}_h, \quad a_h(\boldsymbol{v}, \boldsymbol{v}) \geq \alpha_1 \, ||| \, \boldsymbol{v} \, |||^2 \, .$$

Proof. One has by definition that

$$a_h(\boldsymbol{v}, \boldsymbol{v}) = ||| \, \boldsymbol{v} \, |||^2 + A_1(\boldsymbol{v}, \boldsymbol{v})$$

so one only has to control the term

$$\text{(11)} \qquad A_1(\boldsymbol{v}, \boldsymbol{v}) = -2\mu \sum_{e \in \varepsilon_h} \int_e \left\{ \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{n}_e} \right\} \cdot [\boldsymbol{v}] \, ds = -2\mu \sum_{e \in \varepsilon_h} \int_e \left\{ \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{n}_e} \right\} \cdot [\boldsymbol{\pi}_{k-1}\boldsymbol{v}] \, ds.$$

Thanks to the Cauchy-Schwarz inequality, it follows that

$$A_1(\boldsymbol{v},\boldsymbol{v}) \geq -2\mu \left(\sum_{e\in\varepsilon_h} |e| \left\| \left\{ \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{n}_e} \right\} \right\|_{0,e}^2 \right)^{1/2} \left(\sum_{e\in\varepsilon_h} \frac{1}{|e|} \left\| [\boldsymbol{\pi}_{k-1}\boldsymbol{v}] \right\|_{0,e}^2 \right)^{1/2}.$$

A classical scaling argument together with the equivalence of norms in finite dimensional spaces yields, for any polynomial function w, that :

$$\sqrt{|e|}\,\|w\|_{0,e} \leq c\,\|w\|_{0,T}$$

where c is a constant independent of the discretization. Now let $e \in \varepsilon_h^{int}$ such that $\{e\} = \partial T_1 \cap \partial T_2$ with $T_1, T_2 \in \mathcal{T}_h$. Obviously,

$$\sqrt{|e|} \left\| \left\{ \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{n}_e} \right\} \right\|_{0,e} \leq \frac{\sqrt{|e|}}{2} \sum_{i=1}^{2} \left\| (\underline{\nabla} \boldsymbol{v})_{/T_i} \right\|_{0,e} \leq \frac{c}{2}(|\boldsymbol{v}|_{1,T_1} + |\boldsymbol{v}|_{1,T_2}).$$

A similar argument holds on a boundary edge $e \in \varepsilon_h^{\partial}$. So by summing upon all edges it follows that

$$A_1(\boldsymbol{v},\boldsymbol{v}) \geq -2c\sqrt{\mu}|\boldsymbol{v}|_{1,h}\sqrt{J(\boldsymbol{v},\boldsymbol{v})}$$

and finally,

$$a_h(\boldsymbol{v},\boldsymbol{v}) \geq \mu|\boldsymbol{v}|_{1,h}^2 + \gamma J(\boldsymbol{v},\boldsymbol{v}) - \frac{2c}{\sqrt{\gamma}} \left(\sqrt{\mu}|\boldsymbol{v}|_{1,h}\right) \left(\sqrt{\gamma J(\boldsymbol{v},\boldsymbol{v})}\right).$$

In conclusion, $a_h(\cdot,\cdot)$ is positive definite for $\gamma > c^2$ so the statement is established.

Lemma 5.2. *For γ large enough, there exists a constant $\alpha_2 > 0$ independent of h and μ such that*

$$\forall \boldsymbol{v} \in \boldsymbol{V}_h, \quad c_h(\boldsymbol{v},\boldsymbol{v}) \geq \alpha_2 \, [[\boldsymbol{v}]]^2 .$$

Proof. The proof is completely similar to the one of Lemma 5.1. Noting that

$$c_h(\boldsymbol{v},\boldsymbol{v}) = [[\boldsymbol{v}]]^2 - 4\mu \sum_{e\in\varepsilon_h} \int_e \{\underline{D}(\boldsymbol{v})\boldsymbol{n}_e\} \cdot [\boldsymbol{v}]\, ds,$$

one immediately gets

$$c_h(\boldsymbol{v},\boldsymbol{v}) \geq [[\boldsymbol{v}]]^2 - 4 \left(\mu \sum_{e\in\varepsilon_h} |e| \, \|\{\underline{D}(\boldsymbol{v})\boldsymbol{n}_e\}\|_{0,e}^2 \right)^{1/2} \sqrt{J(\boldsymbol{v},\boldsymbol{v})}.$$

Using that

$$\sqrt{|e|}\,\|\{\underline{D}(\boldsymbol{v})\boldsymbol{n}_e\}\|_{0,e} \leq c \left(\|\underline{D}(\boldsymbol{v})\|_{0,T_1} + \|\underline{D}(\boldsymbol{v})\|_{0,T_2} \right)$$

on $\{e\} = \partial T_1 \cap \partial T_2$ finally leads to the desired result, for γ sufficiently large.

Let us now focus on the inf-sup condition for $b_h(\cdot,\cdot)$, with respect to both norms $|||\cdot|||$ and $[[\cdot]]$.

Lemma 5.3. *There exists a constant $\beta_1 > 0$ independent of h, μ and γ such that :*

$$\inf_{q\in Q_h} \sup_{\boldsymbol{v}\in\boldsymbol{V}_h} \frac{b_h(q,\boldsymbol{v})}{\|q\|_{0,\Omega}\, |||\, \boldsymbol{v}\, |||} \geq \frac{\beta_1}{\sqrt{\mu}}.$$

Proof. The proof is rather classical. With any $q \in Q_h$, we shall associate $\boldsymbol{w} \in \boldsymbol{V}_h$ satisfying :

$$b_h(q,\boldsymbol{w}) = \|q\|_{0,\Omega}^2 \quad \text{and} \quad |||\, \boldsymbol{w}\, ||| \leq c\|q\|_{0,\Omega}.$$

For this purpose, we make use of the continuous inf-sup condition for the Stokes problem (see for instance [16]). So let $q \in Q_h \subset L_0^2(\Omega)$ and let $\boldsymbol{z} \in \boldsymbol{H}_0^1(\Omega)$ such that :

$$\begin{cases} \nabla \cdot \boldsymbol{z} = q \\ \|\boldsymbol{z}\|_{1,\Omega} \leq c\|q\|_{0,\Omega} \end{cases}.$$

Then we put $\boldsymbol{w} = \boldsymbol{I}_h \boldsymbol{z} \in \boldsymbol{V}_h$. By construction, we have according to (5) on every $T \in \mathcal{T}_h$ and every $e \in \varepsilon_h$:

$$\int_T q \nabla \cdot \boldsymbol{I}_h \boldsymbol{z} dx = \int_T q \nabla \cdot \boldsymbol{z} dx = \|q\|_{0,\Omega}^2,$$
$$\int_e \{q\} [\boldsymbol{I}_h \boldsymbol{z} \cdot \boldsymbol{n}_e] ds = 0$$

so $b_h(q, \boldsymbol{w}) = \|q\|_{0,\Omega}^2$. Next, let us point out that

$$\begin{aligned} J(\boldsymbol{w}, \boldsymbol{w}) &= \mu \sum_{e \in \varepsilon_h} \frac{1}{|e|} \int_e [\boldsymbol{\pi}_{k-1} \boldsymbol{w}] \cdot [\boldsymbol{\pi}_{k-1} \boldsymbol{w}] ds \\ &= \mu \sum_{e \in \varepsilon_h} \frac{1}{|e|} \int_e [\boldsymbol{\pi}_{k-1} \boldsymbol{w}] \cdot [\boldsymbol{I}_h \boldsymbol{z}] ds = 0, \end{aligned}$$

thanks to (5). Therefore, using the interpolation estimate (6) we get

$$||| \, \boldsymbol{w} \, ||| = \sqrt{\mu} \, |\boldsymbol{I}_h \boldsymbol{z}|_{1,h} \le c \sqrt{\mu} \, |\boldsymbol{z}|_{1,h} \le c \sqrt{\mu} \|q\|_{0,\Omega}$$

which finally yields

$$\sup_{\boldsymbol{v} \in \boldsymbol{V}_h} \frac{b_h(q, \boldsymbol{v})}{||| \, \boldsymbol{v} \, |||} \ge \frac{b_h(q, \boldsymbol{w})}{||| \, \boldsymbol{w} \, |||} \ge \frac{\|q\|_{0,\Omega}^2}{c \sqrt{\mu} \|q\|_{0,\Omega}} \ge \frac{\beta_1}{\sqrt{\mu}} \|q\|_{0,\Omega}.$$

Lemma 5.4. *There exists a constant $\beta_2 > 0$ independent of h, μ and γ such that :*

$$\inf_{q \in Q_h} \sup_{\boldsymbol{v} \in \boldsymbol{V}_h} \frac{b_h(q, \boldsymbol{v})}{\|q\|_{0,\Omega} \, [[\boldsymbol{v}]]} \ge \frac{\beta_2}{\sqrt{\mu}}.$$

Proof. We closely follow the proof of Lemma 5.3. With an arbitrary $q \in Q_h$, we associate the same function $\boldsymbol{w} = \boldsymbol{I}_h \boldsymbol{z} \in \boldsymbol{V}_h$ as previously and we next prove that

$$[[\boldsymbol{w}]] = \left(2\mu \sum_{T \in \mathcal{T}_h} \|\underline{D}(\boldsymbol{w})\|_{0,T}^2 + \gamma_1 J_1(\boldsymbol{w}, \boldsymbol{w}) \right)^{1/2} \le c \sqrt{\mu} \|q\|_{0,\Omega}.$$

For $k = 2$ or 3, one has that $J_1(\boldsymbol{w}, \boldsymbol{w}) = 0$, since $J(\boldsymbol{w}, \boldsymbol{w}) = 0$. So the result is obvious, since by (6)

$$\sum_{T \in \mathcal{T}_h} \|\underline{D}(\boldsymbol{w})\|_{0,T}^2 \quad \le \quad c \, |\boldsymbol{z}|_{1,h} \le c \|q\|_{0,\Omega}.$$

For $k = 1$, we still have to bound the term

$$J_1(\boldsymbol{w}, \boldsymbol{w}) = \mu \sum_{e \in \varepsilon_h^{int}} \frac{1}{|e|} \, \|[\boldsymbol{w} \cdot \boldsymbol{n}_e]\|_{0,e}^2 \, .$$

This is achieved by combining the following ingredients : the fact that $[\boldsymbol{z} \cdot \boldsymbol{n}_e] = 0$, the trace inequality on $\{e\} = \partial T_1 \cap \partial T_2$:

$$\frac{1}{\sqrt{|e|}} \, \|[(\boldsymbol{I}_h \boldsymbol{z} - \boldsymbol{z}) \cdot \boldsymbol{n}_e]\|_{0,e}$$

$$\le c \left(\tfrac{1}{h_{T_1}} \|\boldsymbol{I}_h \boldsymbol{z} - \boldsymbol{z}\|_{0,T_1} + \tfrac{1}{h_{T_2}} \|\boldsymbol{I}_h \boldsymbol{z} - \boldsymbol{z}\|_{0,T_2} + |\boldsymbol{I}_h \boldsymbol{z} - \boldsymbol{z}|_{1,T_1} + |\boldsymbol{I}_h \boldsymbol{z} - \boldsymbol{z}|_{1,T_2} \right) \tag{12}$$

and the interpolation properties of the Crouzeix-Raviart operator $\boldsymbol{I}_h$. Thus, we get

$$J_1(\boldsymbol{w}, \boldsymbol{w}) \le c \mu \, |\boldsymbol{z}|_{1,h}^2$$

which allows us to conclude.

Remark. Note that the constants β_1 and β_2 are independent of γ, which is not the case when considering the stabilization term $J^*(\cdot, \cdot)$ of [17]. We obtain the same coercivity constant α_1 as in [17], but with different energy norms (we shall see in Lemma 7.2 that $J^*(\boldsymbol{v}_h, \boldsymbol{v}_h) \le c \, |||\boldsymbol{v}_h|||$ on $\boldsymbol{V}_h$).

In order to establish the continuity of the bilinear form $a_h(\cdot,\cdot)$, it is sufficient to notice that the following bounds hold true :

$$\begin{aligned}
\mu \sum_{e\in\varepsilon_h} \int_e \left\{\frac{\partial \boldsymbol{u}_h}{\partial \boldsymbol{n}_e}\right\} \cdot [\boldsymbol{v}_h]\, ds &= \mu \sum_{e\in\varepsilon_h} \int_e \left\{\frac{\partial \boldsymbol{u}_h}{\partial \boldsymbol{n}_e}\right\} \cdot [\boldsymbol{\pi}_{k-1}\boldsymbol{v}_h]\, ds \\
&\leq \left(\mu \sum_{e\in\varepsilon_h} |e|\, \|\underline{\nabla}\boldsymbol{u}_h\|_{0,e}^2\right)^{1/2} J(\boldsymbol{v}_h,\boldsymbol{v}_h)^{1/2} \\
&\leq \left(\mu\, |\boldsymbol{u}_h|_{1,h}^2\right)^{1/2} J(\boldsymbol{v}_h,\boldsymbol{v}_h)^{1/2} \leq |||\boldsymbol{u}_h|||\;|||\boldsymbol{v}_h|||\,.
\end{aligned}$$

Here above, we have applied to $\varphi = \underline{\nabla}\boldsymbol{u}_h$ the next trace inequality on $e \subset \partial T$ combined with an inverse inequality :

$$\frac{1}{\sqrt{|e|}}\, \|\varphi\|_{0,e} \leq c(\frac{1}{h_T}\, \|\varphi\|_{0,T} + |\varphi|_{1,T}) \leq \frac{c}{h_T}\, \|\varphi\|_{0,T}\,.$$

Similar arguments yield the continuity of $c_h(\cdot,\cdot)$:

$$\mu \sum_{e\in\varepsilon_h} \int_e \{\underline{D}(\boldsymbol{u}_h)\boldsymbol{n}_e\}\cdot[\boldsymbol{v}_h]\, ds \leq \left(\mu \sum_{T\in T_h} \|\underline{D}(\boldsymbol{u}_h)\|_{0,T}^2\right)^{1/2} J(\boldsymbol{v}_h,\boldsymbol{v}_h)^{1/2} \leq [[\boldsymbol{u}_h]]\,[[\boldsymbol{v}_h]]$$

and of $b_h(\cdot,\cdot)$:

$$\begin{aligned}
\sum_{e\in\varepsilon_h} \int_e \{q_h\}\,[\boldsymbol{v}_h\cdot\boldsymbol{n}_e]\, ds &= \sum_{e\in\varepsilon_h} \int_e \{q_h\}\,[\boldsymbol{\pi}_{k-1}\boldsymbol{v}_h\cdot\boldsymbol{n}_e]\, ds \\
&\leq \frac{1}{\sqrt{\mu}}\left(\sum_{e\in\varepsilon_h} |e|\, \|q_h\|_{0,e}^2\right)^{1/2} J(\boldsymbol{v}_h,\boldsymbol{v}_h)^{1/2} \leq \frac{1}{\sqrt{\mu}}\, \|q_h\|_{0,T}\, J(\boldsymbol{v}_h,\boldsymbol{v}_h)^{1/2}.
\end{aligned}$$

The continuity of the linear forms $l_h(\cdot)$, $f_h(\cdot)$ and $g_h(\cdot)$ follows with the same arguments.

We are now able to state the main results of this section.

Theorem 5.5. *For γ sufficiently large, each of the mixed problems (8) and (9) has a unique solution.*

Proof. According to Lemmas 5.1 and 5.3 for problem (8), respectively Lemmas 5.2 and 5.4 for (9), the hypotheses of the Babŭska-Brezzi theorem are satisfied (cf. [7]). Therefore, these mixed variational formulations are well-posed.

Theorem 5.6. *For γ sufficiently large, problem (10) has a unique solution. Moreover, its solution is $(\boldsymbol{U}_h, P_h, 2\mu\underline{D}(\boldsymbol{U}_h))$ where $(\boldsymbol{U}_h, P_h)$ is the unique solution of (9).*

Proof. It is obvious that $(\boldsymbol{U}_h, P_h, 2\mu\underline{D}(\boldsymbol{U}_h))$ belongs to $\boldsymbol{V}_h \times Q_h \times \underline{X}_h$ and satisfies the variational problem (10). Since we are dealing with finite dimensional spaces, this equally ensures the uniqueness of the solution.

6. Robustness with respect to the stabilization parameter

For the sake of simplicity, we suppose in what follows that $\boldsymbol{g} = \boldsymbol{0}$. We study here the behaviour of our dG method when $\gamma \to \infty$ and we prove its robustness with respect to large stabilization parameters. More precisely, we prove that our dG method is robust with respect to large stabilization parameters since its solution converges, when $\gamma \to \infty$, towards the solution of the $\boldsymbol{P}_k \times P_{k-1}$ nonconforming finite element approximation of the Stokes problem.

Let us first consider the formulation (1). Its nonconforming approximation is given by

$$(13)\qquad \left\{\begin{array}{lll}
(\boldsymbol{u}_h^*, p_h^*) \in \boldsymbol{H}_h \times Q_h, & & \\
A_0(\boldsymbol{u}_h^*, \boldsymbol{v}_h) + b_h(p_h^*, \boldsymbol{v}_h) & = l_h(\boldsymbol{v}_h) & \forall \boldsymbol{v}_h \in \boldsymbol{H}_h \\
b_h(q_h, \boldsymbol{u}_h^*) & = 0 & \forall q_h \in Q_h
\end{array}\right.$$

where

$$\begin{aligned} \boldsymbol{H}_h &= \{\boldsymbol{v} \in \boldsymbol{L}^2(\Omega);\ (\boldsymbol{v}_h)_{/T} \in \boldsymbol{P}_k,\ \ \forall T \in \mathcal{T}_h, \\ &\quad \boldsymbol{v}_h \text{ continuous (resp. null) at the } k \text{ Gauss points of } e \in \varepsilon_h^{int} \text{ (resp. } \varepsilon_h^{\partial})\Big\}. \end{aligned}$$

For $k = 1$, $\boldsymbol{H}_h$ is the well-known $\boldsymbol{P}_1$ nonconforming space of Crouzeix-Raviart [11]; $k = 2$ corresponds to the $\boldsymbol{P}_2$ nonconforming finite elements of Fortin-Soulie [15], whereas for $k = 3$ we retrieve the $\boldsymbol{P}_3$ nonconforming finite elements of Crouzeix-Falk [10]. It is a well-known result that (13) is well-posed for $k = 1,\ 2,\ 3$, thanks to a discrete Poincaré inequality on $\boldsymbol{H}_h$.

It is important to notice that our choice of the stabilization yields

$$Ker_h J = \{\boldsymbol{v}_h \in \boldsymbol{V}_h;\ [\boldsymbol{\pi}_{k-1}\boldsymbol{v}_h]_{/e} = \boldsymbol{0},\ \forall e \in \varepsilon_h\} = \boldsymbol{H}_h. \tag{14}$$

Theorem 6.1. *Let $(\boldsymbol{u}_h, p_h)$ be the solution of (8) and $(\boldsymbol{u}_h^*, p_h^*)$ the solution of (13). Then one has :*

$$\lim_{\gamma \mapsto \infty} (|||\ \boldsymbol{u}_h - \boldsymbol{u}_h^*\ ||| + \|p_h - p_h^*\|_{0,\Omega}) = 0.$$

Proof. Let us first show that the sequence $(\boldsymbol{u}_h, p_h)_\gamma$ is bounded with respect to γ.

By taking $\boldsymbol{v}_h = \boldsymbol{u}_h$ as test-function in (8) and by using the second variational equation, one classically gets that

$$a_h(\boldsymbol{u}_h, \boldsymbol{u}_h) = l_h(\boldsymbol{u}_h) \leq \|\boldsymbol{f}\|_{0,\Omega}\|\boldsymbol{u}_h\|_{0,\Omega}.$$

Following the proof of Lemma 5.1, we have that

$$a_h(\boldsymbol{u}_h, \boldsymbol{u}_h) \geq \mu|\boldsymbol{u}_h|_{1,h}^2 + \gamma J(\boldsymbol{u}_h, \boldsymbol{u}_h) - 2c\sqrt{\mu}|\boldsymbol{u}_h|_{1,h}\sqrt{J(\boldsymbol{u}_h, \boldsymbol{u}_h)}$$

with c a constant independent of h, μ and γ. We next recall a Poincaré-Friedrichs inequality for discontinuous finite element spaces (see Brenner [5]) :

$$\|\boldsymbol{v}\|_{0,\Omega} \leq c\left(|\boldsymbol{v}|_{1,h}^2 + \sum_{e \in \varepsilon_h^{int}} \frac{1}{|e|}\|[\boldsymbol{\pi}_0\boldsymbol{v}]\|_{0,e}^2 + \phi(\boldsymbol{v})\right)^{1/2}, \quad \forall \boldsymbol{v} \in \boldsymbol{V}_h \tag{15}$$

where $\phi : \boldsymbol{H}^1(\Omega) \to \mathbb{R}$ is a continuous semi-norm such that for a constant function $\boldsymbol{c}$, $\phi(\boldsymbol{c}) = 0$ if and only if $\boldsymbol{c} = \boldsymbol{0}$. We choose

$$\phi(\boldsymbol{v}) = \sum_{e \in \varepsilon_h^{\partial}} \|\boldsymbol{\pi}_0\boldsymbol{v}\|_{0,e}^2 \tag{16}$$

and we use that $\|\boldsymbol{\pi}_0\boldsymbol{v}\|_{0,e} \leq \|\boldsymbol{\pi}_{k-1}\boldsymbol{v}\|_{0,e}$, since

$$\|\boldsymbol{\pi}_0\boldsymbol{v}\|_{0,e}^2 = \int_e \boldsymbol{\pi}_0\boldsymbol{v} \cdot \boldsymbol{v} ds = \int_e \boldsymbol{\pi}_0\boldsymbol{v} \cdot \boldsymbol{\pi}_{k-1}\boldsymbol{v} ds \leq \|\boldsymbol{\pi}_{k-1}\boldsymbol{v}\|_{0,e}\|\boldsymbol{\pi}_0\boldsymbol{v}\|_{0,e}.$$

Then we can deduce a slightly different Poincaré-Friedrichs inequality, which will be employed in the rest of the paper :

$$\|\boldsymbol{v}\|_{0,\Omega} \leq c\left(|\boldsymbol{v}|_{1,h}^2 + \frac{1}{\mu}J(\boldsymbol{v}, \boldsymbol{v})\right)^{1/2}, \quad \forall \boldsymbol{v} \in \boldsymbol{V}_h. \tag{17}$$

Gathering together the last inequalities yields, for γ large enough, that $|||\ \boldsymbol{u}_h\ ||| \leq \frac{C}{\sqrt{\mu}}$, with C independent of h, μ and γ.

The inf-sup condition of Lemma 5.3 together with the continuity of $a_h(\cdot,\cdot)$ now leads to :

$$\begin{aligned} \|p_h\|_{0,\Omega} &\leq \frac{\sqrt{\mu}}{\beta_1} \sup_{\boldsymbol{v}_h \in \boldsymbol{V}_h} \frac{b_h(p_h, \boldsymbol{v}_h)}{|||\, \boldsymbol{v}_h \,|||} \\ &= \frac{\sqrt{\mu}}{\beta_1} \sup_{\boldsymbol{v}_h \in \boldsymbol{V}_h} \frac{l_h(\boldsymbol{v}_h) - a_h(\boldsymbol{u}_h, \boldsymbol{v}_h)}{|||\, \boldsymbol{v}_h \,|||} \\ &\leq \frac{\sqrt{\mu}}{\beta_1} \left(\|f\|_{0,\Omega} \sup_{\boldsymbol{v}_h \in \boldsymbol{V}_h} \frac{\|\boldsymbol{v}_h\|_{0,\Omega}}{|||\, \boldsymbol{v}_h \,|||} + 2 \,|||\, \boldsymbol{u}_h \,||| \right) \end{aligned}$$

so according to (17), one gets for γ large enough that $\|p_h\|_{0,\Omega} \leq C$ with C independent of h , μ and γ.

Therefore, there exist two subsequences of $(\boldsymbol{u}_h)_\gamma$ and $(p_h)_\gamma$ which converge as $\gamma \to \infty$ towards $\boldsymbol{u}_h^\infty \in \boldsymbol{V}_h$, respectively $p_h^\infty \in Q_h$. From the variational problem (8) one next deduces that

$$\boldsymbol{u}_h^\infty \quad \in \quad Ker_h J = \boldsymbol{H}_h.$$

Using that

$$A_1(\boldsymbol{u}_h, \boldsymbol{v}_h) = -\mu \sum_{e \in \varepsilon_h} \left(\int_e \left\{ \frac{\partial \boldsymbol{u}_h}{\partial \boldsymbol{n}_e} \right\} \cdot [\boldsymbol{\pi}_{k-1} \boldsymbol{v}_h] \, ds + \int_e \left\{ \frac{\partial \boldsymbol{v}_h}{\partial \boldsymbol{n}_e} \right\} \cdot [\boldsymbol{\pi}_{k-1} \boldsymbol{u}_h] \, ds \right) = 0$$

for any $\boldsymbol{u}_h, \boldsymbol{v}_h \in \boldsymbol{H}_h$, it follows by passing to the limit in (8) that $(\boldsymbol{u}_h^\infty, p_h^\infty)$ satisfies the limit problem (13). The well-posedness of (13) implies on the one hand, that

$$(\boldsymbol{u}_h^\infty, p_h^\infty) = (\boldsymbol{u}_h^*, p_h^*)$$

and on the other hand, that the whole sequences $(\boldsymbol{u}_h)_\gamma$ and $(p_h)_\gamma$ are convergent.

Remark. If the stabilization term $J(\cdot,\cdot)$ is replaced by $J^*(\cdot,\cdot)$ of [17], then the limit $(\boldsymbol{u}_h^\infty, p_h^\infty)$ belongs to $Ker_h J^* \times Q_h$, that is to the ($\boldsymbol{P}_k$-continuous)$\times$(P_{k-1}-discontinuous) finite element spaces. This is not a stable pair of spaces for the Stokes problem, hence the dG method developed in [17] is not robust for any k as γ tends towards infinity.

We are now interested in the robustness of the formulation (9). Let us consider the discretization of the variational problem (2) by means of $\boldsymbol{P}_k \times P_{k-1}$ nonconforming finite elements, which reads as follows :

$$(18) \qquad \left\{ \begin{array}{lll} (\boldsymbol{U}_h^*, P_h^*) \in \boldsymbol{H}_h \times Q_h & & \\ C_0(\boldsymbol{U}_h^*, \boldsymbol{v}_h) + \gamma_1 J_1(\boldsymbol{U}_h^*, \boldsymbol{v}_h) + b_h(P_h^*, \boldsymbol{v}_h) & = f_h(\boldsymbol{v}_h) & \forall \boldsymbol{v}_h \in \boldsymbol{H}_h \\ b_h(q_h, \boldsymbol{U}_h^*) & = 0 & \forall q_h \in Q_h. \end{array} \right.$$

Remark. In the case $k = 2$ or 3, one actually has that $J_1(\boldsymbol{u}_h, \boldsymbol{v}_h) = 0$ for all $\boldsymbol{u}_h,\ \boldsymbol{v}_h \in \boldsymbol{H}_h$.

Since for any $\boldsymbol{u}_h, \boldsymbol{v}_h \in \boldsymbol{H}_h$ one has that

$$c_h(\boldsymbol{u}_h, \boldsymbol{v}_h) = C_0(\boldsymbol{u}_h, \boldsymbol{v}_h) + \gamma_1 J_1(\boldsymbol{u}_h, \boldsymbol{v}_h),$$

one can immediately deduce the well-posedness of the mixed formulation (18), thanks to the Babŭska-Brezzi theorem and to Lemmas 5.2 and 5.4.

Theorem 6.2. *Let $(\boldsymbol{U}_h, P_h)$ be the solution of (9) and $(\boldsymbol{U}_h^*, P_h^*)$ the solution of (18). Then, for γ_1 fixed, one has that :*

$$\lim_{\gamma \mapsto \infty} ([[\boldsymbol{U}_h - \boldsymbol{U}_h^*]] + \|P_h - P_h^*\|_{0,\Omega}) = 0$$

Proof. The proof is very similar to the one of the previous theorem. The only difference concerns the discrete Poincaré inequality (17) on $\boldsymbol{V}_h$, which has to be replaced here by a Korn's type inequality. For this purpose, we recall the folowing result, first established by Brenner in [6] in a stronger form and then improved by Mardal and Winther in [24] :

$$|\boldsymbol{v}|_{1,h} \leq c \left(\sum_{T \in \mathcal{T}_h} \|\underline{D}(\boldsymbol{v})\|^2_{0,T} + \frac{1}{\mu} J_1(\boldsymbol{v}, \boldsymbol{v}) + \phi(\boldsymbol{v}) \right)^{1/2}, \quad \forall \boldsymbol{v} \in \boldsymbol{V}_h \tag{19}$$

where now $\phi : \boldsymbol{H}^1(\Omega) \to \mathbb{R}$ is a continuous semi-norm such that if $\phi(\boldsymbol{v}) = 0$ for a rigid motion $\boldsymbol{v}$, then $\boldsymbol{v}$ is a constant vector. With the same choice of ϕ as in (16), we can now deduce the following Korn inequalities on $\boldsymbol{V}_h$:

$$|\boldsymbol{v}|_{1,h} \leq c \left(\sum_{T \in \mathcal{T}_h} \|\underline{D}(\boldsymbol{v})\|^2_{0,T} + \sum_{e \in \varepsilon_h^\partial} \frac{1}{|e|} \|\boldsymbol{\pi}_{k-1} \boldsymbol{v}\|^2_{0,e} + \frac{1}{\mu} J_1(\boldsymbol{v}, \boldsymbol{v}) \right)^{1/2}, \tag{20}$$

$$\|\boldsymbol{v}\|_{0,\Omega} \leq c \left(\sum_{T \in \mathcal{T}_h} \|\underline{D}(\boldsymbol{v})\|^2_{0,T} + \frac{1}{\mu} J(\boldsymbol{v}, \boldsymbol{v}) + \frac{1}{\mu} J_1(\boldsymbol{v}, \boldsymbol{v}) \right)^{1/2}. \tag{21}$$

The last estimate allows us to conclude as in Theorem 6.1.

As regards the three-fields formulation (10), it is obvious that for fixed γ_1,

$$\lim_{\gamma \mapsto \infty} \left\| \underline{\tau}_h^{'} - \underline{\tau}_h^* \right\|_{0,\Omega} = 0$$

where $\underline{\tau}_h^{'} = 2\mu \underline{D}(\boldsymbol{U}_h)$ and $\underline{\tau}_h^* = 2\mu \underline{D}(\boldsymbol{U}_h^*)$. Note that $(\boldsymbol{U}_h^*, P_h^*, \underline{\tau}_h^*)$ is the unique solution of the $\boldsymbol{P}_k \times P_{k-1} \times \underline{P}_{k-1}$ nonconforming approximation of the three-fields formulation (3), namely :

$$\left\{ \begin{array}{llll} (\boldsymbol{U}_h^*, P_h^*, \underline{\tau}_h^*) \in \boldsymbol{H}_h \times Q_h \times \underline{X}_h & & & \\ \gamma_1 J_1(\boldsymbol{U}_h^*, \boldsymbol{v}_h) + b_h(P_h^*, \boldsymbol{v}_h) + d_h(\underline{\tau}_h^*, \boldsymbol{v}_h) & & = f_h(\boldsymbol{v}_h) & \forall \boldsymbol{v}_h \in \boldsymbol{H}_h \\ b_h(q_h, \boldsymbol{U}_h^*) & & = 0 & \forall q_h \in Q_h \\ d_h(\underline{\theta}_h, \boldsymbol{U}_h^*) & - e(\underline{\theta}_h, \underline{\tau}_h^*) & = 0 & \forall \underline{\theta}_h \in \underline{X}_h. \end{array} \right.$$

7. A priori error estimates

This section is devoted to the derivation of optimal *a priori* error estimates for both the velocity and the pressure, for the two dG formulations (8) and (9). We will first consider the energy norm of the error and then the L^2-norm of the velocity error, in order to prove a $O(h^{k+1})$ convergence rate for the latter. Let us first establish some auxiliary results.

Lemma 7.1. *The solution $(\boldsymbol{u}, p)$ of the continuous Stokes problem satisfies the consistency properties :*

$$\begin{array}{rcl} a_h(\boldsymbol{u}, \boldsymbol{v}_h) + b_h(p, \boldsymbol{v}_h) & = & l_h(\boldsymbol{v}_h), \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h \\ b_h(q_h, \boldsymbol{u}) & = & 0, \quad \forall q_h \in Q_h. \end{array}$$

Proof. The proof is classical (see also [17]), therefore we only give here a sketch of the proof, for the sake of clarity. The second relation is obvious, since $\nabla \cdot \boldsymbol{u} = 0$ on any triangle and $[\boldsymbol{u} \cdot \boldsymbol{n}_e] = 0$ across any internal edge e. The first consistency property is obtained after integrating by parts on each element and using the following regularity of the Stokes problem (cf. [18]) with data $\boldsymbol{f} \in \boldsymbol{L}^{4/3}(\Omega)$ on a Lipschitz polygon : $(\boldsymbol{u}, p) \in \boldsymbol{W}^{2,4/3}(\Omega) \times W^{1,4/3}(\Omega)$. This ensures that both $\underline{\nabla} \boldsymbol{u}$ and p have a trace on each line segment e, which moreover belongs to $L^2(e)$. Since $\mu \underline{\nabla} \boldsymbol{u} - p\underline{I} \in \underline{H}(div, \Omega)$ and $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$, it follows that $\mu \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{n_e}} - p\boldsymbol{n}$ and $\boldsymbol{u}$ are continuous across any internal edge, so the desired result holds true.

Lemma 7.2. *There exists a constant $c > 0$ independent of h and μ such that :*

$$\left(\sum_{e\in\varepsilon_h} \frac{1}{|e|}\|[\boldsymbol{v}_h]\|_{0,e}^2\right)^{1/2} \leq \frac{c}{\sqrt{\mu}}\,|||\boldsymbol{v}_h|||\,, \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h.$$

Proof. We obviously have, on any $e \in \mathcal{E}_h$:

$$\|[\boldsymbol{v}_h]\|_{0,e} \leq \|[\boldsymbol{v}_h - \boldsymbol{\pi}_{k-1}\boldsymbol{v}_h]\|_{0,e} + \|[\boldsymbol{\pi}_{k-1}\boldsymbol{v}_h]\|_{0,e}.$$

Let T be a triangle such that $e \subset \partial T$. Then

$$\frac{1}{\sqrt{|e|}}\|\boldsymbol{v}_h - \boldsymbol{\pi}_{k-1}\boldsymbol{v}_h\|_{0,e} \quad\leq\quad \frac{1}{\sqrt{|e|}}\|\boldsymbol{v}_h - \boldsymbol{\pi}_{k-1}^T\boldsymbol{v}_h\|_{0,e} \leq \frac{c}{h_T}\|\boldsymbol{v}_h - \boldsymbol{\pi}_{k-1}^T\boldsymbol{v}_h\|_{0,T} \leq c|\boldsymbol{v}_h|_{1,T}$$

where $\boldsymbol{\pi}_{k-1}^T$ denotes the $\boldsymbol{L}^2(T)$-orthogonal projection on $\boldsymbol{P}_{k-1}$ on the element T. These inequalities yield the desired statement, for $\gamma \geq 1$.

Lemma 7.3. *Suppose $\boldsymbol{u} \in \boldsymbol{H}^{k+1}(\Omega)$. Then there exists a constant $c > 0$ independent of h and μ such that*

$$\forall \boldsymbol{v}_h \in \boldsymbol{V}_h, \quad |a_h(\boldsymbol{u} - I_h\boldsymbol{u}, \boldsymbol{v}_h)| \leq c\sqrt{\mu}h^k|||\boldsymbol{v}_h||||\boldsymbol{u}|_{k+1,\Omega}.$$

Proof. We recall that $a_h(\cdot,\cdot) = A_0(\cdot,\cdot) + A_1(\cdot,\cdot) + \gamma J(\cdot,\cdot)$. The Cauchy-Schwarz inequality immediately gives that

$$A_0(\boldsymbol{u} - \boldsymbol{I}_h\boldsymbol{u}, \boldsymbol{v}_h) \leq \mu\,|\boldsymbol{v}_h|_{1,h}\,|\boldsymbol{u} - \boldsymbol{I}_h\boldsymbol{u}_h|_{1,h} \leq c\sqrt{\mu}h^k|||\boldsymbol{v}_h||||\boldsymbol{u}|_{k+1,\Omega}.$$

The property (5) of the interpolation operator $\boldsymbol{I}_h$ gives on the one hand, that

$$J(\boldsymbol{u} - \boldsymbol{I}_h\boldsymbol{u}, \boldsymbol{v}_h) = \mu\sum_{e\in\varepsilon_h}\frac{1}{|e|}\int_e [\boldsymbol{\pi}_{k-1}\boldsymbol{I}_h\boldsymbol{u}]\cdot[\boldsymbol{\pi}_{k-1}\boldsymbol{v}_h]ds = 0$$

and on the other hand, that

$$\sum_{e\in\varepsilon_h}\int_e \left\{\frac{\partial \boldsymbol{v}_h}{\partial \boldsymbol{n}_e}\right\}\cdot[\boldsymbol{u} - \boldsymbol{I}_h\boldsymbol{u}]\,ds = 0.$$

So we only have to bound the remaining term $\mu\sum_{e\in\varepsilon_h}\int_e\left\{\frac{\partial(\boldsymbol{u}-\boldsymbol{I}_h\boldsymbol{u})}{\partial\boldsymbol{n}_e}\right\}\cdot[\boldsymbol{v}_h]\,ds$. For this purpose, let us introduce the classical Lagrange interpolation operator of polynomial degree k, denoted by $\boldsymbol{L}_h$. Then we can write that

$$\begin{aligned}\int_e\left\{\frac{\partial(\boldsymbol{u}-\boldsymbol{I}_h\boldsymbol{u})}{\partial\boldsymbol{n}_e}\right\}\cdot[\boldsymbol{v}_h]ds &= \int_e\left\{\frac{\partial(\boldsymbol{u}-\boldsymbol{L}_h\boldsymbol{u})}{\partial\boldsymbol{n}_e}\right\}\cdot[\boldsymbol{v}_h]ds + \int_e\left\{\frac{\partial(\boldsymbol{L}_h\boldsymbol{u}-\boldsymbol{I}_h\boldsymbol{u})}{\partial\boldsymbol{n}_e}\right\}\cdot[\boldsymbol{\pi}_{k-1}\boldsymbol{v}_h]ds\\ &\leq \|[\boldsymbol{v}_h]\|_{0,e}\|\{\underline{\nabla}(\boldsymbol{u}-\boldsymbol{L}_h\boldsymbol{u})\}\|_{0,e} + \|[\boldsymbol{\pi}_{k-1}\boldsymbol{v}_h]\|_{0,e}\|\{\underline{\nabla}(\boldsymbol{L}_h\boldsymbol{u}-\boldsymbol{I}_h\boldsymbol{u})\}\|_{0,e}.\end{aligned}$$

Let e be an internal edge common to the triangles T_1 and T_2; the proof is completely similar for a boundary edge. Thanks to the trace inequality (12) and to classical interpolation estimates for $\boldsymbol{L}_h$, one obtains :

$$\begin{aligned}\frac{1}{\sqrt{|e|}}\|\{\underline{\nabla}(\boldsymbol{u}-\boldsymbol{L}_h\boldsymbol{u})\}\|_{0,e} &\leq c\left(\frac{1}{h_{T_1}}\|\underline{\nabla}(\boldsymbol{u}-\boldsymbol{L}_h\boldsymbol{u})\|_{0,T_1} + \frac{1}{h_{T_2}}\|\underline{\nabla}(\boldsymbol{u}-\boldsymbol{L}_h\boldsymbol{u})\|_{0,T_2} + |\underline{\nabla}(\boldsymbol{u}-\boldsymbol{L}_h\boldsymbol{u})|_{1,T_1\cup T_2}\right)\\ &\leq ch^{k-1}|\boldsymbol{u}|_{k+1,T_1\cup T_2}.\end{aligned}$$

It follows, thanks to Lemma 7.2, that

$$\sum_{e\in\varepsilon_h}\int_e\left\{\frac{\partial(\boldsymbol{u}-\boldsymbol{L}_h\boldsymbol{u})}{\partial\boldsymbol{n}_e}\right\}\cdot[\boldsymbol{v}_h]ds \leq ch^k\left(\sum_{e\in\varepsilon_h}\frac{1}{|e|}\|[\boldsymbol{v}_h]\|_{0,e}^2\right)^{1/2}|\boldsymbol{u}|_{k+1,\Omega} \leq \frac{c}{\sqrt{\mu}}h^k|||\boldsymbol{v}_h||||\boldsymbol{u}|_{k+1,\Omega}.$$

Next, using that $\boldsymbol{L}_h\boldsymbol{u} - \boldsymbol{I}_h\boldsymbol{u}$ is piecewise polynomial, we obtain by means of a scaling argument that :

$$\frac{1}{\sqrt{|e|}}\|\{\underline{\nabla}(\boldsymbol{L}_h\boldsymbol{u} - \boldsymbol{I}_h\boldsymbol{u})\}\|_{0,e} \leq c\left(\frac{1}{h_{T_1}}\|\underline{\nabla}(\boldsymbol{L}_h\boldsymbol{u} - \boldsymbol{I}_h\boldsymbol{u})\|_{0,T_1} + \frac{1}{h_{T_2}}\|\underline{\nabla}(\boldsymbol{L}_h\boldsymbol{u} - \boldsymbol{I}_h\boldsymbol{u})\|_{0,T_2}\right).$$

The triangle inequality implies, on any triangle T,

$$|\boldsymbol{L}_h\boldsymbol{u} - \boldsymbol{I}_h\boldsymbol{u}|_{1,T} \leq |\boldsymbol{L}_h\boldsymbol{u} - \boldsymbol{u}|_{1,T} + |\boldsymbol{u} - \boldsymbol{I}_h\boldsymbol{u}|_{1,T} \leq ch_T^k|\boldsymbol{u}|_{k+1,\Delta_T}$$

so finally,

$$\sum_{e\in\varepsilon_h}\int_e\left\{\frac{\partial(\boldsymbol{L}_h\boldsymbol{u} - \boldsymbol{I}_h\boldsymbol{u})}{\partial\boldsymbol{n}_e}\right\}\cdot[\boldsymbol{\pi}_{k-1}\boldsymbol{v}_h]ds \leq \frac{c}{\sqrt{\mu}}h^k\sqrt{J(\boldsymbol{v}_h,\boldsymbol{v}_h)}|\boldsymbol{u}|_{k+1,\Omega}.$$

It is now sufficient to gather together the previous estimates in order to end the proof.

Lemma 7.4. *Suppose $p \in H^k(\Omega)$. Then there exists a constant $c > 0$ independent of h and μ such that*

$$\forall \boldsymbol{v}_h \in \boldsymbol{V}_h, \quad |b_h(p - i_hp, \boldsymbol{v}_h)| \leq \frac{c}{\sqrt{\mu}}h^k|||\boldsymbol{v}_h||||p|_{k,\Omega}.$$

Proof. We recall that

$$\begin{aligned} b_h(p - i_hp, \boldsymbol{v}_h) &= -\sum_{T\in\mathcal{T}_h}\int_T(p - i_hp)\nabla\cdot\boldsymbol{v}_h dx + \sum_{e\in\varepsilon_h}\int_e\{p - i_hp\}[\boldsymbol{v}_h\cdot\boldsymbol{n}_e]ds \\ &= \sum_{e\in\varepsilon_h}\int_e\{p - i_hp\}[\boldsymbol{v}_h\cdot\boldsymbol{n}_e]ds. \end{aligned}$$

Taking into account Lemma 7.2, we get for $\gamma \geq 1$ that

$$|b_h(p - i_hp, \boldsymbol{v}_h)| \leq \frac{c}{\sqrt{\mu}}\left(\sum_{e\in\varepsilon_h}|e|\,\|\{p - i_hp\}\|_{0,e}^2\right)^{1/2}|||\boldsymbol{v}_h|||\,.$$

On each edge $e \in \varepsilon_h$ such that $e \subset \partial T$, we bound the term $\|p - i_hp\|_{0,e}$ as in the proof of Lemma 7.3. Denoting by l_h the Lagrange interpolation operator on P_{k-1} for $k = 2$ or 3, we obtain :

$$\begin{aligned} \sqrt{|e|}\|p - i_hp\|_{0,e} &\leq \sqrt{|e|}\left(\|p - l_hp\|_{0,e} + \|l_hp - i_hp\|_{0,e}\right) \\ &\leq c\left(\|p - l_hp\|_{0,T} + h_T\,|p - l_hp|_{1,T} + \|l_hp - i_hp\|_{0,T}\right) \\ &\leq c \quad h^k|p|_{k,T}. \end{aligned}$$

For $k = 1$, we directly have :

$$\sqrt{|e|}\|p - i_hp\|_{0,e} \leq c\left(\|p - i_hp\|_{0,T} + h_T\,|p - i_hp|_{1,T}\right) \leq ch|p|_{1,T}.$$

So the announced result holds.

Remark. For $\gamma < 1$, the two previous lemmas also hold true, but with constants dependent of γ.

Theorem 7.5. *Let $(\boldsymbol{u}, p) \in \boldsymbol{H}^{k+1}(\Omega)\times H^k(\Omega)$ be the solution of the continuous Stokes problem and let γ be sufficiently large (as in Lemma 5.1). Then the solution $(\boldsymbol{u}_h, p_h)$ of the discrete problem (8) satisfies the following a priori error bounds :*

$$|||\,\boldsymbol{u} - \boldsymbol{u}_h\,||| \leq ch^k(\sqrt{\mu}|\boldsymbol{u}|_{k+1,\Omega} + \frac{1}{\sqrt{\mu}}|p|_{k,\Omega}) \tag{22}$$

$$\|p - p_h\|_{0,\Omega} \leq ch^k(\mu|\boldsymbol{u}|_{k+1,\Omega} + |p|_{k,\Omega}) \tag{23}$$

with a constant c independent of h and μ.

Proof. According to Lemma 5.1, one has for γ large enough that

$$\begin{aligned}\alpha_1 \; |||\, \boldsymbol{u}_h - \boldsymbol{I}_h\boldsymbol{u} \,|||^2 &\leq a_h(\boldsymbol{u}_h - \boldsymbol{I}_h\boldsymbol{u}, \boldsymbol{u}_h - \boldsymbol{I}_h\boldsymbol{u}) \\ &= a_h(\boldsymbol{u}_h, \boldsymbol{u}_h - \boldsymbol{I}_h\boldsymbol{u}) - a_h(\boldsymbol{u}, \boldsymbol{u}_h - \boldsymbol{I}_h\boldsymbol{u}) + a_h(\boldsymbol{u} - \boldsymbol{I}_h\boldsymbol{u}, \boldsymbol{u}_h - \boldsymbol{I}_h\boldsymbol{u}) \\ &= l_h(\boldsymbol{u}_h - \boldsymbol{I}_h\boldsymbol{u}) - b_h(p_h, \boldsymbol{u}_h - \boldsymbol{I}_h\boldsymbol{u}) - a_h(\boldsymbol{u}, \boldsymbol{u}_h - \boldsymbol{I}_h\boldsymbol{u}) + a_h(\boldsymbol{u} - \boldsymbol{I}_h\boldsymbol{u}, \boldsymbol{u}_h - \boldsymbol{I}_h\boldsymbol{u}).\end{aligned}$$

Thanks to the consistency property stated in Lemma 7.1, one next gets that

$$\alpha_1 \; |||\, \boldsymbol{u}_h - \boldsymbol{I}_h\boldsymbol{u} \,|||^2 \leq b_h(p - p_h, \boldsymbol{u}_h - \boldsymbol{I}_h\boldsymbol{u}) + a_h(\boldsymbol{u} - \boldsymbol{I}_h\boldsymbol{u}, \boldsymbol{u}_h - \boldsymbol{I}_h\boldsymbol{u}). \tag{24}$$

On the one hand, the continuity property proved in Lemma 7.3 implies that

$$a_h(\boldsymbol{u} - \boldsymbol{I}_h\boldsymbol{u}, \boldsymbol{u}_h - \boldsymbol{I}_h\boldsymbol{u}) \leq c\sqrt{\mu} h^k |\boldsymbol{u}|_{k+1,\Omega} |||\boldsymbol{u}_h - \boldsymbol{I}_h\boldsymbol{u}|||.$$

On the other hand, we write that

$$b_h(p - p_h, \boldsymbol{u}_h - \boldsymbol{I}_h\boldsymbol{u}) = b_h(p - i_h p, \boldsymbol{u}_h - \boldsymbol{I}_h\boldsymbol{u}) + b_h(i_h p - p_h, \boldsymbol{u}_h - \boldsymbol{I}_h\boldsymbol{u})$$

and we notice that, thanks to the second equation of (8) and to the interpolation properties of $\boldsymbol{I}_h$,

$$b_h(i_h p - p_h, \boldsymbol{u}_h - I_h\boldsymbol{u}) = -b_h(i_h p - p_h, I_h\boldsymbol{u}) = -b_h(i_h p - p_h, \boldsymbol{u}) = 0.$$

Using now Lemma 7.4, it follows that

$$b_h(p - p_h, \boldsymbol{u}_h - \boldsymbol{I}_h\boldsymbol{u}) \leq \frac{c}{\sqrt{\mu}} h^k |p|_{k,\Omega} |||\boldsymbol{u}_h - \boldsymbol{I}_h\boldsymbol{u}|||.$$

From (24), we obtain that

$$|||\, \boldsymbol{u}_h - \boldsymbol{I}_h\boldsymbol{u} \,||| \leq c h^k (\sqrt{\mu} |\boldsymbol{u}|_{k+1,\Omega} + \frac{1}{\sqrt{\mu}} |p|_{k,\Omega}). \tag{25}$$

Finally, by means of the triangle inequality and thanks to the relation

$$J(\boldsymbol{u} - \boldsymbol{I}_h\boldsymbol{u}, \boldsymbol{u} - \boldsymbol{I}_h\boldsymbol{u}) = J(\boldsymbol{I}_h\boldsymbol{u}, \boldsymbol{I}_h\boldsymbol{u}) = 0, \tag{26}$$

we deduce (22). In order to establish the error estimate for the pressure, we write that

$$\|p - p_h\|_{0,\Omega} \leq \|p - i_h p\|_{0,\Omega} + \|i_h p - p_h\|_{0,\Omega}.$$

According to the discrete inf-sup condition (see Lemma 5.3), one has that

$$\|i_h p - p_h\|_{0,\Omega} \leq \frac{\sqrt{\mu}}{\beta_1} \sup_{\boldsymbol{v}_h \in \boldsymbol{V}_h} \frac{b_h(i_h p - p_h, \boldsymbol{v}_h)}{|||\boldsymbol{v}_h|||} = \frac{\sqrt{\mu}}{\beta_1} \sup_{\boldsymbol{v}_h \in \boldsymbol{V}_h} \frac{b_h(i_h p - p, \boldsymbol{v}_h) + b_h(p - p_h, \boldsymbol{v}_h)}{|||\boldsymbol{v}_h|||}.$$

The continuity property of Lemma 7.4 gives :

$$b_h(i_h p - p, \boldsymbol{v}_h) \leq \frac{c}{\sqrt{\mu}} h^k |||\boldsymbol{v}_h||| |p|_{k,\Omega}$$

whereas Lemma 7.1 together with the first variational equation of (8) yield

$$b_h(p - p_h, \boldsymbol{v}_h) = -a_h(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{v}_h), \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h.$$

One can then show, using Lemma 7.3 and the continuity of $a_h(\cdot,\cdot)$ on $\boldsymbol{V}_h$, that

$$\begin{aligned}a_h(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{v}_h) &= a_h(\boldsymbol{u} - \boldsymbol{I}_h\boldsymbol{u}, \boldsymbol{v}_h) + a_h(\boldsymbol{I}_h\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{v}_h) \\ &\leq c\sqrt{\mu} h^k |\boldsymbol{u}|_{k+1,\Omega} |||\boldsymbol{v}_h||| + 2 \; |||\, \boldsymbol{I}_h\boldsymbol{u} - \boldsymbol{u}_h \,||| \; |||\boldsymbol{v}_h|||.\end{aligned}$$

In conclusion, we obtain that

$$\|i_h p - p_h\|_{0,\Omega} \leq c h^k (\mu |\boldsymbol{u}|_{k+1,\Omega} + |p|_{k,\Omega}) + c\sqrt{\mu} \; |||\, \boldsymbol{I}_h\boldsymbol{u} - \boldsymbol{u}_h \,|||,$$

which together with (25) imply the desired estimate (23).

Theorem 7.6. *Suppose Ω is convex, $(\boldsymbol{u}, p) \in \boldsymbol{H}^{k+1}(\Omega) \times H^k(\Omega)$ and γ is sufficiently large. Then there exists a constant c independent of h and μ such that*

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{0,\Omega} \le ch^{k+1}(|\boldsymbol{u}|_{k+1,\Omega} + \frac{1}{\mu}|p|_{k,\Omega}).$$

Proof. We adapt Nitsche's argument to our dG discretization. We follow the same steps as in [17], but with a different norm $|||\cdot|||$ resulting from a different stabilization.

Let us first recall that, due to the convexity of Ω, the Stokes problem

$$\begin{cases} -\mu\Delta\boldsymbol{\phi} + \nabla\xi &= \boldsymbol{\psi} \quad \text{in } \Omega \\ \nabla\cdot\boldsymbol{\phi} &= 0 \quad \text{in } \Omega \\ \boldsymbol{\phi} &= \boldsymbol{0} \quad \text{on } \partial\Omega \end{cases}$$

admits a unique solution $(\boldsymbol{\phi}, \xi) \in \boldsymbol{H}^2(\Omega) \times H^1(\Omega)$ which moreover satisfies :

$$\mu\,\|\boldsymbol{\phi}\|_{2,\Omega} + \|\xi\|_{1,\Omega} \le c\,\|\boldsymbol{\psi}\|_{0,\Omega}\,. \tag{27}$$

We next consider the dual problem with $\boldsymbol{\psi} = \boldsymbol{u}_h - \boldsymbol{u}$ and we write, thanks to an integration by parts on each triangle $T \in \mathcal{T}_h$ and to the regularity of $\boldsymbol{\phi}$ and ξ, that

$$\begin{aligned} \|\boldsymbol{u} - \boldsymbol{u}_h\|^2_{0,\Omega} &= \int_\Omega (\boldsymbol{u}_h - \boldsymbol{u})\cdot(-\mu\Delta\boldsymbol{\phi} + \nabla\xi)dx \\ &= a_h(\boldsymbol{\phi}, \boldsymbol{u}_h - \boldsymbol{u}) + b_h(\xi, \boldsymbol{u}_h - \boldsymbol{u}). \end{aligned}$$

By using the orthogonality equations (cf. Lemma 7.1)

$$\begin{aligned} a_h(\boldsymbol{u}_h - \boldsymbol{u}, \boldsymbol{v}_h) + b_h(p_h - p, \boldsymbol{v}_h) &= 0, \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h \\ b_h(q_h, \boldsymbol{u}_h - \boldsymbol{u}) &= 0, \quad \forall q_h \in Q_h, \end{aligned}$$

we obtain, by choosing $\boldsymbol{v}_h = \boldsymbol{I}_h\boldsymbol{\phi}$ and $q_h = i_h\xi$:

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|^2_{0,\Omega} = a_h(\boldsymbol{\phi} - \boldsymbol{I}_h\boldsymbol{\phi}, \boldsymbol{u}_h - \boldsymbol{u}) + b_h(\xi - i_h\xi, \boldsymbol{u}_h - \boldsymbol{u}) - b_h(p_h - p, \boldsymbol{I}_h\boldsymbol{\phi}). \tag{28}$$

In what follows, we estimate each of the three righthand side terms.

The last one can be bounded exactly as in [17]. Let us give some details, for the sake of clarity. First, thanks to the properties of $\boldsymbol{\phi}$ and of the interpolation operators $\boldsymbol{I}_h$ and i_h, we write that

$$b_h(p_h - p, \boldsymbol{I}_h\boldsymbol{\phi}) = b_h(i_hp - p, \boldsymbol{I}_h\boldsymbol{\phi} - \boldsymbol{\phi}) = \sum_{e\in\varepsilon_h}\int_e \{i_hp - p\}[(\boldsymbol{I}_h\boldsymbol{\phi} - \boldsymbol{\phi})\cdot\boldsymbol{n}_e]ds.$$

Then we employ interpolation estimates (see also Lemma 7.4) and the regularity stated in (27) in order to conclude that

$$b_h(p_h - p, \boldsymbol{I}_h\boldsymbol{\phi}) \le \frac{c}{\mu}h^{k+1}\,|p|_{k,\Omega}\,\|\boldsymbol{u} - \boldsymbol{u}_h\|_{0,\Omega}\,. \tag{29}$$

Concerning the second righthand side term of (28), we can write thanks to (4) and to the properties of $\boldsymbol{u}$ that

$$\begin{aligned} b_h(\xi - i_h\xi, \boldsymbol{u}_h - \boldsymbol{u}) &= \sum_{e\in\varepsilon_h}\int_e \{\xi - i_h\xi\}[(\boldsymbol{u}_h - \boldsymbol{L}_h\boldsymbol{u})\cdot\boldsymbol{n}_e]ds \\ &\le ch\,\|\boldsymbol{u} - \boldsymbol{u}_h\|_{0,\Omega}\left(\sum_{e\in\varepsilon_h}\frac{1}{|e|}\,\|[\boldsymbol{u}_h - \boldsymbol{L}_h\boldsymbol{u}]\|^2_{0,e}\right)^{1/2}. \end{aligned}$$

By using Lemma 7.2, the fact that $[\boldsymbol{u}] = [\boldsymbol{L}_h\boldsymbol{u}] = \boldsymbol{0}$ as well as the error estimate (22), it follows that

$$\left(\sum_{e\in\varepsilon_h}\frac{1}{|e|}\,\|[\boldsymbol{u}_h - \boldsymbol{L}_h\boldsymbol{u}]\|^2_{0,e}\right)^{1/2} \le c\left(|\boldsymbol{u} - \boldsymbol{L}_h\boldsymbol{u}|_{1,\Omega} + \frac{1}{\sqrt{\mu}}\,|||\boldsymbol{u} - \boldsymbol{u}_h|||\right) \le ch^k(|\boldsymbol{u}|_{k+1,\Omega} + \frac{1}{\mu}|p|_{k,\Omega})$$

so finally,

$$b_h(\xi - i_h\xi, \boldsymbol{u}_h - \boldsymbol{u}) \le ch^{k+1}(|\boldsymbol{u}|_{k+1,\Omega} + \frac{1}{\mu}|p|_{k,\Omega})\, \|\boldsymbol{u} - \boldsymbol{u}_h\|_{0,\Omega}\,. \tag{30}$$

We still have to estimate

$$a_h(\boldsymbol{\phi} - \boldsymbol{I}_h\boldsymbol{\phi}, \boldsymbol{u}_h - \boldsymbol{u}) = A_0(\boldsymbol{\phi} - \boldsymbol{I}_h\boldsymbol{\phi}, \boldsymbol{u}_h - \boldsymbol{u}) + A_1(\boldsymbol{\phi} - \boldsymbol{I}_h\boldsymbol{\phi}, \boldsymbol{u}_h - \boldsymbol{u}) + \gamma J(\boldsymbol{\phi} - \boldsymbol{I}_h\boldsymbol{\phi}, \boldsymbol{u}_h - \boldsymbol{u}).$$

The following bound is straightforward :

$$\begin{aligned} A_0(\boldsymbol{\phi} - \boldsymbol{I}_h\boldsymbol{\phi}, \boldsymbol{u}_h - \boldsymbol{u}) + \gamma J(\boldsymbol{\phi} - \boldsymbol{I}_h\boldsymbol{\phi}, \boldsymbol{u}_h - \boldsymbol{u}) &\le |||\boldsymbol{\phi} - \boldsymbol{I}_h\boldsymbol{\phi}|||\; |||\boldsymbol{u}_h - \boldsymbol{u}||| \\ &\le ch^{k+1}(|\boldsymbol{u}|_{k+1,\Omega} + \frac{1}{\mu}|p|_{k,\Omega})\, \|\boldsymbol{u} - \boldsymbol{u}_h\|_{0,\Omega}\,, \end{aligned}$$

so let us next consider

$$A_1(\boldsymbol{\phi} - \boldsymbol{I}_h\boldsymbol{\phi}, \boldsymbol{u}_h - \boldsymbol{u}) = -\mu \sum_{e\in\varepsilon_h} \left(\int_e \left\{ \frac{\partial(\boldsymbol{\phi} - \boldsymbol{I}_h\boldsymbol{\phi})}{\partial \boldsymbol{n}_e} \right\} \cdot [\boldsymbol{u}_h - \boldsymbol{u}]\, ds + \int_e \left\{ \frac{\partial(\boldsymbol{u}_h - \boldsymbol{u})}{\partial \boldsymbol{n}_e} \right\} \cdot [\boldsymbol{\phi} - \boldsymbol{I}_h\boldsymbol{\phi}]\, ds \right).$$

Similarly to the proof of Lemma 7.3, we obtain that

$$\begin{aligned} \mu \sum_{e\in\varepsilon_h} \int_e \left\{ \frac{\partial(\boldsymbol{\phi} - \boldsymbol{I}_h\boldsymbol{\phi})}{\partial \boldsymbol{n}_e} \right\} \cdot [\boldsymbol{u}_h - \boldsymbol{u}]\, ds &\le c\mu h |\boldsymbol{\phi}|_{2,\Omega} \left(\sum_{e\in\varepsilon_h} \frac{1}{|e|} \|[\boldsymbol{u}_h - \boldsymbol{L}_h\boldsymbol{u}]\|_{0,e}^2 \right)^{1/2} \\ &\le ch^{k+1}(|\boldsymbol{u}|_{k+1,\Omega} + \frac{1}{\mu}|p|_{k,\Omega})\, \|\boldsymbol{u} - \boldsymbol{u}_h\|_{0,\Omega}\,, \end{aligned}$$

whereas we can write, thanks to (5) and to the fact that $\left\{ \frac{\partial(\boldsymbol{u}_h - \boldsymbol{L}_h\boldsymbol{u})}{\partial \boldsymbol{n}_e} \right\} \in \boldsymbol{P}_{k-1}$ on every edge $e \in \varepsilon_h$, that :

$$\begin{aligned} \mu \sum_{e\in\varepsilon_h} \int_e \left\{ \frac{\partial(\boldsymbol{u}_h - \boldsymbol{u})}{\partial \boldsymbol{n}_e} \right\} \cdot [\boldsymbol{\phi} - \boldsymbol{I}_h\boldsymbol{\phi}]\, ds &= \mu \sum_{e\in\varepsilon_h} \int_e \left\{ \frac{\partial(\boldsymbol{L}_h\boldsymbol{u} - \boldsymbol{u})}{\partial \boldsymbol{n}_e} \right\} \cdot [\boldsymbol{\phi} - \boldsymbol{I}_h\boldsymbol{\phi}]\, ds \\ &\le c\mu h^{k+1}\, |\boldsymbol{\phi}|_{2,\Omega}\, |\boldsymbol{u}|_{k+1,\Omega} \le ch^{k+1}\, |\boldsymbol{u}|_{k+1,\Omega}\, \|\boldsymbol{u} - \boldsymbol{u}_h\|_{0,\Omega}\,. \end{aligned}$$

The previous estimates yield that

$$a_h(\boldsymbol{\phi} - \boldsymbol{I}_h\boldsymbol{\phi}, \boldsymbol{u}_h - \boldsymbol{u}) \le ch^{k+1}(|\boldsymbol{u}|_{k+1,\Omega} + \frac{1}{\mu}|p|_{k,\Omega})\, \|\boldsymbol{u} - \boldsymbol{u}_h\|_{0,\Omega}\,. \tag{31}$$

By gathering together (28), (29), (30) and (31), one now obtains the announced result.

One can equally establish optimal error bounds for the other two-fields formulation (9).

Theorem 7.7. *Let $(\boldsymbol{u}, p) \in \boldsymbol{H}^{k+1}(\Omega) \times H^k(\Omega)$ be the solution of the continuous Stokes problem and let γ be sufficiently large (as in Lemma 5.1). Then the solution $(\boldsymbol{U}_h, P_h)$ of (9) satisfies :*

$$\begin{aligned} [[\boldsymbol{u} - \boldsymbol{U}_h]] &\le ch^k(\sqrt{\mu}|\boldsymbol{u}|_{k+1,\Omega} + \frac{1}{\sqrt{\mu}}|p|_{k,\Omega}) \\ \|p - P_h\|_{0,\Omega} &\le ch^k(\mu|\boldsymbol{u}|_{k+1,\Omega} + |p|_{k,\Omega}) \end{aligned}$$

with a constant c independent of h and μ. Moreover, if Ω is convex then

$$\|\boldsymbol{u} - \boldsymbol{U}_h\|_{0,\Omega} \le ch^{k+1}(|\boldsymbol{u}|_{k+1,\Omega} + \frac{1}{\mu}|p|_{k,\Omega}).$$

Proof. The proof is similar to those of Theorem 7.5 and Theorem 7.6.

Concerning the error in the energy norm, one only needs to note the following additional estimates, for any $\boldsymbol{v}_h \in \boldsymbol{V}_h$:

$$\begin{aligned} J_1(\boldsymbol{u} - \boldsymbol{I}_h \boldsymbol{u}, \boldsymbol{v}_h) &= 0 \quad \text{for } k = 2,\ 3 \\ J_1(\boldsymbol{u} - \boldsymbol{I}_h \boldsymbol{u}, \boldsymbol{v}_h) &\leq c\sqrt{\mu} \left(\sum_{e \in \varepsilon_h^{int}} \frac{1}{|e|} \| [\boldsymbol{u} - \boldsymbol{I}_h \boldsymbol{u}] \|_{0,e}^2 \right)^{1/2} [[\boldsymbol{v}_h]] \\ &\leq c\sqrt{\mu} h^k |\boldsymbol{u}|_{k+1,\Omega} \, [[\boldsymbol{v}_h]] \quad \text{for } k = 1 \\ |||\boldsymbol{v}_h||| &\leq c \, [[\boldsymbol{v}_h]] \, , \end{aligned}$$

with c independent of h and μ. The first one results from the trace inequality (12) and the interpolation error (6) while the second one is an immediate consequence of (20).

As regards the $\boldsymbol{L}^2$-norm of the velocity error, the main change is that one now has to bound the term $\sum_{e \in \varepsilon_h} \frac{1}{\sqrt{|e|}} \| [\boldsymbol{U}_h - \boldsymbol{L}_h \boldsymbol{u}] \|_{0,e}$. Thanks to Lemma 7.2 and to the relations $[\boldsymbol{L}_h \boldsymbol{u}] = [\boldsymbol{u}] = \boldsymbol{0}$, this can be done as follows :

$$\begin{aligned} \sum_{e \in \varepsilon_h} \frac{1}{\sqrt{|e|}} \| [\boldsymbol{U}_h - \boldsymbol{L}_h \boldsymbol{u}] \|_{0,e} &\leq \frac{c}{\sqrt{\mu}} |||\boldsymbol{U}_h - \boldsymbol{L}_h \boldsymbol{u}||| \leq \frac{c}{\sqrt{\mu}} [[\boldsymbol{U}_h - \boldsymbol{L}_h \boldsymbol{u}]] \\ &\leq c \left(\| \underline{D}(\boldsymbol{u}) - \underline{D}(\boldsymbol{L}_h \boldsymbol{u}) \|_{0,\Omega} + \frac{1}{\sqrt{\mu}} [[\boldsymbol{u} - \boldsymbol{U}_h]] \right) \\ &\leq c h^k (|\boldsymbol{u}|_{k+1,\Omega} + \frac{1}{\mu} |p|_{k,\Omega}). \end{aligned}$$

8. A posteriori error estimates

Our analysis follows the idea of Kim [21], who proposed an *a posteriori* error indicator for the Laplace equation based on the reconstruction of a locally conservative $H(div, \Omega)$-conforming vector approximation.

We perform here only the *a posteriori* analysis of problem (8); analoguous results can be established for the formulation (9). For this purpose, we put

$$\underline{H}(div, \Omega) = \left\{ \underline{\theta} \in \underline{L}^2(\Omega);\ div \underline{\theta} \in \boldsymbol{L}^2(\Omega) \right\}$$

and we introduce the Raviart-Thomas finite element space (cf. [26])

$$\underline{\Sigma}_h = \left\{ \underline{\theta}_h \in \underline{H}(div, \Omega);\ (\underline{\theta}_h)_{/T} \in \underline{RT}_{k-1},\ \ \forall T \in \mathcal{T}_h \right\}$$

where $\underline{RT}_{k-1} = \underline{P}_{k-1} + \boldsymbol{P}_{k-1} \otimes \boldsymbol{x}$. Then we construct a tensor $\underline{\sigma}_h \in \underline{\Sigma}_h$ from the solution $(\boldsymbol{u}_h, p_h)$ of (8) by specifying its degrees of freedom as follows :

$$\underline{\sigma}_h \boldsymbol{n}_e = \mu \left\{ \frac{\partial \boldsymbol{u}_h}{\partial \boldsymbol{n}_e} \right\} - \frac{\mu \gamma}{|e|} [\boldsymbol{\pi}_{k-1} \boldsymbol{u}_h] - \{p_h\} \, \boldsymbol{n}_e, \quad \forall e \in \varepsilon_h$$

and for $k = 2$ or 3,

$$\int_T \underline{\sigma}_h : \underline{r} dx = \int_T (\mu \underline{\nabla} \boldsymbol{u}_h - p_h \underline{I}) : \underline{r} dx, \quad \forall T \in \mathcal{T}_h \text{ and } \forall \underline{r} \in \underline{P}_{k-2}.$$

By taking as test-function $\boldsymbol{v}_h$ in the dG formulation (8) a piecewise polynomial of degree $k-1$, it follows that

$$\int_T (div \underline{\sigma}_h + \boldsymbol{f}) \cdot \boldsymbol{r} dx = 0, \quad \forall T \in \mathcal{T}_h \text{ and } \forall \boldsymbol{r} \in \boldsymbol{P}_{k-1} \tag{32}$$

so one has $(div \underline{\sigma}_h)_{/T} = -\boldsymbol{\pi}_{k-1}^T \boldsymbol{f}$ on every triangle $T \in \mathcal{T}_h$. Obviously, $\underline{\sigma}_h$ is locally conservative. It is useful to introduce $\underline{\sigma} = \mu \underline{\nabla} \boldsymbol{u} - p \underline{I}$ which clearly belongs to $\underline{H}(div, \Omega)$.

We next define, following [21], a residual-type error estimator by

$$\eta_h^2 = \frac{1}{\mu} \sum_{T\in\mathcal{T}_h} \|\underline{\sigma}_h - \mu\underline{\nabla}\boldsymbol{u}_h + p_h\underline{I}\|_{0,T}^2 + J^*(\boldsymbol{u}_h, \boldsymbol{u}_h).$$

In order to establish the reliability of η_h, let us denote by $S(\cdot,\cdot)$ the bilinear form of the continuous Stokes problem (1), which we extend on $(\boldsymbol{H}_0^1(\Omega) + \boldsymbol{V}_h) \times L_0^2(\Omega)$ as follows :

$$S((\boldsymbol{u}, p), (\boldsymbol{v}, q)) = \mu \sum_{T\in\mathcal{T}_h} \int_T \underline{\nabla}\boldsymbol{u} : \underline{\nabla}\boldsymbol{v} dx - \sum_{T\in\mathcal{T}_h} \int_T p\nabla\cdot\boldsymbol{v} dx + \sum_{T\in\mathcal{T}_h} \int_T q\nabla\cdot\boldsymbol{u} dx$$

and let $(\boldsymbol{\phi}, \xi) \in \boldsymbol{H}_0^1(\Omega) \times L_0^2(\Omega)$ be the unique solution of

$$S((\boldsymbol{\phi}, \xi), (\boldsymbol{v}, q)) = S((\boldsymbol{u}_h, p_h), (\boldsymbol{v}, q)), \quad \forall (\boldsymbol{v}, q) \in \boldsymbol{H}_0^1(\Omega) \times L_0^2(\Omega). \tag{33}$$

Then we split the error by means of the triangle inequality :

$$\sqrt{\mu}\,|\boldsymbol{u} - \boldsymbol{u}_h|_{1,\Omega} + \tfrac{1}{\sqrt{\mu}}\,\|p - p_h\|_{0,\Omega} \le$$
$$\left(\sqrt{\mu}\,|\boldsymbol{u} - \boldsymbol{\phi}|_{1,\Omega} + \tfrac{1}{\sqrt{\mu}}\,\|p - \zeta\|_{0,\Omega}\right) + \left(\sqrt{\mu}\,|\boldsymbol{u}_h - \boldsymbol{\phi}|_{1,\Omega} + \tfrac{1}{\sqrt{\mu}}\,\|p_h - \zeta\|_{0,\Omega}\right)$$

and in what follows, we bound each righthand-side term with respect to η_h.

Lemme 1. There exists $c > 0$ depending on Ω and k such that

$$\sqrt{\mu}\,|\boldsymbol{u} - \boldsymbol{\phi}|_{1,\Omega} + \tfrac{1}{\sqrt{\mu}}\,\|p - \zeta\|_{0,\Omega} \le$$
$$c\left(\tfrac{1}{\sqrt{\mu}}\,\|\underline{\sigma}_h - \mu\underline{\nabla}\boldsymbol{u}_h + p_h\underline{I}\|_{0,\Omega} + \textstyle\sum_{T\in\mathcal{T}_h} \tfrac{h_T}{\sqrt{\mu}}\,\left\|\boldsymbol{f} - \boldsymbol{\pi}_{k-1}^T\boldsymbol{f}\right\|_{0,T} + \sqrt{J^*(\boldsymbol{u}_h, \boldsymbol{u}_h)}\right).$$

Proof. The well-posedness of the continuous Stokes problem implies that :

$$\sqrt{\mu}\,|\boldsymbol{u} - \boldsymbol{\phi}|_{1,\Omega} + \frac{1}{\sqrt{\mu}}\,\|p - \zeta\|_{0,\Omega} \le c \sup_{(\boldsymbol{v},q)\in\boldsymbol{H}_0^1(\Omega)\times L_0^2(\Omega)} \frac{S((\boldsymbol{u} - \boldsymbol{\phi}, p - \xi), (\boldsymbol{v}, q))}{\sqrt{\mu}\,|\boldsymbol{v}|_{1,\Omega} + \frac{1}{\sqrt{\mu}}\,\|q\|_{0,\Omega}}$$

where the constant depends on Ω. Using now that

$$S((\boldsymbol{u} - \boldsymbol{\phi}, p - \xi), (\boldsymbol{v}, q)) = S((\boldsymbol{u} - \boldsymbol{u}_h, p - p_h), (\boldsymbol{v}, q))$$
$$= \textstyle\int_\Omega(\underline{\sigma} - \underline{\sigma}_h) : \nabla\boldsymbol{v} dx + \sum_{T\in\mathcal{T}_h} \int_T(\underline{\sigma}_h - \mu\underline{\nabla}\boldsymbol{u}_h + p_h\underline{I}) : \nabla\boldsymbol{v} dx$$
$$+ \textstyle\sum_{T\in\mathcal{T}_h} \int_T q\nabla\cdot(\boldsymbol{u} - \boldsymbol{u}_h) dx$$

it follows, after integrating by parts and after using (32), that

$$\begin{aligned} S((\boldsymbol{u} - \boldsymbol{\phi}, p - \xi), (\boldsymbol{v}, q)) \quad &= \quad \sum_{T\in\mathcal{T}_h} \left(\int_T (\underline{\sigma}_h - \mu\underline{\nabla}\boldsymbol{u}_h + p_h\underline{I}) : \nabla\boldsymbol{v} dx - \int_T q\nabla\cdot\boldsymbol{u}_h dx\right) \\ &\quad + \sum_{T\in\mathcal{T}_h} \int_T (\boldsymbol{f} - \boldsymbol{\pi}_{k-1}^T\boldsymbol{f})\cdot\boldsymbol{v} dx \end{aligned}$$

which yields

$$\sqrt{\mu}\,|\boldsymbol{u} - \boldsymbol{\phi}|_{1,\Omega} + \tfrac{1}{\sqrt{\mu}}\,\|p - \zeta\|_{0,\Omega} \le$$
$$\frac{c}{\sqrt{\mu}} \sum_{T\in\mathcal{T}_h} \left(h_T\left\|\boldsymbol{f} - \boldsymbol{\pi}_{k-1}^T\boldsymbol{f}\right\|_{0,T} + \|\underline{\sigma}_h - \mu\underline{\nabla}\boldsymbol{u}_h + p_h\underline{I}\|_{0,T} + \mu\,\|\nabla\cdot\boldsymbol{u}_h\|_{0,T}\right)$$

In order to bound $\|\nabla\cdot\boldsymbol{u}_h\|_{0,T}$, let us take as test-function q_h in (8) the function given by $\nabla\cdot\boldsymbol{u}_h$ on T and 0 elsewhere. Then

$$\begin{aligned}\|\nabla\cdot\boldsymbol{u}_h\|_{0,T}^2 &= \frac{1}{2}\sum_{e\subset\partial T}\int_e(\nabla\cdot\boldsymbol{u}_h)[\boldsymbol{u}_h\cdot\boldsymbol{n}_e]ds\\ &\le c\,\|\nabla\cdot\boldsymbol{u}_h\|_{0,T}\left(\sum_{e\subset\partial T}\frac{1}{|e|}\,\|[\boldsymbol{u}_h]\|_{0,e}^2\right)^{1/2},\end{aligned}$$

where c depends on the polynomial degree k. So it follows that

$$\left(\sum_{T\in\mathcal{T}_h}\mu\,\|\nabla\cdot\boldsymbol{u}_h\|_{0,T}^2\right)^{1/2}\le c\sqrt{J^*(\boldsymbol{u}_h,\boldsymbol{u}_h)}$$

which finally leads to the announced estimate.

Lemme 2. There exists $c>0$ depending on k and Ω such that :

$$\sqrt{\mu}\,|\boldsymbol{u}_h-\boldsymbol{\phi}|_{1,h}+\frac{1}{\sqrt{\mu}}\,\|p_h-\zeta\|_{0,\Omega}\le c\sqrt{J^*(\boldsymbol{u}_h,\boldsymbol{u}_h)}.$$

Proof. The continuous inf-sup condition on $b(\cdot,\cdot)$ implies that

$$c\,\|p_h-\zeta\|_{0,\Omega}\le\sup_{\boldsymbol{v}\in\boldsymbol{H}_0^1(\Omega)}\frac{b(p_h-\xi,\boldsymbol{v})}{|\boldsymbol{v}|_{1,\Omega}}=\sup_{\boldsymbol{v}\in\boldsymbol{H}_0^1(\Omega)}\frac{a(\boldsymbol{u}_h-\boldsymbol{\phi},\boldsymbol{v})}{|\boldsymbol{v}|_{1,\Omega}}\le\mu\,|\boldsymbol{u}_h-\boldsymbol{\phi}|_{1,h}$$

therefore it is sufficient to bound $\sqrt{\mu}\,|\boldsymbol{u}_h-\boldsymbol{\phi}|_{1,h}$.

A simple calculation together with (33) yield, for any $\boldsymbol{v}\in\boldsymbol{H}_0^1(\Omega)$:

$$\begin{aligned}&\mu\,|\boldsymbol{u}_h-\boldsymbol{\phi}|_{1,h}^2-\mu\,|\boldsymbol{u}_h-\boldsymbol{v}|_{1,h}^2\\ &=S((\boldsymbol{u}_h-\boldsymbol{\phi},p_h-\xi),(\boldsymbol{u}_h-\boldsymbol{\phi},p_h-\xi))-S((\boldsymbol{u}_h-\boldsymbol{v},p_h-\xi),(\boldsymbol{u}_h-\boldsymbol{v},p_h-\xi))\\ &=S((\boldsymbol{u}_h-\boldsymbol{\phi},p_h-\xi),(\boldsymbol{v}-\boldsymbol{\phi},0))+S((\boldsymbol{v}-\boldsymbol{\phi},0),(\boldsymbol{u}_h-\boldsymbol{\phi},p_h-\xi))\\ &\quad-S((\boldsymbol{\phi}-\boldsymbol{v},0),(\boldsymbol{\phi}-\boldsymbol{v},0))\\ &=2\textstyle\sum_{T\in\mathcal{T}_h}\int_T(p_h-\xi)\nabla\cdot(\boldsymbol{v}-\boldsymbol{\phi})dx-\mu\,|\boldsymbol{\phi}-\boldsymbol{v}|_{1,\Omega}^2\\ &\le 2\textstyle\sum_{T\in\mathcal{T}_h}\int_T(p_h-\xi)\nabla\cdot(\boldsymbol{v}-\boldsymbol{u}_h)dx.\end{aligned}$$

Then it follows, with $C=1/c$, that

$$|\boldsymbol{u}_h-\boldsymbol{\phi}|_{1,h}^2\le|\boldsymbol{u}_h-\boldsymbol{v}|_{1,h}^2+2C\,|\boldsymbol{u}_h-\boldsymbol{\phi}|_{1,h}\,|\boldsymbol{u}_h-\boldsymbol{v}|_{1,\Omega}$$

and hence,

$$|\boldsymbol{u}_h-\boldsymbol{\phi}|_{1,h}\le(C+\sqrt{1+C^2})\,|\boldsymbol{u}_h-\boldsymbol{v}|_{1,h}\,,\quad\forall\boldsymbol{v}\in\boldsymbol{H}_0^1(\Omega).$$

We proceed as in [21] (see also [20]) and we take for $\boldsymbol{v}\in\boldsymbol{C}^0(\overline{\Omega})$ the piecewise $\boldsymbol{P}_k$ function defined by its values at the Lagrangian nodes $\boldsymbol{z}$ as follows :

$$\boldsymbol{v}(\boldsymbol{z})=(\boldsymbol{u}_h)_{/T}(\boldsymbol{z}),$$

where $T\in\mathcal{T}_h$ is such that $\boldsymbol{z}\in\overline{T}$. Then one can establish (cf. [21]) that :

$$|\boldsymbol{u}_h-\boldsymbol{v}|_{1,h}\le c\left(\sum_{e\in\varepsilon_h}\frac{1}{\sqrt{|e|}}\|[\boldsymbol{u}_h]\|_{0,e}\right)$$

where c depends on k. This ends the lemma's proof.

The two previous lemmas together with the obvious inequality :

$$J(\boldsymbol{u}_h,\boldsymbol{u}_h)\le J^*(\boldsymbol{u}_h,\boldsymbol{u}_h)$$

allow us to conclude to the reliability of η_h, stated in the next theorem.

Theorem 8.1. *There exists a constant c depending on k and Ω such that :*

$$|||\boldsymbol{u}-\boldsymbol{u}_h||| + \frac{1}{\sqrt{\mu}}\|p-p_h\|_{0,\Omega} \leq (c+\sqrt{\gamma})\eta_h + c\sum_{T\in\mathcal{T}_h}\frac{h_T}{\sqrt{\mu}}\left\|\boldsymbol{f}-\boldsymbol{\pi}_{k-1}^T\boldsymbol{f}\right\|_{0,T}.$$

We are now interested in the efficiency of the *a posteriori* error indicator. Let us introduce for any triangle $T\in\mathcal{T}_h$ the local contribution

$$\eta_T^2 = \frac{1}{\mu}\|\underline{\sigma}_h - \mu\underline{\nabla}\boldsymbol{u}_h + p_h\underline{I}\|_{0,T}^2 + \sum_{e\subset\partial T}\frac{\mu}{|e|}\|[\boldsymbol{u}_h]\|_{0,e}^2.$$

It is useful to recall some results of Verfürth [29]. Let b_T denote the cubic bubble function on $T\in\mathcal{T}_h$ and b_e the quadratic bubble function on $e\in\varepsilon_h$, satisfying $0\leq b_T\leq 1=\max b_T$ and $0\leq b_e\leq 1=\max b_e$. Then there exists a constant C depending only on the minimum angle of $\mathcal{T}_h$ and on the polynomial degree r such that, for any $v\in P_r(T)$:

$$\|v\|_{0,T}\leq C\left\|b_T^{1/2}v\right\|_{0,T},\quad \|v\|_{0,e}\leq C\left\|b_e^{1/2}v\right\|_{0,e}. \tag{34}$$

Moreover, there exists an operator P_e which extends any function defined on $e\in\varepsilon_h$ to the triangle T and satisfies :

$$C_1\sqrt{|e|}\,\|v\|_{0,e}\leq\|b_eP_ev\|_{0,T}\leq C_2\sqrt{|e|}\,\|v\|_{0,e},\quad \forall v\in P_r(e). \tag{35}$$

Then we have :

Theorem 8.2. *There exists a constant c depending on the minimum angle of $\mathcal{T}_h$ and on k such that, for any $T\in\mathcal{T}_h$, one has*

$$\eta_T^2\leq c\left(\mu(\sum_{S\in\omega_T}|\boldsymbol{u}-\boldsymbol{u}_h|_{1,S}^2)+\frac{1}{\mu}\|p-p_h\|_{0,\omega_T}^2+\sum_{e\subset\partial T}\frac{(1+\gamma)^2\mu}{|e|}\|[\boldsymbol{\pi}_{k-1}\boldsymbol{u}_h]\|_{0,e}^2\right)$$

where ω_T is the set of all elements sharing an edge with T. Consequently,

$$\eta_h\leq c\left(\sqrt{\mu}\,|\boldsymbol{u}-\boldsymbol{u}_h|_{1,\Omega}+(1+\gamma)\sqrt{J(\boldsymbol{u}_h,\boldsymbol{u}_h)}+\frac{1}{\sqrt{\mu}}\|p-p_h\|_{0,\Omega}\right).$$

Proof. In order to bound the term $\frac{1}{\sqrt{\mu}}\|\underline{\sigma}_h-\mu\underline{\nabla}\boldsymbol{u}_h+p_h\underline{I}\|_{0,T}$, we point out that $\underline{\sigma}_h-\mu\underline{\nabla}\boldsymbol{u}_h+p_h\underline{I}$ belongs to $\underline{RT}_{k-1}(T)$ and for $k\geq 2$, its $\underline{L}^2(T)$-orthogonal projection on $\underline{P}_{k-2}$ is zero. Therefore $\|\underline{\sigma}_h-\mu\underline{\nabla}\boldsymbol{u}_h+p_h\underline{I}\|_{0,T}$ and $\|(\underline{\sigma}_h-\mu\underline{\nabla}\boldsymbol{u}_h+p_h\underline{I})\boldsymbol{n}_e\|_{0,\partial T}$ are equivalent norms. A scaling argument yields

$$\|\underline{\sigma}_h-\mu\underline{\nabla}\boldsymbol{u}_h+p_h\underline{I}\|_{0,T}\leq c\sum_{e\subset\partial T}\sqrt{|e|}\,\|(\underline{\sigma}_h-\mu\underline{\nabla}\boldsymbol{u}_h+p_h\underline{I})\boldsymbol{n}_e\|_{0,e}$$

with a constant c depending on k. For a given edge $e\subset\partial T$, one next has

$$(\underline{\sigma}_h-\mu\underline{\nabla}\boldsymbol{u}_h+p_h\underline{I})\boldsymbol{n}_e=\pm\left[\mu\underline{\nabla}\boldsymbol{u}_h-p_h\underline{I}\right]\boldsymbol{n}_e-\frac{\mu\gamma}{|e|}\left[\boldsymbol{\pi}_{k-1}\boldsymbol{u}_h\right]$$

the sign depending on the orientation of $\boldsymbol{n}_e$ with respect to T. So

$$\|\underline{\sigma}_h-\mu\underline{\nabla}\boldsymbol{u}_h+p_h\underline{I}\|_{0,T}\leq$$
$$c\textstyle\sum_{e\subset\partial T}\left(\sqrt{|e|}\,\|\left[\mu\underline{\nabla}\boldsymbol{u}_h-p_h\underline{I}\right]\boldsymbol{n}_e\|_{0,e}+\frac{\mu\gamma}{\sqrt{|e|}}\|[\boldsymbol{\pi}_{k-1}\boldsymbol{u}_h]\|_{0,e}\right).$$

Let us remark that an integration by parts gives, for any $\boldsymbol{w}\in\boldsymbol{H}_0^1(\Omega)$, that

$$\begin{aligned}\textstyle\sum_{T\in\mathcal{T}_h}\int_T(\underline{\sigma}-\mu\underline{\nabla}\boldsymbol{u}_h+p_h\underline{I}):\underline{\nabla}\boldsymbol{w}dx+\sum_{T\in\mathcal{T}_h}\int_T div\,(\underline{\sigma}-\mu\underline{\nabla}\boldsymbol{u}_h+p_h\underline{I})\cdot\boldsymbol{w}dx\\ =\textstyle\sum_{T\in\mathcal{T}_h}\int_{\partial T}(\underline{\sigma}-\mu\underline{\nabla}\boldsymbol{u}_h+p_h\underline{I})\boldsymbol{n}\cdot\boldsymbol{w}ds\\ =-\textstyle\sum_{e\in\varepsilon_h}\int_e\left[\mu\underline{\nabla}\boldsymbol{u}_h-p_h\underline{I}\right]\boldsymbol{n}_e\cdot\boldsymbol{w}ds.\end{aligned}$$

We now employ the argument used by Verfüsrth in [29], which is based on the weighted norms by the bubble functions and on inverse inequalities. By taking $\boldsymbol{w} = b_T div(\underline{\sigma} - \mu\underline{\nabla}\boldsymbol{u}_h + p_h\underline{I})$ and using (34), we first get that

$$h_T \left\| div(\underline{\sigma} - \mu\underline{\nabla}\boldsymbol{u}_h + p_h\underline{I})\right\|_{0,T} \le c \left\|\underline{\sigma} - \mu\underline{\nabla}\boldsymbol{u}_h + p_h\underline{I}\right\|_{0,T}.$$

We next take $\boldsymbol{w} = b_e P_e([\mu\underline{\nabla}\boldsymbol{u}_h - p_h\underline{I}]\,\boldsymbol{n}_e)$ and we obtain, thanks to (34) and (35),

$$\sqrt{|e|}\,\left\|[\mu\underline{\nabla}\boldsymbol{u}_h - p_h\underline{I}]\,\boldsymbol{n}_e\right\|_{0,e} \le c \left\|\underline{\sigma} - \mu\underline{\nabla}\boldsymbol{u}_h + p_h\underline{I}\right\|_{0,T_1\cup T_2},$$

with c now depending on the minimum angle of $\mathcal{T}_h$ and on k. This finally implies that :

$$\frac{1}{\sqrt{\mu}}\left\|\underline{\sigma}_h - \mu\underline{\nabla}\boldsymbol{u}_h + p_h\underline{I}\right\|_{0,T} \le$$
$$c\left(\sqrt{\mu}\sum_{S\in\omega_T}|\boldsymbol{u}-\boldsymbol{u}_h|_{1,S} + \frac{1}{\sqrt{\mu}}\|p-p_h\|_{0,\omega_T} + \gamma\sum_{e\subset\partial T}\frac{\sqrt{\mu}}{\sqrt{|e|}}\left\|[\boldsymbol{\pi}_{k-1}\boldsymbol{u}_h]\right\|_{0,e}\right).$$

We still have to bound the remaining term $\frac{\sqrt{\mu}}{\sqrt{|e|}}\|[\boldsymbol{u}_h]\|_{0,e}$ of η_T, on every edge $e\subset\partial T$. For this purpose, we proceed similarly to the proof of Lemma 7.2 and, using that $[\boldsymbol{u}]=0$, we write that:

$$\|[\boldsymbol{u}_h]\|_{0,e} \le \|[(\boldsymbol{u}_h-\boldsymbol{u}) - \boldsymbol{\pi}_{k-1}(\boldsymbol{u}_h-\boldsymbol{u})]\|_{0,e} + \|[\boldsymbol{\pi}_{k-1}\boldsymbol{u}_h]\|_{0,e}.$$

Let S a triangle of $\mathcal{T}_h$ such that $e\subset\partial S$. The trace inequality together with the interpolation properties of $\boldsymbol{\pi}_{k-1}^S$ next give that

$$\frac{1}{\sqrt{|e|}}\|(\boldsymbol{u}_h-\boldsymbol{u}) - \boldsymbol{\pi}_{k-1}(\boldsymbol{u}_h-\boldsymbol{u})\|_{0,e} \le \frac{1}{\sqrt{|e|}}\|(\boldsymbol{u}_h-\boldsymbol{u}) - \boldsymbol{\pi}_{k-1}^S(\boldsymbol{u}_h-\boldsymbol{u})\|_{0,e}$$
$$\le c\left(\frac{1}{h_S}\|(\boldsymbol{u}_h-\boldsymbol{u}) - \boldsymbol{\pi}_{k-1}^S(\boldsymbol{u}_h-\boldsymbol{u})\|_{0,S} + |(\boldsymbol{u}_h-\boldsymbol{u}) - \boldsymbol{\pi}_{k-1}^S(\boldsymbol{u}_h-\boldsymbol{u})|_{1,S}\right)$$
$$\le c|\boldsymbol{u}_h-\boldsymbol{u}|_{1,S}$$

with c depending on k. So it follows that

$$\sum_{e\subset\partial T}\frac{\sqrt{\mu}}{\sqrt{|e|}}\|[\boldsymbol{u}_h]\|_{0,e} \le c\sqrt{\mu}\big(\sum_{S\in\omega_T}|\boldsymbol{u}_h-\boldsymbol{u}|_{1,S}\big) + \sum_{e\subset\partial T}\frac{\sqrt{\mu}}{\sqrt{|e|}}\|[\boldsymbol{\pi}_{k-1}\boldsymbol{u}_h]\|_{0,e}$$

which concludes the proof.

Finally, we show in what follows that the *a posteriori* error indicator η_h tends, as $\gamma\to\infty$, towards an *a posteriori* error estimator of the nonconforming discretization (13) similar to the one developed in [12] . For simplicity, we restrict ourselves to the case $k=1$ and piecewise constant right-hand side.

Theorem 8.3. *Let $k=1$ and the right-hand side $\boldsymbol{f}$ be piecewise constant with respect to $\mathcal{T}_h$. Then we have with $\boldsymbol{u}_h^*$ the nonconforming finite element solution of (13):*

$$\lim_{\gamma\to\infty}\eta_h^2 = \frac{1}{4\mu}\sum_{T\in\mathcal{T}_h}\|\boldsymbol{f}\otimes(\boldsymbol{x}-\boldsymbol{x}_T)\|_{0,T}^2 + J^*(\boldsymbol{u}_h^*,\boldsymbol{u}_h^*). \tag{36}$$

Proof. Following the idea of Marini, we define on each triangle T an element of $\underline{RT}_0$ by

$$\underline{\sigma}_h^* = \mu\underline{\nabla}\boldsymbol{u}_h^* - p_h^*\underline{I} - \frac{1}{2}\boldsymbol{f}\otimes(\boldsymbol{x}-\boldsymbol{x}_T).$$

Let any $\boldsymbol{v}_h\in\boldsymbol{V}_h$. Then obviously $div\underline{\sigma}_h^* = -\boldsymbol{f} = div\underline{\sigma}_h$, $\underline{\sigma}_h^*\in\underline{\Sigma}_h$ and also, using the fact that $\int_T(\boldsymbol{x}-\boldsymbol{x}_T)dx = \boldsymbol{0}$,

$$\sum_{e\in\varepsilon_h}\int_e\underline{\sigma}_h^*\boldsymbol{n}_e\cdot[\boldsymbol{v}_h]\,ds = \int_\Omega div\underline{\sigma}_h^*\cdot\boldsymbol{v}_h dx + \int_\Omega\underline{\sigma}_h^*:\underline{\nabla}\boldsymbol{v}_h dx$$
$$= \sum_{T\in\mathcal{T}_h}\Big(-\int_T\boldsymbol{f}\cdot\boldsymbol{v}_h dx + \mu\int_T\underline{\nabla}\boldsymbol{u}_h^*:\underline{\nabla}\boldsymbol{v}_h dx - \int_T p_h^*\nabla\cdot\boldsymbol{v}_h dx\Big).$$

Meanwhile, a simple computation yields, thanks to problem (8) :

$$\sum_{e\in\varepsilon_h}\int_e \underline{\sigma}_h \boldsymbol{n}_e\cdot[\boldsymbol{v}_h]\,ds = -\int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}_h dx$$
$$+\sum_{T\in\mathcal{T}_h}(\mu\int_T \underline{\nabla}\boldsymbol{u}_h\underline{\nabla}\boldsymbol{v}_h dx - \int_T p_h\nabla\cdot\boldsymbol{v}_h dx) - \mu\sum_{e\in\varepsilon_h}\int_e\left\{\frac{\partial\boldsymbol{v}_h}{\partial\boldsymbol{n}_e}\right\}\cdot[\boldsymbol{u}_h]\,ds.$$

Theorem 6.1 next gives that

$$\begin{aligned}\lim_{\gamma\to\infty}\sum_{e\in\varepsilon_h}\int_e \underline{\sigma}_h\boldsymbol{n}_e\cdot[\boldsymbol{v}_h]\,ds &= -\int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}_h dx + \mu\int_\Omega \underline{\nabla}\boldsymbol{u}_h^* : \underline{\nabla}\boldsymbol{v}_h dx - \int_\Omega p_h^*\nabla\cdot\boldsymbol{v}_h dx \\ &= \sum_{e\in\varepsilon_h}\int_e \underline{\sigma}_h^*\boldsymbol{n}_e\cdot[\boldsymbol{v}_h]\,ds.\end{aligned}$$

By taking as $\boldsymbol{v}_h$ the restriction on T of the basis function of the Crouzeix-Raviart space associated with $e\subset\partial T$ and zero elsewhere, it follows that

$$\lim_{\gamma\to\infty}\|(\underline{\sigma}_h-\underline{\sigma}_h^*)\boldsymbol{n}_e\|_{0,e}=0,\quad \forall e\in\varepsilon_h$$

so $\lim_{\gamma\to\infty}\|\underline{\sigma}_h-\underline{\sigma}_h^*\|_{0,\Omega}=0$. Passing to the limit in the expression of η_h yields (36).

Remarque 1. Thanks to the continuity condition of the nonconforming space, the term $J^*(\boldsymbol{u}_h^*,\boldsymbol{u}_h^*)$ can be bounded by the term $\mu\sum_{e\in\varepsilon_h}|e|\int_e[\frac{\partial\boldsymbol{u}_h}{\partial\boldsymbol{t}_e}]^2\,ds$ where $\boldsymbol{t}_e$ denotes the tangent of edge e. This yields the form of the nonconformity error known from the literature, see for example [12].

In the general case $\boldsymbol{f}\neq\boldsymbol{\pi}_0\boldsymbol{f}$, we obtain an additionnal higher order term.

9. Extensions

Several extensions of the proposed dG method for the steady Stokes equations can be envisaged. In what follows, we briefly discuss two of them.

We are first interested in the treatement of more general boundary conditions and, in view of the generalization to non-Newtonian fluids, we focus on the second variational formulation (2). We recall that $\underline{\tau}=2\mu\underline{D}(\boldsymbol{u})$. Besides the usual Dirichlet boundary condition, one may want to prescribe a Neumann condition $\underline{\tau}\boldsymbol{n}-p\boldsymbol{n}=\boldsymbol{\chi}$. Nevertheless, for certain applications it is important to dispose of a larger panel of boundary conditions. For instance, the exact solution of the Poiseuille flow does not satisfy the previous Neumann condition on the outlet boundary. So we consider a partition of the boundary $\partial\Omega=\Gamma_1\cup\Gamma_2\cup\Gamma_3\cup\Gamma_4$ such that $\Gamma_2\neq\partial\Omega$ and we impose :

$$\begin{aligned}\boldsymbol{u}\cdot\boldsymbol{t}=g_t,\quad \boldsymbol{u}\cdot\boldsymbol{n}=g_n &\qquad \text{on } \Gamma_1 \quad \text{(pure Dirichlet)}\\ \underline{\tau}\boldsymbol{n}\cdot\boldsymbol{n}-p=\chi_n,\quad \underline{\tau}\boldsymbol{n}\cdot\boldsymbol{t}=\chi_t &\qquad \text{on } \Gamma_2 \quad \text{(pure Neumann)}\\ \boldsymbol{u}\cdot\boldsymbol{t}=g_t,\quad \underline{\tau}\boldsymbol{n}\cdot\boldsymbol{n}-p=\chi_n &\qquad \text{on } \Gamma_3\\ \boldsymbol{u}\cdot\boldsymbol{n}=g_n,\quad \underline{\tau}\boldsymbol{n}\cdot\boldsymbol{t}=\chi_t &\qquad \text{on } \Gamma_4.\end{aligned}$$

Then, since the boundary terms are treated by means of Nitsche's method, the bilinear forms $J(\cdot,\cdot)$, $C_1(\cdot,\cdot)$ and $b_h(\cdot,\cdot)$ have to be changed as follows :

$$\begin{aligned}
J(\boldsymbol{u}_h,\boldsymbol{v}_h) &= \mu \sum_{e\in\varepsilon_h^{int}\cup\varepsilon_h^1\cup\varepsilon_h^3} \frac{1}{|e|}\int_e [\boldsymbol{\pi}_{k-1}\boldsymbol{u}_h\cdot\boldsymbol{t}_e][\pi_{k-1}\boldsymbol{v}_h\cdot\boldsymbol{t}_e]ds \\
&\quad +\mu \sum_{e\in\varepsilon_h^{int}\cup\varepsilon_h^1\cup\varepsilon_h^4} \frac{1}{|e|}\int_e [\boldsymbol{\pi}_{k-1}\boldsymbol{u}_h\cdot\boldsymbol{n}_e][\pi_{k-1}\boldsymbol{v}_h\cdot\boldsymbol{n}_e]ds \\
C_1(\boldsymbol{u}_h,\boldsymbol{v}_h) &= -2\mu \sum_{e\in\varepsilon_h^{int}\cup\varepsilon_h^1\cup\varepsilon_h^3} \left(\int_e \{\underline{D}(\boldsymbol{v}_h)\boldsymbol{n}_e\cdot\boldsymbol{t}_e\}[\boldsymbol{u}_h\cdot\boldsymbol{t}_e]ds + \int_e \{\underline{D}(\boldsymbol{u}_h)\boldsymbol{n}_e\cdot\boldsymbol{t}_e\}[\boldsymbol{v}_h\cdot\boldsymbol{t}_e]ds\right) \\
&\quad -2\mu \sum_{e\in\varepsilon_h^{int}\cup\varepsilon_h^1\cup\varepsilon_h^4} \left(\int_e \{\underline{D}(\boldsymbol{v}_h)\boldsymbol{n}_e\cdot\boldsymbol{n}_e\}[\boldsymbol{u}_h\cdot\boldsymbol{n}_e]ds + \int_e \{\underline{D}(\boldsymbol{u}_h)\boldsymbol{n}_e\cdot\boldsymbol{n}_e\}[\boldsymbol{v}_h\cdot\boldsymbol{n}_e]ds\right) \\
b_h(q_h,\boldsymbol{v}_h) &= -\sum_{T\in\mathcal{T}_h}\int_T q_h\nabla\cdot\boldsymbol{v}_h dx + \sum_{e\in\varepsilon_h^{int}\cup\varepsilon_h^1\cup\varepsilon_h^4}\int_e \{q_h\}[\boldsymbol{u}_h\cdot\boldsymbol{n}_e]ds.
\end{aligned}$$

The linear forms are modified accordingly :

$$\begin{aligned}
f_h(\boldsymbol{v}_h) &= \sum_{T\in\mathcal{T}_h}\int_T \boldsymbol{f}\cdot\boldsymbol{v}_h dx + \sum_{e\in\varepsilon_h^2\cup\varepsilon_h^3}\int_e \chi_1\boldsymbol{v}_h\cdot\boldsymbol{n}_e ds + \sum_{e\in\varepsilon_h^2\cup\varepsilon_h^4}\int_e \chi_2\boldsymbol{v}_h\cdot\boldsymbol{t}_e ds \\
&\quad -2\mu\Big(\sum_{e\in\varepsilon_h^1\cup\varepsilon_h^3}\int_e \underline{D}(\boldsymbol{v}_h)\boldsymbol{n}_e\cdot\boldsymbol{t}_e g_1 ds + \sum_{e\in\varepsilon_h^1\cup\varepsilon_h^4}\int_e \underline{D}(\boldsymbol{v}_h)\boldsymbol{n}_e\cdot\boldsymbol{n}_e g_2 ds\Big) \\
&\quad +\mu\gamma\Big(\sum_{e\in\varepsilon_h^1\cup\varepsilon_h^3}\frac{1}{|e|}\int_e \boldsymbol{\pi}_{k-1}g_1\pi_{k-1}\boldsymbol{v}_h\cdot\boldsymbol{t}_e ds + \sum_{e\in\varepsilon_h^1\cup\varepsilon_h^4}\frac{1}{|e|}\int_e \boldsymbol{\pi}_{k-1}g_2\pi_{k-1}\boldsymbol{v}_h\cdot\boldsymbol{n}_e ds\Big) \\
g_h(q_h) &= \sum_{e\in\varepsilon_h^1\cup\varepsilon_h^4}\int_e q_h g_2 ds.
\end{aligned}$$

Next, we propose to extend our discretization to the Navier-Stokes equations by following the approach of Girault et al. [17], which yields *a priori* error estimates. For the sake of completeness, we recall here below the discretization of the nonlinear convective term given in [17]. The authors adapted to the case of discontinuous velocities the upwind scheme introduced by Lesaint and Raviart [22] and considered :

$$\begin{aligned}
c_{NS}(\boldsymbol{u},\boldsymbol{v}) &= \sum_{T\in\mathcal{T}_h}\left(\int_T \boldsymbol{u}\cdot\nabla\boldsymbol{u}\boldsymbol{v}dx + \int_{\partial T^-}|\{\boldsymbol{u}\}\cdot\boldsymbol{n}_T|(\boldsymbol{u}^{int}-\boldsymbol{u}^{ext})\cdot\boldsymbol{v}^{int}ds\right) \\
&\quad +\frac{1}{2}\sum_{T\in\mathcal{T}_h}\int_T(\nabla\cdot\boldsymbol{u})\boldsymbol{u}\cdot\boldsymbol{v}dx - \frac{1}{2}\sum_{e\in\varepsilon_h}\int_e[\boldsymbol{u}]\cdot\boldsymbol{n}_e\{\boldsymbol{u}\cdot\boldsymbol{v}\}ds
\end{aligned}$$

where $\partial T^- = \{\boldsymbol{x}\in\partial T;\ \{\boldsymbol{x}\}\cdot\boldsymbol{n}_T<0\}$. It is important, for the mathematical analysis, to note that $c_{NS}(\boldsymbol{u},\boldsymbol{u})\geq 0$.

10. Numerical tests

In this section, we present several numerical experiments in order to confirm the theoretical results. We are interested in the convergence rate and in the influence of the stabilization parameter. Comparisons with the dG method proposed in [17] are carried out, illustrating the robustness of our method. The developed codes are written in C++ and use the in-house C++ library CONCHA (*http://uppa-inria.univ-pau.fr/concha*).

10.1. **Mesh convergence.**

10.1.1. *First two-fields formulation.* We first study the behavior of the numerical scheme (8) with respect to mesh refinement, for $k = 1$, 2 and 3. We consider the exact solution of the Stokes problem with non-homogeneous Dirichlet conditions :

$$\boldsymbol{u}(x,y) = \begin{pmatrix} \pi\cos(\pi x)\sin(\pi y) \\ -\pi\sin(\pi x)\cos(\pi y) \end{pmatrix}, \quad p(x,y) = \sin(\pi x)\sin(\pi y) \tag{37}$$

on the square $\Omega = [-1,1] \times [-1,1]$.

The triangulation is obtained by first meshing the domain into quadrilaterals and then decomposing each quadrilateral into triangles as in the figure below. One thus ends up with a "criss-cross" mesh, cf. Fig. 1. At each refinement step, the discretization parameter h is divided by 2; we denote by ne the total number of triangles.

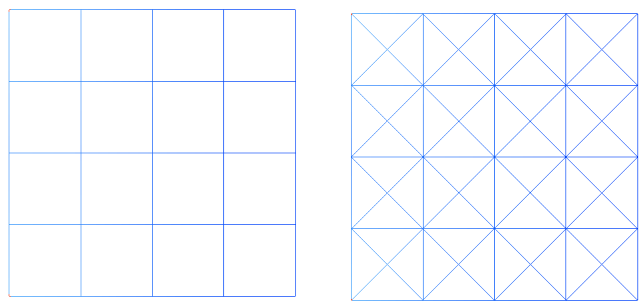
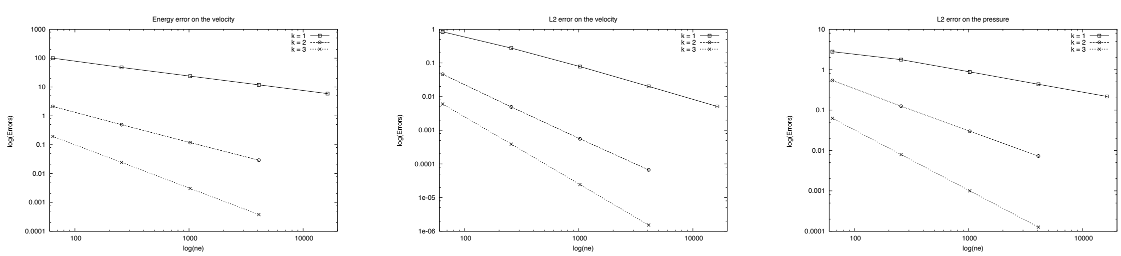


FIGURE 1. Triangular mesh from quadrilateral one

We have represented in Fig. 2 the logarithm of the errors in terms of the logarithm of ne, for all k. As expected, we numerically obtain a convergence rate $\mathcal{O}(h^k)$ for the pressure and for the velocity in the energy norm, and an improved order of convergence $\mathcal{O}(h^{k+1})$ for the velocity in the L^2-norm. The results below are obtained for $\gamma = 10$ and $\mu = 1$.



(a) V-error in energy norm (b) V - error in L^2 - norm (c) Pressure error

FIGURE 2. First formulation : convergence rates for different k

For $k = 3$, we have first taken $\gamma = 10$ as in the previous tests, but then the method didn't converge. This is not in contradiction with the theoretical results since according to Lemma 5.1, γ has to be large enough and it depends on the polynomial degree. The influence of the stabilization parameter will be studied more extensively in the next subsection; for the moment, we have simply performed similar tests with a larger value of γ ($\gamma = 100$).

We present in Tables 1, 2 and 3 the values of the errors on different meshes for k equal to 1, 2 and 3 respectively and, for each error, the ratio between the value computed on the previous (coarse) mesh and on the actual (refined) one.

ne	$\|\boldsymbol{u}-\boldsymbol{u}_h\|_{0,\Omega}$	ratio	$\|\|\|\boldsymbol{u}-\boldsymbol{u}_h\|\|\|$	ratio	$\|p-p_h\|_{0,\Omega}$	ratio
64	0.843959	—	10.010565	—	2.79255	—
256	0.276895	3.04793	4.767698	2.09966	1.77575	1.572603
1024	0.078143	3.54341	2.382578	2.00107	0.884179	2.008360
4096	0.020192	3.86998	1.188162	2.00526	0.43601	2.027886
16384	0.005090	3.96664	0.592460	2.00547	0.216991	2.009346
65536	0.001275	3.99174	0.295707	2.00354	0.108361	2.002482

Table 1: Ratio of the errors on succesive meshes for $k = 1$ ($\gamma = 10$)

ne	$\|\boldsymbol{u}-\boldsymbol{u}_h\|_{0,\Omega}$	ratio	$\|\|\|\boldsymbol{u}-\boldsymbol{u}_h\|\|\|$	ratio	$\|p-p_h\|_{0,\Omega}$	ratio
64	0.046359	—	2.122048	—	0.539482	—
256	0.004927	9.40802	0.492963	4.30468	0.125013	4.315407
1024	0.000557	8.83647	0.118451	4.16172	0.029860	4.186553
4096	6.645e-05	8.39157	0.029019	4.08185	0.007281	4.100626

Table 2 : Ratio of the errors on succesive meshes for $k = 2$ ($\gamma = 10$)

ne	$\|\boldsymbol{u}-\boldsymbol{u}_h\|_{0,\Omega}$	ratio	$\|\|\|\boldsymbol{u}-\boldsymbol{u}_h\|\|\|$	ratio	$\|p-p_h\|_{0,\Omega}$	ratio
64	0.006025	—	0.193471	—	0.062737	—
256	0.000387	15.5564	0.024415	7.9241	0.007919	7.921984
1024	2.443e-05	15.8513	0.003050	8.00418	0.001001	7.181947
4096	1.528e-06	15.9876	0.000380	8.02131	0.000126	7.907046

Table 3 : Ratio of the errors on succesive meshes for $k = 3$ ($\gamma = 100$)

A similar behavior of the error was observed for different values of the viscosity. To illustrate this point, we present the results obtained for the same test-case, but with $\mu = 100$. As predicted by the theory, the viscosity has no influence on the convergence rate (see Fig. 3).

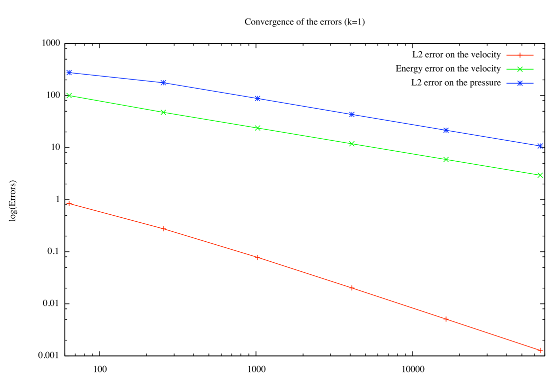


FIGURE 3. Convergence rate of the errors for $k = 1$ and $\mu = 100$

10.1.2. *Second two-fields formulation.* We now perform the same test as in the previous paragraph, but we employ the variational formulation (9). We recall that the additionnal stabilization term $J_1(\cdot,\cdot)$, whose role is to ensure the discrete Korn inequality, is necessary only in the case $k=1$. For this reason, we have chosen to illustrate the convergence rate of the dG method only for $k=1$.

We have represented in Fig. 4 the corresponding error curves in *log* scale, for the velocity (in the energy norm and in the L^2-norm) and for the pressure. They are in agreement with the theoretical results, that is :

$$\|\boldsymbol{u}-\boldsymbol{u}_h\|_{0,\Omega} = O(h^2), \quad [[\boldsymbol{u}-\boldsymbol{u}_h]] = O(h), \quad \|p-p_h\|_{0,\Omega} = O(h).$$

ne	$\|\boldsymbol{u}-\boldsymbol{u}_h\|_{0,\Omega}$	ratio	$\vert\vert\vert\boldsymbol{u}-\boldsymbol{u}_h\vert\vert\vert$	ratio	$\|p-p_h\|_{0,\Omega}$	ratio
64	0.732828	—	16.625998	—	3.06364	—
256	0.156187	4.69197	7.740131	2.14803	1.0945	2.799122
1024	0.037076	4.2126	3.750639	2.06368	0.470646	2.325527
4096	0.009035	4.1034	1.847880	2.0297	0.212294	2.216953
16384	0.002228	4.05541	0.916672	2.01586	0.100356	2.115409
65536	0.000553	4.02883	0.456425	2.00837	0.048785	2.057082

Table 4 : Ratio of the errors on succesive meshes for $k=1$ ($\gamma=10$, $\mu=1$)

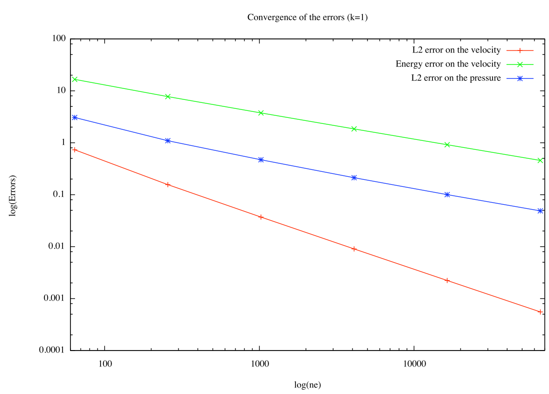
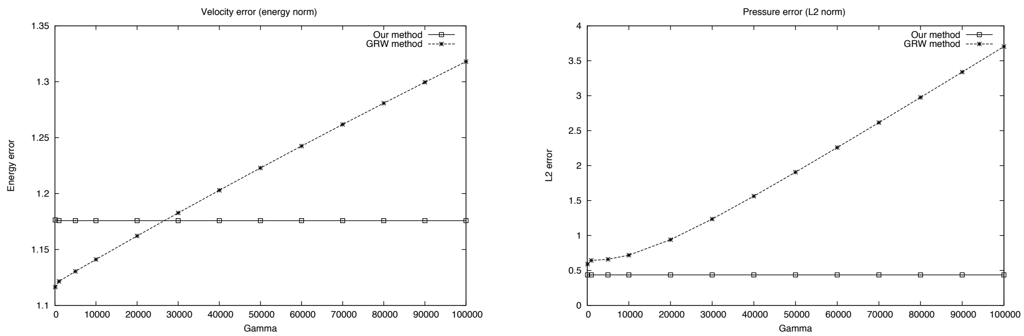


FIGURE 4. Second formulation : convergence rate of the errors for $k=1$

10.2. **Behavior with respect to the stabilization parameter.** In this subsection, we let γ vary and compare the results given by our numerical method for (8) with those given by the scheme of [17], which we call GRW in what follows. We first consider the previous test-case for which the exact solution is known, and then we treat the Poiseuille flow. We are interested in the computed errors and solutions for large γ, on a fixed mesh.

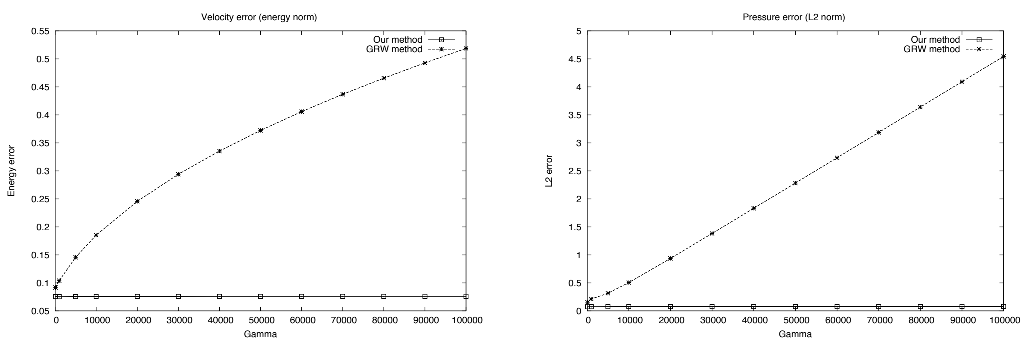
10.2.1. *Comparison of errors.* Let the exact solution be given by (37). We employ a mesh consisting of 4096 elements. We next compare the velocity errors in energy norm and the pressure errors computed by the two methods (ours in continuous lines, the one of [17] in dotted lines), for different values of γ. Fig.5 corresponds to the case $k=1$, Fig. 6 to $k=2$ while Fig. 7 corresponds to $k=3$. Different values of the viscosity have been chosen, in order to emphasize the independence of the schemes on this parameter. We recall that the energy norm of [17] associated with the velocity is different from ours, and is given by :

$$\left(\mu\,|\boldsymbol{v}|^2_{1,h} + \gamma\mu \sum_{e\in\varepsilon_h} \frac{1}{|e|}\,\|[\boldsymbol{v}]\|^2_{0,e}\right)^{1/2}.$$



(a) Velocity error in energy norm (b) Pressure error

FIGURE 5. Behavior of the errors with respect to γ for $k = 1$ ($\mu = 1$)



(a) Velocity error in energy norm (b) Pressure error

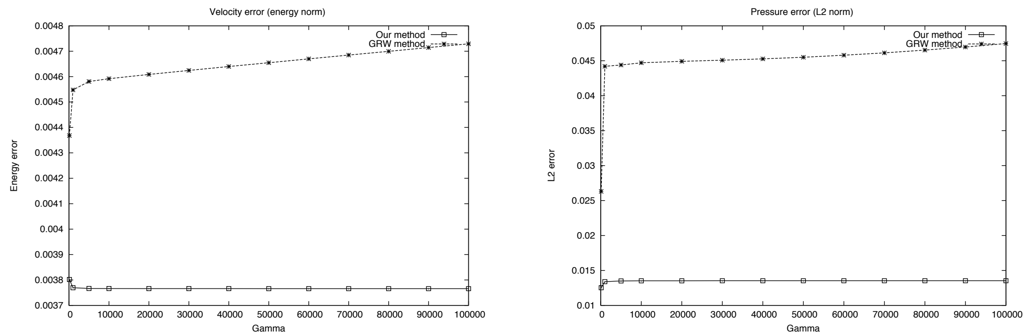
FIGURE 6. Behavior of the errors with respect to γ for $k = 2$ ($\mu = 10$)

One can notice that, contrarily to the stabilization of [17], ours yields a stable scheme independently of γ. The method of [17] leads to bigger errors, which increase with γ .

10.2.2. *Comparison of solutions for a Poiseuille flow.* We now consider a Poiseuille flow in the domain $\Omega = [0; 0.06] \times [-0.01; 0.01]$. On the inflow, we set $\boldsymbol{u} \cdot \boldsymbol{t} = 0$ and $\boldsymbol{u} \cdot \boldsymbol{n}$, whereas on the outflow we impose a homogeneous Neumann condition : $\mu(\underline{\nabla}\boldsymbol{u})\boldsymbol{n} - p\boldsymbol{n} = \boldsymbol{0}$.

We first set a parabolic velocity on the inflow $\boldsymbol{u} \cdot \boldsymbol{n} = a(0.01^2 - y^2)$, which yields the following exact solution of the Stokes problem: $\boldsymbol{u} = (a(0.01^2 - y^2), 0)$, $p = bx + c$. For $k \geq 2$ both dG codes give the exact solution, as expected. We now let the stabilization parameter γ vary and we compare the two solutions obtained for $k = 1$. The numerical tests are carried out on a unstructured mesh consisting of 10954 triangles. We have obtained similar results for the velocity field, for γ between 10 and 10000, even though the method of [17] presents some instabilities at large γ (see Fig. 9 and 10). Nevertheless, significant differences between the two methods appear as regards the computation of the pressure, therefore we have performed a more detailed study with respect to γ. One may see in Fig. 11 that the method of [17] is clearly less accurate and less stable than ours.

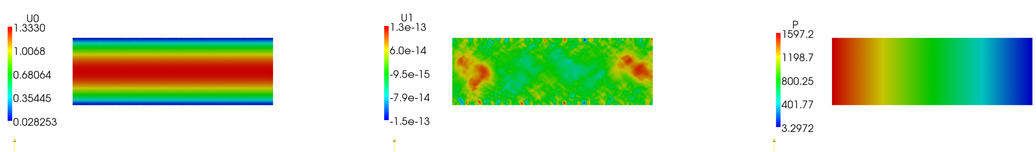
We have equally tested the two dG methods on non-smooth solutions, by imposing in the previous test-case $\boldsymbol{u} \cdot \boldsymbol{n} = 1$ on the inflow boundary. Note that the exact velocity does not belong to $\boldsymbol{H}^1(\Omega)$.



(a) Velocity error in energy norm

(b) Pressure error

FIGURE 7. Behavior of the errors with respect to γ for $k = 3$ ($\mu = 100$)

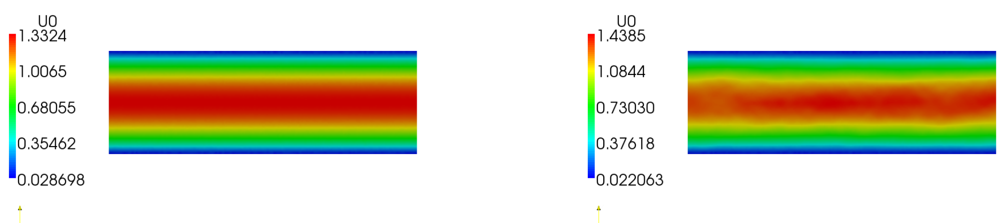


(a) U_0

(b) U_1

(c) P

FIGURE 8. Exact solution of the Poiseuille flow computed by $\boldsymbol{P}_2 \times P_1$ elements



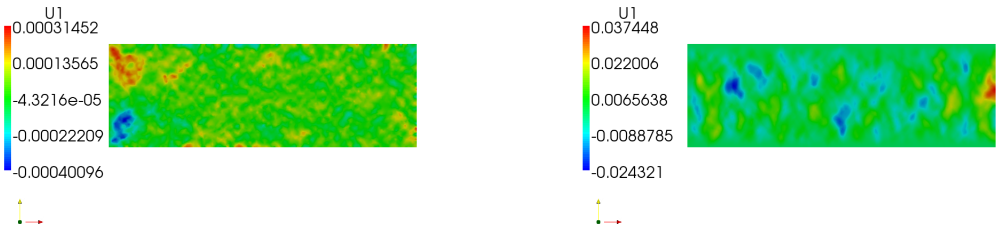
(a) Our method

(b) Method of [17]

FIGURE 9. Comparison of U_0 for a Poiseuille flow at $\gamma = 100000$

In order to dispose of a reference solution, we have computed it by means of nonconforming finite elements of Crouzeix-Raviart (see Fig. 12). We have now employed a criss-cross mesh consisting of 18432 triangles.

One can notice again a lack of accuracy of the method of [17], which becomes visible at rather small values of the stabilization parameter, such as $\gamma = 20$. As γ increases, the pressure computed with the method of [17] gets worse whereas our method is robust.



(a) Our method

(b) Method of [17]

FIGURE 10. Comparison of U_1 for a Poiseuille flow at $\gamma = 10000$

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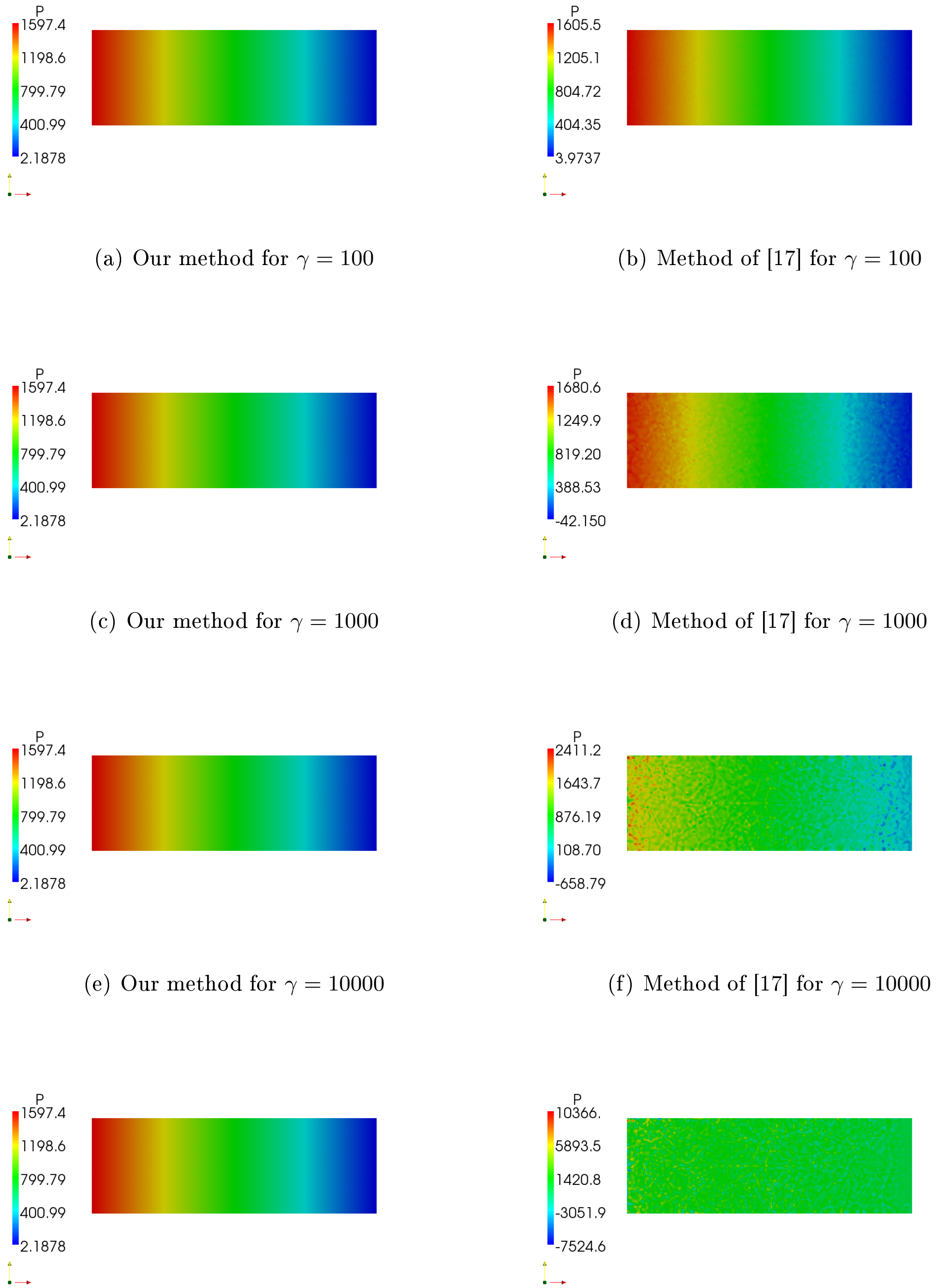

(a) Our method for $\gamma = 100$

(b) Method of [17] for $\gamma = 100$

(c) Our method for $\gamma = 1000$

(d) Method of [17] for $\gamma = 1000$

(e) Our method for $\gamma = 10000$

(f) Method of [17] for $\gamma = 10000$

(g) Our method for $\gamma = 100000$

(h) Method of [17] for $\gamma = 100000$

FIGURE 11. Poiseuille flow : comparison of pressures for different γ

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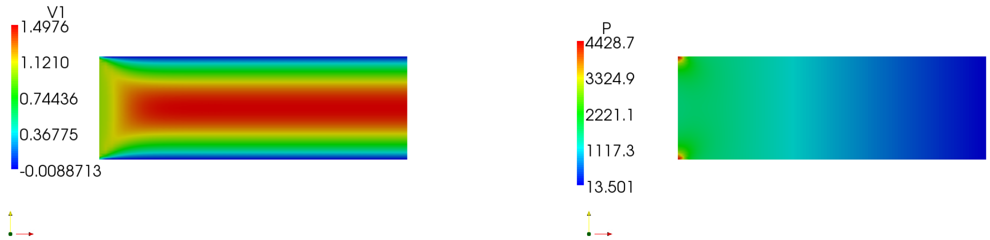

(a) First component of the velocity

(b) Pressure

FIGURE 12. Non-smooth solution obtained by nonconforming $\boldsymbol{P}_1 \times P_0$ elements

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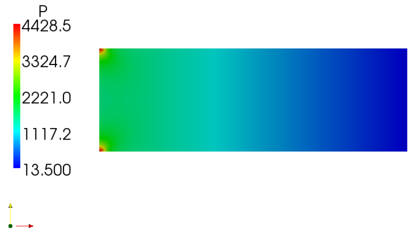
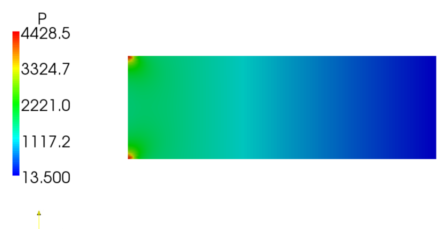


(a) Our method for $\gamma = 10$

(b) Method of [17] for $\gamma = 10$

(c) Our method for $\gamma = 100$

(d) Method of [17] for $\gamma = 100$



(e) Our method for $\gamma = 1000$

(f) Method of [17] for $\gamma = 1000$

(g) Our method for $\gamma = 10000$

(h) Method of [17] for $\gamma = 10000$

FIGURE 13. Comparison of pressures for a non-smooth solution