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Michel FLIESS * **Cédric JOIN** ** **Wilfrid PERRUQUETTI** ***

* *Michel FLIESS is with INRIA-ALIEN & LIX (UMR-CNRS 7161), École polytechnique, 91128 Palaiseau, France. E-mail:* Michel.Fliess@polytechnique.edu
** *Cédric JOIN is with INRIA-ALIEN & CRAN (UMR-CNRS 7039), Nancy-Université, BP 239, 54506 Vandœuvre-lès-Nancy, France. E-mail:* Cedric.Join@cran.uhp-nancy.fr
*** *Wilfrid PERRUQUETTI is with INRIA-ALIEN & LAGIS (UMR-CNRS 8146), École Centrale de Lille, BP 48, Cité Scientifique, 59651 Villeneuve-d'Ascq, France. E-mail:* wilfrid.perruquetti@ec-lille.fr.

Abstract We extend previous works of Fliess et al. [2008] on the estimation of the switching signal and of the state for switching linear systems to the perturbed case when the perturbation is structured that is when the perturbation is unknown but known to satisfy a certain differential equation (for example if the perturbation is constant then its time-derivative is zero). We characterize also singular inputs and/or perturbations for which the switched systems become undistinguishable. Several convincing numerical experiments are illustrating our techniques which are easily implementable.

Keywords: Linear systems, switched systems, hybrid systems, state estimation, switching signal estimation, numerical differentiation.

1. INTRODUCTION

Hybrid systems (coupling between a continuous dynamical system and a discrete state system), encountered in practice, can exhibit switchings between several subsystems, both as a result of controller design, such as in switching supervisory control, and inherently by nature, such as when a physical plant has the capability of undergoing several operational modes.

Hybrid systems is an active ongoing research fields for at least two reasons: the first one being for economic and social needs and the second one because it offer us a stimulating playground. The first reason find its roots in embedded systems which are more and more becoming part of our everyday-life: their control is becoming much more complex because of: the coupling between continuous dynamics and decision making process, the increasing requested performances and the process networking.

Switched systems may be viewed as higher–level abstractions of hybrid systems, obtained by neglecting the details of the discrete behavior. Informally, a switched system is composed of a family of dynamical subsystems (linear or nonlinear), and a rule, called the switching law, that orchestrates the switching between them. In recent years, there has been increasing interest in the control problems of switched systems due to their significance from both a theoretical and practical point of view and also because of their inherently interdisciplinary nature.

So several important results for switched systems have been achieved, including various results on *stability* (see Agrachev and Liberzon [2001], Arapostathis and Broucke [2007], Blanchini and Savorgnan [2008], Boscain [2006], Branicky [1994, 1998], Hespanha and Morse [1999], Liberzon and Morse [1999], Vu and Liberzon. [2005], Mancilla-Aguilar and García [2000], Mancilla-Aguilar et al. [2005], Pettersson and Lennartson [1996], Skafidas et al. [1999]) (with many applications see, for example, Buisson et al. [2005] for application to electrical power converters), *stabilization* (see Bourdais et al. [2006, 2008], Moulay et al. [2007], De Persis et al. [2003, 2004], Pettersson [1999], Wicks et al. [1994], Wicks and DeCarlo [1997], Wicks et al. [1998], Zhai et al. [2003], Wang et al. [2004]), *tracking* (see Bourdais et al. [2007]), *controllability* results (see Sun et al. [2002], Xie et al. [2002]), and *input-to-state properties*, ... See, e.g., Branicky [1993], Brockett [1993], Decarlo et al. [2000], Liberzon and Morse [1999], Liberzon [2003], Sun and Ge [2005] for a survey of this type of results. A large number of these results highlight the central role played by the knowledge of the switching function.

More over observability and state estimation is a key problem for such systems, where discrete and continuous parts are mixed. In Ackerson and Fu [1970], the notion of state estimation for switched systems is introduced. A generic setting for the observability of switched linear systems in a continuous setting has been given in Babaali and Pappas [2005]. In Vidal et al. [2003] the observability of switched linear systems in the case of deterministic switching signal was carried out. The unobserved switching

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case was analyzed in De Santis et al. [2003]. In most of the cases, the hybrid observer consists of two parts: an index estimator of the current active sub-model and a continuous observer that estimates the continuous state of the hybrid system.

But concerning the estimation of the switching function, up to now very few results are concerned with such estimation problem let us mention Millnert [1980] on identification of abruptly changing systems using least-square methods (with an example of switching between two linear systems), Vu and Liberzon [2008] on left invertibility [1] and Balluchi et al. [2002], Ragot et al. [2003], Saadaoui et al. [2006], Fliess et al. [2008] that are providing switching function estimators. Following the previous obtained results by the authors (see Fliess et al. [2008]) we investigate here the structured perturbed case that is when the sub-models are perturbed with an exogenous signal which is unknown but which is known to satisfy a given differential equation. As mentioned in Fliess et al. [2008] it may happen that singular inputs may lead to a situation where it is not possible to distinguish the output of two different sub-models: in this situation it is not possible to determine the active subsystem. As we can easily imagine such situation may also happen for some particular perturbations. Thus the paper is structured as follows: section 2 is formulating the problem to be solve and gives necessary and sufficient, in terms of easy to check algebraic conditions, for distinguishability independently of the structured perturbation ; section 3 gives a real time algorithm for the estimation of the switching function when the problem is solvable (that is when all the sub-models are distinguishable whatever is the structured perturbation). The algorithm being convincingly illustrated through a simple simulation example in subsection 3.3.

2. DISTINGUISHABILITY

2.1 Problem formulation: an Input-output behavior

Consider SISO linear switched systems of the form:

$$\begin{aligned} \dot{x} &= A_{\sigma(t)}x + B_{\sigma(t)}u + E_{\sigma(t)}p, \\ y &= C_{\sigma(t)}x + D_{\sigma(t)}u + F_{\sigma(t)}p, \\ & x \in \mathbb{R}^{n_{x_{\sigma(t)}}}, u \in \mathbb{R}, y \in \mathbb{R}, p \in \mathbb{R} \end{aligned} \tag{1}$$

where $\sigma(t)$ is the switching signal taking value within the index set $I_M = \{1, ..., M\}$ (M is a finite integer), x is the state of eventually variable dimension ($n_{x_{\sigma(t)}}$), u the known input of the system, y is the measured output (eventually a noisy measurement) and p is the unknown perturbation acting on the system. In the rest we address the problem of the reconstruction of the switching signal $\sigma(t)$ in "real-time", and of the state variables (not detailed here, see Fliess et al. [2008] for the details in the unperturbed case which can be under some reasonable assumptions extended to our concerned setting)).

Assume that the perturbation is an exogenous signal of known structure that is p is the solution of a known differential equation. Let us mention some examples $p = c$ (constant) : $\dot{p} = 0$, $p = \sum_{i=1}^{n_p} a_i t^i$: $p^{(n_p+1)} = 0$ and $p = a\cos(\omega t)$: $\ddot{p} + \omega^2 p = 0$ which are the more common encountered perturbations in physics. Since no state jump is assumed, we adopt an i/o behavior representation:

$$\mathfrak{a}_i \left(\frac{d}{dt}\right) y_i = \mathfrak{b}_i \left(\frac{d}{dt}\right) u + \mathfrak{c}_i \left(\frac{d}{dt}\right) p, \quad i \in I_M, \tag{2}$$

$$\mathfrak{c}_{M+1} \left(\frac{d}{dt}\right) p = 0 \tag{3}$$

where $\mathfrak{a}_i, \mathfrak{b}_i, \mathfrak{c}_i$ belongs to $\mathbb{R}[\frac{d}{dt}]$ (the ring of polynomials in the variable $\frac{d}{dt}$). Using the known input and the measured output without knowing exactly the perturbation we want to estimate σ (once this is done one can get x in real-time).

2.2 Distinguishability

Let us consider two monovariable linear systems described by (2 with $M = 2$, $\mathfrak{a}_i, \mathfrak{b}_i$ are relatively prime (for $i = 1, 2$) and the structured perturbation satisfies (3) with $M = 2$.

Let us assume that $\mathfrak{a}_i, \mathfrak{b}_i, \mathfrak{c}_i$ for $i = 1, 2$ and $\mathfrak{c}_3$ are known. It is clear that we cannot distinguish their i/o behaviors if, and only if, u, p and $y = y_1 = y_2$ satisfy the matrix differential equation

$$\begin{pmatrix} \mathfrak{a}_1 & -\mathfrak{b}_1 & -\mathfrak{c}_1 \\ \mathfrak{a}_2 & -\mathfrak{b}_2 & -\mathfrak{c}_2 \\ 0 & 0 & \mathfrak{c}_3 \end{pmatrix} \begin{pmatrix} y \\ u \\ p \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{4}$$

Classic algebraic manipulations show that Eq. (4) is equivalent to

$$\mathfrak{A}\left(\frac{d}{dt}\right) y = 0 \;\; \mathfrak{B}\left(\frac{d}{dt}\right) u = 0 \;\; \mathfrak{C}\left(\frac{d}{dt}\right) p = 0. \tag{5}$$

where $\mathfrak{A}, \mathfrak{B}, \mathfrak{C} \in \mathbb{R}[\frac{d}{dt}]$, $\mathfrak{A}\mathfrak{B} \neq 0$ ($\mathfrak{B} \neq 0$ otherwise the two systems have the same transfer function).

Definition 1. The two systems are said to be *strongly distinguishable* if for any non-zero input and any perturbation the two systems have distinguishable behavior that is if, the polynomials $\mathfrak{A}$ and $\mathfrak{B}$ are constant polynomials. If not the two systems are said to be *weakly distinguishable*, that is there exist a non zero input and a perturbation for which the two systems have distinguishable behavior.

The next result summarizes the above computations.

Theorem 2. In (5) $\mathfrak{A}$ and $\mathfrak{B}$ are given by [2]

$$\begin{aligned} \mathfrak{A} &= \mathfrak{c}_3' \gcd(\mathfrak{b}_1\mathfrak{p}_1^1, \mathfrak{b}_2\mathfrak{p}_1^2), \\ \mathfrak{B} &= \mathfrak{c}_3' \left(\mathfrak{a}_2\mathfrak{b}_1' - \mathfrak{a}_1\mathfrak{b}_2'\right), \\ \mathfrak{C} &= \mathfrak{c}_3, \end{aligned} \tag{6}$$

where $\mathfrak{a} = \gcd(\mathfrak{a}_1, \mathfrak{a}_2)$, $\mathfrak{a}_1 = \mathfrak{a}\mathfrak{a}_1'$, $\mathfrak{a}_2 = \mathfrak{a}\mathfrak{a}_2'$, $\mathfrak{b} = \gcd(\mathfrak{b}_1, \mathfrak{b}_2)$, $\mathfrak{b}_1 = \mathfrak{b}\mathfrak{b}_1'$, $\mathfrak{b}_2 = \mathfrak{b}\mathfrak{b}_2'$, $\mathfrak{c} = \gcd(\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{c}_3)$, $\mathfrak{c}_3 = \mathfrak{c}\mathfrak{c}_3'$, $\mathfrak{p}^1 = \mathrm{lcm}(\mathfrak{a}_1', \mathfrak{a}_2'\mathfrak{b}_1' - \mathfrak{a}_1'\mathfrak{b}_2')$, and $\mathfrak{p}^1 = \mathfrak{p}_1^1\mathfrak{a}_1' = \mathfrak{p}_2^1(\mathfrak{a}_2'\mathfrak{b}_1' - \mathfrak{a}_1'\mathfrak{b}_2')$, $\mathfrak{p}^2 = \mathrm{lcm}(\mathfrak{a}_2', \mathfrak{a}_2'\mathfrak{b}_1' - \mathfrak{a}_1'\mathfrak{b}_2')$, and $\mathfrak{p}^2 = \mathfrak{p}_1^2\mathfrak{a}_2' = \mathfrak{p}_2^2(\mathfrak{a}_2'\mathfrak{b}_1' - \mathfrak{a}_1'\mathfrak{b}_2')$. Strong distinguishability is equivalent to $\mathfrak{c}_3'\left(\mathfrak{a}_2\mathfrak{b}_1' - \mathfrak{a}_1\mathfrak{b}_2'\right) \in \mathbb{R}\setminus\{0\}$ and $\mathfrak{c}_3' \gcd(\mathfrak{b}_1\mathfrak{p}_1^1, \mathfrak{b}_2\mathfrak{p}_1^2) \in \mathbb{R}\setminus\{0\}$.

Proof. Using $\mathfrak{a} = \gcd(\mathfrak{a}_1, \mathfrak{a}_2)$, $\mathfrak{b} = \gcd(\mathfrak{b}_1, \mathfrak{b}_2)$ and $\mathfrak{c} = \gcd(\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{c}_3)$, (4) reads as:

$$\begin{pmatrix} \mathfrak{a}\mathfrak{a}_1' & -\mathfrak{b}\mathfrak{b}_1' & -\mathfrak{c}\mathfrak{c}_1' \\ \mathfrak{a}\mathfrak{a}_2' & -\mathfrak{b}\mathfrak{b}_2' & -\mathfrak{c}\mathfrak{c}_2' \\ 0 & 0 & \mathfrak{c}\mathfrak{c}_3' \end{pmatrix} \begin{pmatrix} y \\ u \\ p \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{7}$$

[1] "From the knowledge of $y(t)$ can we recover the input and the switching function" which is closely related to our problem of interest.

[2] Where gcd (resp. lcm) is the usual notation of the greatest common divisor (resp. least common multiple).

Some manipulations on the lines of (7) leads to:

$$\mathfrak{b}_1\left(\frac{d}{dt}\right)\mathfrak{p}_1^1\left(\frac{d}{dt}\right)\mathfrak{c}_3'\left(\frac{d}{dt}\right)u=0$$

$$\left[\mathfrak{a}_1\left(\frac{d}{dt}\right)\mathfrak{b}_2'\left(\frac{d}{dt}\right)-\mathfrak{a}_2\left(\frac{d}{dt}\right)\mathfrak{b}_1'\left(\frac{d}{dt}\right)\right]\mathfrak{c}_3'\left(\frac{d}{dt}\right)y=0$$

$$\mathfrak{c}_3\left(\frac{d}{dt}\right)p=0$$

And to

$$\mathfrak{b}_2\left(\frac{d}{dt}\right)\mathfrak{p}_1^2\left(\frac{d}{dt}\right)\mathfrak{c}_3'\left(\frac{d}{dt}\right)u=0$$

$$\left[\mathfrak{a}_1\left(\frac{d}{dt}\right)\mathfrak{b}_2'\left(\frac{d}{dt}\right)-\mathfrak{a}_2\left(\frac{d}{dt}\right)\mathfrak{b}_1'\left(\frac{d}{dt}\right)\right]\mathfrak{c}_3'\left(\frac{d}{dt}\right)y=0$$

$$\mathfrak{c}_3\left(\frac{d}{dt}\right)p=0$$

The conclusion follows. ■

Remark 3. We have the same conditions as the ones obtained before in Fliess et al. [2008] except that $\mathfrak{a}_i, \mathfrak{b}_i$ are replaced by $\mathfrak{c}_3'\mathfrak{a}_i, \mathfrak{c}_3'\mathfrak{b}_i$.

2.3 Example

In order to illustrate the above obtained result, let us consider the first two sub-systems of example 3.3:

$i=1: \ \dot{x}_1=-x_1+u+p, \ y=x_1$: (i/o) $\dot{y}+y=u+p$,
$i=2: \ \dot{x}_1=x_2+p, \ \dot{x}_2=-x_1-x_2+u, \ y=x_1+x_2$: (i/o) $\ddot{y}+\dot{y}+y=\dot{u}+u+\dot{p}$,

to determine the singular situation(s) (such analysis can be done for each pair of sub-systems leading to table 1). We assume that the structured perturbation is a constant one that is $\dot{p}=0$ leading to $\mathfrak{c}_3(\frac{d}{dt})=\frac{d}{dt}$. Since $\mathfrak{a}_1=\frac{d}{dt}+1, \mathfrak{b}_1=1, \mathfrak{c}_1=1$ (for the first sub-model) and $\mathfrak{a}_2=\left(\frac{d}{dt}\right)^2+\frac{d}{dt}+1, \mathfrak{b}_1=\frac{d}{dt}+1, \mathfrak{c}_1=\frac{d}{dt}$ (for the second sub-model), one obtains $\mathfrak{a}=\gcd(\mathfrak{a}_1,\mathfrak{a}_2)=1$, $\mathfrak{a}_1=\mathfrak{a}_1'$, $\mathfrak{a}_2=\mathfrak{a}_2'$, $\mathfrak{b}=\gcd(\mathfrak{b}_1,\mathfrak{b}_2)=1$, $\mathfrak{b}_1=\mathfrak{b}_1'$, $\mathfrak{b}_2=\mathfrak{b}_2'$, $\mathfrak{c}=\gcd(\mathfrak{c}_1,\mathfrak{c}_2,\mathfrak{c}_3)=1$, $\mathfrak{c}_3=\mathfrak{c}_3'$, $\mathfrak{p}^1=\mathrm{lcm}(\mathfrak{a}_1',\mathfrak{a}_2'\mathfrak{b}_1'-\mathfrak{a}_1'\mathfrak{b}_2')$, and $\mathfrak{p}^1=\mathfrak{p}_1^1\mathfrak{a}_1'=\mathfrak{p}_2^1(\mathfrak{a}_2'\mathfrak{b}_1'-\mathfrak{a}_1'\mathfrak{b}_2')$, $\mathfrak{p}^2=\mathrm{lcm}(\mathfrak{a}_2',\mathfrak{a}_2'\mathfrak{b}_1'-\mathfrak{a}_1'\mathfrak{b}_2')$, and $\mathfrak{p}^2=\mathfrak{p}_1^2\mathfrak{a}_2'=\mathfrak{p}_2^2(\mathfrak{a}_2'\mathfrak{b}_1'-\mathfrak{a}_1'\mathfrak{b}_2')$. This is $\mathfrak{p}_1^1=\frac{d}{dt}=-\mathfrak{p}_1^2$,

Leading to

$$\mathfrak{A}=\mathfrak{c}_3'\gcd(\mathfrak{b}_1\mathfrak{p}_1^1,\mathfrak{b}_2\mathfrak{p}_1^2)=\frac{d^2}{dt^2}, \tag{8}$$

$$\mathfrak{B}=\mathfrak{c}_3'\left(\mathfrak{a}_2\mathfrak{b}_1'-\mathfrak{a}_1\mathfrak{b}_2'\right)=\frac{d^2}{dt^2}, \tag{9}$$

$$\mathfrak{C}=\mathfrak{c}_3=\frac{d}{dt}. \tag{10}$$

Which implies that a singular situation occurs when u,y,p are of the form $u(t)=u_0+t\dot{u}_0, y(t)=y_0+t\dot{y}_0, p=p_0$ the constants $(u_0,\dot{u}_0,y_0,\dot{y}_0,p_0)$ being found such that the two sub-models equations holds $\dot{y}+y=u+p$ and $\ddot{y}+\dot{y}+y=\dot{u}+u+\dot{p}$. Which finally leads to $u(t)=u_0+tp_0, y(t)=u_0+tp_0, p=p_0$. Thus, these two systems are weakly distinguishable.

3. SWITCHING SIGNAL ESTIMATION

From now on, assume that all the subsystems models are known and that any pair is strongly distinguishable independently of the perturbation. Let us consider a switching system defined by a finite collection of i/o behaviors driven by LTI satisfying the previous assumptions. As soon as the system is not at rest, for the given control, the measured output can be used to determine which subsystem is active. From now on, we want to obtain effective real-time algorithm to determine the current "i". If one is able to construct in real time the following quantities

$$r_i(t)=\mathfrak{a}_i\left(\frac{d}{dt}\right)y_i-\mathfrak{b}_i\left(\frac{d}{dt}\right)u-\mathfrak{c}_i\left(\frac{d}{dt}\right)p,$$

it is clear that the current "i" is such that $r_i(t)=0$ on a sub-set of $\mathbb{R}$ with non zero measure. The problem is thus reduced to the real-time computation of time derivative of the output and input despite the noise.

$r_i(t)$ are not the right quantities to consider because they involve p (which is unknown), there is a natural way suggested by the previous theorem which is to multiply $r_i(t)$ by a differential operator such that the term p disappear namely by $\mathfrak{c}_i'\left(\frac{d}{dt}\right)$ where $\mathfrak{c}_i'=\gcd(\mathfrak{c}_i,\mathfrak{c}_{M+1})$. Thus if we consider

$$\begin{aligned}\overline{r}_i(t)&=\mathfrak{c}_i'\left(\frac{d}{dt}\right)r_i(t)\\&=\mathfrak{c}_i'\left(\frac{d}{dt}\right)\left(\mathfrak{a}_i\left(\frac{d}{dt}\right)y_i-\mathfrak{c}_i'\left(\frac{d}{dt}\right)\mathfrak{b}_i\left(\frac{d}{dt}\right)u\right)\end{aligned} \tag{11}$$

one can distinguish any active subsystem: the active one being such that $\overline{r}_i(t)=0$.

The numerical differentiation techniques introduced below are of non asymptotic nature, and the desired estimation can be obtained instantaneously (there is a singularity at time $t=0$). But in practice they are numerically implemented with discrete measured data, thus from a practical point of view, it will be necessary that the sampling time should be small enough with respect to the duration time between two successive switchings [3].

3.1 Numerical differentiation

This algebraic setting for numerical differentiation of noisy signals was introduced in Fliess et al. [2004] and analysed in Mboup et al. [2007, 2009] (see also Nöthen [2007] for interesting discussions and comparisons). The reader may find additional theoretical foundations in Fliess [2006], Fliess and Sira-Ramírez [2003]. Consider a signal $y(t)=\sum_{l=0}^{\infty}y^{(l)}(0)\frac{t^l}{l!}$ which is assumed to be analytic around $t=0$ and its truncated Taylor expansion

$$y_N(t)=\sum_{l=0}^{N}y^{(l)}(0)\frac{t^l}{l!}$$

at order N. The usual rules of symbolic calculus in Schwartz's distribution theory (Schwartz [1966]) yield

$$y_N^{(N+1)}(t)=y(0)\delta^{(N)}+\ldots+y^{(N)}(0)\delta,$$

where δ is the Dirac measure at zero. Multiply both sides by $(-t)^l$:

$$(-t)^l y_N^{(N+1)}(t)=(-t)^l\left(y(0)\delta^{(N)}+\ldots+y^{(N)}(0)\delta\right),$$

and apply the rules $t\delta=0$, $t\delta^{(l)}=-l\delta^{(l-1)}$, $l\geq 1$. We obtain a triangular system of linear equations from which the derivatives $y^{(l)}(0)$ can be obtained $(1\leq l\leq N)$

[3] In practice at least 30 times smaller, Zeno phenomenon are thus excluded.

$$(-t)^l y_N^{(N+1)}(t) = \frac{N!}{(N-l)!}\delta^{(N-l)}y(0) + \ldots + \delta y^{(N-l)}(0). \tag{12}$$

It means that the coefficients $y(0), \ldots, y^{(N)}(0)$ are *linearly identifiable* (see Fliess and Sira-Ramírez [2003, 2008]). The time derivatives of $y_N(t)$, the Dirac measures and its derivatives are removed by integrating with respect to time both sides of Eq. (12) at least ν times ($\nu > N$):

$$\int_0^t \int_0^{t_{\nu-1}} \cdots \int_0^{t_1} (-\tau)^l y_N^{(N+1)} dt_{\nu-1} \cdots dt_1 d\tau =$$

$$\frac{N!}{(N-l)!}\frac{t^{\nu-N-l-1}}{(\nu-N-l-1)!}y(0) + \ldots + \frac{t^{\nu-1}}{(\nu-1)!}y^{(N-l)}(0).$$

The iterated integrals may be replaced by

$$\int_0^t \int_0^{t_{\nu-1}} \cdots \int_0^{t_1} \tau^\alpha x(\tau) dt_{\nu-1} \cdots dt_1 d\tau =$$

$$\int_0^t \frac{(t-\tau)^{\nu-1}}{(\nu-1)!}\tau^\alpha x(\tau) d\tau. \tag{13}$$

It is clear that the numerical estimation relies on

$$\lim_{N \to +\infty} [y_N^{(l)}(0)]_{\text{estim}}(t) = y^{(l)}(0).$$

Remark 4. These iterated integrals are low pass filters which attenuate the noises, which are viewed as highly fluctuating phenomena (see Fliess [2006] for more details). The above formulae may easily be extended to sliding time windows in order to obtain real time estimates (see Mboup et al. [2007, 2009] for further details).

3.2 Algorithm

<u>Off line:</u>

(1) determine the highest order of differentiation to be estimated: with respect to the output $ky_{\max} = \max_i(\text{degree}(\mathfrak{c}_i'(s)\mathfrak{a}_i(s)))$ and w.r.t the input $ku_{\max} = \max_i(\text{degree}(\mathfrak{c}_i'(s)\mathfrak{b}_i(s)))$,
(2) test distinguishability using (6) which will provide the "bad" inputs (let us note that the second relation of (6) can be used to check if the input is a "bad" one just by checking if it satisfies the differential relation).

<u>On line:</u>

(1) using our techniques (see section 3.1) compute $y, \dot{y}, \ldots, y^{(ky_{\max})}; u, \dot{u}, \ldots, u^{(ku_{\max})}$,
(2) check if $\overline{r}_i(t)$ is zero for some time interval then the corresponding active subsystem is the "i-th" one.

3.3 Example

Let us consider the following switching system (2) where

$i = 1:\ \dot{x}_1 = -x_1 + u + p,\ y = x_1$: (i/o) $\dot{y} + y = u + p$,
$i = 2:\ \dot{x}_1 = x_2 + p,\ \dot{x}_2 = -x_1 - x_2 + u,\ y = x_1 + x_2$: (i/o) $\ddot{y} + \dot{y} + y = \dot{u} + u + \dot{p}$,
$i = 3:\ \dot{x}_1 = -\frac{1}{2}x_1 + u + \frac{3}{2}p,\ y = x_1$: (i/o) $2\dot{y} + y = 2u + 3p$,
$i = 4:\ \dot{x}_1 = x_2 + p, \dot{x}_2 = -2x_1 - x_2 + u,\ y = x_1 + x_2$: (i/o) $\ddot{y} + \dot{y} + 2y = \dot{u} + u + \dot{p} - p$,

where the structured perturbations are constant that is when (3) is

$$\dot{p} = 0. \tag{14}$$

For the first order systems, in that follows, x_2 is enforced to zero. Moreover, the output continuity is ensured between two systems whereas initial condition of derivative output is randomly chosen in $[-0.5, +0.5]$.
Residuals associated to previous systems are

$$i = 1: \overline{r}_i = [\ddot{y}]_e + [\dot{y}]_e - [\dot{u}]_e$$
$$i = 2: \overline{r}_i = [\ddot{y}]_e + [\dot{y}]_e + [y]_e - [\dot{u}]_e - u$$
$$i = 3: \overline{r}_i = 2[\ddot{y}]_e + [\dot{y}]_e - 2[\dot{u}]_e$$
$$i = 4: \overline{r}_i = [y^{(3)}]_e + [\ddot{y}]_e + 2[\dot{y}]_e - [\ddot{u}]_e - [\dot{u}]_e$$

where $[\bullet]_e$ is the estimation of $\bullet$ and $[y]_e$ corresponds the y denoised signal. Table 1 gives the singular i/o for which distinuishability is lost for a constant perturbation $p = p_0$.

$i\backslash j$	**1**	**2**
1	X	$\begin{cases} u = u_0 + tp_0 \\ y = u_0 + tp_0 \end{cases}$
2	$\begin{cases} u = u_0 + tp_0 \\ y = u_0 + tp_0 \end{cases}$	X
3	$\begin{cases} u = -2p_0 \\ y = -p_0 \end{cases}$	$\begin{cases} u = -3p_0 + u_0'(\exp(t)) \\ y = -3p_0 + \frac{2}{3}u_0'(\exp(t)) \end{cases}$
4	$\begin{cases} u = p_0(1 - 2\exp(t)) \\ y = -p_0\exp(t) \end{cases}$	$\begin{cases} u = -2p_0 \\ y = -p_0 \end{cases}$

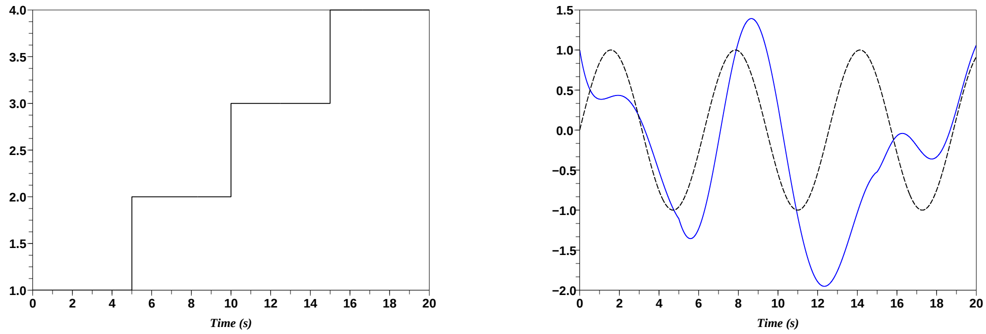
$i\backslash j$	**3**	**4**
1	$\begin{cases} u = -2p_0 \\ y = -p_0 \end{cases}$	$\begin{cases} u = p_0(1 - 2\exp(t)) \\ y = -p_0\exp(t) \end{cases}$
2	$\begin{cases} u = -3p_0 + u_0'(\exp(t)) \\ y = -3p_0 + \frac{2}{3}u_0'(\exp(t)) \end{cases}$	$\begin{cases} u = -2p_0 \\ y = -p_0 \end{cases}$
3	X	$\begin{cases} u = 2p_0 + \frac{1}{3}u_0'(\exp(3t)) \\ y = p_0 + \frac{2}{9}u_0'(\exp(3t)) \end{cases}$
4	$\begin{cases} u = 2p_0 + \frac{1}{3}u_0'(\exp(3t)) \\ y = p_0 + \frac{2}{9}u_0'(\exp(3t)) \end{cases}$	X

Table 1. Singular situations for each pair of sub-systems

Figure 1, system behavior is given. A sinusoidal function is applied as an input and no noise is added to the output (see figure 1-(b)). The switching signal σ is given in figure 1-(a).
In noisy free case, residuals can be evaluated according to Euler's method. The corresponding results, given in figure 2-(a), are perfect, i.e. residuals are null when the associated system is active and becomes non zero in other cases.
In the noisy case (additive output noise $N(0, 0.001)$ (figure 2-(b)) or $N(0, 0.005)$ (figure 2-(c)), Euler's method is not available. We propose to apply recent results on derivative estimation (see Mboup et al. [2007, 2009]) in order to evaluate residuals. To detect quickly all system changes, derivative estimation is obtained using a very short sliding window. However, sometimes this choice implies some difficulties as for the last switch figure 2-(c) to evaluate residuals. In Fliess et al. [2008], authors improve this particular point.

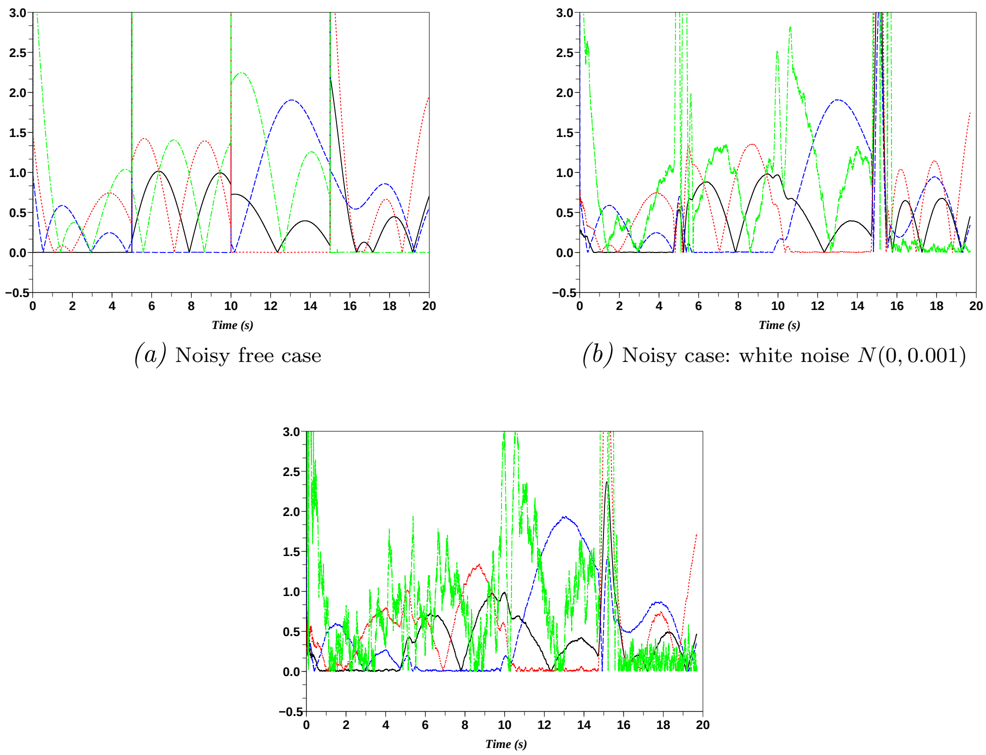
CONCLUSION

Distinguishability has been investigated for LTI switched systems. Easy to check necessary and sufficient conditions were obtained, which provide "bad input/perturbations" to be avoided in order to be able to detect the active mode. Real time estimation of the current index are obtained using differentiation techniques robust w.r.t. noises. Some



(a) Switching signal σ *(b)* input (- -) ; Output (–)

Figure 1. Free noise behavior with a sinusoidal input and an unknown constant perturbation $p = -0.5$



(a) Noisy free case *(b)* Noisy case: white noise $N(0, 0.001)$

(c) Noisy case: white noise $N(0, 0.005)$

Figure 2. Residuals : $|r_1|$ (–); $|r_2|$ (- -); $|r_3|$ (. .); $|r_4|$ (- .)

extension to nonlinear switching systems is under investigation.

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