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**Abstract:** A $\mathcal{V}$-coloring of a graph $G$ is a coloring of the nodes of $G$ with $\mathcal{V}$ colors in such a way any two neighboring nodes have different colors. We prove that there exists a $O(\Delta \log n)$ time distributed algorithm computing a $O(\Delta)$-coloring for unit disc graphs under the signal-to-interference-plus-noise ratio (SINR)-based physical model ($\Delta$ is the maximum degree of the graph). We also show that, for a well defined constant $d$, a $d$-hop $O(\Delta)$-coloring allows us to schedule an interference free MAC protocol under the physical SINR constraints. For instance this allows us to prove that any point-to-point message passing algorithm with running time $\tau$ can be simulated in the SINR model in $O(\Delta(\log n + \tau))$ time using messages of well chosen size. All our algorithms are proved to be correct with high probability.

**Key-words:** Node coloring, Distributed algorithms, SINR interference model, time complexity, Radio networks.

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# A Note on Node Coloring in the SINR Model

**Résumé :** A $\mathcal{V}$-coloring of a graph $G$ is a coloring of the nodes of $G$ with $\mathcal{V}$ colors in such a way any two neighboring nodes have different colors. We prove that there exists a $O(\Delta \log n)$ time distributed algorithm computing a $O(\Delta)$-coloring for unit disc graphs under the signal-to-interference-plus-noise ratio (SINR)-based physical model ($\Delta$ is the maximum degree of the graph). We also show that, for a well defined constant $d$, a $d$-hop $O(\Delta)$-coloring allows us to schedule an interference free MAC protocol under the physical SINR constraints. For instance this allows us to prove that any point-to-point message passing algorithm with running time $\tau$ can be simulated in the SINR model in $O(\Delta(\log n + \tau))$ time using messages of well chosen size. All our algorithms are proved to be correct with high probability.

**Mots-clés :** Node coloring, Distributed algorithms, SINR interference model, time complexity, Radio networks.

# 1 Introduction

**Motivation** We are interested in structuring wireless multi-hop radio networks formed of autonomous nodes communicating via radio and having no fixed built-in communication infrastructure. In such networks, the nodes must organize themselves distributively and establish some kind of structure allowing them to communicate and to perform their intended task efficiently.

Structuring radio networks in a local manner is of great practical and theoretical importance [KMW04]. Among the most useful structures allowing radio devices to self-organize their communications efficiently is coloring [MW08, MW05]. Many kinds of coloring may exist for different purposes. In fact, there is an impressive amount of work concerning network coloring (with several variants and applications) witnessing of the quintessential of this task, e.g., [BKM+06, SGP08, SGP05]. For instance, assigning different colors to neighboring vertices and associating colors with different time-slots in a time-division multiple access (TDMA) scheme is nothing else than a medium access control (MAC) protocol without direct interferences.

Many distributed algorithms computing good colorings can be found in the literature. Although a lot of advance has been made, the existing distributed algorithms still suffer from some weaknesses. For instance, some existing distributed algorithms are proved to be efficient only through simulations or for some particular network topologies. More importantly, some other distributed algorithms assume rather simplified models of radio communication which allows a good theoretical understanding of the algorithmic issues but do not lead to practical implementations in real radio networks under different deployment scenarios. In particular, the best time efficient coloring algorithm [MW08, MW05] is described and analyzed in the *graph-based* model. In the graph based model and some of its variant, e.g., *protocol* model, the interference experienced by a node is modeled as local binary function. Typically in the classical graph based model, a node experiences an interference if more than two of its neighbors are sending at the same time slot. This model allows us to concentrate on the algorithmic issues and to give rigorous analysis of designed algorithms. However, it abstracts away the interferences experienced by a node in a realistic environment, thus the obtained algorithms cannot be guaranteed to work under more sophisticated interference constraints.

The aim of this paper is to study distributed coloring algorithms under the more realistic signal-to-interference-plus-noise ratio (SINR)-based physical model [GK00]. In this model, a node experience an interference if the ratio of the received signal strength and the sum of the interference caused by all nodes sending simultaneously in the network, plus noise is less then a hardware-defined threshold $\beta$, where the signal fades with the distance to the power of some path-loss exponent $\alpha$.

**Contribution** In this paper, we adapt the best state-of-the-art coloring algorithm given initially by [MW08, MW05] to compute with high probability a $O(\Delta)$-coloring in $O(\Delta \log n)$ time under the SINR physical constraints. The number of colors produced by the algorithm is optimal up to a constant factor, and the time complexity is optimal up to a $\log n$ factor. To the extent of our knowledge, this the first time where an almost optimal coloring algorithm is given in the SINR model.

As stated previously, one important application of coloring algorithms is to design interference free MAC protocols. However, it is well known that a $O(\Delta)$-coloring is not sufficient to design such interference free protocols even under the simple graph based model. In this paper we show that under the SINR model a MAC layer without direct interference can be scheduled in $O(\Delta)$ time using a $d$-hop $O(\Delta)$-coloring for a well chosen constant $d$. A $d$-hop $\mathcal{V}$-coloring is a coloring of nodes with at most $\mathcal{V}$ colors such that every node has a different color than the nodes in its $d$-hop

Euclidian neighborhood[1]. As a direct application, we obtain upper bounds on the simulation of any point-to-point message passing algorithm in the SINR model.

## 2 Model and Definitions

We assume that nodes are placed arbitrarily in the 1 dimensional Euclidean space[2], i.e., the plane. Given two nodes $u$ and $v$, we denote by $d(u,v)$ the distance between nodes $u$ and $v$. We consider the signal-to-interference-plus-noise ratio model [GK00]. In this model, a node $u$ successfully receives a message from a sender $v$ if and only if the following condition holds:

$$\frac{\frac{P_v}{d(u,v)^\alpha}}{N \; + \; \sum_{w \in V \setminus \{v\}} \frac{P_w}{d(u,w)^\alpha}} \quad \geq \quad \beta$$

where $P_v$ is the power level of the transmission of node $v$, $\alpha > 2$ is the path-loss exponent, which depends on external conditions of the medium, $\beta \geq 1$ denotes the minimum signal to interference ratio required for a message to be successfully received, $N$ is the ambient noise, and $\sum_{w \in V \setminus \{v\}} \frac{P_w}{d(u,w)^\alpha}$ is the total amount of interference experienced by receiver $u$ and caused by all simultaneously transmitting nodes in the network.

We assume that all nodes are transmitting with the same power $P$, i.e., uniform power assignment scheme. Let $R_T$ be the transmission range of nodes. Let $B_v$ be the transmission region of node $v$ that is the disc of radius $R_T$ around $v$. From the definition of the SINR model, we have that $R_T \leq (\frac{P}{N\beta})^{1/\alpha}$. We assume that the ambient noise level $N$ is upper bounded by a fraction of the maximum tolerable interference level for a successful broadcast. More precisely, we assume that $R_T = (\frac{P}{2N\beta})^{1/\alpha}$, i.e., $N = \frac{P}{2\beta R_T^\alpha}$. Thus, the network can be viewed as a 'unit' disc graph $G = (V, E)$ where two nodes are neighbors (i.e., they can potentially communicate) if they are at distance $R_T$.

Given the transmission range $R_T$ of nodes, we define $\phi(R)$ to be size of the largest independent set (with respect to $R_T$) in the disc of radius $R$ around any node of the graph[3].

For every node $v$, let $I_v$ be the disc of radius $R_I$ around $v$ where $R_I$ is defined as following ($\rho > 1$ is a constant to be fixed later):

$$R_I = \left(96 \; \rho \; \beta \cdot \frac{\alpha - 1}{\alpha - 2}\right)^{1/(\alpha-2)} \cdot 2R_T$$

Note that $R_I \geq 2R_T$. For any constants $\rho > 1$ and $c_2 \geq c_1 \geq 5$, we define the following variables:

$$\lambda \;=\; \left(1 - \frac{1}{\rho}\right) \cdot \frac{1}{e^{\phi(R_I)/\phi(R_I+R_T)}} \cdot \left(1 - \frac{\phi(R_I)}{\phi(R_I+R_T)^2 \cdot \Delta}\right) \cdot \left(1 - \frac{1}{\phi(R_I+R_T)}\right)^{\phi(R_I)}$$

$$\lambda' \;=\; \left(1 - \frac{1}{\rho}\right) \cdot \frac{1}{e \cdot \phi(R_I+R_T)} \cdot \left(1 - \frac{1}{\phi(R_I+R_T)\Delta}\right) \cdot \left(1 - \frac{1}{\phi(R_I+R_T)}\right)^{\phi(R_I+R_T)}$$

[1] Here, the $d$-hop neighborhood of a given node $v$ in the plane is the set of nodes at distance at most $d \cdot R_v$ where $R_v$ is the transmission range of $v$ that is the distance up to which a node can hear $v$'s transmissions in absence of interferneces, i.e., in a clear environment.

[2] We remark that our results can be proved to hold under any bounded independence graph. The assumption that nodes are placed in the plane is only made for the sake of clarity.

[3] An independent set with respect to a given transmission range $R_T$ is a set of nodes mutually at distance $d > R_T$. Notice that $\phi(R)$ can be roughly bounded as following: $\phi(R) \leq \frac{\pi(R+R_T/2)^2}{\pi(R_T/2)^2} = \left(\frac{2R}{R_T}+1\right)^2$. Notice also that in our proofs, knowing only an upper bound on $\phi(R)$ affects our bound only by a constant, i.e., knowing the exact value of $\phi(R)$ is not required to prove our results.

$$\sigma = \frac{2c_2}{\lambda'} \quad , \quad \gamma = \frac{c_1 \cdot \phi(R_I + R_T)}{\lambda}$$
$$q_\ell = \frac{1}{\phi(R_I + R_T)} \quad , \quad q_s = \frac{1}{\phi(R_I + R_T)\Delta}$$

By a routine computation, one can easily verify that $\sigma \geq 2\gamma$. Finally, we shall use any constant variables $\eta$ and $\mu$ verifying: $\eta \geq 2\gamma\phi(2R_T) + \sigma + 1$ and $\mu \geq \gamma$.

# 3 The coloring algorithm of [MW08, MW05]

The algorithm of [MW08] is decomposed in three parts according to the states of nodes; see figures 1, 2 and 3[4]. It is important to remark that we do *not* use the same constant used in [MW08]. In fact, the value of constants $\gamma$, $\sigma$, $\eta$ and $\mu$ used by the algorithm below are those defined in the previous section. As argued in [MW08], the choice of those constants is fundamental to prove the correctness of the coloring.

To make the paper self-contained, we recall the general idea of the algorithm. Each node can be in three state classes $\mathcal{A}$, $\mathcal{R}$ or $\mathcal{C}$. Upon wake up, a node enters state $\mathcal{A}_0$ and executes the algorithm of Fig. 1. Whenever a node enters state $\mathcal{R}$ it executes the algorithm of Fig. 2. Whenever a node enters state $\mathcal{C}_i$ it executes the algorithm of Fig. 3. As explained in [MW08, MW05], the general idea of the algorithm is as follows. First, the algorithm attempts to compute an independent set of the graph: nodes in state $\mathcal{A}_0$ compete in order to be in the independent set. Once a node becomes a leader it enters state $\mathcal{C}_0$. The algorithm also computes a clustering of the graph, i.e., each node is associated with one node in the independent set, each node of the independent set is the leader of its cluster. Then, every leader attempts to assign a different color to each node in its cluster. Nodes in state $\mathcal{R}$ are nodes waiting to receive a color from their leader. Once a node receives a color from its leader it must verify that no neighbors belonging to another cluster have been assigned the same color. Nodes verifying their color are in state $\mathcal{A}_i$ with $i > 0$.

$P_v := \varnothing$; $\quad \zeta_i := \begin{cases} 1 & \text{if } i = 0 \\ \Delta & \text{if } i > 0 \end{cases}$; $\quad \mathcal{A}_{suc} := \begin{cases} \mathcal{R} & \text{if } i = 0 \\ \mathcal{A}_{i+1} & \text{if } i > 0 \end{cases}$;
**for** $\lceil \eta\Delta \ln n \rceil$ *time slots* **do**
  **for** *each* $w \in P_v$ **do** $d_v(w) := d_v(w) + 1$;
  **if** $M^i_{\mathcal{A}}(w, c_w)$ *received* **then** $P_v := P_v \cup \{w\}$; $d_v(w) := c_w$;
  **if** $M^i_{\mathcal{C}}(w)$ *received* **then** *state* $:= \mathcal{A}_{suc}$; $L(v) := w$;
$c_v := \chi(P_v)$, where $\chi(P_v)$ is the maximum value s.t.,
$\chi(P_v) \notin \{d_v(w) - \lceil \gamma\zeta_i \ln n \rceil, \ldots, d_v(w) + \lceil \gamma\zeta_i \ln n \rceil\}$ for each $w \in P_v$, and $\chi(P_v) \leq 0$;
**while** *state* $= \mathcal{A}_i$ **do**
  $c_v := c_v + 1$;
  **for** *each* $w \in P_v$ **do** $d_v(w) := d_v(w) + 1$;
  **if** $c_v \geq \lceil \sigma\Delta \ln n \rceil$ **then** *state* $:= \mathcal{C}_i$;
  transmit $M^i_{\mathcal{A}}(v, c_v)$ with probability $q_s$;
  **if** $M^i_{\mathcal{C}}(w)$ *received* **then** *state* $:= \mathcal{A}_{suc}$; $L(v) := w$;
  **if** $M^i_{\mathcal{A}}(w, c_w)$ *received* **then**
    $P_v := P_v \cup \{w\}$; $d_v(w) := c_w$;
    **if** $|c_v - c_w| \leq \lceil \gamma\zeta_i \ln n \rceil$ **then** $c_v := \chi(P_v)$

Figure 1: Coloring Algorithm: code for node $v$ in state $\mathcal{A}_i$

[4] Note that we have used the same notations used in [MW08].

**while** $state = \mathcal{R}$ **do**
  transmit $M_{\mathcal{R}}(v, L(v))$ with probability $q_s$;
  **if** $M_{\mathcal{C}}^{0}(L(v), v, tc_v)$ *received* **then**
    $state := \mathcal{A}_{tc_v \cdot (\phi(2R_T)+1)}$;

Figure 2: Coloring Algorithm: code for node $v$ in state $\mathcal{R}$

color$_v$ := i;
**if** $i > 0$ **then**
  **repeat** transmit $M_{\mathcal{C}}^{i}(v)$ with probability $q_s$ **until** *protocol stopped;*
**else if** $i = 0$ **then**
  $tc := 0$; $\mathcal{Q} := \varnothing$;
  **repeat**
    **if** $M_{\mathcal{R}}(w, v)$ *received and* $w \notin \mathcal{Q}$ **then** add $w$ to $\mathcal{Q}$;
    **if** $\mathcal{Q}$ *is empty* **then**
      transmit $M_{\mathcal{C}}^{0}(v)$ with probabilty $q_\ell$;
    **else**
      $tc := tc + 1$;
      Let $w$ be the first element in $\mathcal{Q}$;
      **for** $\lceil \mu \ln n \rceil$ *time slots* **do** transmit $M_{\mathcal{C}}^{0}(v, w, tc)$ with probability $q_\ell$;
      Remove $w$ from $\mathcal{Q}$;
  **until** *protocol stopped*;

Figure 3: Coloring Algorithm: code for node $v$ in state $\mathcal{C}_i$

# 4 Analysis in the SINR model

Given a time slot and two nodes $u$ and $v$, we denote $p_v$ the sending probability of node $v$ at that time slot and $\Psi_u^v = p_v / d(u,v)^\alpha$ the probabilistic interference caused by $v$ to node $u$. For every node $u$, we define the probabilistic interference induced by nodes outside some region $R$ in a given time slot as following :

$$\Psi_u^{v \notin R} = P \cdot \sum_{v \notin R} \Psi_u^v$$

The algorithm analysis follows the same scheme than the analysis made in [MW08, MW05]. The correctness of the algorithm is based on the fact that at any time of the execution of the algorithm the set of node in state $\mathcal{C}_i$ forms an independent set. This is announced in Theorem 1 below. In order to prove the theorem, we need to prove some intermediary properties expressed in Lemmas 1 and 2[5]. These two lemmas provide a bound on the time needed for a node in some particular state to communicate without interferences. To prove Lemmas 1 and 2 in the SINR model, we need to bound the interference experienced by a node at any step of the algorithm execution. This is the aim of Lemma 3 below. In the following, we start proving Lemma 3 and then come back to Lemmas 1 and 2.

**Theorem 1** *For all $i$, with high probability the color class $\mathcal{C}_i$ forms an independent set throughout the execution of the algorithm.*

**Lemma 1** *Assume $\mathcal{C}_0$ forms an independent set. Consider nodes $u$, $u'$, and $v$ such that $d(u,v) \leq R_T$, $d(u',v) \leq R_T$, and $u \in V \setminus \mathcal{C}_0$ and $u \in \mathcal{C}_0$. Let $J$ and $J'$ be time intervals of length $\gamma \Delta \ln n$*

[5]These two lemmas corresponds to Lemmas 2, 3 and 4 proved in [MW08] under the simple graph based interference model.

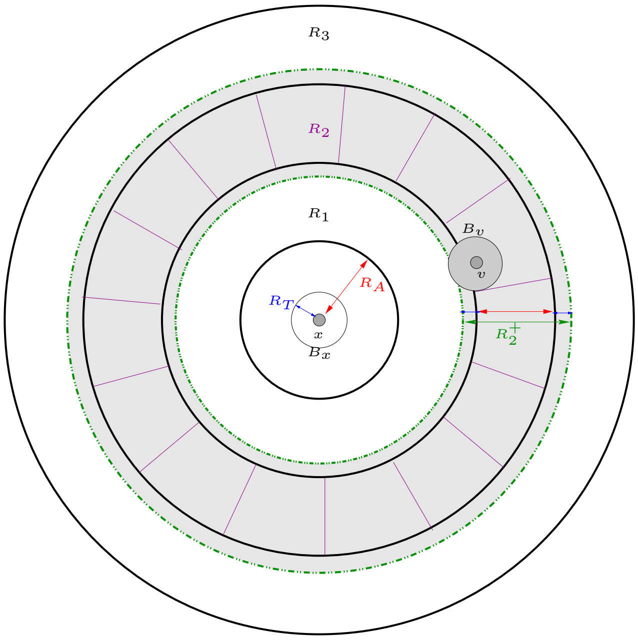


Figure 4: The rings around node $x$

*and $\gamma \ln n$, respectively. The probability $\mathbb{P}_{no}$ (resp. $\mathbb{P}'_{no}$) that $v$ does not get a message from $u$ (resp. $u'$) during an interval $J$ (resp. $J'$) is upper bounded by $1/n^{c_1}$.*

**Lemma 2** *Assume $\mathcal{C}_0$ forms an independent set. Consider a node $v \in \mathcal{A}_i$ for an arbitrary $i$. Further, let $J$ be a time interval of length $|J| = \frac{\sigma}{2}\Delta \ln n$. With probability at least $1 - 1/n^{c_2}$, there is a time slot $t \in J$ such that, at least one node $w \in B_v \cap \mathcal{A}_i$ sends successfully.*

**Lemma 3** *Assume $\mathcal{C}_0$ forms an independent set. Then, for every node $u$, the probabilistic interference $\Psi_u^{v \notin I_u}$ verifies: $\Psi_u^{v \notin I_u} \leq P/(2\rho\beta R_T^\alpha)$.*

**Proof of Lemma 3** Consider a fixed time slot and let $p_w$ the sending probability of node $w$. From the algorithm, if $w \in \mathcal{C}_0$ then $p_w = q_\ell$ otherwise $p_w = q_s$. Thus, the sum of transmitting probabilities over all nodes in a given region $B_v$ can be bounded as following:

$$\begin{aligned} \sum_{w \in B_v} p_w &= \sum_{w \in B_v \cap \mathcal{C}_0} q_\ell + \sum_{w \in B_v \cap V \setminus \mathcal{C}_0} q_s \\ &\leq \sum_{w \in B_v \cap \mathcal{C}_0} \frac{1}{\phi(R_I + R_T)} + \sum_{w \in B_v \cap V \setminus \mathcal{C}_0} \frac{1}{\phi(R_I + R_T)\Delta} \\ &\leq 2 \end{aligned} \tag{1}$$

Given a node $u$, let $R_\ell = \{v \in V \text{ such that } \ell R_I \leq d(u,v) \leq (\ell+1)R_I\}$. Consider an independent set $\mathcal{I}$ of maximum size in $R_\ell$. It is clear that $\cup_{z \in \mathcal{I}} B_z$ (the union of the broadcast regions of nodes in $\mathcal{I}$) covers entirely the ring $R_\ell$ (otherwise we can add a new node to $\mathcal{I}$ leading to a contradiction since the size of $\mathcal{I}$ is maximum). Since nodes in $\mathcal{I}$ are independent, the discs of radius $R_T/2$ around nodes in $\mathcal{I}$ are mutually disjoint (otherwise the independence of $\mathcal{I}$ is violated). Note that these discs are located inside the extended region $R_\ell^+$ defined by $R_\ell^+ = \{v \in V \text{ such that } \ell R_I - R_T/2 \leq d(u,v) \leq (\ell+1)R_I + R_T/2\}$ (Fig. 4 gives an example). Thus, $|\mathcal{I}| \leq \dfrac{Area\,(R_+^\ell)}{Area\,(Disc(R_T/2))}$. Thus, the probabilistic interference caused by nodes inside $R^\ell$ can be bounded as following:

$$\begin{aligned}
\Psi_u^{R_\ell} &= P \cdot \sum_{v \in R_\ell} \Psi_u^v \\
&\leq \max_{v \in R_\ell} \left\{ \frac{Area\left(R_+^\ell\right)}{Area\left(Disc(R_T/2)\right)} \cdot P \sum_{w \in B_v \cap R_\ell} \frac{p_w}{(\ell R_I)^\alpha} \right\} \\
&\leq_{\text{By Eq. 1}} \frac{Area\left(R_+^\ell\right)}{Area\left(Disc(R_T/2)\right)} \cdot \frac{2P}{\ell^\alpha R_I^\alpha} \\
&= \frac{\pi\left(((\ell+1)R_I + R_T/2)^2 - (\ell R_I - R_T/2)^2\right)}{\pi (R_T/2)^2} \cdot \frac{2P}{\ell^\alpha R_I^\alpha} \\
&= \frac{4(2\ell+1)\left(R_I^2 + R_I R_T\right)}{R_T^2} \cdot \frac{2P}{\ell^\alpha R_I^\alpha} \\
&\leq \frac{1}{\ell^{\alpha-1}} \cdot \frac{48 P R_I^2}{R_T^2 \cdot R_I^\alpha}
\end{aligned}$$

Thus,

$$\begin{aligned}
\Psi_u^{v \notin I_u} &= \sum_{\ell=1}^{\infty} \Psi_u^{R_\ell} \\
&\leq \frac{48 P R_I^2}{R_T^2 \cdot R_I^\alpha} \cdot \sum_{\ell=1}^{\infty} \frac{1}{\ell^{\alpha-1}} \\
&\leq P \cdot \frac{48 R_I^{2-\alpha}}{R_T^2} \cdot \frac{\alpha-1}{\alpha-2} \\
&\leq \frac{P}{2\rho\beta R_T^\alpha}
\end{aligned}$$

The last inequality holds from the definition of $R_I$. ∎

**Proof of Lemma 1** Consider a given time slot of the algorithm. Suppose that node $u$ is the only node inside $I_v$ that transmits at that time slot. Then, we can show that node $v$ is likely to hear the message sent by $u$, i.e., the SINR condition is likely to be verified. In fact, from the Markov inequality and using Lemma 3, we have that the probability that the interferences caused by nodes outside $I_v$ exceeds $\rho \cdot \Psi_v^{w \notin I_v}$ is at most $1/\rho$. Thus, given that $u$ is the only sending node in $I_v$, with probability at least $1 - 1/\rho$, the SINR at node $v$ can be bounded as following:

$$\frac{\frac{P}{d(u,v)^\alpha}}{\rho \cdot \Psi_v^{w \notin I_v} + N} \geq \frac{\frac{P}{d(u,v)^\alpha}}{\frac{P}{2\beta R_T^\alpha} + \frac{P}{2\beta R_T^\alpha}} \geq \beta \tag{2}$$

Thus the probability $\mathbb{P}((u \longrightarrow v))$ that node $v$ hears a message sent by $u$ at a given time slot can be bounded as following:

$$\begin{aligned}
\mathbb{P}((u \longrightarrow v)) &\geq \left(1 - \frac{1}{\rho}\right) \cdot p_u \cdot \prod_{w \in I_v \setminus \{u\}} (1 - p_w) \\
&= \left(1 - \frac{1}{\rho}\right) \cdot p_u \cdot \prod_{w \in I_v \cap \mathcal{C}_0} (1 - q_\ell) \cdot \prod_{w \in I_v \cap V \setminus \mathcal{C}_0} (1 - q_s) \\
&\geq \left(1 - \frac{1}{\rho}\right) \cdot p_u \cdot \left(1 - \frac{1}{\phi(R_I + R_T)}\right)^{\phi(R_I)} \cdot \left(1 - \frac{1}{\phi(R_I + R_T) \cdot \Delta}\right)^{\phi(R_I) \cdot \Delta} \\
&\geq \left(1 - \frac{1}{\rho}\right) \cdot p_u \cdot \left(1 - \frac{1}{\phi(R_I + R_T)}\right)^{\phi(R_I)} \\
&\quad \cdot \left(\left(1 - \frac{\phi(R_I)}{\phi(R_I + R_T)^2 \cdot \Delta}\right) e^{-\phi(R_I)/\phi(R_I + R_T)}\right) \\
&= \lambda \cdot p_u
\end{aligned}$$

Thus, if $u \in I_v \cap \mathcal{C}_0$, i.e., $p_u = q_s = 1/(\phi(R_I + R_T)\Delta)$, then we get:

$$\mathbb{P}_{no} \leq (1 - \mathbb{P}((u \longrightarrow v)))^{|J|} = \left(1 - \frac{\lambda}{\phi(R_I + R_T)\Delta}\right)^{\frac{c_1 \phi(R_I+R_T)}{\lambda}\Delta \ln(n)} \leq n^{-c_1}$$

Similarly, for node $u' \in I_v \cap \mathcal{C}_0$, we get

$$\mathbb{P}_{no} \leq (1 - \mathbb{P}((u \longrightarrow v)))^{|J'|} = \left(1 - \frac{\lambda}{\phi(R_I + R_T)}\right)^{\frac{c_1 \phi(R_I+R_T)}{\lambda}\ln(n)} \leq n^{-c_1}$$

∎

**Proof of Lemma 2** Let $v \in \mathcal{A}$ a node and $w \in B_v \cap \mathcal{A}$ a neighbor of $v$.

Consider a node $u$ such that $u$ is a neighbor of $w$. We remark that if $w$ is the only sending node in the region $I_u$ around $u$, then $u$ is likely to hear the message sent by $w$, i.e., the SINR condition is likely to be verified for $u$. Thus, if $w$ is the only sending node in $\cup_{u \in B_w} I_u$, then the message sent by $w$ is likely to be received by $w$'s neighbors.

More precisely, let $(w \longrightarrow \star)$ be the event "all $w$'s neighbors hear a message from $w$". Using Lemma 3, we have that $\Psi_w^{x \notin \cup_{u\in B_w} I_u} \leq P/(2\rho\beta R_T^\alpha)$. Thus, provided that $w$ is the only sending node in $\cup_{u \in B_w} I_u$, and using the Markov inequality (as for Equation 2), it is not difficult to see that the SINR condition holds for all $w$'s neighbor with probability at least $1 - 1/\rho$, i.e., if $w$ is the only sending node in $\cup_{u\in B_w} I_u$ then for every node $u \in B_w$ the SINR condition holds and $u$ hears $w$'s message with probability at least $1 - 1/\rho$. Thus, we have:

$$\begin{aligned}
\mathbb{P}\left((w \longrightarrow \star)\right) &\geq (1 - \frac{1}{\rho}) \cdot p_w \cdot \prod_{r \in \left(\bigcup_{u\in B_w} I_u\right)\setminus\{w\}} (1 - p_r) \\
&= (1 - \frac{1}{\rho}) \cdot p_w \cdot \prod_{r \in \left(\left(\bigcup_{u\in B_w} I_u\right)\setminus\{w\}\right)\cap \mathcal{C}_0} (1 - q_\ell) \\
&\quad \cdot \prod_{r \in \left(\left(\bigcup_{u\in B_w} I_u\right)\setminus\{w\}\right)\cap (V\setminus\mathcal{C}_0)} (1 - q_s) \\
&\geq (1 - \frac{1}{\rho}) \cdot \frac{1}{\phi(R_I+R_T)\Delta} \cdot \left(1 - \frac{1}{\phi(R_I+R_T)}\right)^{\phi(R_I+R_T)} \\
&\quad \cdot \left(1 - \frac{1}{\phi(R_I+R_T)\Delta}\right)^{\phi(R_I+R_T)\cdot\Delta} \\
&\geq (1 - \frac{1}{\rho}) \cdot \frac{1}{\phi(R_I+R_T)\Delta} \cdot \left(1 - \frac{1}{\phi(R_I+R_T)}\right)^{\phi(R_I+R_T)} \\
&\quad \cdot \left(1 - \frac{1}{\phi(R_I+R_T)\Delta}\right) \cdot e^{-1} \\
&\geq \frac{\lambda'}{\Delta}
\end{aligned}$$

Thus the probability that $w$ fails sending successfully during the interval $J$ is at most:

$$(1 - \mathbb{P}\left((w \longrightarrow \star)\right))^{|J|} = \left(1 - \frac{\lambda'}{\Delta}\right)^{\frac{c_2}{\lambda'}\Delta \ln(n)} \leq n^{-c_2}$$

∎

Having Lemma 1 and Lemma 2 at hand, Theorem 1 is straightforwardly proved using the same arguments than those in [MW08]'s proof. In fact, apart from using Lemma 1 and Lemma 2, the reasoning in [MW08] simply needs that $\sigma > 2\gamma$ which is actually verified by our constants.

Having proved Theorem 1, Lemmas 1 and 2, the analysis of the correctness and time complexity of the coloring algorithm is basically the same than the analysis made in [MW08][6]. Therefore we can state the following theorem (For a full proof we refer the reader to the proof of Theorem 5 in [MW08]).

**Theorem 2** *With high probability, the algorithm produces a correct coloring with at most* $\phi(2R_T)\Delta$ *colors within at most* $c\Delta \ln n$ *time slots after nodes wake up, where* $c$ *is a constant verifying* $c = O(\phi(2R_T)^2 \cdot \sigma)$.

# 5 MAC layer in the SINR model

In this section, we show how to use the coloring algorithm to design a generic MAC Layer allowing to simulate classical interference free message passing algorithms in the SINR model.

Let us consider the classical graph-based point-to-point message passing model. In other words, we consider a distributed model where every two neighboring nodes are connected by a private channel allowing them to communicate in a bi-directional manner without any interferences, i.e., no interference can prevent a message sent by a node to arrive to the second node. For simplicity let us assume that in such a model, any algorithm proceeds into *rounds*. In each round, a node can receive messages, do some local computation and send messages. We consider two classes of algorithms: the *uniform* model and the *general* model. In the former, each node is allowed to send the same message to all its neighbors at a given round, e.g. broadcast algorithms. In the latter, a node can send a different message to each neighbor.

We first consider a *uniform* algorithm. Our goal is to transform the algorithm so we can run it in the SINR physical model and obtain the same output. The idea is to simulate each round of the algorithm and to attempt to deliver each original message to destination while avoiding interference. For that purpose, we shall use the previous coloring algorithm as a MAC layer by associating each color with a time slot where a node can transmit. However, given a node $u$ having two neighbors $v$ and $w$ with the same color $c_v = c_w$, if $v$ and $w$ both transmit a message during the time slot corresponding to color $c_v$, then an interference occurs at node $u$, i.e., node $u$ does not receive any message. This is a well known problem which is commonly solved by using a distance 2 coloring as a MAC layer, that is a coloring such that each node have a different color than his 2-hop neighbors. Unfortunately, such a coloring do not allow us to avoid interferences in the SINR physical model.

In the following we show how to schedule a MAC layer without direct interferences in the SINR model. In fact, suppose we have constructed a $(d+1)$-hop $\mathcal{V}$-coloring for some parameter $d$, that is a coloring using at most $\mathcal{V}$ colors such that each node have a different color than his $(d+1)$-hop neighbors (by $t$-hop neighborhood, we mean nodes at distance $t \cdot R_T$ in the Euclidean plane). We associate each color $c$ with a time slot $t_c$ where nodes having color $c$ can transmit in time slot $t_c$. Using this simple protocol, we obtain the following:

**Theorem 3** *For* $d = \left(32 \cdot \frac{\alpha - 1}{\alpha - 2} \cdot \beta\right)^{1/\alpha}$, *a* $(d+1)$*-hop* $\mathcal{V}$*-coloring defines a scheduled MAC layer protocol allowing every node to successfully send a message to its neighbors within at most* $\mathcal{V}$ *time.*

**Proof.** Consider a time slot $t_c$ and a node $u$ having a neighbor $v$ with color $c_v = c$. Let us compute the amount of interference $\Phi_{u \setminus v}$ experienced by $u$ and caused by nodes $w \in V \setminus \{v\}$ during time slot $t_c$.

Let $H_{\ell,d} = \{w \in V \text{ such that } \ell d R_T \leq d(u,v) \leq (\ell+1) d R_T\}$. The $d$ hop neighborhood of any node $w \in H_\ell$ must be located in an extended region defined as following: $H^+_{\ell,d} = \{w \in V \text{ such that } (\ell - 1/2) d R_T \leq d(u,v) \leq (\ell + 3/2) d R_T\}$. Let $Disc(r)$ any disc of radius $r$.

[6] The analysis made in [MW08] needs that $\eta \geq 2\gamma\phi(2R_T) + \sigma + 1$ and $\mu \geq \sigma$, which is actually verified by our constants.

Let us denote by $\phi'(\ell, d)$ the maximum number of nodes with the same color in some region $H_{\ell,d}$. Since the $d/2$ hop neighborhoods of any two nodes $w$ and $w'$ with the same color $c$ are disjoint, we can bound $\phi'(\ell, d)$ as following:

$$\begin{aligned}
\phi'(\ell, d) &\leq \frac{Area(H^+_{\ell,d})}{Area(Disc(dR_T/2))} \\
&= \frac{\pi\left((\ell+3/2)dR_T\right)^2 - \pi\left((\ell-1/2)dR_T\right)^2}{\pi d^2 R_T^2/4} \\
&= 4 \cdot \left((\ell+3/2)^2 - (\ell-1/2)^2\right) \\
&\leq 4(4\ell - 2) \\
&\leq 16\ell
\end{aligned}$$

At time slot $t_c$ corresponding to color $c$, we have that:

$$\begin{aligned}
\Phi_{u\setminus v} &= \sum_{w \in V\setminus\{v\}} \frac{P}{d(w,u)^\alpha} &&= \sum_{w \in V\setminus\{v\}\ s.t.,\ c_w = c} \frac{P}{d(w,u)^\alpha} \\
&= P \cdot \sum_{\ell=1}^{\infty} \sum_{w \in H_{\ell,d}} \frac{1}{(\ell \cdot d \cdot R_T)^\alpha} &&\leq P \cdot \sum_{\ell=1}^{\infty} \frac{\phi'(\ell,d)}{(\ell \cdot d \cdot R_T)^\alpha} \\
&\leq \frac{16P}{d^\alpha R_T^\alpha} \cdot \sum_{\ell=1}^{\infty} \frac{1}{\ell^{\alpha-1}} &&\leq \frac{16P}{d^\alpha R_T^\alpha} \cdot \frac{\alpha-1}{\alpha-2} \\
&\leq \frac{P}{2\beta R_T^\alpha}
\end{aligned}$$

Thus the SINR at node $u$ verifies:

$$SINR_u = \frac{\frac{P}{R_T^\alpha}}{\Phi_{u\setminus v} + N} \geq \frac{\frac{P}{R_T^\alpha}}{\frac{P}{2\beta R_T^\alpha} + \frac{P}{2\beta R_T^\alpha}} = \beta$$

Thus, node $u$ receives the message sent by $v$. In other words, a message sent by node $v$ at time slot $t_{c_v}$ is correctly received by all its neighbors and the theorem is proved. ∎

Now, suppose that we have an algorithm $A$ computing a 1-hop $(c\Delta)$-coloring of a given (unit disc) graph $G$ where $c$ is a constant. It is not difficult to adapt algorithm $A$ to compute a $d$-hop coloring of $G$ as following. In fact, we remark that a 1-hop coloring of $G^d$ (where $G^d$ is the graph obtained from $G$ by adding an edge between any tow nodes at distance at most $d \cdot R_T$ in the Euclidean plane) is also a $d$-hop $\Delta(G^d)$-coloring of $G$ where $\Delta(G^d)$ is the maximum degree of $G^d$. The maximum degree of $G^d$ can be upper bounded as following $\Delta(G^d) \leq \phi(d \cdot R_T)\Delta \leq (2d+1)^2\Delta$. Thus, running algorithm $A$ on $G^d$ produces a $d$-hop $(c \cdot (2d+1)^2 \cdot \Delta)$-coloring of $G$.

To summarize, a simple idea to obtain a $d$-hop $O(\Delta)$-coloring of $G$ is to compute a 1-hop coloring algorithm of $G^d$ on top of $G$. One simple idea to achieve that is to initially increase the transmission power of every node from $P$ to $d \cdot P$. Since $\Delta(G^d) = O(\Delta)$, the time needed to output the coloring is still $O(\Delta \log n)$.

From the previous discussion, we get the following:

**Corollary 1** *Consider a uniform (resp. general) point-to-point message passing algorithm $A$ (resp. $A'$) running on an $n$-node graph in $\tau$ time and using messages of size at most $s$ bits. With high probability,*

- *Algorithm $A$ can be simulated in the SINR physical model in $O(\Delta \cdot (\log n + \tau))$ time using messages of size at most $O(s \log n)$ bits.*

- *Algorithm $A'$ can be simulated in the SINR physical model in $O(\Delta \cdot (\log n + \tau))$ (resp. $O(\Delta \cdot \log n + \Delta^2 \cdot \tau)$) time using messages of size at most $O(s\Delta \log n)$ (resp. $O(s \log n)$) bits.*

The latter corollary can be of special interest, for instance, for some random geometric graphs with polylogaritmic expected maximum degree.

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