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Abstract

We consider stochastic equations of the prototype $du(t,x) = \left(\Delta u(t,x) + u(t,x)^{1+\beta}\right) dt + \kappa u(t,x)\, dW_t$ on a smooth domain $D \subset \mathbb{R}^d$, with Dirichlet boundary condition, where $\beta,\ \kappa$ are positive constants and $\{W_t,\ t \geq 0\}$ is a one-dimensional standard Wiener process. We estimate the probability of finite time blowup of positive solutions, as well as the probability of existence of non-trivial positive global solutions.

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1 Introduction

Let $D \subset \mathbb{R}^d$ be a bounded domain with smooth boundary ∂D. We consider a semilinear equation of the form

$$\begin{aligned} du(t,x) &= (\Delta u(t,x) + G(u(t,x)))\, dt + \kappa u(t,x)\, dW_t, \quad t > 0, \\ u(0,x) &= f(x) \geqq 0, \quad x \in D, \\ u(t,x) &= 0, \quad t \geqq 0, \quad x \in \partial D, \end{aligned} \tag{1}$$

where $G : \mathbb{R} \to \mathbb{R}_+$ is locally Lipschitz and satisfies

$$G(z) \geq C z^{1+\beta} \quad \text{for all} \ \ z > 0, \tag{2}$$

C, β and κ are given positive numbers, $\{W_t,\ t \geq 0\}$ is a standard one-dimensional Brownian motion on a stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t, t \geqq 0), P)$, and $f : D \to \mathbb{R}_+$ is of class C^2 and not identically zero. We assume (2) in sections 1 to 3 only; it is replaced by (11) in section 4. Since we do not assume G to be Lipschitz, blowup of the solution (1) in finite time can not be excluded, and our aim is to give estimates of the probability of blowup and conditions for the existence of a global solution of (1). A (random) time T is called blowup time of u if

$$\lim_{t \nearrow T} \sup_{x \in D} |u(t,x)| = +\infty \quad P-\text{a.s. on } \{T < +\infty\}.$$

In the classical (deterministic) case where $G(z) = z^{1+\beta}$ and $\kappa = 0$, it is well-known that for a nonnegative $f \in L^2(D)$, the condition

$$\int_D f(x)\psi(x)\,dx > \lambda_1^{1/\beta} \tag{3}$$

already implies finite time blowup of (1). Here $\lambda_1 > 0$ is the first eigenvalue of the Laplacian on D, and ψ the corresponding eigenfunction normalized so that $\|\psi\|_{L^1} = 1$.

The existence, uniqueness and trajectorial regularity of global solutions of parabolic equations perturbed by a time-homogenous white noise have been investigated by different methods (see e.g. Chueshov and Vuillermot [4], Denis et al. [5], Gyöngy and Rovira [9], Krylov [11], Lototski and Rozovskii [12], Mikulevicius and Pragarauskas [14]). Several types of solutions have been proposed (see especially the last cited reference for strong solutions), and the regularity results show that the solution is much smoother in the space variable than for equations perturbed by space-dependent white noise.

Let us recall the notions of weak and mild solutions of (1) we are going to use here. Let $\tau \leq +\infty$ be a stopping time. A continous $\mathcal{F}_t$-adapted random field $u = \{u(t,x),\ t \geq 0,\ x \in D\}$ is a *weak solution* of (1) on the interval $]0,\tau[$ provided that, for every $\varphi \in C^2(D)$ vanishing on ∂D, there holds

$$\begin{aligned}\int_D u(t,x)\varphi(x)\,dx &= \int_D f(x)\varphi(x)\,dx + \int_0^t \int_D [u(s,x)\Delta\varphi(x) + G(u(s,x))\varphi(x)]\,dx\,ds \\ &\quad + \kappa \int_0^t \int_D u(s,x)\varphi(x)\,dx\,dW_s \quad P-\text{a.s.}\end{aligned}$$

for all $t \in [0,\tau[$. Let $\{S_t,\ t \geqq 0\}$ be the semigroup of d-dimensional Brownian motion killed at the boundary of D. A continous $\mathcal{F}_t$-adapted random field $u = \{u(t,x),\ t \geqq 0,\ x \in D\}$ is a *mild solution* of (1) on the interval $]0,\tau[$ if it satisfies

$$u(t,x) = S_t f(x) + \int_0^t [S_{t-r}(G(u(r,\cdot))(x)\,dr + \kappa S_{t-r}(u(r,\cdot))(x)\,dW_r] \quad P\text{-a.s. and } x\text{-a.e. in } D$$

for all $t \in]0,\tau[$ (see e.g. [16], Chapter IV). We refer to [9] for background on existence of weak and mild solutions, and for their equivalence under local Lipschitz conditions on G. Let us note that the results in [9] hold for a more general class of second order differential operators which includes the Laplacian as a special case. The positivity of the solution of (1) follows from comparison theorems (see e.g. Bergé et al. [2] or Manthey and Zausinger [13]).

Our aim in this communication is to study the blowup behaviour of u by means of a related random partial differential equation (see (4) below). In section 3 we describe the blowup behaviour of the solution v of this random partial differential equation in terms of the first eigenvalue and the first eigenfunction of the Laplace operator on D. This is done by solving explicitly a stochastic equation in the time variable which is obtained from the weak form of (4). The solution of this differential equation can be written in terms of integrals of exponential Brownian motion with drift. The results of Dufresne [6] and Yor [18] on the law of these integrals easily imply estimates for the probability of existence of a global solution, or of blowup in finite time of u and v. In section 4 sufficient conditions for v to be a global solution are given in terms of the semigroup of the Laplace operator using recent sharp results on its transition density. These conditions show in particular that the initial condition f has to be small enough in order to avoid for a given G the blowup of v, and that the presence of noise may help to prevent blowup. The results of section 4 can be used to investigate the blowup behavior of u by means of conditions (2) and (11).

2 A related random partial differential equation

In this section we investigate the random partial differential equation

$$\begin{aligned} \frac{\partial v}{\partial t}(t,x) &= \Delta v(t,x) - \frac{\kappa^2}{2}v(t,x) + e^{-\kappa W_t}G(e^{\kappa W_t}v(t,x)), \quad t>0, \quad x\in D, \\ v(0,x) &= f(x), \quad x\in D, \\ v(t,x) &= 0, \quad x\in\partial D. \end{aligned} \tag{4}$$

This equation is understood trajectorywise and classical results for partial differential equations of parabolic type apply to show existence, uniqueness and positivity of a solution up to eventual blowup (see e.g. Friedman [7] Chapter 7, Theorem 9).

Proposition 1 *Let u be a weak solution of (1). Then the function v defined by*

$$v(t,x) = e^{-\kappa W_t}u(t,x), \quad t\geq 0, \quad x\in D.$$

solves (4).

Remark Proposition 1 implies in particular that (1) possesses a strong local solution.

Proof. Recall that Itô's formula states that $\{e^{-\kappa W_t},\, t\geq 0\}$ is the semimartingale given by

$$e^{-\kappa W_t} = 1 - \kappa\int_0^t e^{-\kappa W_s}dW_s + \frac{\kappa^2}{2}\int_0^t e^{-\kappa W_s}\,ds.$$

Let us write $u(t,\varphi)\equiv\int_D u(t,x)\varphi(x)\,dx$. Then a weak solution of (1) can be written as

$$u(t,\varphi) = u(0,\varphi) + \int_0^t u(s,\Delta\varphi)\,ds + \int_0^t G(u)(s,\varphi)\,ds + \kappa\int_0^t u(s,\varphi)\,dW_s.$$

Therefore, for φ fixed, $\{u(t,\varphi)1_{[0,\tau[}(t),\, t\geq 0\}$ is again a semimartingale. By applying the integration by parts formula (see e.g. Klebaner [10], Ch. 8) we get

$$\begin{aligned} v(t,\varphi) &:= \int_D v(t,x)\varphi(x)\,dx \\ &= v(0,\varphi) + \int_0^t e^{-\kappa W_s}\,du(s,\varphi) + \int_0^t u(s,\varphi)\left(-\kappa e^{-\kappa W_s}\,dW_s + \frac{\kappa^2}{2}e^{-\kappa W_s}\,ds\right) \\ &\quad + \left[e^{-\kappa W_\cdot}, u(\cdot,\varphi)\right](t), \end{aligned}$$

where the quadratic variation is given by

$$\left[e^{-\kappa W_\cdot}, u(\cdot,\varphi)\right](t) = -\int_0^t \kappa^2 e^{-\kappa W_s} u(s,\varphi)\, ds, \quad t \geq 0.$$

Therefore,

$$\begin{aligned} v(t,\varphi) &= v(0,\varphi) + \int_0^t e^{-\kappa W_s}\left(u(s,\Delta\varphi) + G(u)(s,\varphi)\right) ds + \kappa \int_0^t e^{-\kappa W_s} u(s,\varphi)\, dW_s \\ &\quad -\kappa \int_0^t e^{-\kappa W_s} u(s,\varphi)\, dW_s + \frac{\kappa^2}{2}\int_0^t e^{-\kappa W_s} u(s,\varphi)\, ds - \kappa^2 \int_0^t e^{-\kappa W_s} u(s,\varphi)\, ds \\ &= v(0,\varphi) + \int_0^t \left[v(s,\Delta\varphi) + e^{-\kappa W_s} G(e^{\kappa W_\cdot} v)(s,\varphi) - \frac{\kappa^2}{2} v(s,\varphi)\right] ds. \end{aligned}$$

Moreover, by self-adjointness of the Laplacian, and the fact that $\varphi(x) = 0$ for $x \in \partial D$,

$$v(s,\Delta\varphi) = \int_D v(s,x)\Delta\varphi(x), dx = \int_D \Delta v(s,x)\varphi(x)\, dx = \Delta v(s,\varphi).$$

■

3 An estimate of the probability of blowup

Without loss of generality, let us assume that $C = 1$ in (2). Let ψ be the eigenfunction corresponding to the first eigenvalue λ_1 of the Laplacian on D, normalized by $\int_D \psi(x)\, dx = 1$. It is well-known that ψ is strictly positive on D. Due to Proposition 1 we have that

$$v(t,\psi) = v(0,\psi) + \int_0^t \left[v(s,\Delta\psi) - \frac{\kappa^2}{2} v(s,\psi)\right] ds + \int_0^t e^{-\kappa W_s} G(e^{\kappa W_\cdot} v)(s,\psi)\, ds.$$

Moreover,

$$v(s,\Delta\psi) = -\lambda_1 v(s,\psi), \tag{5}$$

and, due to (2),

$$\int_D e^{-\kappa W_s} G(e^{\kappa W_s} v(s,x))\psi(x)\, dx \;\geq\; e^{\kappa\beta W_s} \int_D v(s,x)^{1+\beta}\psi(x)\, dx. \tag{6}$$

By Jensen's inequality

$$\int_D v(s,x)^{1+\beta}\psi(x)\, dx \;\geq\; \left[\int_D v(s,x)\psi(x)\, dx\right]^{1+\beta} \;=\; v(s,\psi)^{1+\beta}, \tag{7}$$

and therefore

$$\frac{d}{dt} v(t,\psi) \geq -\left(\lambda_1 + \frac{\kappa^2}{2}\right) v(t,\psi) + e^{\kappa\beta W_t} v(t,\psi)^{1+\beta}.$$

Hence $v(t,\psi) \geq I(t)$ for all $t \geq 0$, where $I(\cdot)$ solves

$$\frac{d}{dt}I(t) = -\left(\lambda_1 + \frac{\kappa^2}{2}\right) I(t) \; + e^{\kappa\beta W_s} I(t)^{1+\beta}\,, \quad I(0) = v(0,\psi),$$

and is given by

$$I(t) \;=\; e^{-(\lambda_1+\kappa^2/2)t}\left[v(0,\psi)^{-\beta} - \beta\int_0^t e^{-(\lambda_1+\kappa^2/2)\beta s+\kappa\beta W_s}\,ds\right]^{-\frac{1}{\beta}}, \quad 0 \leq t < \tau,$$

with

$$\tau := \inf\left\{ t \geq 0 \;\middle|\; \int_0^t e^{-(\lambda_1+\kappa^2/2)\beta s+\kappa\beta W_s}\,ds \geq \frac{1}{\beta}v(0,\psi)^{-\beta} \right\}. \tag{8}$$

It follows that I exhibits finite time blowup on the event $[\tau < \infty]$. Since $I \leqq v(\cdot,\psi)$, τ is an upper bound for the blow-up time of $v(\cdot,\psi)$, and therefore for the blowup times of v and u.

Remark 2 The same formula for the blow-up time of a stochastic differential equation, containing a stochastic integral with respect to W, has been obtained in Bandle et al [1]. The argument based on the first eigenvalue (and the corresponding eigenfunction) of the Laplace operator on D is applied there directly to u, and leads to a stochastic differential inequality for $u(t,\psi)$. The associated stochastic differential equation can again be solved explicitly by means of the Itô calculus, and the same formula as above is obtained for the blowup time of the solution of this equation. Both approaches are therefore equivalent, but the approach in [1] requires a more complicated comparison theorem for stochastic differential inequalities.

Let us now give an estimate for the probability of blowup in finite time of v. From (8),

$$\begin{aligned} P[\tau = +\infty] &= P\left[\int_0^t \exp(-(\lambda_1+\kappa^2/2)\beta s + \kappa\beta W_s)\,ds < \frac{1}{\beta}v(0,\psi)^{-\beta} \text{ for all } t > 0\right] \\ &= P\left[\int_0^\infty \exp(-(\lambda_1+\kappa^2/2)\beta s + \kappa\beta W_s)\,ds \leqq \frac{1}{\beta}v(0,\psi)^{-\beta}\right] \\ &= P\left[\int_0^\infty \exp(2\widehat{\beta}W_s^{(\mu)})\,ds \leqq \frac{1}{\beta}v(0,\psi)^{-\beta}\right], \end{aligned} \tag{9}$$

where $W_s^{(\mu)} := \mu s + W_s$, $\mu := -(\lambda_1+\kappa^2/2)/\kappa$, and $\hat{\beta} := \kappa\beta/2$. Setting $\widehat{\mu} = \mu/\widehat{\beta}$ we get

$$P[\tau = +\infty] = P\left[\frac{4}{\kappa^2\beta^2}\int_0^\infty \exp(2W_s^{(\widehat{\mu})})\,ds \leqq \frac{1}{\beta}v(0,\psi)^{-\beta}\right]. \tag{10}$$

It follows from [18] (Chapter 6, Corollary 1.2) that

$$\int_0^\infty \exp(2W_s^{(\widehat{\mu})})\,ds \;=\; \frac{1}{2Z_{-\widehat{\mu}}}$$

in distribution, where $Z_{-\widehat{\mu}}$ is a random variable with law $\Gamma(-\widehat{\mu})$, i.e. $P(Z_{-\widehat{\mu}} \in dy) = \frac{1}{\Gamma(-\widehat{\mu})} e^{-y} y^{-\widehat{\mu}-1} dy$. We get therefore (see also formula 1.10.4(1) in [3])

$$P[\tau = +\infty] = \int_0^{\frac{1}{\beta} v(0,\psi)^{-\beta}} h(y) dy,$$

where

$$h(y) = \frac{(\kappa^2 \beta^2 y/2)^{(2\lambda_1 + \kappa^2)/\kappa^2 \beta}}{y \Gamma((2\lambda_1 + \kappa^2)/(\kappa^2 \beta))} \exp\left(-\frac{2}{\kappa^2 \beta^2 y} \right).$$

In this way we have proved the following

Proposition 3 *The probability that the solution of (1) blows up in finite time is lower bounded by* $\int_{\frac{1}{\beta} v(0,\psi)^{-\beta}}^{+\infty} h(y)\, dy$.

Remark 4.1 Notice that formula 1.10.4(1) in [3] expresses the probability density function of $\int_0^t \exp(-(\lambda_1 + \kappa^2/2)\beta s + \kappa\beta W_s)\, ds$ in terms of the Kummer functions for $\widehat{\mu} < 2$.

Remark 4.2 By putting $\kappa = 0$ we get $v = u$ and, moreover, in (9) we obtain that $P[\tau = +\infty] = 0$ or 1 according to $\int_D f(x)\psi(x)\, dx > \lambda_1^{1/\beta}$ or $\int_D f(x)\psi(x)\, dx \leq \lambda_1^{1/\beta}$, which is a probabilistic counterpart to (3).

4 Non explosion of v

We consider again equation (4), but we assume now that $\kappa \neq 0$ and that $G : \mathbb{R}_+ \to \mathbb{R}_+$ satisfies $G(0) = 0$, $G(z)/z$ is increasing and

$$G(z) \leq \Lambda z^{1+\beta} \quad \text{for all} \quad z > 0, \tag{11}$$

where Λ and β are certain positive numbers. Let $\{S_t,\, t \geq 0\}$ again denote the semigroup of d-dimensional Brownian motion killed at the boundary of D. Recall that Equation (4) can be re-written as

$$v(t,x) = e^{-\kappa^2 t/2} S_t f(x) + \int_0^t e^{-\kappa^2 (t-r)/2} S_{t-r} \left(e^{-\kappa W_r} G\left(e^{\kappa W_r} v(r,\cdot) \right) \right)(x)\, dr. \tag{12}$$

We give now a sufficient condition for the existence of a global solution of (4).

Theorem 5 *Assume that f satisfies*

$$\Lambda\beta \int_0^\infty e^{\kappa\beta W_r} \| e^{-\kappa^2 r/2} S_r f \|_\infty^\beta \, dr < 1. \tag{13}$$

Then Equation (4) admits a global solution $v(t,x)$ that satisfies

$$0 \leq v(t,x) \leq \frac{e^{-\kappa^2 t/2} S_t f(x)}{\left(1 - \Lambda\beta \int_0^t e^{\kappa\beta W_r} \| e^{-\kappa^2 r/2} S_r f \|_\infty^\beta \, dr \right)^{\frac{1}{\beta}}}, \qquad t \geq 0. \tag{14}$$

Proof. Defining

$$B(t) = \left(1 - \Lambda\beta \int_0^t e^{\kappa\beta W_r} \|e^{-\kappa^2 r/2} S_r f\|_\infty^\beta \, dr\right)^{-\frac{1}{\beta}}, \quad t \geq 0,$$

we get $B(0) = 1$ and

$$\frac{dB}{dt}(t) = \Lambda e^{\kappa\beta W_t} \|e^{-\kappa^2 t/2} S_t f\|_\infty^\beta B^{1+\beta}(t),$$

which implies

$$B(t) = 1 + \Lambda \int_0^t e^{\kappa\beta W_r} \|e^{-\kappa^2 r/2} S_r f\|_\infty^\beta B^{1+\beta}(r) \, dr.$$

Suppose now that $(t, x) \mapsto V_t(x)$ is a nonnegative continuous function such that $V_t(\cdot) \in C_0(D)$, $t \geq 0$, and

$$e^{-\kappa^2 t/2} S_t f(x) \leq V_t(x) \leq B(t) e^{-\kappa^2 t/2} S_t f(x), \quad t \geq 0, \quad x \in D. \tag{15}$$

Let

$$R(V)(t, x) := e^{-\kappa^2 t/2} S_t f(x) + \int_0^t e^{-\kappa W_r} e^{-\kappa^2 (t-r)/2} S_{t-r} \left(G(e^{\kappa W_r} V_r(\cdot))\right)(x) \, dr.$$

Then,

$$\begin{aligned}
&R(V)(t, x) \\
&= e^{-\kappa^2 t/2} S_t f(x) + \int_0^t e^{-\kappa W_r} e^{-\kappa^2 (t-r)/2} S_{t-r} \left(\frac{G(e^{\kappa W_r} V_r(\cdot))}{V_r(\cdot)} V_r(\cdot)\right)(x) \, dr \\
&\leq e^{-\kappa^2 t/2} S_t f(x) + \int_0^t e^{-\kappa W_r} e^{-\kappa^2 (t-r)/2} S_{t-r} \left(\frac{G(e^{\kappa W_r} B(r) \|e^{-\kappa^2 r/2} S_r f\|_\infty)}{B(r) \|e^{-\kappa^2 r/2} S_r f\|_\infty} V(r)\right)(x) \, dr \\
&\leq e^{-\kappa^2 t/2} S_t f(x) + \Lambda \int_0^t e^{\kappa\beta W_r} B^{1+\beta}(r) \|e^{-\kappa^2 r/2} S_r f\|_\infty^\beta e^{-\kappa^2 (t-r)/2} S_{t-r}(e^{-\kappa^2 r/2} S_r f)(x) \, dr \\
&= e^{-\kappa^2 t/2} S_t f(x) \left[1 + \Lambda \int_0^t e^{\kappa\beta W_r} B^{1+\beta}(r) \|e^{-\kappa^2 r/2} S_r f\|_\infty^\beta \, dr\right] = e^{-\kappa^2 t/2} S_t f(x) B(t),
\end{aligned} \tag{16}$$

where to obtain the first inequality we used the rightmost inequality in (15) and the fact that $G(z)/z$ is increasing, and to obtain the second inequality we used (11). Consequently,

$$e^{-\kappa^2 t/2} S_t f(x) \leq R(V)(t, x) \leq B(t) e^{-\kappa^2 t/2} S_t f(x), \quad t \geq 0, \quad x \in D.$$

Let

$$v_t^0(x) := e^{-\kappa^2 t/2} S_t f(x) \quad \text{and} \quad v_t^{n+1}(x) = R(v^n)(t, x), \quad n = 0, 1, 2, \ldots.$$

Letting $n \to \infty$ yields, for $t \geq 0$ and $x \in D$,

$$0 \leq v(t, x) = \lim_{n \to \infty} v_t^n(x) \leq B(t) e^{-\kappa^2 t/2} S_t f(x) \leq \frac{e^{-\kappa^2 t/2} S_t f(x)}{\left(1 - \Lambda\beta \int_0^t e^{\kappa\beta W_r} \|e^{-\kappa^2 r/2} S_r f\|_\infty^\beta dr\right)^{1/\beta}}.$$

Hence, $v(t,x)$ is a global solution of (12) due to the monotone convergence theorem. ■

Remark. If we modify (4) and (12) by replacing $G(e^{\kappa W_r} v(t,x))$ by $G(v(t,x))$, then a global positive solution still exists for all f small enough, even if the inequality in (11) holds only for $z \in (0, C^*)$, where C^* is some positive constant. In fact, if f satisfies

$$\|f\|_\infty \leq C^* \left(1 - \Lambda\beta \int_0^\infty e^{-\kappa W_r} \|e^{-\kappa^2 r/2} S_r f\|_\infty^\beta \, dr\right)^{\frac{1}{\beta}}, \tag{17}$$

then Theorem 5 still holds if we replace the factor $e^{\kappa\beta W_r}$ in (13) and (14) by the factor $e^{-\kappa W_r}$. We only have to verify that assuming $z \in (0, C^*)$ in (11) already implies the second inequality in (16):

$$\begin{aligned}
\|e^{-\kappa^2 t/2} S_t f\|_\infty &\leq \|f\|_\infty \\
&\leq C^* \left(1 - \Lambda\beta \int_0^t e^{-\kappa W_r} \|e^{-\kappa^2 r/2} S_r f\|_\infty^\beta \, dr\right)^{\frac{1}{\beta}} = \frac{C^*}{B^*(t)} \quad \text{for all } t \geqq 0,
\end{aligned}$$

where

$$B^*(t) = \left(1 - \Lambda\beta \int_0^t e^{-\kappa W_r} \|e^{-\kappa^2 r/2} S_r f\|_\infty^\beta \, dr\right)^{-\frac{1}{\beta}}.$$

This yields

$$B^*(t) \|e^{-\kappa^2 t/2} S_t f\|_\infty \in (0, C^*) \tag{18}$$

for all $t \geq 0$, since $f \not\equiv 0$.

Let us now proceed to derive a sufficient condition for (13) in terms of the transition kernels $\{p_t(x,y),\, t > 0\}$ of $\{S_t,\, t \geq 0\}$ and the first eigenvalue λ_1 and corresponding eigenfunction ψ. We recall the following sharp bounds for $\{p_t(x,y),\, t > 0\}$, which we borrowed from Ouhabaz and Wang [15].

Theorem 6 *Let $\psi > 0$ be the first Dirichlet eigenfunction on a connected bounded $C^{1,\alpha}$-domain in $\mathbb{R}^d$, where $\alpha > 0$ and $d \geq 1$, and let $p_t(x,y)$ be the corresponding Dirichlet heat kernel. There exists a constant $c > 0$ such that, for any $t > 0$,*

$$\max\left\{1, \frac{1}{c} t^{-(d+2)/2}\right\} \leq e^{\lambda_1 t} \sup_{x,y} \frac{p_t(x,y)}{\psi(x)\psi(y)} \leq 1 + c(1 \wedge t)^{-(d+2)/2} e^{-(\lambda_2 - \lambda_1)t},$$

where $\lambda_2 > \lambda_1$ are the first two Dirichlet eigenvalues. This estimate is sharp for both short and long times.

The above theorem is useful in verifying condition (13). Indeed, let the initial value $f \geq 0$ be chosen so that

$$f(y) \leq K S_\eta \psi(y), \quad y \in D, \tag{19}$$

where $\eta \geq 1$ is fixed and $K > 0$ is a sufficiently small constant to be specified later on. Therefore $S_t f \leq K S_{t+\eta}\psi$, and for any $t > 0$,

$$\begin{aligned}
S_t f(x) &\leq K \int_D p_{t+\eta}(x,y)\psi(y)\,dy \\
&= K \int_D e^{\lambda_1(t+\eta)} \frac{p_{t+\eta}(x,y)}{\psi(x)\psi(y)} e^{-\lambda_1(t+\eta)}\psi(x)\psi^2(y)\,dy \\
&\leq K \left(\sup_{x\in D}\psi(x)\right)^2 \int_D e^{\lambda_1(t+\eta)} \sup_{x,y\in D} \frac{p_{t+\eta}(x,y)}{\psi(x)\psi(y)} e^{-\lambda_1(t+\eta)}\psi(y)\,dy \\
&\leq K \left(\sup_{x\in D}\psi(x)\right)^2 \int_D \left(1 + c(1\wedge(t+\eta))^{-(d+2)/2} e^{-(\lambda_2-\lambda_1)(t+\eta)}\right) e^{-\lambda_1(t+\eta)}\psi(y)\,dy \\
&= K \left(\sup_{x\in D}\psi(x)\right)^2 \left(e^{-\lambda_1(t+\eta)} + c e^{-\lambda_2(t+\eta)}\right) \int_D \psi(y)\,dy \\
&\leq K(1+c)e^{-\lambda_1\eta} \left(\sup_{x\in D}\psi(x)\right)^2 e^{-\lambda_1 t} \int_D \psi(y)\,dy,
\end{aligned}$$

which is independent of x. Since the function $(t,x) \mapsto S_t f(x)$ is uniformly bounded in x, condition (13) is satisfied provided that

$$\Lambda\beta \left[K(1+c)e^{-\lambda_1\eta}\left(\sup_{x\in D}\psi(x)\right)^2 \int_D \psi(y)\,dy\right]^\beta \int_0^\infty dr\, e^{\kappa\beta W_r - (\lambda_1+\kappa^2/2)\beta r} < 1,$$

or

$$\int_0^\infty dr\, e^{\kappa\beta W_r - (\lambda_1+\kappa^2/2)\beta r} < \frac{e^{\lambda_1\beta\eta}}{\Lambda\beta\left[K(1+c)\left(\sup_{x\in D}\psi(x)\right)^2 \int_D \psi(y)\,dy\right]^\beta}. \tag{20}$$

Notice that condition (17) is satisfied if K in (19) is sufficiently small.

We have proved the following

Theorem 7 *Let G satisfy (11), and let D be a connected, bounded $C^{1,\alpha}$-domain in $\mathbb{R}^d$, where $\alpha > 0$. If (19) and (20) hold for some $\eta > 0$ and $K > 0$, then the solution of Equation (12) is global.*

Remarks. 1) The integral on the left of (20) coincides with the corresponding integral in sections 2 and 3. The same type of bounds as in Section 3 can therefore be applied to estimate the probability of existence of a global positive solution. By means of the law of the iterated logarithm for W we see from (20) that the presence of a noise may help to prevent blowup in finite time.

2) If $G(z) = \Lambda z^{1+\beta}$, the results of this section can be applied to the solution u of equation (1) since $v(t,x) = e^{-\kappa W_t}u(t,x)$, $t \geq 0$, $x \in D$.

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