Claw-free circular-perfect graphs

Arnaud Pecher, Xuding Zhu

Claw-free circular-perfect graphs

Arnaud Pêcher [1,3]

University of Bordeaux (LaBRI, INRIA)
351 cours de la Libération
33405 Talence, France

Xuding Zhu [2,4]

Department of Applied Mathematics
National Sun Yat-sen University
Kaohsiung, Taiwan
and
National Center for Theoretical Sciences

Abstract

The circular chromatic number of a graph is a well-studied refinement of the chromatic number. Circular-perfect graphs is a superclass of perfect graphs defined by means of this more general coloring concept. This paper studies claw-free circular-perfect graphs. A consequence of the strong perfect graph theorem is that minimal circular-imperfect graphs G have $\min\{\alpha(G), \omega(G)\} = 2$. In contrast to this result, it is shown in [9] that minimal circular-imperfect graphs G can have arbitrarily large independence number and arbitrarily large clique number. We prove that claw-free minimal circular-imperfect graphs G have $\min\{\alpha(G), \omega(G)\} \leq 3$.

Keywords: circular-perfect graph, claw-free graph

Let $G = (V, E)$ be a graph with vertex set V and edge set E, then a *k-coloring* of G is a mapping $f : V \to \{1, \ldots, k\}$ with $f(u) \neq f(v)$ if $uv \in E$, i.e., adjacent vertices receive different colors. The minimum k for which G admits a k-coloring is called the *chromatic number* $\chi(G)$. The *clique number* $\omega(G)$ (resp. *independence number* $\alpha(G)$) of G is the order of a largest clique (resp. independent set) of G, i.e., the maximum number of pairwise adjacent (resp. non-adjacent) vertices of G.

The circular chromatic number and circular clique number of graphs are refinements of the chromatic number and the clique number. Suppose $G = (V, E)$ is a graph with at least one edge, and $k \geq 2d$ are positive integers. A *(k, d)-circular coloring* of G is a mapping $f : V \to \{0, \ldots, k-1\}$ with $d \leq |f(u) - f(v)| \leq k - d$ if $uv \in E$. The *circular chromatic number* $\chi_c(G)$ is the minimum $\frac{k}{d}$ taken over all (k, d)-circular colorings of G. Since every $(k, 1)$-circular coloring is a usual k-coloring of G, we have $\chi_c(G) \leq \chi(G)$. On the other hand, it is known [13] and easy to see that for any graph G, $\chi_c(G) > \chi(G) - 1$, and hence $\chi(G) = \lceil \chi_c(G) \rceil$. So $\chi_c(G)$ is a refinement of $\chi(G)$.

Let $K_{k/d}$ with $k \geq 2d$ denote the graph with the k vertices $0, \ldots, k-1$ and edges ij such that $d \leq |i - j| \leq k - d$. The graphs $K_{k/d}$ are called *circular cliques*. Circular cliques include all cliques $K_t = K_{t/1}$, all odd antiholes $\overline{C}_{2t+1} = K_{(2t+1)/2}$, and all odd holes $C_{2t+1} = K_{(2t+1)/t}$. The *circular clique number* is defined as $\omega_c(G) = \max\{\frac{k}{d} : K_{k/d} \subseteq G, \gcd(k, d) = 1\}$. It follows from the definition that $\omega(G) \leq \omega_c(G)$. It is also known [17] that for any graph G, $\omega_c(G) < \omega(G) + 1$, and hence $\omega(G) = \lfloor \omega_c(G) \rfloor$. Therefore $\omega_c(G)$ is a refinement of $\omega(G)$.

Obviously $\omega(G)$ is a lower bound for $\chi(G)$. A graph G is *perfect* if each induced subgraph $G' \subseteq G$ has $\omega(G') = \chi(G')$. Similarly, $\omega_c(G)$ is a lower bound for $\chi_c(G)$. A graph G is called *circular-perfect* [17] if each induced subgraph $G' \subseteq G$ has $\chi_c(G') = \omega_c(G')$.

Perfect graphs have been studied extensively since the concept and two conjectures (the weak and the strong perfect graph conjectures) were proposed by Berge [1] in 1961. The weak perfect graph conjecture was settled by Lovász [8] in 1972. Recently, the strong perfect graph conjecture has been settled by Chudnovsky, Robertson, Seymour and Thomas in [2], which gives a characterization of perfect graphs by means of forbidden induced subgraphs: a graph G is perfect if and only if G contains neither chordless odd cycles C_{2k+1} with $k \geq 2$, nor their complements $\overline{C}_{2k+1}$.

It follows from the definitions that for any graph G, $\omega(G) \leq \omega_c(G) \leq \chi_c(G) \leq$

[1] Email:Arnaud.Pecher@labri.fr
[2] Email:zhu@math.nsysu.edu.tw
[3] Supported by the french-taiwanese partnership CNRS/NSC number 1872
[4] Supported in part by the National Science Council of ROC under grant NSC96-2115-M-110-001

$\chi(G)$. Therefore every perfect graph is circular-perfect. However, odd cycles and their complements are circular-perfect graphs but not perfect graphs. So the class of circular-perfect graphs is a proper superclass of the class of perfect graphs.

Is there a simple characterization of circular-perfect graphs by means of forbidden induced subgraphs? It is shown in [14] that the line graph $L(G)$ of a cubic graph G is circular-perfect if and only if G is 3-edge colourable. Thus such a characterization of circular-perfect graphs implies a characterization of critically non-3-edge colourable cubic graphs, which is known to be a difficult problem. So it is unlikely that there is a simple forbidden induced subgraph characterization of circular-perfect graphs. Some sufficient conditions for a graph to be circular-perfect were obtained in [16,17]. Classes of (minimal) circular-imperfect graphs were constructed in [9,11,12,15]. Minimal circular-imperfect line graphs were studied in [14]. In this paper, we study claw-free circular-perfect graphs.

A graph G is *claw-free* if $K_{1,3}$ is not an induced subgraph of G. Claw-free graphs is a superclass of line graphs and has been studied extensively in the litterature. Recently, Chudnovsky and Seymour [4,3] presented a structural characterization of claw-free graphs. A graph G for which the neighbourhood of each vertex can be covered by two cliques is called a *quasi-line graph*. We use their characterization, restricted to quasi-line graphs, to prove a structural property of minimal circular-imperfect graphs. One consequence of the strong perfect graph theorem is that minimal imperfect graphs G have $\min\{\omega(G), \alpha(G)\} = 2$. It was asked in [11] whether $\min\{\omega(G), \alpha(G)\}$ is bounded for minimal circular-imperfect graphs G. This question was answered in the negative in [9], where it is proved that for any positive integer k, there is a minimal circular-imperfect graph G with $\min\{\omega(G), \alpha(G)\} \geq k$.

We show that if restricted to claw-free graphs, the question above has a positive answer: if G is a claw-free minimal circular-imperfect graph, then $\min\{\omega(G), \alpha(G)\} \leq 3$.

Before the strong perfect graph conjecture becomes a theorem, the conjecture was confirmed for claw-free graphs in [7][10]. Our result above implies an alternative proof of this result, of course without making use of the strong perfect graph theorem [2].

Sketch of the proof

Suppose G is a claw-free graph with independence number at least 3. It was proved by Fouquet [6] that for any vertex x of G, the neighborhood $N_G(x)$ of x either contains an induced C_5, or can be covered with two cliques. If G is circular-perfect, then $N_G(x)$ does not contain an induced C_5, for otherwise G contains the odd wheel

W_5, which is circular-imperfect. Thus we have the following observation: *if G is a claw-free circular-perfect graph with independence number at least* 3*, then* G *is a quasi-line graph.*

It turns out that claw-free circular-perfect graphs with an induced odd antihole of size at least 7 have a basic structure:

Theorem 1. *If* G *is a connected claw-free circular-perfect graph with an induced odd antihole* H *of size at least* 7 *then* $G \setminus H$ *is a clique. Furthermore* $\alpha(G) = 2$*.*

Since every minimal circular-imperfect graph is 2-connected, we have the following corollary:

Corollary 1. *If* G *is a claw-free minimal circular-imperfect graph and contains an induced odd antihole* H *of size at least 7, then* $\alpha(G) \leq 3$*.*

It remains to study claw-free circular perfect graphs G that do not contain an odd antihole of order at least 7. Due to Theorem 1, G has independence number at least 3, and is therefore, as mentioned above, quasi-line. We establish that if G has clique number at least 4 then G has an independent set I that intersects each maximum clique.

We prove a stronger statement:

Theorem 2. *If* G *is a quasi-line graph,* $\omega(G) = k \geq 4$ *and for every vertex* x*,* $G - x$ *has a* k*-colouring, then either* G *is the complement of a circular clique or* G *has a stable set which intersects every maximum clique of* G*.*

As a consequence, a claw-free graph G with $\omega(G) = k \geq 4$ and $\alpha(G) \geq 4$ can not be minimal circular-imperfect. Because otherwise, G is not the complement of a circular-clique [5], is quasi-line since it does not contain the odd wheel W_5 (which is already minimal circular-imperfect). Hence there is an independent set I intersecting each maximum clique. Since $G - I$ is circular-perfect, we have $\omega_c(G - I) = \chi_c(G - I)$. Due to Corollary 1, G, and thus $G - I$, does not contain $K_{(2p+1)/2}$ for $p \geq 3$. It follows that $\omega_c(G - I) = \omega(G - I) = k - 1$ and hence $\chi(G - I) = \chi_c(G - I) = k - 1$. But then $\chi(G) = \omega(G) = k$, and hence G is circular-perfect.

References

[1] C. Berge. Farbung von graphen deren sietliche bzw. deren ungerade kreise starr sind. *Wiss. Z. Martin-Luther Univ. Halle-Wittenberg*, pages 114–115, 1961.

[2] M. Chudnovsky, N. Robertson, P. Seymour, and R. Thomas. The strong perfect graph theorem. *Ann. of Math. (2)*, 164(1):51–229, 2006.

[3] M. Chudnovsky and P. Seymour. Claw-free graphs vi. the structure of quasi-line graphs. *manuscript*, 2004.

[4] M. Chudnovsky and P. Seymour. The structure of claw-free graphs. In B.S. Webb, editor, *Surveys in combinatorics*, volume 327 of *London. Math. Soc. Lecture Notes*, pages 153–171, 2005.

[5] S. Coulonges, A. Pêcher, and A. Wagler. On strongly circular-perfectness. In *VIIth International Conference on Graph Theory*, volume 22 of *Electonic Notes in Discrete Mathematics*, pages 369–374. Elsevier, oct 2005.

[6] J. L. Fouquet. A strengthening of Ben Rebea's lemma. *J. Comb. Theory Ser. B*, 59(1):35–40, 1993.

[7] R. Giles, L.E. Trotter jr, and A. Tucker. The strong perfect graph theorem for a class of partitionable graphs. *Annals of Discrete Mathematics*, 21:161–167, 1984.

[8] L. Lovász. A characterization of perfect graphs. *J. of Combinatorial Theory B*, 13:95–98, 1972.

[9] Z. Pan and X. Zhu. Minimal circular-imperfect graphs of large clique number and large independence number. manuscript, 2006.

[10] K.R. Parthasarathy and G. Ravindra. The strong perfect graph conjecture is true for $k_{1,3}$-free graphs. *J. Combin. Theory Ser. B*, 21:212–223, 1976.

[11] A. Pêcher, A. Wagler, and X. Zhu. Three classes of minimal circular imperfect graphs. In *Proceedings of GRACO05*, volume 19 of *Electonic Notes in Discrete Mathematics*, pages 9–15. Elsevier, avril 2005.

[12] A. Pêcher and X. Zhu. On the circular chromatic number of circular partitionable graphs. *Journal of Graph Theory*, 52(4):294–306, aug 2006.

[13] A. Vince. Star chromatic number. *J. of Graph Theory*, 12:551–559, 1988.

[14] B. Xu. On minimally circular-imperfect graphs. manuscript.

[15] B. Xu. Some results on circular perfect graphs and perfect graphs. *J. Syst. Sci. Complex.*, 18:167–173, 2005.

[16] X. Zhu. Perfect graphs for generalized colouring—circular perfect graphs. In *Graphs, morphisms and statistical physics*, volume 63 of *DIMACS Ser. Discrete Math. Theoret. Comput. Sci.*, pages 177–193. Amer. Math. Soc., Providence, RI, 2004.

[17] X. Zhu. Circular perfect graphs. *J. of Graph Theory*, 48(1–3):186–209, 2005.