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DANIEL ALPAY AND MAMADOU MBOUP

Abstract. We introduce multi-scale filtering by the way of certain double convolution systems. We prove stability theorems for these systems and make connections with function theory in the poly-disc. Finally, we compare the framework developed here with the white noise space framework, within which a similar class of double convolution systems has been defined earlier.

Contents

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1. Introduction

A wide class of causal discrete time-invariant linear systems can be given in terms of convolution in the form

$$y_n = \sum_{m=0}^{n} h_{n-m}u_m, \quad n = 0, 1, \ldots \tag{1.1}$$

where (h_n) is the impulse response and where the input sequence (u_m) and output sequence (y_m) belong to some sequences spaces. The Z transform of the sequence (h_n)

$$h(z) = \sum_{n=0}^{\infty} z^n h_n$$

is called the transfer function of the system, and there are deep relationships between properties of h and of the system. For instance the system will be dissipative in the sense that for all ℓ_2 inputs (u_n) it holds that

$$\sum_{n=0}^{\infty} |y_n|^2 \le \sum_{n=0}^{\infty} |u_n|^2,$$

if and only if h is analytic and contractive in the open unit disc $\mathbb{D}$. The function h is then called a *Schur function*. Similarly, the system will be $\ell_1 - \ell_2$ bounded if for every ℓ_1 entry, the output is in ℓ_2 and there is a $M > 0$ independent of the input such that

$$\left(\sum_{n=0}^{\infty} |y_n|^2\right)^{1/2} \le M \sum_{n=0}^{\infty} |u_n|.$$

As is well known, the system is $\ell_1 - \ell_2$ bounded if and only if h belongs to the Hardy space $\mathbf{H}_2(\mathbb{D})$. Systems of the form (1.1), the Hardy space $\mathbf{H}_2(\mathbb{D})$ and Schur functions have been generalized to a number of situations in the theory of N-dimensional systems and beyond; see for instance the works [1], [10] [11], [12], and the references therein.

Another generalization of systems of the form (1.1) occurs when h_n and u_n are not complex numbers, but belong to some space with a product, say $\star$:

$$y_n = \sum_{m=0}^{n} h_{n-m} \star u_m, \quad n = 0, 1, \ldots. \tag{1.2}$$

Of special interest is the case where $\star$ is a convolution as, for instance in [4], and the system is then called a *double convolution* system. Therein, the first named author together with David Levanony considered an example of such a double convolution system, when both h_n and u_n are random variables, which belong to the white noise space, or more generally to the Kondratiev space. The product $h_{n-m}u_m$ in (1.1) is then replaced by the Wick product. The Wick product takes the form of a convolution with respect to an appropriate basis, and we have an example of a double convolution system. Using the Hermite transform, one can define a generalized transfer function, which is a function analytic in z and in a countable number of other variables (these variables take into account the randomness). The white noise space setting is reviewed in the last section of this paper, with purpose the comparison between the present paper and [4].

In the present work we study another type of double convolution system, which arises in the theory of multi-scale systems. We use the approach of the second named author presented at the *Mathematical Theory of Networks and Systems* conference in 2006 in Kyoto, see [22], to define the multi-scale version of the systems (1.1). Let

$$\boldsymbol{\varphi} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SU(1,1) \quad \text{and let} \quad \varphi(z) = \frac{az+b}{cz+d}$$

be the corresponding automorphism of $\mathbb{D}$. Following [22], consider the map

$$T_{\boldsymbol{\varphi}}(f)(z) = \frac{1}{cz+d} f(\varphi(z))$$

where f is analytic in $\mathbb{D}$. Then $T_{\boldsymbol{\varphi}}f$ is also analytic in the open unit disc. Let

$$f(z) = \sum_{n=0}^{\infty} f_n z^n \quad \text{and} \quad (T_{\boldsymbol{\varphi}}f)(z) = \sum_{n=0}^{\infty} f_{n,\boldsymbol{\varphi}} z^n$$

be the Taylor expansions of f and $T_{\boldsymbol{\varphi}}f$ respectively. Let

$$\mathbb{N} = \{1, 2, \ldots\} \quad \text{and} \quad \mathbb{N}_0 = \{0, 1, 2, \ldots\}.$$

The map which associates to the sequence $(f_n)_{n\in\mathbb{N}_0}$ the sequence $(f_{n,\boldsymbol{\varphi}})_{n\in\mathbb{N}_0}$ is called the *scaling operation* (see the precise definition in Section 2). Consider now a subgroup Γ of $SU(1,1)$, which represents the scales we will use to study the signals and systems. One associates to the sequence $(f_n)_{n\in\mathbb{N}_0}$ its *scale transform* $(f_n(\gamma))_{n\in\mathbb{N}_0,\gamma\in\Gamma}$, which is a function of $n \in \mathbb{N}_0$ and $\gamma \in \Gamma$ and where we have written $f_n(\gamma)$ for $f_{n,\boldsymbol{\gamma}}$. In the case which we will consider, Γ will be indexed by $\mathbb{Z}^p$, but the resulting setting is quite different from classical ND theory.

The scale transform is the starting point of our approach (initiated in [22]) to multi-scale analysis in discrete time. In opposition to wavelets, we propose a transform which has on the same level both the time and the scale aspect. Let us now elaborate on the differences of our approach to wavelets. Recall that in continuous time the wavelet transform of a signal f is defined by

$$Wf(u,s)=\int_{\mathbb{R}} f(t)\frac{1}{\sqrt{s}}\psi(\frac{t-u}{s})^*dt,$$

where s is the scale parameter and ψ is the mother wavelet; see [21, pp. 78-79]. The discrete time version of the transform is obtained, for discrete scales $s=a^j$, by discretizing the above integral as in

$$Wf[n,a^j]=\sum_k f_k\psi^*_{k-n,j},$$

where $\psi_{n,j}=\frac{1}{\sqrt{a^j}}\psi\left(\frac{n}{a^j}\right)$ and $f_k=f(k)$, assuming a sampling period normalized by 1 (see [21, pp. 88-89]). The decomposition provides an appropriate mathematical tool for signal analysis. In particular, it makes it possible to extract the components of a given discrete-time signal at a given scale on a discrete grid. However, the question of defining the scale shift operator (dilation) for purely discrete-time signals is dodged somehow. Another point of departure from our approach is that the wavelet transform (either continuous- or discrete-time) has only one convolution as compared to the double time and scale convolution considered in the present work (see equation (1.3) below).

We define linear systems as expressions of the form (1.2):

$$y_n=\sum_{m=0}^{n} h_{n-m}\star u_m,$$

where $\star$ denotes the convolution in Γ. Thus,

$$y_n(\gamma)=\sum_{m=0}^{n}\left(\sum_{\varphi\in\Gamma} h_{n-m}(\gamma\circ\varphi^{-1})u_m(\varphi)\right),\quad \gamma\in\Gamma. \tag{1.3}$$

When Γ is trivial we recover (1.1).

There are parallel and analogies between the theory of linear stochastic systems presented in [4] and the theory developed here. These parallels are pointed out in the sequel, and serve as guide and motivation for some of the proofs in the present paper. Still, there are some differences between the statements and the proofs of the stability theorems in [4] and the proofs given here. These differences are pointed out in the text. We note that a general theory of double convolution systems is in preparation, [5]. We also note that some of the results presented here have been announced in [7].

We now turn to the outline of this paper. It consists of 10 sections besides the introduction. In Section 2 we review the approach to discrete multi-scale systems presented in [22]. In Section 3 we define the systems which we will study in the paper, and the related notion of transfer function. Section 4 is of a review nature. We discuss the trigonometric moment problem and related reproducing kernel Hilbert spaces of the type introduced by de Branges and Rovnyak. In Section 5 we consider the case where the sub-group has one generator and is infinite. We use the classical one dimensional moment problem to associate to the Haar measure of the dual group a uniquely defined measure on the unit circle. This allows us to use function theory on the disc and on the bi-disc. In Section 6 we review some deep results of Mihai Putinar, see [23], on positive polynomials on compact semi-algebraic sets and their use to solve the trigonometric moment problem on the poly-disc. In Section 7 we consider the case where Γ is no more cyclic but has a finite number of generators. Although the results in Section 5 are particular cases of the ones in Section 7 we have chosen to present both for ease of exposition. The next three sections consider stability results: BIBO stability is considered in Section 8, dissipative systems are studied in Section 9 and Section 10 is devoted to $\ell_1 - \ell_2$ stability. In the last section we review the white noise space setting, and present a table and some remarks, which point out the analogies between the setting in [4] and the present work.

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2. Scaling operator for discrete-time signals

We briefly summarize the approach to multi-scale systems presented in [22]. We first note the following: If $F(s)$, $\Re(s) \geqslant 0$, denotes the Laplace transform of a continuous-time signal $f(t)$, $t \geqslant 0$, then, for any $\alpha = 1/\beta > 0$, $\sqrt{\alpha}F(\alpha s)$ is the Laplace transform of $f(\beta t)$. Therefore, time scaling has a similar form in the frequency domain. As opposed to the continuous-time case, time scaling is not clearly defined in the discrete-time setting. Nevertheless, the preceding remark is the key step to define a scaling operator for discrete-time signal. Consider the

Möbius transformation

$$G_\theta(s)=\frac{e^{i\theta}-s}{e^{-i\theta}+s},\quad |\theta|<\frac{\pi}{2}$$

which maps conformally the open right half-plane $\mathbb{C}_+$ onto the open unit disc. To recall our definition of the scaling operator (see [22] and also [8]), we note that the scale shift in $\mathbb{C}_+$,

$$S_\alpha : s\mapsto S_\alpha(s)=\alpha s,\quad \alpha>0$$

translates in the unit disc, via $G_\theta(s)$, into the hyperbolic transformation

$$\gamma_{\{\alpha\}}(z)=(G_\theta\circ S_\alpha\circ G_\theta^{-1})(z)=\frac{\left(e^{i\theta}+\alpha e^{-i\theta}\right)\,z+(1-\alpha)}{(1-\alpha)z+\left(e^{-i\theta}+\alpha e^{i\theta}\right)}. \tag{2.1}$$

Any such transformation maps the open unit disc (resp. the unit circle) into itself. Now, the most general linear transformation which maps the open unit disc (resp. the unit circle) into itself has the form

$$\gamma(z)=\frac{\gamma_1 z+\gamma_2}{\gamma_2^* z+\gamma_1^*},\quad |\gamma_1|^2-|\gamma_2|^2=1. \tag{2.2}$$

If $|\Re(\gamma_1)|>1$, then the transformation is hyperbolic [15] and it takes the form

$$\frac{\gamma(z)-\xi_1}{\gamma(z)-\xi_2}=\alpha_\gamma\frac{z-\xi_1}{z-\xi_2},\quad \alpha_\gamma>0, \tag{2.3}$$

where $\xi_1=\frac{\sqrt{[\Re(\gamma_1)]^2-1}+i\Im(\gamma_1)}{\gamma_2^*}\triangleq\frac{\lambda_\gamma}{\gamma_2^*}$ and $\xi_2=-\frac{\lambda_\gamma^*}{\gamma_2^*}$ are the two fixed points. The constant α_γ is called the multiplier of the transformation, see [15, p. 15], and is given by

$$\alpha_\gamma=\frac{\Re(\gamma_1)-\sqrt{[\Re(\gamma_1)]^2-1}}{\Re(\gamma_1)+\sqrt{[\Re(\gamma_1)]^2-1}}.$$

Noting that $|\xi_1|=|\xi_2|=1$, one may rearrange (2.3) to obtain

$$\frac{\lambda_\gamma-\lambda_\gamma^* e^{i\xi_\gamma}\gamma(z)}{1+e^{i\xi_\gamma}\gamma(z)}=\alpha_\gamma\frac{\lambda_\gamma-\lambda_\gamma^* e^{i\xi_\gamma}z}{1+e^{i\xi_\gamma}z},$$

where $e^{i\xi_\gamma}=\frac{\lambda_\gamma}{\gamma_2}$. Dividing both sides of this equality by $|\lambda_\gamma|$ and setting $e^{i\theta_\gamma}=\frac{\lambda_\gamma}{|\lambda_\gamma|}$, we recover (2.1) up to a rotation:

$$e^{i\xi_\gamma}\gamma(z)=(G_{\theta_\gamma}\circ S_{\alpha_\gamma}\circ G_{\theta_\gamma}^{-1})(e^{i\xi_\gamma}z). \tag{2.4}$$

Any hyperbolic transformation γ of the form (2.2) is thus (conformally) equivalent to a scale shift S_{α_γ} in $\mathbb{C}_+$.

In the sequel, we will be interested in Abelian subgroups of hyperbolic transformations, but the remainder of this section deals with general linear transformations

$$\varphi(z) = \frac{az+b}{cz+d}, \text{ with } ad-bc=1,$$

from the open unit disk onto itself. To each such transformation, we associate (in bold letters)

$$\boldsymbol{\varphi} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SU(1,1),$$

and we define

$$(T_{\boldsymbol{\varphi}} f)(z) = \frac{1}{cz+d} f(\varphi(z)).$$

Lemma 2.1. *Let $\boldsymbol{\varphi}_1$ and $\boldsymbol{\varphi}_2$ belong to $SU(1,1)$. Then*

$$T_{\boldsymbol{\varphi}_2} \circ T_{\boldsymbol{\varphi}_1} = T_{\boldsymbol{\varphi}_1 \boldsymbol{\varphi}_2}, \tag{2.5}$$

and in particular for every $\boldsymbol{\varphi} \in SU(1,1)$ it holds that

$$T_{\boldsymbol{\varphi}^{-1}} = (T_{\boldsymbol{\varphi}})^{-1}. \tag{2.6}$$

Proof: We have

$$(T_{\boldsymbol{\varphi}_2}(T_{\boldsymbol{\varphi}_1} f))(z) = \frac{1}{c_2 z + d_2} \frac{1}{c_1 \varphi_2(z) + d_1} f(\varphi_1(\varphi_2(z))).$$

But

$$\begin{aligned} \frac{1}{c_2 z + d_2} \frac{1}{c_1 \varphi_2(z) + d_1} &= \frac{1}{c_2 z + d_2} \frac{1}{c_1 \frac{a_2 z + b_2}{c_2 z + d_2} + d_1} \\ &= \frac{1}{(c_1 a_2 + d_1 c_2) z + c_1 b_2 + d_1 d_2}, \end{aligned}$$

which ends the proof since the second row of

$$\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$$

is equal to

$$\begin{pmatrix} c_1 a_2 + d_1 c_2 & c_1 b_2 + d_1 d_2 \end{pmatrix}.$$

$\square$

Let f be analytic in the open unit disc. Then $T_{\boldsymbol{\varphi}} f$ is also analytic in the open unit disc. The mapping $T_{\boldsymbol{\varphi}}$ induces a mapping from the space of sequences coefficients of power series of functions analytic in a neighborhood of the origin into itself: if $f(z) = \sum_{n=0}^{\infty} z^n x_n$ is the power series expansion at the origin of the function f analytic in $\mathbb{D}$, then

$$(T_{\boldsymbol{\varphi}}f)(z)=\sum_{n=0}^{\infty}x_{n,\boldsymbol{\varphi}}z^n.$$

In view of (2.6) we have:

Proposition 2.2. *Let $\boldsymbol{\varphi}\in SU(1,1)$, and let $(a_n)_{n\in\mathbb{N}_0}$ be a sequence of complex numbers such that $\limsup_{n\longrightarrow\infty}|a_{n+1}|^{1/(n+1)}\le 1$. Then there exists a sequence of complex numbers $(b_n)_{n\in\mathbb{N}_0}$ such that*

$$\limsup_{n\longrightarrow\infty}|b_{n+1}|^{1/(n+1)}\le 1,$$

and

$$a_n=b_{n,\boldsymbol{\varphi}},\quad n=0,1,\ldots$$

Proof: Let $a(z)=\sum_{n=0}^{\infty}a_nz^n$. It suffices to define a series $b_{n,\boldsymbol{\varphi}}$ via the formula

$$\sum_{n=0}^{\infty}b_{n,\boldsymbol{\varphi}}z^n=(T_{\boldsymbol{\varphi}^{-1}}a)(z),$$

and use (2.6). □

Theorem 2.3. *The operator $T_{\boldsymbol{\varphi}}$ is unitary from $\mathbf{H}_2(\mathbb{D})$ onto itself with norm equal to 1. It is also continuous from $\mathbf{H}_\infty(\mathbb{D})$ into itself with norm equal to $1/(|d|-|c|)$.*

Proof: First recall the formula

$$\frac{1-\varphi(z)\varphi(w)^*}{1-zw^*}=\frac{1}{(cz+d)(cw+d)^*}\tag{2.7}$$

where z,w are in the domain of definition of φ. Furthermore, recall that a function f defined in $\mathbb{D}$ is analytic there and belongs to $\mathbf{H}_2(\mathbb{D})$, with $\|f\|_{\mathbf{H}_2(\mathbb{D})}\le 1$, if and only if the kernel

$$\frac{1}{1-zw^*}-f(z)f(w)^*\tag{2.8}$$

is positive in $\mathbb{D}$; see for instance [1, Theorem 2.6.6]. We now compute for

$$\Delta(z,w)=\frac{1}{1-zw^*}-(T_{\boldsymbol{\varphi}}f)(z)(T_{\boldsymbol{\varphi}}f(w))^*$$

for $z, w \in \mathbb{D}$. Using (2.7) we can write:

$$\begin{aligned}\Delta(z,w) &= \frac{1}{(1-\varphi(z)\varphi(w)^*)(cz+d)(cw+d)^*} - \\ &\quad - \frac{1}{(cz+d)(cw+d)^*} f(\varphi(z))f(\varphi(w))^* \\ &= \frac{1}{(cz+d)(cw+d)^*} \times \\ &\quad \times \left\{ \frac{1}{1-\varphi(z)\varphi(w)^*} - f(\varphi(z))f(\varphi(w))^* \right\}.\end{aligned}$$

The kernel

$$\frac{1}{(1-\varphi(z)\varphi(w)^*)} - f(\varphi(z)f(\varphi(w))^*$$

is positive in $\mathbb{D}$ since the kernel (2.8) is positive there. It follows that $\Delta(z,w)$ is positive in the open unit disc, and thus, by [1, Theorem 2.6.6], the function $T_{\boldsymbol{\varphi}}(f)$ belongs to $\mathbf{H}_2(\mathbb{D})$ and has norm less or equal to 1. To prove that the norm is indeed equal to 1 we use (2.5) and (2.6), which imply that

$$1 \le \|T_{\boldsymbol{\varphi}}\| \cdot \|T_{\boldsymbol{\varphi}^{-1}}\|,$$

which, together with the fact that both $T_{\boldsymbol{\varphi}}$ and $T_{\boldsymbol{\varphi}^{-1}}$ have norm less that 1 implies that $\|T_{\boldsymbol{\varphi}}\| = 1$. We now show that $T_{\boldsymbol{\varphi}}$ is unitary. Let $f \in \mathbf{H}_2(\mathbb{D})$. We have:

$$\begin{aligned}\|f\|_{\mathbf{H}_2(\mathbb{D})} &= \|T_{\boldsymbol{\varphi}^{-1}}T_{\boldsymbol{\varphi}}(f)\|_{\mathbf{H}_2(\mathbb{D})} \\ &\le \|T_{\boldsymbol{\varphi}}(f)\|_{\mathbf{H}_2(\mathbb{D})} \\ &\le \|f\|_{\mathbf{H}_2(\mathbb{D})},\end{aligned}$$

since both $T_{\boldsymbol{\varphi}}$ and $T_{\boldsymbol{\varphi}^{-1}}$ are contractive. It follows that $T_{\boldsymbol{\varphi}}$ is unitary.

The second claim is easily verified. □

We therefore associate to the signal $\boldsymbol{x} = \{x_n\}_{n\in\mathbb{N}_0} \in \ell_2$ the signal $\{x_{n,\boldsymbol{\varphi}}\}_{\substack{n\in\mathbb{N}_0 \\ \boldsymbol{\varphi}\in SU(1,1)}}$ indexed by $\mathbb{N}_0 \times SU(1,1)$. In the sequel, we will simplify the notation by writing $x_n(\varphi)$ in place of $x_{n,\boldsymbol{\varphi}}$. For fixed m, $\{x_m(\varphi)\}$, with $\boldsymbol{\varphi} \in SU(1,1)$, represents a scale signal, that is, the observation of the signal $\{x_n\}$ at time m through the scales $\boldsymbol{\varphi} \in SU(1,1)$. In the sequel, we will consider (by convention) the zooming as corresponding to the "positive" scales.

Definition 2.4. *The scale-causal projection of* $\{x_m(\varphi)\}, \boldsymbol{\varphi} \in SU(1,1)$ *is given by the restriction of* $\{x_m(\varphi)\}$ *to the scales* $\boldsymbol{\varphi}$ *for which the multiplier is strictly less than one:* $\alpha_\varphi < 1$*.*

Definition 2.5. *Given a discrete subgroup* Γ *of* $SU(1,1)$ *we denote by* Γ_+ *the set of transformations consisting of the identity and of the scales* $\boldsymbol{\varphi}$ *for which the multiplier is strictly less than one:* $\alpha_\varphi < 1$*. The system* (1.2) *will be scale-causal if the elements* $h_n \in \ell_2(\Gamma_+)$*.*

Given γ and φ two elements of Γ, we will say that γ succeeds φ and will note $\varphi \preccurlyeq \gamma$, if $\gamma \circ \varphi^{-1} \in \Gamma_+$ that is:

$$\varphi \preccurlyeq \gamma \qquad \Longleftrightarrow \alpha_{\gamma\circ\varphi^{-1}} \leqslant 1.$$

Proposition 2.6. *The relation* $\preccurlyeq$ *defines a total order in* Γ*.*

Proof: Since we assume that Γ is Abelian, all the transformations must have the same fixed points. The parameters ξ_γ and θ_γ in (2.4) are therefore constant. The proof then follows upon noting that the multiplier $\alpha_{\gamma\circ\varphi}$ is given by: $\alpha_{\gamma\circ\varphi} = \alpha_\gamma\alpha_\varphi$. $\square$

With this order we obtain a bijection

$$\gamma \mapsto \varrho(\gamma)$$

between Γ and $\mathbb{Z}$, and one can identify $\ell_2(\Gamma)$ and $\ell_2(\mathbb{Z})$ and $\ell_2(\Gamma_+)$ and $\ell_2(\mathbb{N}_0)$.

Remark 2.7. Using the isomorphism we introduce the following definition:

Definition 2.8. *The function* $u(\gamma)$ *from* Γ_+ *into* $\mathbb{C}$ *has finite support if*

$$N(u) \stackrel{\triangle}{=} \max\{\varrho(\gamma) \text{ such that } u(\gamma) \neq 0\} < \infty.$$

The support of the function u *is the interval* $[0, N(u)] \subset \mathbb{N}_0$*.*

The results of this section remain valid if we replace the Hardy space $\mathbf{H}_2(\mathbb{D})$ by its vector-valued version $\mathbf{H}_2(\mathbb{D})\otimes\mathcal{H}$, where $\mathcal{H}$ is some Hilbert space. One can define in particular the scaling of random sequences when $\mathcal{H}$ is a probability space.

Remark 2.9. In [22, equation (17)] another kind of systems are considered, with only one convolution. The main issue there is the notion of scale-invariance in a stronger form, which will not be considered here; see also [6] for related work.

Finally, we note that one could consider systems non-causal with respect to n, that is of the form

$$y_n = \sum_{\mathbb{Z}} h_{n-m} \star u_m.$$

Thus, there are really four possibilities for the various stability theorems we present, depending on whether we have time causality or not, and scale-causality or not. In this paper we only give part of all possible results.

3. Discrete-scale invariant systems and signals

The scaling operators $T_{\boldsymbol{\varphi}}$ form a group of operators from the Hardy space $\mathbf{H}_2(\mathbb{D})$ onto itself. From now on, we discretize the scale axis and restrict $\boldsymbol{\varphi}$ to a discrete subgroup Γ of $SU(1,1)$. We will take Γ Abelian (cyclic) and consisting of hyperbolic transformations, and we denote by $\widehat{\Gamma}$ its dual group. Recall that $\widehat{\Gamma}$ is formed by the set of functions

$$\sigma : \Gamma \to \mathbb{T} \text{ such that } \sigma(\iota) = 1 \text{ and } \forall\ \gamma, \varphi,\ \sigma(\gamma \circ \varphi) = \sigma(\gamma)\sigma(\varphi),$$

where ι stands for the identity transformation. The elements of $\widehat{\Gamma}$ are called characters of the group Γ (see [16]). We denote by $\widehat{\mu}$ the Haar measure of $\widehat{\Gamma}$, which is compact by the Pontryagin duality [16]. We recall the definition of the Fourier transform on Γ and of its inverse:

$$\widehat{x}(\sigma) = \sum_{\gamma \in \Gamma} x(\gamma)\sigma(\gamma)^*,$$

$$x(\gamma) = \int_{\widehat{\Gamma}} \widehat{x}(\sigma)\sigma(\gamma) d\widehat{\mu}(\sigma).$$

The Haar measure $d\widehat{\mu}$ is normalized so that Plancherel's theorem holds:

$$\|f\|^2_{\ell_2(\Gamma)} \stackrel{\triangle}{=} \sum_{\gamma \in \Gamma} |f(\gamma)|^2 = \int_{\widehat{\Gamma}} |\widehat{f}(\sigma)|^2 d\widehat{\mu}(\sigma) \stackrel{\triangle}{=} \|\widehat{f}\|^2_{\mathbf{L}_2(d\widehat{\mu})}.$$

See [14, Theorem 8.4.2 p. 123].

Definition 3.1. *A signal will be a sequence $\{u_n(\cdot)\}_{n \in \mathbb{N}_0}$ of elements of $\ell_2(\Gamma)$, and such that the condition*

$$\sup_{n=0,1,\ldots} \|u_n(\cdot)\|_{\ell_2(\Gamma)} < \infty \tag{3.1}$$

holds.
A scale-causal signal will be a sequence $\{u_n(\cdot)\}_{n \in \mathbb{N}_0}$ of elements of

$\ell_2(\Gamma_+)$, *and such that the condition*

$$\sup_{n=0,1,\ldots} \|u_n(\cdot)\|_{\ell_2(\Gamma_+)} < \infty \tag{3.2}$$

holds.

In the sequel, we will impose the following stronger norm constrains on a signal, besides (3.1) or (3.2), namely:

$$\sum_{n=0}^{\infty} \|u_n(\cdot)\|^2_{\ell_2(\Gamma)} < \infty, \tag{3.3}$$

or

$$\sum_{n=0}^{\infty} \|u_n(\cdot)\|_{\ell_2(\Gamma)} < \infty, \tag{3.4}$$

and similarly for scale-causal signals.

We note the following: a dissipative filter cannot be effective at all scales. At some stage, details cannot be seen. These intuitive facts are made more precise in the following proposition.

Proposition 3.2.
(1) *Assume that the supports of the* u_n *are uniformly bounded. Then,* (3.3) *is in force.*
(2) *Assume that the support of* u_n *is infinite for all* n*. Then, the sum on the left side of* (3.3) *diverges.*

Proof: Let N be such that the support of all the functions $\gamma \mapsto u_n(\gamma)$ is inside $[0, N]$. Then,

$$\begin{aligned}
\sum_{n=0}^{\infty} \|u_n(\cdot)\|^2_{\ell_2(\Gamma)} &= \sum_{n=0}^{\infty} \sum_{\varrho(\gamma)=0}^{N} |u_n(\gamma)|^2 \\
&= \sum_{\varrho(\gamma)=0}^{N} \sum_{n=0}^{\infty} |u_n(\gamma)|^2 \\
&\leq N \|u_n\|_{\ell_2(\mathbb{N}_0)},
\end{aligned}$$

since (see Theorem 2.3) the maps T_γ are unitary from $\mathbf{H}_2(\mathbb{D})$ onto itself.

The second claim is proved similarly. □

An example of (u_n) satisfying Condition (1) of the preceding proposition has been presented in the paper [22], where the corresponding group Γ is Fuchsian. This was used therein, to define the scale unit-pulse signal. A similar condition was also considered by P. Yuditskii [25] in the description of the direct integral of spaces of character-automorphic functions.

Definition 3.3. *An impulse response (resp. a scale-causal impulse response) will be a sequence $\{h_n(\cdot)\}_{n\in\mathbb{N}_0}$ of elements of $\ell_2(\Gamma)$ (resp. of $\ell_2(\Gamma_+)$) such that for every $n \in \mathbb{N}_0$, the multiplication operator*

$$\mathcal{M}_{h_n}: \quad u \mapsto h_n \star u, \quad n = 0, 1, \dots \tag{3.5}$$

is bounded from $\ell_2(\Gamma)$ into itself (resp. from $\ell_2(\Gamma_+)$ into itself) and such that

$$\sup_{n=0,1,\dots} \|h_n\|_{\ell_2(\Gamma)} < \infty \quad (\textit{resp.} \quad \sup_{n=0,1,\dots} \|h_n\|_{\ell_2(\Gamma_+)} < \infty). \tag{3.6}$$

The systems that we consider here are defined by (1.2), that is, by the double convolution (1.3), that we recall below:

$$y_n(\gamma) = \sum_{m\in\mathbb{Z}} \left(\sum_{\delta\in\Gamma} h_{n-m}(\gamma \circ \delta^{-1}) x_m(\delta) \right). \tag{3.7}$$

In view of (3.6) the series

$$H(z, \sigma) = \sum_{n=0}^{\infty} z^n \widehat{h}_n(\sigma) \tag{3.8}$$

converges in the $\mathbf{L}_2(d\widehat{\mu})$ norm for every $z \in \mathbb{D}$. Taking the Fourier transform (with respect to Γ) of both sides of (1.3) we obtain

$$\widehat{y}_n(\sigma) = \sum_{m=0}^{n} \widehat{h}_{n-m}(\sigma) \widehat{x}_m(\sigma),$$

where the equality is in the $\mathbf{L}_2(d\widehat{\mu})$ sense. Taking now the Z transform we get

$$Y(z, \sigma) = H(z, \sigma) U(z, \sigma), \tag{3.9}$$

where

$$Y(z, \sigma) = \sum_{n=0}^{\infty} z^n \widehat{y}_n(\sigma) \quad \text{and} \quad U(z, \sigma) = \sum_{n=0}^{\infty} z^n \widehat{u}_n(\sigma), \tag{3.10}$$

and where, for every $z \in \mathbb{D}$ the equality in (3.9) is $\widehat{\mu}$-a.e.

The function $H(z, \sigma)$ can be seen as the transfer function of the discrete-time scale-invariant system. Formula (3.8) suggests to define and study hierarchies of transfer functions, for which the functions $\widehat{h_n}$ depend on

σ in some pre-assigned way (for instance, when they are polynomials in σ), or when the function $H(z,\sigma)$ is a rational function of z or of σ. In the next two sections, under the hypothesis that the sub-group Γ has a finite number, say p, of generators, we will associate to the system (1.2) an analytic function of $p+1$ variables, which we will call the *generalized transfer function of the system.*

4. The trigonometric moment problem

We first gather some well known facts on the trigonometric moment problem in form of a theorem.

Theorem 4.1. *Given an infinite sequence* $\ldots, t_{-1}, t_0, t_1, \ldots$ *of complex numbers such that*

$$t_{-n} = t_n^*, \quad n = 0, 1, \ldots,$$

there exists a positive measure $d\nu$ *on* $[0, 2\pi)$ *such that*

$$t_n = \int_0^{2\pi} e^{-in\theta} d\nu(\theta), \quad n \in \mathbb{N}_0,$$

if and only if all the Toeplitz matrices

$$\mathcal{T}_N = (t_{n-m})_{n,m=0,\ldots,N}$$

are non-negative.

See for instance [20, Theorem 2.7 p. 66]. The measure is then unique (when normalized). We also recall that the sequence (t_n) and the measure $d\nu$ are related by

$$t_0 + 2\sum_{n=1}^{\infty} t_n z^n = \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} d\nu(\theta),$$

and thus the function

$$\Phi(z) = t_0 + 2\sum_{n=1}^{\infty} t_n z^n = \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} d\nu(\theta)$$

is analytic and has a positive real part in the open unit disc. Using Stieltjes inverse formula, one can recover ν from Φ via the formula

$$\lim_{r\to 1} \int_a^b \mathrm{Re}\{\Phi(re^{i\theta})\} d\theta = \nu(b) - \nu(a_-),$$

where we assume that ν is right continuous. We also recall that the function

$$\frac{\Phi(z) + \Phi(w)^*}{2(1 - zw^*)} = \int_0^{2\pi} \frac{d\nu(\theta)}{(e^{i\theta} - z)(e^{i\theta} - w)^*}$$

is positive for $z, w \in \mathbb{C} \setminus \mathbb{T}$. We denote by $\mathcal{L}_+(\Phi)$ the associated reproducing kernel Hilbert space when z and w are restricted to the open unit disc. The following result has first been proved by de Branges and Shulman; see [13]. In the statement, $\mathbf{H}_2(d\nu)$ denotes the closed linear span in $\mathbf{L}_2(d\nu)$ of the functions z^m for $m \geq 0$.

Theorem 4.2. *The space $\mathcal{L}_+(\Phi)$ consists of the functions of the form*

$$\widetilde{h}(z) = \int_0^{2\pi} \frac{h(e^{it})e^{it}}{e^{it} - z} d\nu(t), \quad h \in \mathbf{H}_2(d\nu),$$

with norm

$$\|\widetilde{h}\|_{\mathcal{L}_+(\Phi)} = \|h\|_{\mathbf{H}_2(d\nu)}.$$

We now recall some results on the structure of the space $\mathbf{H}_2(d\nu)$.

Theorem 4.3. *Assume that $\mathbf{H}_2(d\nu) \neq \mathbf{L}_2(d\nu)$. Then $\mathbf{H}_2(d\nu)$ is a reproducing kernel Hilbert space, and its reproducing kernel is of the form*

$$\frac{A(z)A(w)^* - B(z)B(w)^*}{1 - zw^*},$$

where $A(z)$ and $B(z)$ are functions analytic off the unit circle.

Proof: We assume that $\mathbf{H}_2(d\nu) \neq \mathbf{L}_2(d\nu)$, and let $h_0 \in \mathbf{L}_2(d\nu) \ominus \mathbf{H}_2(d\nu)$. Let $\alpha \in \mathbb{C} \setminus \mathbb{T}$ be such that

$$\int_0^{2\pi} \frac{h_0(\theta) d\nu(\theta)}{e^{i\theta} - \alpha} \neq 0.$$

Let p be a polynomial; then $R_\alpha p \in \mathbf{H}_2(d\nu)$, where

$$(R_\alpha p)(z) = \frac{p(z) - p(\alpha)}{z - \alpha}.$$

Then

$$\langle R_\alpha p \, , \, h_0 \rangle_{\mathbf{H}_2(d\nu)} = 0,$$

and therefore we obtain

$$p(\alpha) = \frac{\displaystyle\int_0^{2\pi} \frac{p(e^{i\theta})}{e^{i\theta} - \alpha} d\theta}{\displaystyle\int_0^{2\pi} \frac{h_0(\theta) d\nu(\theta)}{e^{i\theta} - \alpha}}.$$

Therefore the map $p \mapsto p(\alpha)$ is continuous on the polynomials, and extends to a continuous map to $\mathbf{H}_2(d\nu)$. Therefore $\mathbf{H}_2(d\nu)$ is a reproducing kernel Hilbert subspace of $\mathbf{L}_2(d\nu)$. The proof is then finished by using [2, Theorem 3.1 p. 600]. $\square$

5. The case of one generator

In this section, we consider the case of a cyclic group Γ, generated by a hyperbolic transformation $\boldsymbol{\gamma}_0 \in SU(1,1)$. Any transformation in Γ is thus of the form $\gamma_0^m \stackrel{\triangle}{=} \underbrace{\gamma_0 \circ \ldots \circ \gamma_0}_{m \text{ times}}$, $m \in \mathbb{Z}$.

Theorem 5.1. *There exists a positive measure $d\nu(\theta)$ on $[0, 2\pi)$ such that*

$$\int_{\widehat{\Gamma}} \sigma(\gamma_0^m) d\widehat{\mu}(\sigma) = \int_0^{2\pi} e^{im\theta} d\nu(\theta), \quad m \in \mathbb{Z}. \tag{5.1}$$

Proof: We use Theorem 4.1. Let

$$t_m = \int_{\widehat{\Gamma}} \sigma(\gamma_0^m)^* d\widehat{\mu}(\sigma), \quad m \in \mathbb{Z}.$$

Since $|\sigma(\gamma_0)| = 1$ for all $\sigma \in \widehat{\Gamma}$ we have

$$t_{\ell-m} = \int_{\widehat{\Gamma}} \sigma(\gamma_0^\ell)^* \sigma(\gamma_0^m) d\widehat{\mu}(\sigma) = \langle \sigma(\gamma_0^m), \sigma(\gamma_0^\ell) \rangle_{\mathbf{L}_2(d\widehat{\mu})},$$

and therefore all the Toeplitz matrices

$$\mathcal{T}_N = (t_{\ell-m})_{\ell,m=0,\ldots N}$$

are non-negative. It follows from Theorem 4.1 that there exists a uniquely defined measure $d\nu$ such that

$$t_m = \int_0^{2\pi} e^{-im\theta} d\nu(\theta), \quad m = 0, 1, 2, \ldots,$$

and hence we obtain (5.1).

□

Remark 5.2. The proof of the previous theorem formalizes the intuitive idea that one can make the “change of variable”

$$\sigma(\gamma_0) = e^{i\theta(\sigma)}.$$

Theorem 5.3. *The linear map $\mathbf{I}$ which to $\sigma(\gamma_0^m)$ associates the function z^m:*

$$\mathbf{I}(\sigma(\gamma_0^m)) = z^m, \quad m \in \mathbb{Z}, \tag{5.2}$$

is an isomorphism from $\mathbf{L}_2(d\widehat{\mu})$ into $\mathbf{L}_2(d\nu)$.

Proof: For a function f of the form

$$f(\sigma) = \sum_{-N}^{M} c_n \sigma(\gamma_0^n) \quad \text{where} \quad N, M \in \mathbb{N}_0 \quad \text{and} \quad c_n \in \mathbb{C}, \tag{5.3}$$

we have

$$\begin{aligned}
\|f\|^2_{\mathbf{L}_2(d\widehat{\mu})} &= \sum_{n,m=-N,\ldots,M} c_n c_m^* t_{m-n}\\
&= \sum_{n,m=-N,\ldots,M} c_n c_m^* \int_0^{2\pi} e^{-i(m-n)\theta} d\nu(\theta)\\
&= \int_0^{2\pi} |\sum_{n=-N}^{M} c_n e^{in\theta}|^2 d\nu(\theta)\\
&= \|\mathbf{I}(f)\|^2_{\mathbf{L}_2(d\nu)}.
\end{aligned}$$

The result follows by continuity since such f are dense in $\mathbf{L}_2(d\widehat{\mu})$. To verify this last claim we note the following: By Plancherel's theorem, the map from $\ell_2(\Gamma)$ onto $\mathbf{L}_2(d\widehat{\mu})$ which to the sequence which consists only of zeros, except the n-th element which is equal to 1, associates the function $\sigma(\gamma_0)^n$, extends to a unitary map. □

We will be interested in particular in the positive powers of $\boldsymbol{\gamma}_0$, which correspond to zooming (we consider that the multiplier of $\boldsymbol{\gamma}_0$, *i.e.* the associated scale α_{γ_0}, is less than 1).

Definition 5.4. *We denote by $\mathbf{H}_2(d\widehat{\mu})$ the closure in $\mathbf{L}_2(d\widehat{\mu})$ of the functions $\sigma(\gamma_0)^n$, $n = 0, 1, 2, \ldots$. Similarly, we denote by $\mathbf{H}_2(d\nu)$ the closure in $\mathbf{L}_2(d\nu)$ of the functions z^n, $n = 0, 1, 2, \ldots$.*

Note that it may happen that $\mathbf{L}_2(d\widehat{\mu}) = \mathbf{H}_2(d\widehat{\mu})$.

Following [4] we introduce the next definition.

Definition 5.5. *The map $\mathbf{I}$ will be called the Hermite transform.*

Recall that $\widehat{\Gamma}$ is compact and therefore

$$\mathbf{L}_2(d\widehat{\mu}) \subset \mathbf{L}_1(d\widehat{\mu}). \tag{5.4}$$

In general the product of two elements f and g in $\mathbf{L}_2(d\widehat{\mu})$ does not belong to $\mathbf{L}_2(d\widehat{\mu})$, and one cannot define $\mathbf{I}(fg)$, let alone compare it with the product $\mathbf{I}(f)\mathbf{I}(g)$. On the other hand, we will need in the sequel only the case where at least one of the elements in the product fg defines a bounded multiplication operator from $\mathbf{L}_2(d\widehat{\mu})$ into itself; see Definition 3.3 and the proof of Theorem 9.3 for instance. This is exploited in the next theorem.

Theorem 5.6. *Let $f \in \mathbf{L}_2(d\widehat{\mu})$ such that the operator of multiplication by f defines a bounded operator from $\mathbf{L}_2(d\widehat{\mu})$ into itself. Then for every*

g in $\mathbf{L}_2(d\widehat{\mu})$ it holds that:

$$\mathbf{I}(fg) = \mathbf{I}(f)\mathbf{I}(g). \tag{5.5}$$

Proof: We note that the multiplicative property (5.5) holds for f and g of the form (5.3). To prove the theorem we first assume that g is of the form (5.3), and consider a sequence (p_n) of elements of the form (5.3), converging to f in the $\mathbf{L}_2(d\widehat{\mu})$ norm. The function g is in particular bounded, and so $fg \in \mathbf{L}_2(d\widehat{\mu})$, and we can write:

$$\|fg - p_n g\|_{\mathbf{L}_2(d\widehat{\mu})} \leq K\|f - p_n\|_{\mathbf{L}_2(d\widehat{\mu})},$$

where $K > 0$ is such that $|g| \leq K$. Thus, fp_n tends in $\mathbf{L}_2(d\widehat{\mu})$- norm to fg.

The function $\mathbf{I}(g)$ is bounded, and $\mathbf{I}(f) \in \mathbf{L}_2(d\nu)$. Therefore:

$$\begin{aligned}\|\mathbf{I}(fg) - \mathbf{I}(f)\mathbf{I}(g)\|_{\mathbf{L}_2(d\nu)} &\leq \|\mathbf{I}(fg) - \mathbf{I}(p_n g)\|_{\mathbf{L}_2(d\nu)} + \\ &\quad + \|\mathbf{I}(p_n g) - \mathbf{I}(f)\mathbf{I}(g)\|_{\mathbf{L}_2(d\nu)} \\ &= \|fg - p_n g\|_{\mathbf{L}_2(d\widehat{\mu})} + \\ &\quad + \|\mathbf{I}(p_n)\mathbf{I}(g) - \mathbf{I}(f)\mathbf{I}(g)\|_{\mathbf{L}_2(d\nu)} \\ &\leq \|fg - p_n g\|_{\mathbf{L}_2(d\widehat{\mu})} + \\ &\quad + K_1\|p_n - g\|_{\mathbf{L}_2(d\widehat{\mu})},\end{aligned}$$

where $K_1 > 0$ is such that $|\mathbf{I}(g)| \leq K_1$, and where we have used that

$$\mathbf{I}(p_n g) = \mathbf{I}(p_n)\mathbf{I}(g)$$

since both p_n and g are of the form (5.3). Hence, we obtain (5.5) for f and g as asserted.

Let now $f, g \in \mathbf{L}_2(d\widehat{\mu})$ be such that $fg \in \mathbf{L}_2(d\widehat{\mu})$. Then, $\mathbf{I}(fg)$ is well defined. Let (q_n) be a sequence of elements of the form (5.3), converging to g in the $\mathbf{L}_2(d\widehat{\mu})$ norm. Then, by the preceding argument,

$$\mathbf{I}(fq_n) = \mathbf{I}(f)\mathbf{I}(q_n), \quad \forall n \in \mathbb{N}.$$

In view of this equation and of the inclusion (5.4) we can write:

$$\begin{aligned}\|\mathbf{I}(fg) - \mathbf{I}(f)\mathbf{I}(g)\|_{\mathbf{L}_1(d\nu)} &\leq \|\mathbf{I}(fg) - \mathbf{I}(f)\mathbf{I}(q_n)\|_{\mathbf{L}_1(d\nu)} + \\ &\quad + \|\mathbf{I}(f)\mathbf{I}(q_n) - \mathbf{I}(f)\mathbf{I}(g)\|_{\mathbf{L}_1(d\nu)} \\ &= \|\mathbf{I}(f(g - q_n))\|_{\mathbf{L}_1(d\nu)} + \\ &\quad + \|\mathbf{I}(f)\mathbf{I}(q_n - g)\|_{\mathbf{L}_1(d\nu)}.\end{aligned}$$

By Cauchy-Schwartz inequality and by the isometry property of $\mathbf{I}$ on $\mathbf{L}_2(d\widehat{\nu})$, we have that

$$\begin{aligned}\|\mathbf{I}(f(g-q_n))\|_{\mathbf{L}_1(d\nu)} &\leq \sqrt{\nu((0,2\pi])}\cdot\|\mathbf{I}(f(g-q_n))\|_{\mathbf{L}_2(d\nu)}\\ &= \sqrt{\nu((0,2\pi])}\cdot\|f(g-q_n)\|_{\mathbf{L}_2(d\widehat{\mu})}.\end{aligned}$$

Since multiplication by f is assumed to define a bounded operator from $\mathbf{L}_2(d\widehat{\mu})$ into itself, there exists a constant $C>0$ such that

$$\|f(g-q_n)\|_{\mathbf{L}_2(d\widehat{\mu})}\leq C\|g-q_n\|_{\mathbf{L}_2(d\widehat{\mu})}\to 0\quad\text{as}\quad n\to\infty.$$

Using once more the Cauchy-Schwartz inequality we have:

$$\begin{aligned}\|\mathbf{I}(f)\mathbf{I}(q_n-g)\|_{\mathbf{L}_1(d\nu)} &\leq \|\mathbf{I}(f)\|_{\mathbf{L}_2(d\nu)}\cdot\|\mathbf{I}(q_n)-\mathbf{I}(g)\|_{\mathbf{L}_2(\nu)}\\ &= \|f\|_{\mathbf{L}_2(d\widehat{\mu})}\cdot\|q_n-g\|_{\mathbf{L}_2(d\widehat{\mu})}\to 0\quad\text{as}\quad n\to\infty.\end{aligned}$$

The claim follows. □

At this stage we need a change of notation; since two (and, in the next section, $p+1$) complex variables appear, we denote by z_1 (and by $z_1,\ldots,z_p$ in the following section) the variables related to the Hermite transform, and keep z for the Z-transform variable (this notation differs from the one in [4], where the Z-transform variable is denoted by ζ).

Definition 5.7. *The function*

$$\mathscr{H}(z,z_1)=\sum_{n=0}^{\infty}z^n\mathbf{I}(\widehat{h_n})(z_1)\tag{5.6}$$

is called the generalized transfer function of the system.

Taking the Hermite transform on both sides of (3.9), or, equivalently, taking the Z transform and the Hermite transform on both sides of (3.7), we obtain

$$\mathscr{Y}(z,z_1)=\mathscr{H}(z,z_1)\mathscr{U}(z,z_1),$$

where $\mathscr{U}(z,z_1)=\sum_{n=0}^{\infty}z^n\mathbf{I}(\widehat{u_n})(z_1)$, and similarly for $\mathscr{Y}(z,z_1)$. The function $\mathscr{H}$ is analytic in a neighborhood of $(0,0)\in\mathbb{C}^2$. It is of interest to relate the properties of $\mathscr{H}$ and of the system. This is done in Sections 9 and 10 of the paper. We first study, in the next section, the case where Γ has a finite number of generators.

6. The trigonometric moment problem in the poly-disc

In [23] a solution is given to the K-moment problem when K is a compact semi-algebraic set. The material is quite deep, and cannot be easily summarized in a short overview here. The purpose of this section is to serve as a guide to the reader to the topic. The starting point is a semi-algebraic subset of $\mathbb{R}^n$, defined by the positivity of κ polynomials

$$K = \{x \in \mathbb{R}^n \,;\, p_j(x) \geq 0\,, j = 1, \ldots \kappa\}\,.$$

Because of the application we have in mind in the next section, we will assume:

Hypothesis 6.1.
a) n is even and we set $n = 2p$

b) The polynomials p_j are of even degree and their highest degree homogeneous parts have only the origin as common zero.

One denotes by $C_+(K)$ the cone of polynomials positive on K. In [23, Theorem 1.4, p. 972] it is proved that, under Hypothesis 6.1, positive polynomials on K belong to the additive cone

$$C = \Sigma^2 + p_1\Sigma^2 + \cdots + p_\kappa\Sigma^2,$$

where Σ^2 denotes the convex cone generated by all squares of polynomials in $\mathbb{C}[x]$. The key result of [23] is:

Theorem 6.2. *([23, Lemma 3.2 p. 978]). A functional L on $\mathbb{R}[x]$ which is positive on C is of the form*

$$L(P) = \int_K P(x)d\nu(x), \quad P \in \mathbb{R}[x],$$

where $d\nu$ is a positive measure on K.

This result gives the solution to the moment problem on K: Let $\alpha_{\ell,m}$ with $\ell, m \in \mathbb{N}_0^p$ be complex numbers. Then there exists a positive measure on K such that

$$\int_K z^\ell z^{*m} d\nu(x) = \alpha_{\ell,m}$$

if and only if the following conditions hold:

$$\begin{aligned} L(|P(z,z^*)|^2) &\geq 0, \qquad \forall P \in \mathbb{C}[x], \\ L(p_j(x)|s(z)|^2) &\geq 0, \qquad \forall s \in \mathbb{C}[z], \quad j = 1, \ldots p, \\ L(p_j(x)|P(z,z^*)|^2) &\geq 0, \quad j = p+1, \ldots \kappa. \end{aligned} \tag{6.1}$$

7. The case of a finite number of generators

We now assume that the Abelian group Γ has a finite number, say p, of generators, which we will denote by $\gamma_1, \ldots, \gamma_p$. We assume that there are independent in the sense that if

$$\gamma_1^{n_1} \circ \cdots \circ \gamma_p^{n_p} = \iota$$

for some integers $n_1, \dots n_p \in \mathbb{Z}$, then $n_1 = \cdots = n_p = 0$. In particular, each generator is of the form (2.1),

$$\gamma_i(z) = \gamma_{\{\alpha_i\}}(z) = (G_\theta \circ S_{\alpha_i} \circ G_\theta^{-1})(z)$$

with θ fixed, and where the set $\{\alpha_i\}_{i=1}^p$ generates a free discrete subgroup of the multiplicative group of positive real numbers. We use in a free way the multi-index notation.

Theorem 7.1. *There is a positive measure $d\nu$ on the distinguished boundary of the poly-disc such that*

$$\int_{\widehat{\Gamma}} \sigma(\gamma_1^{n_1}) \cdots \sigma(\gamma_p^{n_p}) d\widehat{\mu}(\sigma) = \int_{\mathbb{T}^p} e^{in_1\theta_1} \cdots e^{in_p\theta_p} d\nu(\theta_1, \dots, \theta_p).$$

To prove this theorem we specialize the results of the preceding section to the case of the poly-disc $\mathbb{D}^p$. It is a compact algebraic set, with $n = 2p$, $m = 2p$ and polynomials

$$P_1(x) = 1 - |z_1|^2, \dots, P_p(x) = 1 - |z_p|^2,$$

and

$$P_{p+1}(x) = |z_1|^2 - 1, \dots, P_{2p}(x) = |z_p|^2 - 1.$$

Proof of Theorem 7.1: We define a linear form on polynomials in the variables $z_1, \dots, z_p, z_1^*, \dots, z_p^*$ by

$$L(z^\alpha z^{*\beta}) = \int_{\widehat{\Gamma}} (\sigma(\gamma_1))^{\alpha_1 - \beta_1} \cdots (\sigma(\gamma_p))^{\alpha_p - \beta_p} d\widehat{\mu}(\sigma).$$

Let p be a polynomial in the variables $z_1, \dots, z_p, z_1^*, \dots, z_p^*$. We write for short

$$p(z, z^*) = p(z_1, \dots, z_p, z_1^*, \dots, z_p^*).$$

Let $p(z, z^*) = \sum c_{\alpha,\beta} z^\alpha z^{*\beta}$. Then

$$\begin{aligned} L(p(z, z^*)) &= \\ &= \sum c_{\alpha,\beta} \int_{\widehat{\Gamma}} (\sigma(\gamma_1))^{\alpha_1 - \beta_1} \cdots (\sigma(\gamma_p))^{\alpha_p - \beta_p} d\widehat{\mu}(\sigma), \end{aligned}$$

and therefore we have

$$L(p(z, z^*)) = \int_{\widehat{\Gamma}} p(\sigma(\gamma_1), \sigma(\gamma_2), \dots, \sigma(\gamma_1)^*, \sigma(\gamma_2)^*, \dots) d\widehat{\mu}(\sigma). \tag{7.1}$$

Since $|p(z, z^*)|^2$ is still a polynomial in z and z^*, the following conditions hold:

$$\begin{aligned} L\left((1 - z_j z_j^*) p(z, z^*)\right) &= 0, \quad j = 1, 2, \dots p, \\ L\left(|p(z, z^*)|^2\right) &\geq 0. \end{aligned} \tag{7.2}$$

□

Remark 7.2. The fact that the characters are of modulus 1 allows to prove (7.1) and (7.2). It does not seem possible to relate our problems with another moment problem when $p > 1$ (for instance on the ball of $\mathbb{C}^p$).

Definition 7.3. *The Hermite transform of the element*

$$f(\sigma) = \sum_\alpha h_\alpha \sigma(\gamma^\alpha)$$

is

$$\mathbf{I}(f)(z) = \sum_\alpha h_\alpha z^\alpha.$$

Theorem 7.4. *Let $f \in \mathbf{L}_2(d\widehat{\mu})$ be such that the operator of multiplication by f defines a bounded operator from $\mathbf{L}_2(d\widehat{\mu})$ into itself. Then, for every $g \in \mathbf{L}_2(d\widehat{\mu})$:*

$$\mathbf{I}(fg) = \mathbf{I}(f)\mathbf{I}(g). \tag{7.3}$$

The proof is the same as for $p = 1$ (see the proof of Theorem 5.6). As in Definition 5.7, the function of $p+1$ variables

$$\mathscr{H}(z, z_1, \ldots, z_p) = \sum_{n=0}^\infty z^n \mathbf{I}(\widehat{h_n})(z_1, \ldots, z_p)$$

is called the *generalized transfer function* of the system.

8. BIBO stability

The system 1.2 will be called bounded input bounded output (BIBO) if there is an $M > 0$ such that for every $\{u_n(\gamma)\}$ such that

$$\sup_{n\in\mathbb{N}_0} \|u_n(\cdot)\|_{\ell_2(\Gamma)} < \infty \tag{8.1}$$

the output is such that $\{y_n(\gamma)\}_{\gamma\in\Gamma} \in \ell_2(\Gamma)$, $n = 0, 1, \ldots$, and it holds that

$$\sup_{n\in\mathbb{N}_0} \|y_n(\cdot)\|_{\ell_2(\Gamma)} \le M \sup_{n\in\mathbb{N}_0} \|u_n(\cdot)\|_{\ell_2(\Gamma)}. \tag{8.2}$$

The following theorem gives a characterization of BIBO systems. The proof follows the proof of [4, Theorem 3.2]. We note the following difference between the two theorems: in [4] the multiplication operators, that is the counterparts of the operators $\mathcal{M}_{h_n}$ defined here using the Wick product, are automatically bounded. As explained there, this is due to Våge's inequality (see [19, Proposition 3.3.2 p. 118] and (3.1) in [4], and Section 11 below). Here we do not have an analogue of this inequality.

Theorem 8.1. *The system* (1.2) *is bounded input bounded output if and only if the following two conditions hold:*
(*a*) *The multiplication operators* (3.5)

$$\mathcal{M}_{h_n}: \quad u \mapsto h_n \star u, \quad n = 0, 1, \ldots$$

are bounded from $\ell_2(\Gamma)$ *into itself.*
(*b*) *For all* $v(\cdot) \in \ell_2(\Gamma_+)$ *with* $\|v(\cdot)\|_{\ell_2(\Gamma)} = 1$ *it holds that*

$$\sum_{n=0}^{\infty} \|\mathcal{M}^*_{h_n}(v)\|_{\ell_2(\Gamma)} \leq M. \tag{8.3}$$

Proof: That the condition (8.3) is sufficient is readily seen. Indeed, take $v \in \ell_2(\Gamma)$ with $\|v(\cdot)\|_{\ell_2(\Gamma)} = 1$. From (1.2) we have:

$$\langle y_n \,,\, v\rangle_{\ell_2(\Gamma)} = \sum_{m=0}^{n} \langle u_m \,,\, \mathcal{M}^*_{h_{n-m}} v\rangle_{\ell_2(\Gamma)}, \quad n = 0, 1, \ldots, \tag{8.4}$$

and hence

$$\begin{aligned} |\langle y_n \,,\, v\rangle_{\ell_2(\Gamma)}| &\leq \sum_{m=0}^{n} \|u_m(\cdot)\|_{\ell_2(\Gamma)} \|\mathcal{M}^*_{h_{n-m}} v\|_{\ell_2(\Gamma)} \\ &\leq \left(\sup_{m=0,\ldots n} \|u_m(\cdot)\|_{\ell_2(\Gamma)}\right) \left(\sum_{m=0}^{n} \|\mathcal{M}^*_{h_{n-m}} v\|_{\ell_2(\Gamma)}\right) \\ &\leq M \sup_{m \in \mathbb{N}_0} \|u_m(\cdot)\|_{\ell_2(\Gamma)}. \end{aligned}$$

We obtain (8.3) by taking $v = y_n / \|y_n\|_{\ell_2(\Gamma)}$ when $y_n \neq 0$.

We now show that (8.3) is necessary. We assume that the system is bounded input and bounded output. We first note that the multiplication operators $\mathcal{M}_{h_n}$ are necessarily bounded. Indeed, assume that (8.2) is in force and take $u_0 = u \in \ell_2(\Gamma)$ and $u_n = 0$ for $n > 0$. Then,

$$y_n = h_n \star u = \mathcal{M}_{h_n}(u), \quad n = 0, 1, \ldots,$$

and it follows from (8.2) that $\|\mathcal{M}_{h_n}\| \leq M$ for $n = 0, 1, \ldots$

Let us now consider an input sequence (u_n) which satisfies (3.1). For a given n and v choose

$$u_m = 0 \quad \text{if} \quad \mathcal{M}^*_{h_{n-m}} v = 0,$$

and

$$u_m = \frac{\mathcal{M}^*_{h_{n-m}} v}{\|\mathcal{M}^*_{h_{n-m}} v\|_{\ell_2(\Gamma)}} \quad \text{otherwise.}$$

We obtain from (8.4) and (8.2) that

$$\sum_{m=0}^{n} \|\mathcal{M}^*_{h_{n-m}} v\|_{\ell_2(\Gamma)} \le M,$$

from which we get (8.3). □

We now make a number of remarks: first, condition (8.2) is implied by the stronger, but easier to deal with, condition

$$\sum_{n=0}^{\infty} \|\mathcal{M}_{h_n}\| \le M. \tag{8.5}$$

When Γ is the trivial subgroup of $SU(1,1)$, conditions (8.3) or (8.5) reduce to the classical condition

$$\sum_{n=0}^{\infty} |h_n| < \infty.$$

Finally, other versions of this theorem could be given, with non causal systems with respect to the variable n (as in [4]), or with scale-causal signals. We state the last one. The proof is the same as the proof of Theorem 8.1.

Theorem 8.2. *The system* (1.2) *is scale-causal and bounded input bounded output if and only if the following two conditions hold:*
(*a*) *The multiplication operators* (3.5)

$$\mathcal{M}_{h_n} : \quad u \mapsto h_n \star u, \quad n = 0, 1, \ldots$$

are bounded from $\ell_2(\Gamma_+)$ into itself.
(*b*) *For all $v(\cdot) \in \ell_2(\Gamma)$ with $\|v(\cdot)\|_{\ell_2(\Gamma_+)} = 1$ it holds that*

$$\sum_{n=0}^{\infty} \|\mathcal{M}^*_{h_n}(v)\|_{\ell_2(\Gamma_+)} \le M.$$

9. Dissipative systems

We will call the system (1.2) *dissipative* if for every input sequence (u_n) such that

$$\sum_{n=0}^{\infty} \|u_n(\cdot)\|^2_{\ell_2(\Gamma)} < \infty$$

it holds that

$$\sum_{n=0}^{\infty}\|y_n(\cdot)\|^2_{\ell_2(\Gamma)}\le\sum_{n=0}^{\infty}\|u_n(\cdot)\|^2_{\ell_2(\Gamma)}. \tag{9.1}$$

Theorem 9.1. *The system is dissipative if and only if the* $\mathbf{L}(\ell_2(\Gamma))$*-valued function*

$$S(z)=\sum_{n=0}^{\infty}z^n\mathcal{M}_{h_n}$$

is analytic and contractive in the open unit disc.

Proof: Equations (9.1) expresses that the block Toeplitz operator

$$\begin{pmatrix}M_{h_0} & 0 & 0 & \cdots\\ M_{h_1} & M_{h_0} & 0 & \cdots\\ \vdots & \vdots & & \\ & & & \end{pmatrix}$$

is a contraction from $\ell_2(\ell_2(\Gamma))$ into itself, and this is equivalent to the asserted condition on S. $\square$

We consider the case of scale-causal signals (see Definition 2.4).

Definition 9.2. *The system* (1.2) *will be called scale-causal dissipative if the following conditions hold:*
(1) *The operators* M_{h_n} *are bounded from* $\ell_2(\Gamma_+)$ *into itself.*
(2) *Condition* (9.1) *holds, with* $\ell_2(\Gamma)$ *replaced by* $\ell_2(\Gamma_+)$.

Recall that we have denoted by $\mathbf{H}_2(d\nu)$ the closure in $\mathbf{L}_2(d\nu)$ of the powers z^α, where all the components of α are greater or equal to 0. Taking the Fourier and Hermite transforms we have:

Theorem 9.3. *The system is scale-causal dissipative if and only the function*

$$\mathscr{H}(z,z_1,\ldots,z_p)=\sum_{n=0}^{\infty}z^n\mathbf{I}(\widehat{h_n})(z_1,\ldots,z_p) \tag{9.2}$$

is contractive from $\mathbf{H}_2(\mathbb{D})\otimes\mathbf{H}_2(d\nu)$ *into itself. Furthermore, if the space* $\mathbf{H}_2(d\nu)$ *is a reproducing kernel Hilbert space, say with reproducing kernel* $K(z_1,\ldots,w_1,\ldots)$, *condition* (9.2) *is equivalent to the positivity of the kernel*

$$\frac{1-\mathscr{H}(z,z_1,\ldots)\mathscr{H}(w,w_1,\ldots)^*}{1-zw^*}K(z_1,\ldots,w_1,\ldots) \tag{9.3}$$

in $\mathbb{D}^{p+1}$.

Proof: Since the operators M_{h_n} are assumed bounded, we have

$$h_{n-m} \star u_m \in \ell_2(\Gamma_+), \quad m = 0, \ldots, n,$$

for all entries $u_m \in \ell_2(\Gamma_+)$. Thus

$$\widehat{h_{n-m}}\widehat{u_m} \in \mathbf{H}_2(d\widehat{\mu}),$$

and we may apply Theorem 5.6. We can write:

$$\left\| \sum_{n=0}^{m} h_{n-m} \star u_m \right\|_{\ell_2(\Gamma_+)} = \left\| \sum_{n=0}^{m} \mathbf{I}(\widehat{h_{n-m}})\mathbf{I}(\widehat{u_m}) \right\|_{\mathbf{H}_2(d\nu)}.$$

Thus the dissipativity is translated into the contractivity of the block Toeplitz operator

$$\begin{pmatrix} M_{\widehat{h_0}} & 0 & 0 & \cdots \\ M_{\widehat{h_1}} & M_{\widehat{h_0}} & 0 & \cdots \\ \vdots & \vdots & & \\ & & & \end{pmatrix}$$

from $\ell_2(\mathbf{H}_2(d\nu))$ into itself, and hence the claim on $\mathscr{H}$. To prove the second claim, we remark that $\mathbf{H}_2(\mathbb{D}) \otimes \mathbf{H}_2(d\nu)$ is the reproducing kernel Hilbert space with reproducing kernel

$$\frac{1}{1 - zw^*} K(z_1, \cdots, w_1, \cdots).$$

This comes from the fact that the reproducing kernel of a tensor product of reproducing kernel Hilbert spaces is the product of the reproducing kernels; see [9], [24]. Condition (9.3) follows then from the well-known characterization of bounded multipliers in reproducing kernel Hilbert spaces; see for instance [1], and the references therein. □

10. ℓ_1-ℓ_2 BOUNDED SYSTEMS

The system (1.2) will be called ℓ_1-ℓ_2 bounded if there is a $M > 0$ such that for all inputs (u_n) satisfying

$$\sum_{n=0}^{\infty} \|u_n(\cdot)\|_{\ell_2(\Gamma)} < \infty,$$

we have

$$\left(\sum_{n=0}^{\infty} \|y_n(\cdot)\|_{\ell_2(\Gamma)}^2 \right)^{1/2} \leq M \sum_{n=0}^{\infty} \|u_n(\cdot)\|_{\ell_2(\Gamma)}.$$

Taking the Fourier transform, this condition can be rewritten as:

$$\left(\sum_{n=0}^{\infty}\|\widehat{y_n}\|^2_{\mathbf{L}_2(d\widehat{\mu})}\right)^{1/2}\leq M\sum_{n=0}^{\infty}\|\widehat{u_n}\|_{\mathbf{L}_2(d\widehat{\mu})}, \tag{10.1}$$

The system (1.2) will be called *scale-causal* ℓ_1-ℓ_2 *bounded* if it is moreover scale-causal, that is, if the operators M_{h_n} are bounded from $\ell_2(\Gamma_+)$ into itself. Condition (10.1) then becomes:

$$\left(\sum_{n=0}^{\infty}\|\widehat{y_n}\|^2_{\mathbf{H}_2(d\widehat{\mu})}\right)^{1/2}\leq M\sum_{n=0}^{\infty}\|\widehat{u_n}\|_{\mathbf{H}_2(d\widehat{\mu})}, \tag{10.2}$$

from which we obtain, much in the same way as in [4], the following result. For completeness we present a proof.

Theorem 10.1. *A necessary and sufficient condition for the system* (1.2) *to be scalar-causal and* ℓ_1-ℓ_2 *bounded is that the function*

$$H(z,\sigma)=\sum_{n=0}^{\infty}z^n\widehat{h_n}(\sigma)\in\mathbf{H}_2(\mathbb{D})\otimes\mathbf{H}_2(d\widehat{\mu}), \tag{10.3}$$

or, equivalently, that the transfer function

$$\mathscr{H}(z,z_1)=\sum_{n=0}^{\infty}z^n\mathbf{I}(\widehat{h_n})(z_1)\in\mathbf{H}_2(\mathbb{D})\otimes\mathbf{H}_2(d\nu). \tag{10.4}$$

Proof: To see that condition (10.3) is necessary, it suffices to take the sequence

$$\widehat{u_n}=\begin{cases}1 & \text{if}\quad n=0\\ 0 & \text{if}\quad n\neq 0.\end{cases}$$

Then

$$\widehat{y_n}=\widehat{h_n}\quad n=0,1,\ldots,$$

and condition (10.2) implies that $H(z,\sigma)\in\mathbf{H}_2(\mathbb{D})\otimes\mathbf{H}_2(d\widehat{\mu})$. Conversely, assume that the function $H(z,\sigma)\in\mathbf{H}_2(\mathbb{D})\otimes\mathbf{H}_2(d\widehat{\mu})$. From the expression (3.10), and using the Cauchy-Schwarz inequality on

$$Y(z,\sigma)=H(z,\sigma)U(z,\sigma)=\sum_{n=0}^{\infty}H(z,\sigma)\left(z^n\widehat{u_n}\right),$$

we have

$$\begin{aligned}\|Y(z,\sigma)\|_{\mathbf{H}_2(\mathbb{D})\otimes\mathbf{H}_2(d\widehat{\mu})}&\leq\sum_{n=0}^{\infty}\|H(z,\sigma)z^n\widehat{u_n}\|_{\mathbf{H}_2(\mathbb{D})\otimes\mathbf{H}_2(d\widehat{\mu})}\\&\leq\sum_{n=0}^{\infty}\|z^n\widehat{u_n}\|_{\mathbf{H}_2(\mathbb{D})\otimes\mathbf{H}_2(d\widehat{\mu})}\|H(z,\sigma)|\|_{\mathbf{H}_2(\mathbb{D})\otimes\mathbf{H}_2(d\widehat{\mu})}.\end{aligned}$$

But we have that

$$\|z^n\widehat{u_n}\|_{\mathbf{H}_2(\mathbb{D})\otimes\mathbf{H}_2(d\widehat{\mu})} = \|\widehat{u_n}\|_{\mathbf{H}_2(d\widehat{\mu})},$$

and so we obtain (10.2) with $M = \|H(z,\sigma)|\|_{\mathbf{H}_2(\mathbb{D})\otimes\mathbf{H}_2(d\widehat{\mu})}$. The equivalence with condition (10.4) follows by taking the Hermite transform. □

When $\mathbf{H}_2(d\nu) \neq \mathbf{L}_2(d\nu)$ (recall that Γ is finitely generated), (10.4) can be translated into reproducing kernel conditions. In particular, in the cyclic case, we have:

Theorem 10.2. *Assume that* $\mathbf{H}_2(d\nu) \neq \mathbf{L}_2(d\nu)$, *and let*

$$\frac{A(z_1)A(w_1)^* - B(z_1)B(w_1)^*}{1 - z_1w_1^*}$$

be the reproducing kernel of $\mathbf{H}_2(d\nu)$. *The system* (1.2) *is scale-causal and* $\ell_1 - \ell_2$ *bounded if and only if there is a* $M > 0$ *such that the kernel*

$$\frac{A(z_1)A(w_1)^* - B(z_1)B(w_1)^*}{(1 - zw^*)(1 - z_1w_1^*)} - M\mathscr{H}(z, z_1)\mathscr{H}(w, w_1)^*$$

is positive in the bi-disc.

As in the case of equation (9.3), this comes from the characterization of the reproducing kernel of a tensor product of reproducing kernel Hilbert spaces.

11. The white noise space setting and a table

Another kind of double convolution system, with a setting quite similar to the setting presented here, has been developed in [4], and rely on Hida's theory of the white noise space (see [19], [18], [17] for the latter). We now review the main features of Hida' theory and of [4]. The starting point in Hida's theory is the function

$$K(s_1 - s_2) = e^{-\frac{\|s_1-s_2\|^2_{\mathbf{L}_2(\mathbb{R})}}{2}},$$

which is positive in the sense of reproducing kernels for s_1, s_2 in the Schwartz space $\mathcal{S}$ of real valued rapidly vanishing functions. By the Bochner-Minlos theorem there exists a probability measure P on the dual space $\mathcal{S}'$ of real valued tempered distributions such that

$$e^{-\frac{\|s\|^2_{\mathbf{L}_2(\mathbb{R})}}{2}} = \int_{\mathcal{S}'} e^{i\langle w,s\rangle} dP(w), \quad s \in \mathcal{S}',$$

where we have denoted by $\langle w, s\rangle$ the duality between $\mathcal{S}$ and $\mathcal{S}'$. The white noise space is defined to be the real Hilbert space $\mathbf{L}_2(\mathcal{S}', \mathcal{F}, P)$, where $\mathcal{F}$ denotes the underlying Borelian sigma-algebra.

Among all orthonormal basis of $\mathbf{L}_2(\mathcal{S}', \mathcal{F}, P)$, there is one which plays a special role; it is constructed from the Hermite functions, and is indexed by the set ℓ of infinite sequences $(\alpha_1, \alpha_2, \ldots)$ indexed by $\mathbb{N}$, and with values in $\mathbb{N}_0$, and for which $\alpha_j = 0$ for all j at the exception of at most a finite number of j. See [19, Definition 2.2.1 p. 19]. We will denote by H_α (with $\alpha \in \ell$) the elements of this basis. An element

$$F = \sum_{\alpha \in \ell} f_\alpha H_\alpha, \quad f_\alpha \in \mathbb{R}, \tag{11.1}$$

belongs to the white noise space if

$$\sum_{\alpha \in \ell} \alpha! f_\alpha^2 < \infty.$$

The Wick product is defined by

$$H_\alpha \lozenge H_\beta = H_{\alpha+\beta}, \quad \alpha, \beta \in \ell.$$

The white noise space is not stable under the Wick product, and there is the need to introduce a nuclear space, called the Kondratiev space, within which the Wick product is stable. The Kondratiev space is the projective limit of the real Hilbert spaces $\mathcal{H}_k$ of formal sums of the form (11.1) for which

$$\sum_{\alpha} f_\alpha^2 (2\mathbb{N})^{-q\alpha} < \infty$$

for some $q \in \mathbb{N}$, where we use the notation

$$(2\mathbb{N})^\alpha \overset{\text{def.}}{=} 2^{\alpha_1} \times 4^{\alpha_2} \times 6^{\alpha_3} \cdots$$

One can also consider the complexified versions of these spaces.

We also recall Våge's inequality (see [19, Proposition 3.3.2 p. 118]): Fix some integer $l > 0$, and let $k > l+1$. Consider $h \in \mathcal{H}_l$ and $u \in \mathcal{H}_k$. Then, $h \lozenge u \in \mathcal{H}_k$ and

$$\|h \lozenge u\|_k \leq A(k-l) \|h\|_l \|u\|_k,$$

where

$$A(k-l) = \sum_{\alpha \in \ell} (2\mathbb{N})^{(l-k)\alpha}$$

is a finite number.

We can now introduce the systems considered in [4]. A system will be characterized by a sequence $(h_n)_{n \in \mathbb{Z}}$ of elements in $\mathcal{H}_l$ for some $l \in \mathbb{N}$, and a signal will be a sequence of elements in one of the spaces $\mathcal{H}_k$, with $k > l+1$. Input-output relations are expressions of the form

$$y_n = \sum_{m\in\mathbb{Z}} h_{n-m} \lozenge x_m, \quad n \in \mathbb{Z}.$$

Note that in view of Våge's inequality the output sequence consists also of elements of $\mathcal{H}_k$. Furthermore, decomposing this equation along the basis H_α we obtain the double convolution system

$$y_\alpha(n) = \sum_{m\in\mathbb{Z}} \sum_{\beta\le\alpha} h_{\alpha-\beta}(n-m)u_\beta(m), \quad n \in \mathbb{Z}.$$

The map $\mathbf{I}$ which to H_α associates the polynomial z^α is called the Hermite transform. It is such that

$$\mathbf{I}(f \lozenge g) = \mathbf{I}(f)\mathbf{I}(g), \quad \forall f, g \in S_{-1}.$$

Note that under the Hermite transform the white noise space is mapped onto the reproducing kernel Hilbert space with reproducing kernel $e^{\langle z,w\rangle_{\ell_2}}$, that is, onto the Fock space.

We now give the table presenting the parallels between the white noise space case (as applied in the paper [4]), and the present multi-scale case. The reader might want to look at a similar table in [3], where the analogies between the white noise space case and the hyper-holomorphic case are presented.

The setting	Stochastic case	Multi-scale case
Underlying space	The white noise space	$\ell_2(\Gamma)$
Hermite transform	$\mathbf{I}(H_\alpha) = z^\alpha$	$\mathbf{I}(\sigma(\gamma)^\alpha) = z^\alpha$ (Γ: finitely generated)
Underlying reproducing kernel Hilbert space	The Fock space	The space $\mathbf{H}_2(d\nu)$
Key tool used	Minlos theorem (to build the white noise space)	Moment problem on the poly-disc (to build the Hermite transform)
The product	Wick product	Convolution with respect to Γ.
Double convolution	$y_\alpha(n) = \sum_{m\in\mathbb{Z}}\sum_{\beta\leq\alpha} h_{\alpha-\beta}(n-m)u_\beta(m)$	$y_n(\gamma) = \sum_{m=0}^{n}\sum_{\varphi\in\Gamma} h_{n-m}(\gamma\circ\varphi^{-1})u_m(\varphi)$

Remark 11.1. The pointwise product in $\mathbf{L}_2(\widehat{d\mu})$ is a convolution in $\ell_2(\Gamma)$. Strictly speaking, it would be better to define the Hermite transform as the composition of the Fourier transform and of the map $H_\alpha \mapsto z^\alpha$.

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