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Abstract

We study the variety of common tangents for up to four quadric surfaces in projective three-space, with particular regard to configurations of four quadrics admitting a continuum of common tangents.

We formulate geometrical conditions in the projective space defined by all complex quadric surfaces which express the fact that several quadrics are tangent along a curve to one and the same quadric of rank at least three, and called, for intuitive reasons: *a basket*. Lines in any ruling of the latter will be common tangents.

These considerations are then restricted to spheres in Euclidean three-space, and result in a complete answer to the question *over the reals*: "When do four spheres allow infinitely many common tangents?".

Key words: quadric surfaces, duality, Veronese embedding, Grassmannians, complete quadrilaterals, Desargues configuration, Reye configuration, Kummer surfaces.

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Introduction

Tangents to a non-singular complex projective quadric surface make-up a threefold, namely: the projectivized tangent bundle of the given quadric. After a birational contraction, this threefold can be represented as a *quadratic section* of the Grassmannian $G(2,4)$ of all projective lines in $P_3(C)$ i.e. all 2-subspaces of the vector space C^4.

$G(2,4)$, in its Plücker embedding, is itself a quadric in $P_5(C)$, and it follows that *four non-singular quadric surfaces in general position allow* $2^5 = 32$ *common tangents*.

However, what we may call *degenerate configurations* of four quadrics, would still allow a continuum of common tangents. This obviously happens when the four quadrics have a common curve of intersection, other than a union of less than four lines (or, as we shall observe later, when their *duals* do). In general, a curve of common tangents would give a *ruled surface* in $P_3(C)$, which is tangent along some curve with each of the given quadric surfaces.

We investigate the case when this ruled surface is itself a quadric surface. A simple example (where one can "see" what happens for the *real points*) is that of a hyperboloid of revolution in which one throws four spherical "balls" and lets them rest when reaching a circle of tangency with their "basket". Because of this intuitive background, we introduce:

Definition: Let q_1 and q_2 be distinct quadric surfaces of rank at least three (i.e. non-singular or with at most an isolated conic singularity). We say that q_1 is a **basket** for q_2 (and then, q_2 will be a basket for q_1) when the two quadrics are tangent along a conic. (Thus, the intersection of the two quadrics is represented by twice this conic.)

Convention: *In the sequel, whenever we speak of a common basket b for quadrics q_i, we assume that all quadrics concerned are distinct and of rank at least three.*

We are going to formulate conditions expressing the fact that two, three or four quadrics allow a common basket. These will be geometrical conditions in the space of all quadric surfaces which is a nine-dimensional complex projective space corresponding to lines through zero in the vector space of all 4×4 symmetric matrices with complex entries:

$$P_9(C) = P(Sym_C(4))$$

Indeed, one may identify quadratic forms and symmetric matrices (over C) via the standard bilinear form $<,>$:

$$q(x) = \sum_{i,j} q_{ij} x_i x_j =< x, Qx > \qquad Q = Q^t$$

The rank of the quadric q, as already spoken of, is simply the rank of the matrix Q.

Sometimes, we'll refer to quadrics of rank at most three as *cones*, while those of rank at most two, respectively one, will be called *two-planes*, respectively *double-planes*. Obviously, a two-plane is a cone over a degenerate conic i.e. a two-line.

The closure of the rank three locus is denoted $\mathcal{R}^3_8$, the closure of the rank two locus is denoted $\mathcal{R}^2_6$, and the rank one locus is denoted $\mathcal{R}^1_3$.

To begin with, we give an equivalent of our definition of baskets:

Proposition 0.1 *b is a basket for q if and only if the pencil $[b, q] = \lambda b + \mu q$ meets the rank one locus (i.e. contains a double-plane).*

Proof: If the pencil contains a double plane, the two quadrics intersect in a double conic and must be tangent along it.

Conversely, the double plane through the conic of tangency has the same intersection with b as q (and with q as b), and must belong to the pencil. $\square$

In the same vein, we shall obtain:

Proposition 0.2 *The quasi-projective variety (see our convention above):*

$$B^2_{16} = \{(q_1, q_2, b) \ : \ b \textit{ is a common basket for } q_1 \textit{ and } q_2 \ \} \subset (P_9 - \mathcal{R}^2_6)^3$$

is irreducible, of dimension sixteen, and the pair (q_1, q_2) *of a generic point* (q_1, q_2, b) *is characterized by the property that the pencil* $[q_1, q_2]$ *contains a two-plane.*

Proposition 0.3 *The quasi-projective variety:*

$$B^3_{21} = \{(q_1, q_2, q_3, b) \ : \ b \textit{ is a common basket for } q_1, q_2 \textit{ and } q_3 \ \} \subset (P_9 - \mathcal{R}^2_6)^4$$

is irreducible, of dimension twenty one, and the triple (q_1, q_2, q_3) *of a generic point* (q_1, q_2, q_3, b) *is characterized by the property that the span* $[q_1, q_2, q_3] \approx P_2$ *contains a pencil of cones with the same vertex, and the rank two points in this pencil are precisely where it meets the lines* $[q_i, q_j]$*.*

Proposition 0.4 *The quasi-projective variety:*

$$B^4_{25} = \{(q_1, q_2, q_3, q_4, b) \ : \ b \textit{ is a common basket for } q_i, \ i = 1, ..., 4 \ \} \subset (P_9 - \mathcal{R}^2_6)^5$$

is irreducible, of dimension twenty five.

The quadruple $(q_1, ..., q_4)$ *of a generic point* $(q_1, ..., q_4, b) \in B^4_{25}$ *is characterized by the following property: the span* $[q_1, .., q_4] \approx P_3$ *contains a complete quadrilateral consisting of four pencils of cones; the six vertices* p_{kl} *lie, respectively, on the six lines* $[q_i, q_j]$ *and correspond precisely with the rank two quadrics of the quadrilateral.*

When we present our proofs, we'll examine and characterize all possibilities *in terms of conditions on the configuration* (q_i), indicating how to 'reconstruct' all baskets when the conditions are met.

These results then lead to a proof of *uniqueness*, up to the action of the projective automorphism group $PSL_C(4)$, of a "double-four" configuration, namely: two *quadruples of linearly independent and smooth quadrics*, such that each quadric in one group is a common basket for the quadrics in the other group. This configuration arises quite naturally from the point of view of duality, and is related to the *Reye configuration* $(12_4, 16_3)$.

When seen on the Grassmannian $G(2,4)$, the tangents of a smooth quadric surface define a *degenerate quadratic line complex*: it has singularities along the two conics representing the two rulings of the quadric.

The relation of generic quadratic line complexes with Kummer surfaces is a classical subject [Jes], [Hud], [GH]. In our case, Kummer surfaces appear when intersecting two degenerate quadratic line complexes, that is, when considering common tangents to a generic pair of quadrics. A generalisation to higher-dimensional Calabi-Yau varieties is pursued in [Bor2].

In the last section we use our results on quadrics with common baskets to solve the problem of describing *all possible configurations of four spheres in* R^3 *with infinitely many real common tangents.* The conclusion agrees with intuitive expectations: *the four centers have to be collinear*, and the radii must accommodate one of the following possibilities:

(i) the four spheres intersect in a common circle or point;
(ii) the radii are equal, and there's a common cylindrical basket;
(iii) the four spheres have a common conical basket;
(iv) there's a common basket in the shape of a hyperboloid of revolution with one sheet and axis the line of centers.

This complements results in [MPT] and [ST] on configurations of four spheres with a finite number of common tangents. The effective upper bound is 12.

The material is organized in eight sections:

1. Stratification by rank in $P_9 = P(Sym_C(4))$
2. Two quadrics in a basket
3. Three quadrics in a basket
4. Four quadrics in a basket
5. A double-four example
6. Tangents and Grassmannians
7. Duality
8. Common tangents to four spheres in R^3

1 Stratification by rank in $P_9 = P(Sym_C(4))$

In this section we review some classical facts about the space of all quadric surfaces. More general considerations can be found in [Bor1], or [Har].

The stratification by rank yields, in the case of 4×4 symmetric matrices three determinantal varieties:

$$\mathcal{R}_3^1 \subset \mathcal{R}_6^2 \subset \mathcal{R}_8^3 \subset P_9 = P(Sym_C(4))$$

The *rank at most three locus* $\mathcal{R}_8^3$ is the *degree four hypersurface* defined by all singular quadrics:

$$\mathcal{R}_8^3 = \{Q \in P_9 \ : \ det(Q) = 0\}$$

These singular quadrics are obviously *cones* over conics in some plane $P_2 \subset P_3$ constructed from a vertex outside that plane. Generically, the vertex is the only singularity.

The *rank at most two locus* $\mathcal{R}_6^2$ is codimension two in $\mathcal{R}_8^3$ and represents in fact its singular locus. It is defined in P_9 by the vanishing of all 3×3 minors, and Giambelli's formula gives its degree as ten.

A quadric in $\mathcal{R}_6^2$ is the cone over some degenerate conic, that is: two lines, and so the union of *two planes*.

The *rank one locus* $\mathcal{R}_3^1$ is codimension three in $\mathcal{R}_6^2$ and represents its singular locus. It is defined in P_9 by the vanishing of all 2×2 minors, and can also be described as the *image of the quadratic Veronese embedding*:

$$v : P_3 \to P_9, \quad v(x) = x^t \cdot x$$

where $x = (x_0 : x_1 : x_2 : x_3)$, and x^t stands for its (column) transpose. Clearly, the symmetric matrix $x^t \cdot x$ has rank one, and:

$$\mathcal{R}_3^1 = v(P_3) \subset P_9$$

Similarly, $\mathcal{R}_6^2$ can be identified with the quotient of $P_3 \times P_3$ by the involution $\sigma(x, y) = (y, x)$ using:

$$w : (P_3)^2/\sigma \to P_9, \quad w(x, y) = w(y, x) = \frac{1}{2}(x^t \cdot y + y^t \cdot x)$$

This shows that $\mathcal{R}_6^2$ is swept out by a three parameter (rational) family of projective three-spaces in P_9. It is also swept out by a family of projective two-spaces in P_9, indexed by the Grassmannian $G(2, 4)$: indeed, given $\updownarrow \in G(2, 4)$, which we regard as a two-subspace of C^4, we have a plane in $\mathcal{R}_6^2$ given by:

$$P_2 \approx \{Q \ : \ \updownarrow \subset ker(Q)\} \subset \mathcal{R}_6^2$$

Later considerations will involve various pencils of singular quadrics, and it may be remarked here that the first of the above families provides a seven parameter family of pencils (i.e. lines) in $\mathcal{R}_6^2$, to be denoted by $\mathcal{M}_7^2 \subset G(2, 10)$, while the second family provides a six parameter family of pencils in $\mathcal{R}_6^2$, to be denoted by $\mathcal{F}_6^2 \subset G(2, 10)$.

Here, $G(2,10)$ stands for the Grassmannian of all lines in P_9, where we can ask for the intersection of $\mathcal{M}_7^2$ and $\mathcal{F}_6^2$. In terms of symmetric matrices, pencils in the first family involve quadrics with a given vector in the image, while pencils in the second family involve quadrics with a given two-space in the kernel. Considering that $im(Q) = ker(Q)^\perp$, we see that the intersection is five dimensional:

$$\mathcal{M}_7^2 \cap \mathcal{F}_6^2 = \mathcal{T}_5 \subset G(2,10)$$

and consists of pencils of quadrics parametrized by pairs $(\updownarrow, x) \in G(2,4) \times P_3$, with $x \subset \updownarrow^\perp$, and defined by:

$$P_1 \approx \{Q \ : \ \updownarrow \subset ker(Q), x \subset im(Q)\} \subset \mathcal{R}_6^2$$

These relations can be observed from another point of view, based on the fact that $\mathcal{R}_6^2$ is precisely the *secant variety* of $\mathcal{R}_3^1 \subset P_9$. Indeed, $\mathcal{R}_6^2$ can be obtained as the closure of the union of all lines spanned by two distinct points in $\mathcal{R}_3^1$. The closure brings in the points of all lines tangent to $\mathcal{R}_3^1$, and one can see that the family $\mathcal{F}_6^2$ consists of secants and tangents to $\mathcal{R}_3^1$, while $\mathcal{T}_5 \subset \mathcal{F}_6^2$ retains only the lines tangent to the rank one locus.

We note that an arbitrary line in P_9 can meet the rank one locus in at most two points, since any secant is, in adequate coordinates, of the form: $\lambda_1 x_1^2 + \lambda_2 x_2^2$, and this pencil has no other double-plane.

The family of projective three-spaces sweeping out $\mathcal{R}_6^2$ can be recognized now as the family of all tangent spaces to the rank one locus, and this makes obvious our earlier result:

$$\mathcal{M}_7^2 \cap \mathcal{F}_6^2 = \mathcal{T}_5 \subset G(2,10)$$

since a line in a tangent space to $\mathcal{R}_3^1$ must pass through the point of tangency in order to be in $\mathcal{F}_6^2$, and is then perforce in $\mathcal{T}_5$. $\square$

We can expand our description of pencils of singular quadrics by considering those contained in $\mathcal{R}_8^3$ and not already contained in $\mathcal{R}_6^2$. The generic quadric in such a pencil will have a unique singular point (the vertex of the cone), and we may expect two types of pencils:

(F) with *fixed* vertex, or (M) with *moving* vertex.

The first type obviously arises by choosing a pencil of conics in some $P_2 \subset P_3$ and constructing the cones over the conics in the pencil from a fixed vertex away from our P_2. This yields an eleven parameter family, to be denoted $\mathcal{F}_{11}^3$.

This family is related to the fact that $\mathcal{R}_8^3$ is swept out by a family of projective five-spaces indexed by P_3. Indeed, for $x \in P_3$, we have:

$$P_5 \approx \{Q \ : \ Qx = 0\}$$

which describes all quadrics singular at x. Lines in this P_5 make-up a Grassmannian $G(2,6)$ of dimension eight, hence our eleven dimensional $\mathcal{F}^3_{11}$.

Yet another description of this family is related to the fact that $\mathcal{R}^3_8$ is the *variety of secant planes* of $\mathcal{R}^1_3 \subset P_9$. Indeed, $\mathcal{R}^3_8$ is the closure of the union of all planes spanned by three double-planes. Thus, $\mathcal{R}^3_8$ is also swept out by a nine parameter family of planes, hence our $\mathcal{F}^3_{11}$, made of lines in these planes.

For the second type (M), let us observe first that if we write the pencil with moving vertex as: $\lambda_1 q_1 + \lambda_2 q_2$, with q_i of rank three and with vertex v_i, then all cones in the pencil must contain the line $[v_1, v_2]$. This follows from the fact that the tangent hyperplane to $\mathcal{R}^3_8$ at q_i consists of all quadrics passing through v_i, and our pencil lies in the intersection of these two hyperplanes. Thus, q_i passes through v_j, and contains $[v_i, v_j]$.

Clearly, the vertices in the pencil must move along this line, and the intersection $q_1 \cap q_2$ will consist of a conic and the double line $[v_1, v_2]$.

Thus, a pencil of type (M) arises by considering a fixed conic in some $P_2 \subset P_3$, then choosing a line through a point of the conic, but not contained in its plane, and moving a vertex (linearly) along this line. This yields an eleven parameter family, to be denoted $\mathcal{M}^3_{11}$.

We summarize these results in:

Proposition 1.1 *The variety of lines contained in $\mathcal{R}^3_8 \subset P_9$ consists of two irreducible components of dimension eleven: $\mathcal{F}^3_{11}$ and $\mathcal{M}^3_{11}$, made generically of pencils with fixed singularity, respectively moving singularity.*

The variety of lines contained in $\mathcal{R}^2_6 \subset P_9$ consists of two irreducible components: $\mathcal{F}^2_6$ of dimension six, and $\mathcal{M}^2_7$ of dimension seven.

We have inclusions:

$$\mathcal{F}^2_6 \subset \mathcal{F}^3_{11} \quad \textit{and} \quad \mathcal{M}^2_7 \subset \mathcal{M}^3_{11}$$

and intersections:

$$\mathcal{F}^3_{11} \cap \mathcal{M}^3_{11} = \mathcal{F}^2_6 \cap \mathcal{M}^2_7 = \mathcal{T}_5$$

with $\mathcal{T}_5 \subset G(2,10)$ standing for the five dimensional variety made of tangent lines to the rank one locus $\mathcal{R}^1_3$. $\square$

Remark: We have the following implications:

if $\updownarrow \in \mathcal{M}^3_{11} - \mathcal{M}^2_7$, then $\updownarrow$ has a single two-plane;

if $\updownarrow \in \mathcal{M}^2_7 - \mathcal{T}_5$, then $\updownarrow$ has no double-plane;

if $\updownarrow \in \mathcal{F}^3_{11} - \mathcal{F}^2_6$, then $\updownarrow$ has three two-planes, counting multiplicity;

if $\updownarrow \in \mathcal{F}^2_6 - \mathcal{T}_5$, then $\updownarrow$ has two double-planes.

2 Two quadrics in a basket

In this section we prove and refine Proposition 0.2.

Suppose q_1 and q_2 allow a common basket b. Then, by Proposition 0.1, the pencil $[b, q_i]$ meets the rank one locus in d_i. Then, either b, q_1, q_2 are collinear, and so $[q_1, q_2]$ meets the rank one locus in $d = d_1 = d_2$, or $[q_1, q_2]$ meets the rank two locus where it intersects $[d_1, d_2]$.

Conversely, if $[q_1, q_2]$ meets the rank one locus, then any b (of rank at least three) in the pencil will be a common basket. If $[q_1, q_2]$ meets the rank two locus in p, then there's a secant of the rank one locus through p and two double-planes we label d_i. Then $[q_i, d_i],\ i = 1, 2$ are coplanar and meet in a point b which is a common basket for q_i.

Irreducibility and the dimension count for B^2_{16} follows from the fact that there's an eight parameter family of pencils through each point of $\mathcal{R}^2_6$, and on each pencil, the choice of two points means two more parameters

This already proves Proposition 0.2, but we may refine the statement by observing the 'reconstruction' process of a common basket in more detail.

As a **rule**, whenever the variety of common baskets has positive dimension, we'll describe its *closure*, being understood that, according to our convention, we retain only the generic part made of quadrics of rank at least three for the role of baskets.

Thus, when $[q_1, q_2]$ meets the rank one locus, the whole pencil offers common baskets, but there is one type of situation where we have in addition, another rational curve of common baskets: let us call d the double-plane on $[q_1, q_2]$, and suppose that there's a common basket b away from this pencil. Then, we get double-planes d_i on $[b, q_i]$, and $[q_1, q_2]$ is in the plane $[d, d_1, d_2]$ which lies in $\mathcal{R}^3_8$.

It follows that, in our situation, $[q_1, q_2]$ is a pencil of cones with fixed vertex, and as q_i is also a basket for q_j, the two quadrics are cones from the same vertex over two (non-singular) conics which have two points of tangency. We note that $[q_1, q_2]$ has a rank two point at the intersection with $[d_1, d_2]$, and we may run other secants of the rank one locus through this point and construct other common baskets.

The fact that this leads to another rational curve of common baskets follows from the same argument as in the reconstruction process for the hypothesis of $[q_1, q_2]$ meeting the rank two locus, considered presently.

We'll need some lemmas:

Lemma 2.1 *If a plane $P_2 \subset P_9 = P(Sym_C(4))$ contains four distinct rank one quadrics, then the plane contains a (non-singular) conic of rank one quadrics. More precisely, P_2 is then the span of the Veronese image $v(P_1) \subset P_2$ of some line $P_1 \subset P_3$.*

Proof: When we look at the planes in P_3 corresponding to our four double-planes, we see that they must have a point in common, otherwise they would be projectively equivalent to $x_i^2 = 0, \ i = 0,1,2,3$, and the double-planes would span a $P_3 \subset P_9$. Thus, the problem is reduced to its version in $P_5 = P(Sym_C(3))$.

Now, again, the four double-lines must have a common point in P_2, for otherwise three of them would be projectively equivalent to $x_i^2 = 0, \ i = 0,1,2$ and their span in P_5 has no other double-line.

Thus, the original four double-planes have a common line and are projectively equivalent to four points in the family $(\lambda_0 x_0 + \lambda_1 x_1)^2, \ (\lambda_0 : \lambda_1) \in P_1$ which is the Veronese image of a line. $\square$

Lemma 2.2 *Let $p \in \mathcal{R}_6^2 - \mathcal{R}_3^1$ be a two-plane. The family of secants (and tangents) to the rank one locus which pass through p make up a rational curve.*

In fact, there's a unique plane $P_2 \subset \mathcal{R}_6^2$ passing through p and containing (as a non-singular conic) the Veronese image $v(P_1) \subset P_2$ of a line. Thus, all lines through p in this P_2 make up a rational curve of secants of the rank one locus, with exactly two tangents. $\square$

Now, whenever we have a rank two point $p \in [q_1, q_2]$, we can run all proper secants through p and label, in two ways, the two rank one points on it d_1 and d_2. Each labelling gives a common basket $b = [q_1, d_1] \cap [q_2, d_2]$. Thus, the curve of common baskets is a double covering of P_1 ramified over the two tangents, and hence a rational curve itself. The association $b \mapsto d_i = [b, q_i] \cap v(P_1)$ gives an isomorphism with $v(P_1)$, for $i = 1, 2$.

Actually, the curve of common baskets is the residual intersection of the two cones over $v(P_1)$ with vertex at q_1, respectively q_2, and consequently a *conic* itself.

Thereby we obtain this complement to Proposition 0.2:

Proposition 2.3 *The variety of common baskets for two quadrics q_i is determined by the number of points in the intersection $[q_1, q_2] \cap \mathcal{R}_6^2$, and consists of as many rational curves.*

In order to have more than one rational curve, it is necessary and sufficient that $[q_1, q_2]$ be a pencil in $\mathcal{F}_{11}^3 - \mathcal{F}_6^2$, that is: q_1 and q_2 must be two cones with the same vertex, and in this case we have (counting multiplicities) three rational components. $\square$

3 Three quadrics in a basket

In this section we prove and expand Proposition 0.3.

Lemma 3.1 *Let $\updownarrow$ be a pencil of cones with three distinct rank two points, but not contained in $\mathcal{R}_6^2$. Then there's a unique trio of double-planes $d_i,\ i = 1, 2, 3$ (up to permutation), such that the span $P_2 \approx [d_1, d_2, d_3]$ contains the pencil.*

Proof: From section 1 it follows that $\updownarrow$ consists of cones with a fixed vertex over a pencil of conics with three distinct two-lines. All such pencils of conics are equivalent under projective transformations of the plane (i.e. under $PSL_C(3)$), and thus the pencil $\updownarrow$ can be turned into the diagonal form: $\lambda(x_1^2 + x_2^2) + \mu(x_2^2 + x_3^2)$, which is clearly in the span of the three double-planes $d_i = x_i^2$. This proves the existence part.

For uniqueness (up to permutation), suppose we have another trio of double-planes d_i' with $\updownarrow \subset [d_1', d_2', d_3']$. Then $\updownarrow = [d_1, d_2, d_3] \cap [d_1', d_2', d_3']$ and with proper indexing, the edges $[d_i, d_j]$ and $[d_i', d_j']$ meet $\updownarrow$ in the same point of rank two p_{ij}.

By the reciprocal of Desargues' theorem (in dimension three) the triangles determined by d_i, respectively d_i', are in perspective i.e. the lines $[d_i, d_i']$ meet at a point p, which is necessarily of rank two. But Lemma 2.1 requires then all our double-planes to be on the same conic. This contradiction proves the uniqueness part. $\square$

Let $q_i,\ i = 1, 2, 3$ be three distinct quadrics with a common basket b. Again, the pencils $[b, q_i]$ must meet the rank one locus in double-planes d_i.

Suppose first that q_i are collinear. Then, either b is on the same line and $d_1 = d_2 = d_3$, or b is away from this line and then d_i are distinct and span a plane containing the line.

The latter case means that q_i belong to a pencil $\updownarrow$ of cones with fixed vertex which has its rank two points at the intersections $p_i = \updownarrow \cap [d_j, d_k]$. Obviously the six points p_i, q_j on $\updownarrow$ must satisfy a relation, since q_j are projections of d_j from b.

One can guess this relation from the fact that it comes from a (rational) map $P_2 \cdot\cdot \to (P_1)^3$ whose image should be a surface of multi-degree $(1, 1, 1)$. The formula should also have permutation invariance. Indeed, the relation can be written as:

$$(p_1, p_2; p_3, q_1) + (p_2, p_3; p_1, q_2) + (p_3, p_1; p_2, q_3) = \frac{3}{2} \qquad (c1)$$

where $(a, b; u, v)$ denotes the cross-ratio of four points on a projective line (in our case $\updownarrow \approx P_1$):

$$(a, b; u, v) = \frac{a-u}{a-v} \cdot \frac{b-v}{b-u}$$

Conversely, when condition $(c1)$ is satisfied, the lines $[q_i, d_i]$ are concurrent, yielding the basket b.

This settles the collinear case, and we may assume now that the three quadrics span a plane $P_2 \approx [q_1, q_2, q_3]$.

If b is on one of the edges, say $[q_i, q_j]$, we have a double-point $d = d_i = d_j$ on this line and d_k on $[b, q_k]$. Thus, *the existence of the proper secant (of the rank one locus)* $[d, d_k] \subset [d_1, d_2, d_3]$ *intersecting an edge in a double-plane characterizes this case.*

Henceforth, we shall assume that b is not on the lines $[q_i, q_j]$. Then, the double-planes d_i span a plane $[d_1, d_2, d_3]$, and either it coincides with $[q_1, q_2, q_3]$ (and contains b), or the two planes meet in a line.

Let's take first the case $b \in [q_1, q_2, q_3] = [d_1, d_2, d_3]$. Thus $[q_1, q_2, q_3]$ is a generic secant plane of the rank one locus (i.e. not contained in $\mathcal{R}^2_6$), and we look at the situation where a common basket lies in this same plane, but away from the edges $[q_i, q_j]$.

Then all our quadrics are cones with the same vertex, namely the intersection of the three planes in P_3 corresponding to the double-planes d_i. (The intersection cannot be a line since then $[d_1, d_2, d_3]$ would contain a conic $v(P_1)$ of double-planes, and this would contradict the assumption that q_i are of rank at least three.) Since the triangles $\triangle(q_i)$ and $\triangle(d_i)$ are in perspective, Desargues' theorem says that the corresponding edges meet in collinear points $p_{ij} = [q_i, q_j] \cap [d_i, d_j] \in \updownarrow$.

Thus, $[q_1, q_2, q_3]$ contains a line $\updownarrow$ of singular quadrics, meeting the pencils $[q_i, q_j]$ in three distinct rank two points p_{ij}. In view of Lemma 3.1 and by the reciprocal version of Desargues' theorem, this is enough to ensure that the triangle of our three quadrics is in perspective with the triangle (properly labeled) of the three double-planes, and the basket b is retrieved as the point of perspective.

Note that $[q_i, q_j]$ have themselves three rank two points on them, hence the case under consideration arises only when *there's one more collinearity amongst the nine points* $[q_i, q_j] \cap [d_k, d_l]$ *besides the six edges.*

To conclude, we take up the case of a line intersection $\updownarrow = [q_1, q_2, q_3] \cap [d_1, d_2, d_3]$.

Again, the points $p_{ij} = [q_i, q_j] \cap [d_i, d_j]$ are rank two points on $\updownarrow$, and we have a three-dimensional Desargues configuration.

If $\updownarrow$ is not contained in $\mathcal{R}^2_6$, and this is obviously the *generic* case envisaged in our Proposition 0.3, then Lemma 3.1 and the fact that Desargues' theorem works in both directions (from b to the p_{ij}, and from p_{ij} to b), yield the result that *the existence of a pencil of cones* $\updownarrow \subset [q_1, q_2, q_3]$ *with* $\updownarrow \cap [q_i, q_j]$ *of rank two and the rest of rank three, characterizes this situation (and the associated common basket is uniquely determined by* $\updownarrow$*).*

We are left with the degenerate case where $\updownarrow = [q_1, q_2, q_3] \cap [d_1, d_2, d_3]$ is contained in $\mathcal{R}^2_6$, that is: $\updownarrow \in \mathcal{F}^2_6$. In other words, $\updownarrow$ is a secant (or tangent) to the rank one locus.

Under our assumptions d_i are not on $\updownarrow$, and Lemma 2.1 implies that $[d_1, d_2, d_3]$ is the unique P_2 which is the span of a Veronese curve $v(P_1)$ and contains $\updownarrow$.

The question is whether this pencil of two-planes (with fixed singularity line) $\updownarrow \subset [q_1, q_2, q_3]$, together with its three marked points $p_{ij} = \updownarrow \cap [q_i, q_j]$ of rank two, is sufficient information for finding a common basket.

The answer is given in the following lemma, which yields, counting multiplicity, two common baskets corresponding to $(\updownarrow, p_{ij})$:

Lemma 3.2 *Let $v(P_1) \subset P_2$ be a Veronese conic of double-planes, and $\updownarrow \subset P_2$ a line with three distinct marked points p_{ij} away from the intersection $\updownarrow \cap v(P_1)$.*

If $\updownarrow$ is a proper secant of the Veronese conic, there are exactly two triangles $\triangle(d_i)$ and $\triangle(d'_i)$ with vertices on the conic, and such that their edges $[d_i, d_j]$, respectively $[d'_i, d'_j]$, meet $\updownarrow$ in p_{ij}.

If $\updownarrow$ is tangent to the Veronese conic, there's only one such triangle.

Remark: This is clearly related to Desargues' theorem, but requesting the two triangles in perspective to have their vertices on a conic. We have therefore a five parameter family with a (rational) map to lines in P_2 with three marked points: another five parameter family. One can fairly expect the map to be a birational equivalence, which indeed turns out to be the case.

There's an alternative argument for proving that solutions $\triangle(d_i)$ exist and are at most two, finiteness being rather obvious. With $d_i \in v(P_1) \approx P_1$ as unknowns, the determinantal condition on $(d_i, d_j) \in (P_1)^2$ expressing collinearity with p_{ij} is of type $(2, 2)$ but contains the diagonal as improper solutions. Thus we actually have a $(1, 1)$ condition. On $(P_1)^3$ we intersect accordingly three equations: of type $(1, 1, 0)$, $(0, 1, 1)$ and $(1, 0, 1)$. This yields, counting multiplicity, two solutions.

For the more precise statement in our lemma, we observe that, in the case of a proper secant, there's an involution of P_2, induced from an involution of $P_1 \approx v(P_1)$, and which keeps $\updownarrow$ pointwise fixed. (One extends to P_2 the involution of $P_1 = v(P_1)$ fixing the intersection with $\updownarrow$.) Thus, a triangle solution produces a 'reflected' second solution.

When $\updownarrow$ is tangent, this is no longer the case. Indeed, if we would have two solutions: $\triangle(d_i)$ and $\triangle(d'_i)$, there would be an involution of $P_1 \approx v(P_1)$ taking one onto the other, defined by tracing lines through the perspective point p and exchanging the two intersection points with the conic. The associated transformation of $P_2 = P(Sym_C(2))$ would have to fix $\updownarrow$ pointwise, since it must fix p_{ij}, but it also fixes the line through the tangency points of the two tangents from p to the conic. This is a contradiction and the lemma is proven. $\square$

To conclude this analysis, we do some dimension counts.

The quasi-projective variety:

$$B^3_{21} = \{(q_1, q_2, q_3, b) \ : \ b \text{ is a common basket for } q_1, q_2 \text{ and } q_3 \ \}$$

projects to P_9 (the closure of the space of possible baskets b) by $(q_1, q_2, q_3, b) \mapsto b$. The fibers are irredicible open subsets of the third Cartesian power of the cone from b over the rank one locus. This gives irreducibility and dimension: $9 + 3 \times 4 = 21$ for B^3_{21}.

¿From the perspective of our characterization, B^3_{21} is obtained (up to birational equivalence) by running a plane through a generic pencil of cones with fixed vertex, and then choosing a triangle in this plane with edges passing through the three rank two points. This gives dimension 21 as $11 + 7 + 3$.

Similar counts can be performed for various subvarieties of B^3_{21}. For example, when our pencil degenerates to one contained in $\mathcal{R}^2_6$ (i.e. becomes a point of $\mathcal{F}^2_6$), we are generically in the case addressed by our previous lemma, and the dimension of the corresponding subvariety is: $6 + 7 + 3 \times 2 = 19$.

We can regroup now the main results in this section in the form of a complement to Proposition 0.3 :

Proposition 3.3 *Consider the quasi-projective variety*

$$B^3_{21} = \{(q_1, q_2, q_3, b) \ : \ b \textit{ is a common basket for } q_1, q_2 \textit{ and } q_3 \ \} \subset (P_9 - \mathcal{R}^2_6)^4$$

One can distinguish several closed subvarieties, according to the diagram:

$$\begin{array}{ccccccc} B^3_{21} & \supset & C^3_{19} & \supset & E^3_{15} & \supset & F^3_{14} \\ \cup & & \cup & & \cup & & \\ H^3_{14} & \subset & D^3_{18} & & G^3_{13} & & \end{array}$$

These subvarieties are determined by the following geometrical conditions on the quadrics q_i, $i = 1, 2, 3$:

(B^3_{21}) : *the triple (q_1, q_2, q_3) of a generic point $(q_1, q_2, q_3, b) \in B^3_{21}$ spans a plane $[q_1, q_2, q_3] = P_2$ containing a pencil of singular quadrics with exactly three distinct rank two points $p_i \in [q_j, q_k]$ (and rank three elsewhere);*

(C^3_{19}) : *the triple (q_1, q_2, q_3) of a generic point $(q_1, q_2, q_3, b) \in C^3_{19}$ spans a plane $[q_1, q_2, q_3]$ containing a secant or tangent of the rank one locus (i.e. a line from the family $\mathcal{F}^3_6$);*

(D_{18}^3) : *the triple* (q_1, q_2, q_3) *of any point* $(q_1, q_2, q_3, b) \in D_{18}^3$ *spans a plane* $[q_1, q_2, q_3]$ *containing a proper secant of the rank one locus, and one edge* $[q_i, q_j]$ *passes through a rank one point on this secant;*

(E_{15}^3) : *the triple* (q_1, q_2, q_3) *of a generic point* $(q_1, q_2, q_3, b) \in E_{15}^3$ *spans a proper secant plane of the rank one locus;*

(F_{14}^3) : *the triple* (q_1, q_2, q_3) *of a generic point* $(q_1, q_2, q_3, b) \in F_{14}^3$ *spans a proper secant plane of the rank one locus and the triangle* $\triangle(q_i)$ *is in perspective with the triangle of double-planes in that span;*

(G_{13}^3) : *the triple* (q_1, q_2, q_3) *of any point* $(q_1, q_2, q_3, b) \in G_{13}^3$ *spans a pencil of conics with three rank two points, and condition* $(c1)$ *is satisfied on the pencil for some ordering of these points.*

(H_{14}^3) : *the triple* (q_1, q_2, q_3) *of any point* $(q_1, q_2, q_3, b) \in H_{14}^3$ *spans a pencil which meets the rank one locus.*

Any point of B_{21}^3 *satisfies one of the conditions above i.e. it is either already generic on* B_{21}^3*, in the specified sense, or is to be found on (at least) one of these closed subvarieties.*

The dimension of the subvariety is indicated by the subscript, and the number of points (q_1, q_2, q_3, b) *with the same projection* (q_1, q_2, q_3) for some open dense set *in each subvariety is tabulated below:*

B_{21}^3	C_{19}^3	D_{18}^3	E_{15}^3	F_{14}^3	G_{13}^3	H_{14}^3
1	*2*	*1*	*6*	*7*	*1*	P_1

Remark: All triples involved in the families $E_{15}^3, F_{14}^3, G_{13}^3$ are made of cones with common vertex, and relate to issues of tangency for conics. They are less relevant for questions of common tangents to quadric surfaces because all lines through the common vertex are always common tangents.

4 Four quadrics in a basket

In this section we prove and expand Proposition 0.4.

Let $q_i,\ i = 1, ..., 4$ be four quadrics with a common basket b, and let d_i be the double-plane in the pencil $[b, q_i]$. *Generically,* the q_i's would span a three-space $P_3 \approx [q_1, ..., q_4]$, the d_i's would span another three-space $[d_1, ..., d_4]$, and the two would meet in a plane $P_2 \approx [q_1, ..., q_4] \cap [d_1, ..., d_4]$. This plane contains a *complete quadrilateral* made of the four lines $\ell_i = [q_j, q_k, q_l] \cap [d_j, d_k, d_l]$. These lines are pencils of cones and belong to the family $\mathcal{F}_{11}^3$. Clearly the pencil $[q_i, q_j]$ passes through the intersection point $p_{kl} = \ell_k \cap \ell_l = [q_i, q_j] \cap [d_i, d_j]$, and the six points p_{kl} are all rank two points.

We begin our analysis from this end, and establish some facts about complete quadrilaterals.

Definition: A *complete quadrilateral* consists of four lines in general position in P_2, that is: no three are concurrent.

Equivalently, a *complete quadrilateral* is a projective planar configuration $(6_2, 4_3)$ of six points and four lines, with every point incident to two lines, and every line incident to three points. The above labelling: $p_{ij} = \ell_i \cap \ell_j,\ \{i,j\} \subset \{1,2,3,4\}$ will be normally adopted.

Lemma 4.1 (Cayley) *Every complete quadrilateral can be obtained by intersecting the faces of a tetrahedron in P_3 with a plane P_2 avoiding its vertices.*

Remark: It follows from Desargues' theorem that if two tetrahedra in P_3 cut in this fashion the same complete quadrilateral in a P_2, then they are in perspective, that is: with proper labelling, the four lines through corresponding vertices meet in the same point (the perspective point). $\square$

Lemma 4.2 *Suppose we have a complete quadrilateral in $P_2 \subset P_9$, made of pencils $\ell_i \subset \mathcal{R}_8^3$, with rank two points $p_{ij} = \ell_i \cap \ell_j$ and rank three elsewhere.*

Then each ℓ_i is a pencil of cones with fixed vertex v_i, and either:

i) the v_i's are in general position, or:
ii) all v_i's coincide.

In the former case, if we denote by d_i the double-plane supported by the span $P_2 \approx [v_j, v_k, v_l] \subset P_3$, we obtain the unique tetrahedron of rank one points *which contains the initial $P_2 \subset P_9$ in its span, and hence produces the given complete quadrilateral by the four traces of its faces.*

The latter case, when $P_2 \subset P_5 = P(Sym_C(3)) \subset P_9 = P(Sym_C(4))$, is addressed in the next lemma.

Proof: By Lemma 3.1, each ℓ_i is a pencil of cones with fixed vertex $v_i \in P_3$. We look at the traces of these pencils on a plane $P_2 \subset P_3$ chosen away from the vertices.

ℓ_i traces a pencil of conics passing through *four points in general positions*, and the three rank two points on it $p_{ij},\ j \neq i$ correspond to the three pairs of lines, with complementary pairs of points on it.

These three pairs of rank two conics have consequently *non-collinear* singularities s_{ij}. The line $[v_i, s_{ij}]$ is clearly the singularity axis for p_{ij} and contains therefore v_j. Thus, two of the vertices $v_j,\ v_k$ cannot coincide without being both equal to v_i, and repeating this argument shows that we can either have:

i) $v_i,\ i = 1, ..., 4$ in general position in P_3, or:
ii) all v_i's equal.

We pursue here the first case, and adopt as projective coordinates $(x_1 : ... : x_4) \in P_3$ those corresponding to the reference tetrahedron v_i, that is: $\{x_i = 0\}$ is the face $[v_j, v_k, v_l]$.

Then, the quadrics in the pencil ℓ_i do not involve the variable x_i, and so p_{ij} involves neither x_i, nor x_j.

But $\ell_i = \lambda p_{ij} + \mu p_{ik}$ contains p_{il}, which has neither x_i, nor x_l, and this can happen only when p_{ij} has no $x_k x_l$ term, and p_{ik} has no $x_j x_l$ term. This means: $\ell_i \subset [x_j^2, x_k^2, x_l^2]$, hence our complete quadrilateral is in the span of $d_i = x_i^2,\ i = 1, ..., 4$.

Uniqueness follows from the uniqueness part in Lemma 3.1. $\square$

Remark: Degree considerations imply that a complete quadrilateral made of singular quadrics cannot have a line contained in $\mathcal{R}_6^2$ unless its plane is contained in $\mathcal{R}_8^3$.

When all vertices coincide, projection from the common vertex reduces the problem to the space of conics:

Lemma 4.3 *Suppose we have a complete quadrilateral in $P_2 \subset P_5 = P(Sym_C(3))$, made of pencils of conics ℓ_i, with rank two points $p_{ij} = \ell_i \cap \ell_j$ and rank three somewhere. Suppose further that P_2 is not a secant plane of the rank one locus i.e. not the span of three rank one points.*

Then, up to relabelling, there's a unique tetrahedron of rank one points $d_i,\ i = 1, ..., 4$, which contains the initial plane in its span, and produces the complete quadrilateral by the four traces of its faces.

Proof: The rank stratification in the space of conics reads:

$$P_5 = P(Sym_C(3)) \supset \mathcal{R}_4^2 \supset \mathcal{R}_2^1 = v(P_2)$$

with $\mathcal{R}_4^2$ of degree three, and $\mathcal{R}_2^1$ of degree four.

Thus, under our assumptions, $P_2 \cap \mathcal{R}_4^2$ is a *cubic curve* (but not degenerated into three lines), with six distinct points p_{ij} on it, subject to four collinearity conditions.

The assumption of some rank three point means we can apply the argument in Lemma 3.1 to one of the lines, say ℓ_4, and find a unique trio of rank one conics d_1, d_2, d_3 whose span contains ℓ_4.

Because of our other assumption, P_2 and $[d_1, d_2, d_3]$ span a three-space P_3, which must meet the rank one locus in at least one more point, distinct from the other three. However, there might be a whole conic of rank one points in the intersection.

The case of a single new rank one point d_4 obviously corresponds to a complete quadrilateral with no line in $\mathcal{R}^2_4$, and is thus resolved by the quadruple $d_i,\ i = 1,...,4$.

The alternative requires one line of the complete quadrilateral to be contained in $\mathcal{R}^2_4$, and if we call it ℓ_1, then, with the natural labelling in use, the Veronese conic $v(P_1)$ introduced in the intersection will pass through d_2 and d_3. Thus, ℓ_1 is a secant or tangent to this conic.

The restriction of $\mathcal{R}^2_4$ to our P_3 consists therefore of the plane $P_2 \supset v(P_1)$ and the *cone* from d_1 over $v(P_1)$. (Indeed, d_1 must be singular on the residual quadric.)

Thus, in order to find our d_4 we simply intersect $[p_{23}, d_1]$ with $v(P_1)$.

This yields the desired quadruple of rank one points, clearly unique up to relabelling. $\square$

We need to investigate also the most degenerate case, when the entire plane of the complete quadrilateral lies in $\mathcal{R}^2_4$. Since our lines must be pencils with fixed singularity, we are actually envisaging the case $P_2 = P(Sym_C(2)) \supset \mathcal{R}^1_1 = v(P_1)$.

Thus, the question is: given a complete quadrilateral in $P_2 = P(Sym_C(2))$, when is there a quadruple of rank one points $d_i \in v(P_1)$ such that $p_{ij} \in [d_k, d_l],\ \{i,j\} \cup \{k,l\} = \{1,2,3,4\}$?

The space of complete quadrilaterals in P_2 is birationally equivalent to $(P_2)^4$, hence eight-dimensional. On the other hand, the space of ordered quadruples of double-points is $(v(P_1))^4 \approx (P_1)^4$, and a given quadruple has the required relation with a two-parameter family of complete quadrilaterals. Thus, only a six dimensional subfamily of complete quadrilaterals can be 'solved' in this sense, and we need a 'codimension two' condition ($c2$) satisfied.

In order to streamline some of our statements, we introduce:

Definition: *(Typical, special, and solvable complete quadrilaterals)*

A *complete quadrilateral* in $P_9 = P(Sym_C(4))$ will be called **typical** when defined by the traces of the four faces of a tetrahedron with vertices at rank one points on a sectioning plane avoiding these vertices.

A *complete quadrilateral* will be called **special** when contained in the span of a proper secant plane of the rank one locus (i.e. a plane with exactly three rank one points), and has one vertex of rank one, with the remaining five of rank two.

A *complete quadrilateral* will be called **solvable** when contained in the span of a conic of rank one points (i.e. a plane $P_2 \supset v(P_1)$), and there's a quadruple of rank one points $d_i \in v(P_1)$, such that the six vertices satisfy: $p_{ij} \in [d_k, d_l]$. $\square$

Lemma 4.4 *Given a* solvable *complete quadrilateral in* $P_2 = P(Sym_C(2))$, *there's a* unique *corresponding solution* $d_i,\ i = 1, ..., 4$.

Proof: Two solutions d_i and d_i' would be necessarily in perspective. As in Lemma 3.2, intersecting the conic with lines through the perspective point and exchanging the two intersection points defines an involution of $P_1 \approx v(P_1)$ which exchanges the two solutions. However, the associated involution of P_2 would have to fix the complete quadrilateral and thus be the identity. The contradiction proves the claim. $\square$

Lemma 4.5 *The plane of a* typical complete quadrilateral *belongs to one of the following (disjoint) families:*

(Φ_{15}) : *planes* $P_2 \subset P_9$ *where* $\mathcal{R}_8^3 = \{det(Q) = 0\}$ *restricts to four lines in general position (the complete quadrilateral itself);*

(Ψ_{12}) : *planes* $P_2 \subset P_5 \subset \mathcal{R}_8^3$*, with* $P_5 = P(Sym_C(3))$ *in adequate coordinates, where (relative to the corresponding space of conics)* $\mathcal{R}_4^2$ *restricts to a cubic curve other than three lines.*

Every complete quadrilateral supported in a plane of type (Ψ)*, and with vertices at rank two points, is admissible.*

The tetrahedron of rank one points defining an admissible complete quadrilateral is unique.

Proof: In view of the above discussion, all that remains to be shown is that for type (Φ), the lines of singular quadrics meet at rank two points. But this follows from our investigation of lines in $\mathcal{R}_8^3$ of section 1.

Indeed, the lines in $P_2 \cap \mathcal{R}_8^3$ must belong to one of the families: $\mathcal{F}_{11}^3 - \mathcal{F}_6^2$ or $\mathcal{M}_{11}^3 - \mathcal{M}_7^2$. However, in the latter case, the pencil would have a single rank two point. Yet, there are three points on the pencil which are singularities of the restricted determinantal quartic, namely: its intersections with the other three lines. Since our pencil cannot be tangent to $\mathcal{R}_8^3$ in more than two points, this latter case must be discarded.

This leaves us with a pencil of type (F) i.e. a pencil of cones with fixed vertex, and the three rank two points on such pencils must be, in our case, the intersections with the other three lines. $\square$

Note: *While the definition of a* typical, special, or solvable complete quadrilateral *uses a quadruple of rank one points, we have seen above that these properties can be detected directly from the complete quadrilateral itself, and depend* *essentially on the position of its span with respect to the rank stratification of* $P_9 = P(Sym_C(4))$*. The quadruple of rank one points can be 'reconstructed' from this type of information. Even without searching here for the explicit form of condition* $(c2)$*, we shall use henceforth:* typical, special, or solvable *in the sense of* a property which needs no explicit mention of four rank one points.

We can list now the possible types of configurations for $(q_i)_i, b, (d_i)_i, i = 1, 2, 3, 4$: four quadrics, a common basket, and rank one points $d_i \in [b, q_i]$. We shall see, in the spirit of the above note, that these classes need no explicit mention of b and $(d_i)_i$.

First, for a **three-dimensional span**: $[q_1, q_2, q_3, q_4] = P_3$.

(B): $[d_1, ..., d_4] = P_3$

(C): $[d_1, ..., d_4] = P_2$ because $b \in [q_i, q_j]$ and hence $d_i = d_j$

(D): $[d_1, ..., d_4] = P_2$ because $d_i \in v(P_1),\ i = 1, ..., 4$

Next, for a **two-dimensional span**: $[q_1, q_2, q_3, q_4] = P_2$.

(E): $[b, q_1, ..., q_4] = [d_1, ..., d_4] = P_3$

(F): $b \in [q_i, q_j],\ d_{ij} = d_i = d_j,\ [d_{ij}, d_k, d_l] = P_2$

(G): $[b, q_1, ..., q_4] = P_3,\ [d_1, ..., d_4] = P_2$

(H): $b \in [q_i, q_j] \cap [q_k, q_l]$

(I): $b \in [q_i, q_j, q_k] = P_1$

Lastly, for a **one-dimensional span**: $[q_1, q_2, q_3, q_4] = P_1$.

(J): $b \in [q_1, ..., q_4] = P_1$

This list structures our extension of Proposition 0.4. In order to indicate inclusion, rather than adjacency, we consider our subvarieties as *closed subvarieties of* B^4_{25}, that is: as the closure in B^4_{25} of the locus described by some generic property.

Proposition 4.6 *Consider the quasi-projective variety:*

$$B^4_{25} = \{(q_1, q_2, q_3, q_4, b)\ :\ b \text{ is a common basket for } q_i,\ i = 1, ..., 4\ \} \subset (P_9 - \mathcal{R}^2_6)^5$$

One can distinguish closed subvarieties:

$$\begin{array}{ccccccc}
F^4_{17} & \subset & C^4_{22} & & H^4_{19} & & \\
\cap & & \cap & & \cap & & \\
E^4_{18} & \subset & B^4_{25} & \supset & G^4_{20} & \supset & J^4_{15} = H^4_{19} \cap I^4_{19} \\
 & & \cup & & \cup & & \\
 & & D^4_{21} & & I^4_{19} & &
\end{array}$$

according to the following geometrical conditions:

(B^4_{25}) : *the quadrics* (q_i) *of a generic point* $(q_1, ..., q_4, b) \in B^4_{25}$ *define a reference tetrahedron in a three-space, tracing a* typical complete quadrilateral *on some two-subspace avoiding its vertices;*

(C^4_{23}) : *the quadrics* (q_i) *of a generic point* $(q_1, ..., q_4) \in C^4_{23}$ *define a reference tetrahedron in a three-space, tracing a* special complete quadrilateral *on some two-subspace avoiding its vertices;*

(D^4_{21}) : *the quadrics* (q_i) *of any point* $(q_1, ..., q_4) \in D^4_{21}$ *span a three-space containing a* $P_2 \supset v(P_1)$, *and the faces* $[q_i, q_j, q_k]$ *trace a* solvable complete quadrilateral *in this* P_2*;*

(E^4_{18}) : *the quadrics* (q_i) *of a generic point* $(q_1, ..., q_4, b) \in E^4_{18}$ *span a plane containing a* typical complete quadrilateral *with vertices* $p_{ij} \in [q_k, q_l]$*;*

(F^4_{17}) : *the quadrics* (q_i) *of a generic point* $(q_1, ..., q_4, b) \in F^4_{17}$ *spans a proper secant plane of the rank one locus and contains a* special complete quadrilateral *with vertices* $p_{ij} \in [q_k, q_l]$*;*

(G^4_{20}) : *the quadrics* (q_i) *of a generic point* $(q_1, ..., q_4, b) \in G^4_{20}$ *span a plane containing a secant or a tangent* ℓ *of the rank one locus, and the four quadrics* q_i *lie on a conic which contains the two rank one points* A *and* B *of the secant, or is tangent to* ℓ *at* $T = A = B$ *when* ℓ *is a tangent;*

(H^4_{19}) : *the quadrics* (q_i) *of a generic point* $(q_1, ..., q_4, b) \in H^4_{19}$ *span a plane, and for some* $\{i, j, k, l\} = \{1, 2, 3, 4\}$, *both* $[q_i, q_j]$ *and* $[q_k, q_l]$ *meet the rank one locus;*

(I^4_{19}) : *the quadrics* (q_i) *of a generic point* $(q_1, ..., q_4, b) \in I^4_{19}$ *span a plane; three of them are on a line meeting the rank one locus, and the span contains another double-plane;*

(J^4_{15}) : *the quadrics* (q_i) *of any point* $(q_1, ..., q_4, b) \in J^4_{15}$ *span a line meeting the rank one locus.*

Any quadruple of quadrics allowing a common basket enters one of the configurations listed above as (B) *to* (J), *and satisfies the corresponding property stated above.*

Proof: Since almost all relevant arguments have already been presented, we fill in a few remaining details.

For the G^4_{20} family, the conic through the four points q_i which also contains the rank one points A and B of the secant (or is tangent at $T = A = B$ to ℓ in case of a tangent), is obviously the projection from b to $[q_1, ..., q_4] = P_2$ of the Veronese conic $v(P_1) \subset P_2 \supset \ell$.

The existence of this conic is equivalent to conditions we label ($c3$) for the six points p_{ij} on the secant or tangent. These conditions reflect the fact that $p_{ij} = \ell \cap [d_k, d_l]$ and d_i are on the conic $v(P_1)$.

In case ℓ is a secant, A and B will be the two points of $\ell \cap v(P_1)$. Considering the projections of the conic $v(P_1)$ from d_i, respectively d_j, onto ℓ, we obtain:

$$(A, B; p_{jk}, p_{jl}) = (A, B; p_{ik}, p_{il}), \quad \{i, j, k, l\} = \{1, 2, 3, 4\}$$

and we call ($c3$) the collection of these cross-ratio relations.

In case ℓ is tangent, we have $A = B = T = \ell \cap v(P_1)$. Again, projecting from d_i, then d_j, we obtain:

$$(T, d_j; d_k, d_l) = (T, p_{kl}; p_{lj}, p_{jk})$$

$$(d_i, T; d_k, d_l) = (p_{kl}, T; p_{li}, p_{ik})$$

with the first cross-ratio on the conic, and the second on the tangent. We can eliminate the d_i's from the resulting system by first taking the product of the two left-hand sides above with $(d_i, d_j; d_k, d_l)$, to get:

$$1 = (T, p_{kl}; p_{lj}, p_{jk}) \cdot (p_{kl}, T; p_{li}, p_{ik}) \cdot (d_i, d_j; d_k, d_l)$$

and hence:

$$(T, p_{kl}; p_{lj}, p_{jk}) \cdot (p_{kl}, T; p_{li}, p_{ik}) = (T, p_{ij}; p_{jl}, p_{li}) \cdot (p_{ij}, T; p_{jk}, p_{ki})$$

The condition ($c3$) will be the collection of these equations, should we have tangency, and not a proper secant.

One verifies that, for p_{ij} of rank two, ($c3$) is sufficient for finding d_i's on the conic with $p_{ij} \in [d_k, d_l]$. By the same argument as in Lemma 3.2, there are two solutions in the secant case, and one solution in the tangent case. $\square$

5 A double-four example

In this section we study a particular configuration, made of two groups of four quadrics. Each group has *linearly independent and smooth quadrics*, and the four in one group are common baskets for the four in the other, hence the designation "double-four".

It turns out that, up to projective transformations of P_3, this configuration is *uniquely determined* by the stated property. Our proof uses the criteria developed above, in particular the configuration of points and lines created by four mashed complete quadrilaterals. It will be identified as the *Reye configuration* $(12_4, 16_3)$. [HC-V] [Dol]

We start with the eight 'diagonal' quadrics: $\pm x_1^2 \pm x_2^2 \pm x_3^2 \pm x_4^2$. The two groups of four quadrics are those of positive, respectively negative determinant.

Obviously permutations and changes of two signs preserve the two groups, while changing one sign exchanges the groups. Thus, our labelling here is mostly a matter of convenience. We put:

$$q_i = (\sum_{j=1}^{4} x_j^2) - 2x_i^2, \quad d_i = x_i^2, \quad i = 1, ..., 4$$

$$b^1 = \sum_{j=1}^{4} x_j^2, \; b^\beta = (\sum_{j=1}^{4} x_j^2) - 2(x_\beta^2 + x_{\beta-1}^2), \quad \beta = 2, 3, 4$$

Clearly, we have three tetrahedra, spanning the same three-space:

$$P_3 = [q_1, ..., q_4] = [b^1, ..., b^4] = [d_1, ..., d_4]$$

Intersecting the faces of the first two tetrahedra yields sixteen lines:

$$\ell_i^\alpha = [q_j, q_k, q_l] \cap [b^\beta, b^\gamma, b^\delta]$$

and the *same* collection of sixteen lines obtains from intersecting the faces of the first and last tetrahedra, or second and last. Thus, there are twelve planes, each containing four lines, with each line contained in three planes.

This is obviously the *dual* of a $(12_4, 16_3)$ configuration, but we would rather distinguish a direct $(12_4, 16_3)$ configuration by taking into account the *twelve rank two points* which lie on the sixteen lines, with three points on each line, and each point incident to four lines.

Our labelling is now going to show its bias for the q_i and d_j tetrads, but one should remain aware of the perfectly equivalent role of the third tetrad b^α. We put:

$$p_{ij}^+ = x_k^2 + x_l^2 = [d_k, d_l] \cap [q_i, q_j]$$

$$p_{ij}^- = x_k^2 - x_l^2 = [d_k, d_l] \cap [q_k, q_l]$$

Thus, each edge of our three tetrahedra has exactly two of the twelve rank two points $p_{ij}^\pm$.

One can identify now the collection of points and lines $(p_{ij}^\pm, \ell_k^\alpha)$ with their incidence relations, as the *Reye configuration* $(12_4, 16_3)$. The latter is usually depicted in some affine (real) part $R^3 \subset P_3$ by means of a cube, with $p_{ij}^{\pm}$ corresponding with the eight vertices of the cube, its center, and the three points at infinity determined by the three distinct directions of the edges. The sixteen lines ℓ_k^α are the twelve edges and the four diagonals of the cube. [HC-V][Dol]

Our three tetrahedra $(q_i)_i, (d_j)_j, (b^\alpha)_\alpha$, should be seen in this model as the three tetrads of planes determined, each, by two opposite faces of the cube together with the two diagonal planes of the cube which cut the two diagonals in these faces.

We are going to see this configuration emerging from any "double-four", and obtain:

Theorem 5.1 *Suppose* $(q_i)_i$ *and* $(b^\alpha)_\alpha$ *form a double-four configuration of* smooth quadrics *in* P_3*, that is:*
q_i are linearly independent and are common baskets for $(b^\alpha)_\alpha$, and
b^α are linearly independent and common baskets for $(q_i)_i$.

Then, there is a projective automorphism of P_3 which carries $(q_i)_i$ and $(b^\alpha)_\alpha$ to the standard double-four configuration presented above.

The proof requires a number of lemmas. Throughout, we let q_i, b^α stand for a double-four as in the theorem, but the notation should be understood as separate from the one used in describing the standard double-four.

Since b^α is a basket for d_i, the pencil $[b^\alpha, q_i]$ has a (unique) rank one point d_i^α. Considering $(q_i)_i$ as a quadruple with common basket b^α, linear independence and Proposition 4.6. produce an associated *complete quadrilateral*:

$$\ell_i^\alpha = [q_j, q_k, q_l] \cap [d_j^\alpha, d_k^\alpha, d_l^\alpha], \quad i = 1, ..., 4$$

with six vertices $p_{ij}^\alpha =\in [q_k, q_l]$ at points of rank at most two.

However, an edge $[q_i, q_j]$ has at most two points of rank two, or a single rank one point and no rank two. Thus, the collection $p_{ij}^\alpha,\ \{i, j\} \subset \{1, 2, 3, 4\}, \alpha = 1, ..., 4$ consists of at most twelve *distinct* points. We'll see that there must be precisely twelve distinct points, all of rank two.

The fact that there can be no rank one point p_{ij}^α follows from the observation that if one complete quadrilateral were special, all would be special, with the same rank one point on some edge $[q_i, q_j]$. But this would force all baskets on that same edge, contradicting linear independence.

Lemma 5.2 *The sixteen lines ℓ_i^α are all distinct and none lies in $\mathcal{R}_6^2$.*

Proof: Since the four complete quadrilaterals corresponding to $b^\alpha,\ \alpha = 1, ..., 4$ must be distinct, their planes π_α are distinct.

Thus, two complete quadrilaterals can share at most one line, and if they do share one line, it must be in $\mathcal{R}_6^2$, for otherwise the three rank two points on it would be the same for the two quadrilaterals, and this would already force the two baskets to coincide.

We can make now a first estimate of how many of the points p_{ij}^α should be *distinct*. A first quadrilateral brings in six, a second at least three more, and a third at least one more. We find a *minimum* of ten. Thus, at least four of the edges $[q_i, q_j]$ must meet the rank two locus in two distinct points.

This already rules out the possibility of a solvable complete quadrilateral amongst our tetrad. Indeed, we would then have a Veronese conic $v(P_1) \subset P_2 \subset [q_1, ..., q_4] = P_3$, and the restriction of $\mathcal{R}_8^3$ to this P_3 would decompose into the plane $P_2 \subset \mathcal{R}_6^2$ of the conic, counted twice, and a quadric.

This residual quadric must carry more than one rank two point away from the double P_2, otherwise four edges $[q_i, q_j]$ would be concurrent. But two singularities make the quadric a two-plane. And this is not good enough to allow four edges of the q_i tetrahedron to intersect the rank two locus in two points, unless the two-plane is actually a double-plane made of points of rank at most two. But this forces $[q_1, ..., q_4]$ entirely into $\mathcal{R}_8^3$: a contradiction.

Thus, *all four complete quadrilaterals are typical.*

Let us return to the hypothesis that two of them share a line, say $\ell_i^\alpha = \ell_i^\beta = \pi_\alpha \cap \pi_\beta \subset \mathcal{R}_6^2$. The restriction of $\mathcal{R}_8^3$ to $P_2 = [q_j, q_k, q_l]$ then decomposes into the double-line ℓ_i^α and a conic. Since at least one edge in our face requires a second rank two point on it, the conic must be a two-line. We would arrive at a contradiction, as in the previous argument, should we know that two edges require two rank two points.

But now we can review the estimate of a minimum of ten distinct points p_{ij}^α, and see it based on repeated common lines. Hence, either there are at least eleven distinct points p_{ij}^α and we get our contradiction, or we find two lines contained both in $\mathcal{R}_6^2$ and some face $[q_i, q_j, q_l]$, a contradiction again.

This proves that our four complete quadrilaterals cannot have lines in common. With that, the estimate on the cardinality of the p_{ij}^α collection is lifted to the maximum twelve: six from a first quadrilateral, at least four more from a second, and at least two more from a third. Moreover, any two complete quadrilaterals share *exactly two such points* and no three can share the same point.

With the established fact that all six edges $[q_i, q_j]$ carry two rank two points, we conclude as above that no line ℓ_i^α is contained in $\mathcal{R}_6^2$. $\square$

The proof has given at the same time:

Corollary 5.3 *There are exactly twelve distinct points in the set:*

$$\{p_{ij}^{\alpha} \ : \ \{i,j\} \subset \{1,2,3,4\}, \ \alpha = 1,..,4\}$$

with two of them on each edge $[q_i, q_j]$.

There are three of them on each of the sixteen lines ℓ_i^{α}*, and four of these lines are passing through each point. Thus,* $(p_{ij}^{\alpha}, \ell_k^{\beta})$ *defines a* $(12_4, 16_3)$ *configuration.*

Our aim now is to prove that the span of the q_i's is the same as the span of the b^{α}'s, and the same as the span of the d_i^{α}'s, which turn out in fact to be just four distinct rank one points with symmetric role towards the two other tetrads.

In order to simplify statements, we shall refer to the *twelve points of rank two* p_{ij}^{α} as *marked points.*

Lemma 5.4 *For any edge* $[q_i, q_j]$ *and marked rank two point on it, there's some edge* $[b^{\alpha}, b^{\beta}]$ *meeting it at that point.*

Proof: Each basket b^{α} is associated with one of the two marked points on $[q_i, q_j]$, hence we'll find two baskets, say b^{α} and b^{β}, with $p_{kl}^{\alpha} = p_{kl}^{\beta}$.

If the two proper secants of the rank one locus: $[d_i^{\alpha}, d_j^{\alpha}]$ and $[d_i^{\beta}, d_j^{\beta}]$ through this marked point *were distinct*, we would have a Veronese conic $v(P_1)$ in the span $[q_i, q_j, b^{\alpha}, b^{\beta}] = P_3$, and consequently a decomposition of the restriction of $\mathcal{R}_8^3$ to this span into a double-plane $P_2 \supset v(P_1)$ and the cone from the other marked point of $[q_i, q_j]$ over the Veronese conic.

¿From the version of Corollary 5.3 for the $(b^{\alpha})_{\alpha}$ tetrahedron, we must have two rank two points on each edge, and $[b^{\alpha}, b^{\beta}]$, which has one rank two point in $P_2 \supset v(P_1)$, must have the other on $[q_i, q_j]$ at p_{kl}^{γ}. But this is a contradiction, because it reduces the span $[q_i, q_j, b^{\alpha}, b^{\beta}]$ to a plane $P_2 \supset v(P_1)$.

The source of the contradiction was to suppose that the two secants $[d_i^{\alpha}, d_j^{\alpha}]$ and $[d_i^{\beta}, d_j^{\beta}]$ were distinct. Therefore, we must have: $d_i^{\alpha} = d_i^{\beta}$ and $d_j^{\alpha} = d_j^{\beta}$.

Again, by considering the restriction of $\mathcal{R}_8^3$ to the plane $[q_i, q_j, b^{\alpha}, b^{\beta}]$, we find the double-line $[d_i^{\alpha}, d_j^{\alpha}]$ and the two lines from p_{kl}^{γ} to the rank one points. Thus, $[b^{\alpha}, b^{\beta}]$ has no choice but to pass through p_{kl}^{γ} in order to acquire its second rank two point.

It is clear now that no other edge but $[b^{\gamma}, b^{\delta}]$ can (and does) pass through p_{kl}^{α}. At the same time $d_i^{\gamma} = d_i^{\delta}$ and $d_j^{\gamma} = d_j^{\delta}$, with their pencil running through p_{kl}^{γ}. $\square$

The proof has shown more:

Corollary 5.5 *The two tetrahedra* $(q_i)_i$ *and* $(b^{\alpha})_{\alpha}$ *have the same set of twelve rank two points, distributed by two on each system of edges.*

Lemma 5.6 *The collection of rank one points* (d_i^α); $i, \alpha = 1, ..., 4$ *has exactly four distinct points, to be denoted* $d_s, s = 1, ..., 4$. *Each edge* $[d_s, d_t]$ *has two of the above twelve marked points.*

Thus, we have three tetrahedra with the same span:

$$P_3 = [q_1, ..., q_4] = [b^1, ..., b^4] = [d_1, ..., d_4]$$

and each marked point lies on some trio of edges, one from each tetrahedron.

Proof: We have seen in the proof of the previous lemma that the intersection $[q_i, q_j] \cap [b^\alpha, b^\beta] = p_{kl}^\alpha = p_{kl}^\beta$ implies the intersection $[q_i, q_j] \cap [b^\gamma, b^\delta] = p_{kl}^\gamma = p_{kl}^\delta$. At the same time, the rank one points labelled $d_i^\nu, d_j^\nu; \nu = 1, ..., 4$ are just four distinct points.

But we also have the implication $[q_k, q_l] \cap [b^\alpha, b^\beta] = p_{ij}^\alpha = p_{ij}^\beta$, and exactly four distinct points amongst $d_r^\alpha, d_r^\beta; r = 1, ..., 4$.

We want to prove that the above two tetrads of rank one points are one and the same.

Through p_{kl}^α runs the secant $[d_i^\gamma, d_j^\gamma]$ and also the secant $[d_k^\alpha, d_k^\beta]$. Should they be distinct, we would have a Veronese conic $v(P_1)$ in their span, and thereby a plane $v(P_1) \subset P_2 \subset \mathcal{R}_6^2$ in $[q_1, ..., q_4]$. But this is known from previous considerations to lead to a contradiction.

A similar argument works for the other pair. $\square$

In conclusion, we have three tetrahedra in P_3, with edges meeting by three in twelve points, and with faces meeting by three in sixteen lines. This is the Reye configuration.

Returning to the three-space where our quadrics are surfaces, we can first match d_i with x_i^2 by some projective transformation, and then make three of the marked points/quadrics match marked points/quadrics of the standard double-four, by the action of the torus subgroup which preserves the coordinate tetrahedron. Since Reye configurations must match this way, so must our two other tetrahedra match the two other of the standard double-four. By a final switch, if necessary, the theorem is proven. $\square$

Remark: It may be observed, and it will appear with even more emphasis in the next section on tangents and Grassmannians, that what is of the essence in the double-four example is the presence of *one* tetrahedron with *rank four* vertices $(q_i)_i$, and with precisely *two rank two points on every edge* $[q_i, q_j]$. Assuming no face meets the rank two locus in a conic, this leads to $\mathcal{R}_8^3$ restricting on each face $[q_i.q_j, q_k]$ to four lines, with a total of sixteen lines containing the twelve rank two points. One obtains a $(12_4, 16_3)$ configuration.

In closing this section, we illustrate the fact that the *smoothness assumption* in the theorem is important, by presenting a *double-five* example. The example has *all quadrics singular, with a common singularity* and thus actually belongs to the space of *conics* $P_5 = P(Sym_C(3))$. It will be described as such.

A double-five configuration: conics. *One can find two quintets of conics* $(q_i)_i$ *and* $(b_i)_i$, $i = 0, ..., 4$, *such that any line* $[q_i, b_j]$ *meets the rank one locus.*

Construction: Our two quintets will span the same three-space $P_3 \subset P_5 = P(Sym_C(3))$. This three-space should contain a Veronese conic $v(P_1) \subset P_2$ and just one other rank one point.

For specificity, we'll choose the projective subspace P_3 of 3×3 symmetric matrices S defined by: $s_{13} = s_{23} = 0$. Our Veronese conic is then given by:

$$v(P_1) = \{S \ : \ s_{11}s_{22} = s_{12}^2, \ s_{33} = 0\} \subset P_2 = \{S \ : \ s_{33} = 0\} \subset P_3$$

and the only other point of rank one is $S_\infty : \ s_{33} = 1$, rest 0.

We denote it so, because it will lie on the plane at infinity with respect to an *affine piece* we are about to consider. First, we write the Veronese conic as:

$$\frac{1}{2}(s_{11} + s_{22})^2 = s_{12}^2 + \frac{1}{2}(s_{11} - s_{22})^2$$

and then define the *affine piece* by $\frac{1}{2}(s_{11} + s_{22}) = 1$.

In order to be closer to Euclidean intuition, we change coordinates to:

$$x = \frac{1}{\sqrt{2}}(s_{11} - s_{22}), \ y = s_{12}, \ z = s_{33}$$

so that the Veronese conic becomes the unit circle in the plane $z = 0$.

Since we need (and use) only *real points* for our configuration, there should be no confusion here if we express the two real coordinates (x, y) by a complex number. This facilitates indicating our choice of three points on the unit circle as the roots of unity of order three:

$$\omega_k = e^{\frac{2\pi i}{3}k}, \ k = 1, 2, 3; \quad \text{with} \quad \sum_{k=1}^{3} \omega_k = 0$$

Now, we can present our quintets:

$$q_0 = (0, -1), \ q_k = (2(\omega_i + \omega_j), -1), \ k = 1, 2, 3; \ q_4 = (0, \frac{1}{3}) \in C \times R = R^3 \subset P_3$$

$$b_0 = (0,1),\ b_k = (2(\omega_i + \omega_j), 1),\ k = 1,2,3;\ b_4 = (0, -\frac{1}{3}) \in C \times R = R^3 \subset P_3$$

It is elementary to verify that all lines $[q_i, b_j]$ meet the rank one locus. $\square$

6 Tangents and Grassmannians

In this section we look at tangents to quadric surfaces as points in the Grassmann variety of lines in P_3, that is, two dimensional vector subspaces in C^4: $G(2,4) \subset P_5 = P(\wedge^2(C^4))$. This translates questions about common tangents into questions about intersections of quadrics in P_5. In particular, we find that the variety of common tangents for two smooth quadric surfaces in general position is a K3 surface with 16 nodes in $G(2,4)$, more precisely: a *Kummer surface.*

We begin in arbitrary dimension: $P_{n-1} = P(C^n)$. Again, using the standard bilinear form on C^n, we identify quadrics q in P_{n-1} with symmetric matrices $Q \in P(Sym_C(n)) = P_{\binom{n}{2}-1}$.

The Grassmann variety $G(k,n)$ of all $(k-1)$-projective subspaces in P_{n-1}, i.e. of all k-dimensional vector subspaces of C^n, can be realized in the projective space of the k^{th} *exterior power* $P(\wedge^k(C^n)) = P_{\binom{n}{k}-1}$ as all points corresponding to *decomposable k-vectors*, that is:

$$G(k,n) = \{x \in P(\wedge^k(C^n))\ :\ x = \lambda x_1 \wedge ... \wedge x_k \text{ for some independent set } x_i \in C^n\}$$

Obviously, a k-subspace in C^n is represented by $x_1 \wedge ... \wedge x_k$, for any choice of basis $(x_i)_i$, since a change of basis merely introduces a proportionality factor given by the determinant of the transition matrix.

The conditions expressing the fact that an exterior vector, which is, in general, a linear combination of decomposable vectors, has actually a decomposable form are called the *Grassmann-Plücker relations*, and are all quadratic. The above realization is also called the *Plücker embedding* of the Grassmannian $G(k,n)$. [GH]

In general: $dim_C G(k,n) = k(n-k)$; thus $G(2,4) \subset P_5$ is a smooth quadric fourfold.

Definition: Let q be a quadric in P_{n-1}. A $(k-1)$-projective subspace $P_{k-1} \subset P_{n-1}$, is called **tangent** to q (at $x \in q \cap P_{k-1}$), when the restriction of q to P_{k-1} i.e. $q \cap P_{k-1}$ is singular (at x).

With q seen as a symmetric operator Q on C^n, we can define a symmetric operator $\nu_k(Q) = \wedge^k Q$ on $\wedge^k(C^n)$ by:

$$\nu_k(Q)(x_1 \wedge ... \wedge x_k) = Qx_1 \wedge ... \wedge Qx_k$$

In other words, $\nu_k(Q)$ is a quadric $\nu_k(q)$ in $P(\wedge^k(C^n))$.

Lemma 6.1 *A projective subspace $P_{k-1} \subset P_{n-1}$ is tangent to the quadric q in P_{n-1} if and only if the corresponding point of the Grassmannian $G(k,n) \subset P(\wedge^k(C^n))$ lies on the quadric $\nu_k(q)$.*

Proof: The induced standard bilinear form on $\wedge^k(C^n)$ is:

$$< x_1 \wedge ... \wedge x_k, y_1 \wedge ... \wedge y_k >= det(< x_i, y_j >)_{ij}$$

For $(x_i)_i$ a basis in our C^k with $P_{k-1} = P(C^k)$, we have:

$$< x_1 \wedge ... \wedge x_k, \nu_k(Q)(x_1 \wedge ... \wedge x_k) >= det(< x_i, Qx_j >)_{ij}$$

But the matrix $(< x_i, Qx_j >)_{ij}$ is precisely the restriction of q to our P_{k-1}, expressed in the chosen basis. The lemma follows. $\square$

Corollary 6.2 *The variety of $(k-1)$-projective subspaces tangent to a quadric q in P_{n-1} is the quadratic section of the Grassmannian $G(k,n) \subset P(\wedge^k(C^n))$ given by $\nu_k(q)$.* $\square$

Remark: As we shall see in more detail for $\nu = \nu_2$, the map ν_k is a *projection* of the k^{th} Veronese map v_k defined on the space $P(Sym_C(n))$ by the complete linear system of degree k hypersurfaces. In fact, ν_k corresponds to the linear subsystem of all $k \times k$ minors, with base locus made of quadrics of rank less than k.

We now fix $k = 2$ and $n = 4$, and thereby return to quadric surfaces.

We let $e_i,\ i = 1, ..., 4$ denote the standard basis in C^4, and $e_{ij} = e_i \wedge e_j,\ i < j$ the associated standard basis in $\wedge^2(C^4) = C^6$.

The condition for an exterior 2-vector: $x = \sum \alpha_{ij} e_i \wedge e_j$ to have decomposable form reads: $x \wedge x = 0 \in \wedge^4(C^4) = C$, and in our standard basis e_{ij} gives the *quadric*:

$$g = 2(x_{12}x_{34} - x_{13}x_{24} + x_{14}x_{23})$$

and in matrix form G, with $G^2 = I_6$. This is the *Grassmann-Plücker quadric*, with:

$$G(2,4) = \{x \in P_5 = P(\wedge^2(C^4)) \ : \ g(x) =< x, Gx >= 0\}$$

The *rational map*: $\nu = \nu_2 : P_9 = P(Sym_C(4)) \cdot\cdot \to P_{20} = P(Sym_C(6))$ takes a symmetric 4×4 matrix Q to the symmetric 6×6 matrix $\nu(Q)$ with entries made of all 2×2 minors of Q. Clearly, at the projective level ν is only defined away from the rank one locus $R_3^1 \subset P_9$, and one can eliminate the indeterminacy by *blowing-up* this locus.

Since the components of ν are quadratic, it can also be presented as a *quadratic Veronese map* $v_2 : P_9 \to P_{\binom{9+2}{2}-1} = P_{54}$ followed by some projection.

The direction (or center) of this projection shall be the linear span of the image $v_2(R_3^1)$. We may recall that R_3^1 is itself the image of a quadratic Veronese map $v = v_2 : P_3 \to P_9$, hence $v_2(R_3^1) = v_2(v_2(P_3)) = v_4(P_3)$ is the image of the *quartic Veronese map* on P_3, which spans a projective subspace of P_{54} of dimension $\binom{3+4}{3} - 1 = 34$.

Thus, the projection actually takes place on a P_{19}, indicating the fact that the image of ν lies in a hyperplane of $P_{20} = P(Sym_C(6))$. The ensuing set-up is described in:

Proposition 6.3 *There's a commuting diagram of regular and rational maps:*

$$\begin{array}{rcl} P_3 & \stackrel{v_4}{\to} & P_{54} = P(Sym_C(10)) \\ v_2\downarrow & \stackrel{v_2}{\nearrow} & \downarrow\pi \\ P(Sym_C(4)) = P_9 & \stackrel{\nu}{\to} & P_{19} = \{S \ : \ Tr(SG) = 0\} \subset P_{20} = P(Sym_C(6)) \end{array}$$

where G is the Grassmann-Plücker quadric, and π is the projection along the span of $v_4(P_3)$, which is a subspace $P_{34} \subset P_{54}$.

By blowing-up P_9 along the rank one locus $v_2(P_3) = \mathcal{R}_3^1$ to $\tilde{P}_9$, and P_{54} along $P_{34} = span[v_4(P_3)]$ to $\tilde{P}_{54}$, this yields a diagram of regular maps:

$$\begin{array}{ccccccc} E_8 & \subset & \tilde{P}_9 & \stackrel{\tilde{v}_2}{\to} & \tilde{P}_{54} & \stackrel{\pi}{\to} & P_{19} \\ \beta\downarrow & & \downarrow & & \downarrow & & \\ P_3 & \stackrel{v_2}{\to} & P_9 & \stackrel{v_2}{\to} & P_{54} & & \end{array}$$

where E_8 denotes the exceptional divisor over the rank one locus, with a P_5-bundle structure $\beta : E_8 \to P_3$.

Thus, $\tilde{\nu} = \pi \circ \tilde{v}_2 : \tilde{P}_9 \to P_{19}$ produces a lifting of indeterminacies for ν.

The fact that the image of ν lies in the hyperplane of P_{20} defined by $Tr(SG) = 0$ is obvious for diagonal quadrics Q, which have diagonal $S = \nu(Q)$, and follows in general by the action of $SO_C(4)$ which fixes the Grassmann-Plücker quadric G. $\square$

Remark: If we denote by e, H and h, the *divisor classes* defined on $\tilde{P}_9$ by E_8, (the pull-back by $\tilde{\nu}$ of) a hyperplane in P_{19}, respectively (the pull-back of) a hyperplane in P_9, we obtain the relation:

$$H = 2h - e \quad \text{in the Picard group} \quad Pic(\tilde{P}_9) = H^2(\tilde{P}_9, Z)$$

It follows that a pencil in P_9 which meets the rank one locus transversly in a single point, lifts to $\tilde{P}_9$ and meets E_8 in a single point, and then maps by $\tilde{\nu}$ to a *pencil* in P_{19}.

On the other hand, a secant or a tangent to $\mathcal{R}_3^1 \subset P_9$ lifts to a curve which is contracted to a point by $\tilde{\nu}$. $\square$

For a *smooth quadric surface* $q \in P_9$, we have $q = Q_2 \subset P_3$, and we let $T(q) = T(Q_2)$ denote its tangent bundle. There's a natural commutative diagram:

$$\begin{array}{rcr} P(T(q)) & \subset & Q_2 \times G(2,4) \\ c \downarrow & & \downarrow \\ \tau(q) = \nu(q) \cap G(2,4) & \subset & G(2,4) \end{array}$$

where c is a *birational contraction* from the projectivised tangent bundle of Q_2 onto the variety $\tau(q)$ of *lines in* P_3 *tangent to* Q_2. The image of the exceptional locus consists of the two disjoint conics which represent the two rulings of Q_2 in the Grassmannian. $\tau(q)$ is *singular* along these two conics and transversal codimension one sections acquire *nodes* when crossing them.

Thus, one can look upon $\tau(q)$ as a *degenerate quadratic line complex* [Jes].

Now, we can turn our attention to the *variety of common tangents* for two or more quadrics. We begin with a pair of (distinct) quadrics q_1 and q_2, other than double-planes, and define:

$$K(q_1, q_2) = G(2,4) \cap \nu(q_1) \cap \nu(q_2) \subset G(2,4) \subset P_5$$

This is a *complex surface* in the Grassmannian, made of all common tangents for the two quadrics. When considered with its possible multiple structure, it has degree $2^3 = 8$.

We shall explore its structure in some relevant cases. We refer to [BPVdV] and [GH] for background on compact complex surfaces.

Theorem 6.4 *Suppose* (q_1, q_2) *is a pair of smooth quadrics which is* generic *in one of the subvarieties defined by the following conditions:*

(i) *the pencil* $[q_1, q_2]$ *meets the rank one locus;*

(ii) *the pencil* $[q_1, q_2]$ *meets the rank two locus in two points;*

(iii) *the pencil* $[q_1, q_2]$ *meets the rank two locus (in one point);*

(iv) the pencil $[q_1, q_2]$ is generic.

Then, correspondingly, the surface of common tangents $K(q_1, q_2)$ has the following structure:

(i) $P_1 \times P_1$ embedded in P_5 by a complete linear system of type $(1, 2)$, and with multiplicity two;

(ii) two irreducible components, each isomorphic with a nodal complete intersection of two quadrics in P_4; the two components meet along a skew quadrilateral and have nodes precisely at the four vertices of this quadrilateral;

(iii) a surface birational to a P_1-bundle over an elliptic curve;

(iv) a K3 surface with 16 nodes, more precisely: a Kummer surface.

Proof: (i) The two quadrics are one a basket for the other, and the genericity assumption means in particular that they meet along a double conic $2C = q_1 \cap q_2$, with $C \approx P_1$.

We observe first that if $x_1 \in q_1$ is away from the common conic C, then the tangent plane $T_{x_1}(q_1)$ meets q_2 along a smooth conic, and there are exactly two tangents from x_1 to this conic, namely the two lines through x_1 in the two rulings of q_1. Thus, these tangents are accounted for when we consider all common tangents through points of C. The latter make obviously the projectivized tangent bundle of q_1 (or q_2) restricted to C: $P(T(q_1))_{|C} = P(T(q_2))_{|C}$, which is, in fact, a trivial P_1-bundle over $C \approx P_1$.

Thus, at the *reduced level* $K(q_1, q_2)_{red} = P_1 \times P_1$, and clearly one family of P_1's in this product is plunged in $G(2, 4)$ as a family of lines.

To obtain that the embedding is actually of type $(1, 2)$, we may look now from the point of view of a *'basket sweep'* of $K(q_1, q_2)$, namely: we consider the pencil $[q_1, q_2] = P_1$ as a parameter space of common baskets b (and limits thereof), and as we move $b \in [q_1, q_2]$, the two rulings of b (except at the rank three and rank one points of the pencil) 'sweep' $K(q_1, q_2)$ by pairs of conics in $G(2, 4)$. At the singular points of the pencil we have a single rational curve of common tangents.

Thus, $K(q_1, q_2)$ appears as a P_1-bundle over a *double covering* of the 'basket line' $[q_1, q_2]$, *ramified* over the two singular quadrics (i.e. 'degenerate baskets'). By irreducibility, the double covering is itself a rational curve, and we have a P_1-bundle over P_1 presentation of $K(q_1, q_2)$, with the fibers clearly plunged as conics in $G(2, 4)$. The bundle is trivial by the identification of all fibers with the common conic C.

It follows that, with the proper ordering of the factors, the embedding of $P_1 \times P_1$ is of type $(1, 2)$, and this yields degree four. Hence, the surface $K(q_1, q_2)$ is actually the image of this embedding with multiplicity two.

(ii) The fact that $K(q_1, q_2)$ is reducible whenever the pencil $[q_1, q_2]$ has *two* rank two points is a consequence of the fact that the pencil $[\nu(q_1), \nu(q_2)]$ contains a rank two quadric.

Indeed, ν as a projection of a quadratic Veronese map, takes the pencil $[q_1, q_2]$ to a *conic* in $P(Sym_C(6))$. This conic has two rank one points corresponding to the rank two points on $[q_1, q_2]$. The line through these two rank one points meets the pencil $[\nu(q_1), \nu(q_2)]$ in a rank two point.

The intersection $G(2,4) \cap \nu(q_1) \cap \nu(q_2)$ can therefore be presented as an intersection $G(2,4) \cap \nu(q_1) \cap (P_4^+ \cup P_4^-)$. For the generic case in this class, each component $G(2,4) \cap \nu(q_1) \cap P_4^{\pm}$ is singular at the four points defined by the four lines in $q_1 \cap q_2$. Their common part is a skew quadrilateral with edges connecting the four singularities whenever they are not from the same ruling on q_1 or, equivalently, on q_2.

The presence of two components in $K(q_1, q_2)$ is also transparent from the 'basket sweep' approach, since there are two rational curves of common baskets for q_1 and q_2 in this case.

(iii) We can use again a 'basket sweep'. From section 2 and the genericity assumption, we know that there's *a smooth conic of common baskets*, say $B \approx P_1$. It has two rank one points, corresponding to the double-planes supported respectively by each of the two planes of the rank two point in $[q_1, q_2]$. It will have two other points of rank three. These four points on B are the 'degenerate baskets'.

For a proper basket $b \in B$, its two rulings provide two disjoint conics on $K(q_1, q_2)$, while over the four 'degenerate baskets' we'll have a single rational curve. In fact, over the rank one points we have precisely the tangents along the smooth conic component of $q_1 \cap q_2$ which lies in the respective (double)-plane.

Thus, $K(q_1, q_2)$ is *birationally equivalent* to a P_1-bundle over a double covering of B ramified over the four 'degenerate baskets'. The *irreducibility* of this double covering follows from a limit argument with the rank two point moving towards a rank one point and a case (i) situation. (The two rank one points on B then move towards the single rank one point in the limit, and the two rank three points will coalesce into the one rank three point in the limit). The double covering is therefore an *elliptic curve*.

(iv) In this general case, the two quadrics meet along a degree four elliptic curve $E = q_1 \cap q_2$. Also, by the genericity assumption, the curve of pairs $E^* = \{(x_1, x_2) \in q_1 \times q_2 \ : \ T_{x_1}(q_1) = T_{x_2}(q_2)\}$ is an elliptic curve which projects on each factor as a smooth quadratic section $E_i^* \subset q_i$.

We'll elaborate on the role of *duality* in the next section, but we should remark at this point that E^* is simply the intersection of the dual quadrics.

It is convenient now to consider a *modification* $\tilde{K}(q_1, q_2)$ of our surface of common tangents $K(q_1, q_2)$ (which will turn out in fact to be a resolution of singularities), by taking into account the points of tangency:

$$\tilde{K}(q_1, q_2) = \{(x_1, x_2, t) \in q_1 \times q_2 \times K(q_1, q_2) \ : \ x_i \in t \cap q_i, \ i = 1, 2\}$$

The projection $\rho : \tilde{K}(q_1, q_2) \to K(q_1, q_2)$ is clearly an isomorphism away from the points $t \in K(q_1, q_2)$ which are lines in one ruling of one quadric, and tangent somewhere to the other quadric. Their number is easily counted as follows.

Lines in one ruling of, say q_1, define a conic in $G(2, 4)$; those which are tangent to q_2 correspond to the intersection of this conic with the quadric $\nu(q_2)$, and are four in number; all in all, there are *sixteen points* $t \in K(q_1, q_2)$ *which are replaced by* P_1*'s in* $\tilde{K}(q_1, q_2)$.

Now we can look at one of the projections $\rho_i : \tilde{K}(q_1, q_2) \to q_i$. Away from $E \cup E_i^*$, there are two points in a fiber $\rho_i^{-1}(x)$, namely, the two tangents from x to the smooth conic $T_x(q_i) \cap q_j$. Over points in $E \cup E_i^* - E \cap E_i^*$ there will be a single point.

Let us consider finally one of the eight points $E \cap E_i^*$, say y_i. Then the tangent plane $T_{y_i}(q_i)$ must be tangent to q_j at some different point $z_j \in q_j$, with $[y_i, z_j]$ the common tangent. But with two points already in q_j, the whole line $[y_i, z_j]$ must be in q_j. Thus the fiber $\rho_i^{-1}(y_i)$ is a P_1, and coincides with one of the fibers of ρ.

This means that $\tilde{K}(q_1, q_2)$ (via the Stein factorization of ρ_i) is a resolution of the eight nodes of the double covering of q_i ramified over $E \cup E_i^*$. Thus, $\tilde{K}(q_1, q_2)$ is a smooth K3 surface.

Using Nikulin's theorem in [Nik], we may conclude that the surface of common tangents $K(q_1, q_2)$, obtained by contracting sixteen disjoint rational curves on $\tilde{K}(q_1, q_2)$ to nodes, will be a *Kummer surface*. In fact, we need not rely on this result of Nikulin, because we may verify explicitly that the sum of the sixteen exceptional divisors is divisible by two in $Pic(\tilde{K}(q_1, q_2)) = H^2(\tilde{K}(q_1, q_2), Z)$.

Indeed, this follows from a calculation in the Picard lattice of our K3 surface. Let us denote by τ_i the pull-back by ρ_i of the hyperplane class of P_3, by ϵ_k^i; $k = 1, ..., 8$, the classes of the curves contracted by ρ_i, and by γ the class of the elliptic curve given by all tangents to $E = q_1 \cap q_2$.

We've seen above that *together*, the curves contracted by ρ_1 and ρ_2 amount exactly to the sixteen curves contracted by ρ. Besides, we have:

$$2\tau_i = 2\gamma + \sum_{k=1}^{8} \epsilon_k^i \qquad i = 1, 2$$

hence:

$$\sum_{i=1}^{2}\sum_{k=1}^{8}\epsilon_k^i = 2(\tau_1+\tau_2-2\gamma)$$

completing the argument. $\square$

Remark: The divisibility by two condition verified above comes from the representation of *Kummer surfaces as quotients of Abelian surfaces by the involution* $z \mapsto -z$. The sixteen nodes then correspond with the sixteen order two points on the Abelian surface. Proceeding in the other direction, the Abelian surface is obtained from $K(q_1,q_2)$ by considering the double covering of $\tilde{K}(q_1,q_2)$ ramified over the sixteen exceptional curves. On the covering, these rational curves become (-1)-curves and can be contracted to smooth points. The resulting surface is Abelian.

One can establish further relations in the *algebraic lattice* of the K3 surface $\tilde{K}(q_1,q_2)$ obtained in the general case.

Let us denote by σ the pull-back by ρ of the hyperplane class of $G(2,4)$. The degree $\sigma\gamma = 8$ can be found by an application of the Riemann-Hurwitz formula, and then we have:

$$\sigma^2 = 8,\ \gamma^2 = 0,\ (\epsilon_k^i)^2 = -2,\ \sigma\epsilon_k^i = 0,\ \gamma\epsilon_k^i = 1$$

This leads to:

$$2\sigma = 2\gamma + \sum_{i=1}^{2}\sum_{k=1}^{8}\epsilon_k^i$$

$$\sigma = \tau_1+\tau_2-\gamma$$

The elliptic curve γ is part of an *elliptic fibration* of our surface, where we find as another fiber the elliptic curve γ^*, the proper transform of E_i^* by ρ_i, (and one and the same for $i=1,2$). $\square$

Proposition 6.5 *Let* $q_i,\ i=1,2,3$ *be three distinct quadrics in a* generic pencil $\ell \subset P_9 = P(Sym_C(4))$. *Then their curve of common tangents:*

$$C(q_1,q_2,q_3) = G(2,4)\cap\nu(q_1)\cap\nu(q_2)\cap\nu(q_3)$$

is given by the elliptic curve γ*, made of tangents to the common intersection* $E = q_i\cap q_j,\ \{i,j\}\subset\{1,2,3\}$*, and counted with multiplicity two.*

Proof: The image of our pencil by ν is a conic $\nu(\ell) \subset P(Sym_C(6))$ and any three distinct points on it span its plane.

Thus, regarding q_1 and q_2, we may consider ourselves in the generic case (iv) of the preceding Theorem, and move q_3 as we please along the rest of the pencil. Clearly γ, the curve of tangents to the elliptic curve $E = q_1 \cap q_2$ is always part of the intersection $K(q_1, q_2) \cap \nu(q_3)$. But any other common tangent to q_1 and q_2 can be avoided as a tangent by some choice of $q_3 \in [q_1, q_2] = \ell$.

Thus, at the *reduced level* $C(q_1, q_2, q_3)_{red} = \gamma$. But $C(q_1, q_2, q_3)$ has degree $2^4 = 16$, and therefore γ, with $deg(\gamma) = \sigma\gamma = 8$, has to be taken with multiplicity two. $\square$

Corollary 6.6 *Any four distinct quadrics in a pencil $\ell \subset P_9 - \mathcal{R}_6^2$ have a continuum of common tangents. In the generic case, the reduced locus is the elliptic curve γ.*

Before we close this section, we may have a second look at the *double-four* example in section 5. For either tetrad $(q_i)_i$, or $(b_i)_i$, we have a degree sixteen curve of common tangents, made of *eight conic components*, two for each common basket. We can see an alternative reason for this abundant splitting in the fact that each pencil $[q_i, q_j]$, respectively $[b_i, b_j]$, meets the rank two locus in two points.

7 Duality

In this section we make explicit the *role of duality*. For the general notion we refer to [GKZ], but here we need it only in the case of quadrics.

As in the previous section, we begin in arbitrary dimension $P_{n-1} = P(V_n)$, where V_n is a complex vector space of dimension n.

The *dual projective space* P^*_{n-1} is the space of all *hyperplanes* in P_{n-1}, that is: $P^*_{n-1} = P(V^*_n) = G(n-1, V_n)$, where V^*_n is the dual vector space of V_n, and the Grassmannian notation serves a context where no specific basis has been given. We have a *canonical isomorphism*, also called *'orthogonality' isomorphism*:

$$G(k, V_n) = G(n-k, V^*_n), \quad \text{taking } V_k \subset V_n \text{ to } V_k^{\perp} = V^*_{n-k} \subset V^*_n$$

where $V_k^{\perp} = V^*_{n-k}$ stands for the $(n-k)$-subspace of V^*_n consisting of all linear functionals vanishing on V_k.

In other words, any linear subspace $P_{k-1} \subset P_{n-1}$ has its dual counterpart $P^*_{n-k-1} \subset P^*_{n-1}$: all hyperplanes containing P_{k-1}.

In this 'base-free' context, we have self-dual linear operators: $Q : V_n \to V_n^*$, $Q = Q^*$, rather than symmetric $n \times n$ matrices, and the correspondence with quadrics is given by:

$$Q \mapsto q = \{x \in P(V_n) \ : < x, Qx >= 0\}$$

where $<, >$ is the duality pairing of V_n and V_n^*.

Definition: The **dual** of a *smooth* quadric $q \subset P_{n-1}$ is the subvariety $q^* \subset P_{n-1}^*$ consisting of all hyperplanes tangent to q.

Lemma 7.1 *q^* is the quadric of $P_{n-1}^* = P(V_n^*)$ corresponding to the self-dual operator $Q^{-1} : V_n^* \to V_n$.*

Proof: The hyperplane tangent to q at $x \in q$ is $x^* = Qx$, and it satisfies the equation $< Q^{-1}x^*, x^* >=< x, Qx >= 0$. $\square$

If we denote by $Sym(V_n)$ the self-dual operators from V_n to its dual V_n^*, we obtain a *duality transformation*:

$$P_{\binom{n+1}{2}-1} = P(Sym(V_n)) \cdot \cdot \xrightarrow{*} P(Sym(V_n^*))$$

as a *rational map* from the space of quadrics in $P_{n-1} = P(V_n)$ to the space of quadrics in $P_{n-1}^* = P(V_n^*)$. Clearly $(q^*)^* = q$, so that duality is a *birational equivalence*, with inverse given by duality applied on the target space.

The relevance of duality for our concerns comes from the following, nearly tautological fact:

Proposition 7.2 *Let q be a smooth quadric in P_{n-1}, with dual $q^* \subset P_{n-1}^*$. A projective $(k-1)$-subspace $P_{k-1} \subset P_{n-1}$ is tangent to q if and only if its 'orthogonal' $P_{k-1}^{\perp} = P_{n-k-1}^* \subset P_{n-1}^*$ is tangent to q^*.*

Proof; P_{k-1} is tangent to q if and only if there's an $x \in P_{k-1} \cap q$ with $P_{k-1} \subset T_x(q) = Qx$. The latter condition reads: $y = Qx \in P_{k-1}^{\perp} \cap q^*$. $\square$

Corollary 7.3 *Let $(q_i)_i$ be a collection of smooth quadrics in $P_{n-1} = P(V_n)$. The subvariety of $G(k, V_n)$ consisting of their common tangent $(k-1)$-planes is naturally identified, via the 'orthogonality' isomorphism, with the subvariety of $G(n-k, V_n^*)$ consisting of common tangent $(n-k-1)$-planes for the collection of dual quadrics $(q_i^*)_i$.*

It will be convenient to have a *lifting of indeterminacies* for the birational equivalence determined by duality on quadrics. At this point, we *choose a basis* in V_n, and use the standard bilinear form on C^n for the identification: $V_n = V_n^* = C^n$, and its consequent identifications: $P_{n-1} = P(V_n) = P(V_n^*) = P_{n-1}^*$, and

$P(Sym(V_n)) = P(Sym(V_n^*)) = P(Sym_C(n)) = P_N$, where $N = \binom{n+1}{2} - 1$. Thus, the *duality transformation* becomes a *birational involution* $D : P_N \cdot\cdot\to P_N$, with $D(Q) = Q^{-1}$ on smooth quadrics.

Proposition 7.4 *Let $\mathcal{I}_N \subset P(Sym_C(n)) \times P(Sym_C(n)) = (P_N)^2$ be the projective subvariety defined by:*

$$\mathcal{I}_N = \{(A, B) \ : \ AB = \frac{1}{n} Tr(AB) \cdot I_n\}$$

where I_n denotes the identity $n \times n$ matrix.

Then, the two projections $\pi_i : \mathcal{I}_N \to P_N$ are birational morphisms, and the duality transformation D lifts to the involution *of $\mathcal{I}_N$ given by $D(A, B) = (B, A)$.*

Proof: Clearly, for $(A, B) \in \mathcal{I}_N$ with $Tr(AB) \neq 0$, we must have $B = A^{-1}$, and the closure of the graph of D lies in $\mathcal{I}_N$.

When $Tr(AB) = 0$, we must have $AB = BA = 0$, or equivalently: $Im(B) \subset Ker(A)$, or $Im(A) \subset Ker(B)$. Thus, the fiber $\pi_1^{-1}(A)$ can be identified with quadrics in $P(C^n/Im(A))$, and similarly for the other projection.

It follows that, $\mathcal{I}_N$ can be presented as a union of a dense open set $U_N = \{(Q, Q^{-1}) \ : \ Q$ of rank $n\}$, and closed subvarieties $\mathcal{R}^{i,j}, \ i + j \leq n$:

$$\mathcal{I}_N = U_N \cup \bigcup_{i+j \leq n} \mathcal{R}^{i,j}$$

where:

$$\mathcal{R}^{i,j} = \{(A, B) \in \mathcal{I}_N \ : Im(A) \subset Ker(B), \ rk(A) \leq i, \ rk(B) \leq j\}$$

The lifting of indeterminacies on $\mathcal{I}_N$ is now plain, with the additional relation: $D(\mathcal{R}^{i,j}) = \mathcal{R}^{j,i}$. $\square$

Remark: One can verify that the singularity locus of $\mathcal{I}_N$ is: $Sing(\mathcal{I}_N) = \bigcup_{i+j \leq n-3} \mathcal{R}^{i,j}$. In particular, for $n = 4$, $\mathcal{I}_9$ is smooth.

We now fix $n = 4$, and return to the specifics of duality for quadric surfaces.

Another expression of Proposition 7.2 for tangents to quadrics in P_3 is:

Proposition 7.5 *Let $Q \in P(Sym_C(4))$ be a rank four quadric in P_3. Then:*

$$\nu(Q^{-1}) = det(Q) \cdot G\nu(Q)G$$

where, as in section 6, G is the 6×6 matrix corresponding to the Grassmann-Plücker quadric $G(2, 4) \subset P_5$ in the standard basis.

Proof: With $\wedge^4(C^4) = C$, we have:

$$< x_1 \wedge x_2, G(y_1 \wedge y_2) >= x_1 \wedge x_2 \wedge y_1 \wedge y_2$$

$$x_1 \wedge x_2 \wedge Q^{-1}y_1 \wedge Q^{-1}y_2 = det(Q)y_1 \wedge y_2 \wedge Qx_1 \wedge Qx_2$$

and the statement is the rendering of the last equation in the standard basis $e_{ij} = e_i \wedge e_j,\ i < j$. $\square$

Remark: Conjugation with G is *not* part of the $PSL_C(4)$ action on quadrics in P_5, just as the orthogonality isomorphism $\perp$: $G(2,4) \to G(2,4)$ is *not* part of the action of that group on $G(2,4)$.

It may be observed that duality transforms a *generic pencil* in P_9 into a *rational normal cubic* which meets the rank one locus in four points. Such relations reflect relations in the cohomology of $\mathcal{I}_9$.

Proposition 7.6 *Let H_i denote the pull-back of the hyperplane class by the modification morphism $\pi_i : \mathcal{I}_9 \to P_9$. Then:*

$$4H_1 = \mathcal{R}^{3,1} + 2\mathcal{R}^{2,2} + 3\mathcal{R}^{1,3}$$

$$4H_2 = \mathcal{R}^{1,3} + 2\mathcal{R}^{2,2} + 3\mathcal{R}^{3,1}$$

with the consequence:

$$H_1 + H_2 = \mathcal{R}^{3,1} + \mathcal{R}^{2,2} + \mathcal{R}^{1,3}$$

In particular, pencils which contain smooth quadrics and meet the rank one locus will dualize to likewise pencils, an expression of the *duality invariance of the 'basket property'* relating two smooth quadrics. $\square$

8 Common tangents to four spheres in R^3

In this section we interpret our results on common baskets for the particular case of the family $P_4^{(s)} \subset P_9 = P(Sym_C(4))$ which contains all quadrics whose *real points* are *spheres in* R^3. Then, considering only *real tangents* to spheres, we determine all *degenerate configurations of four spheres in* R^3*, that is: configurations with infinitely many common tangents.*

The generic case of configurations with finitely many common tangents, has been studied in [MPT] and [ST]. The effective upper bound 12 is in fact the complex count, which we review in the sequel.

The affine equation of a sphere in R^3 is:

$$\sum_{i=1}^{3}(x_i - x_i^0)^2 = r^2$$

with $c = (x_1^0, x_2^0, x_3^0) \in R^3$ the center, and $r = |r| > 0$ the radius.

This gives in P_3 the quadric:

$$Q = \begin{pmatrix} a_0 & 0 & 0 & a_1 \\ 0 & a_0 & 0 & a_2 \\ 0 & 0 & a_0 & a_3 \\ a_1 & a_2 & a_3 & a_4 \end{pmatrix}$$

with $c = -\frac{1}{a_0}(a_1, a_2, a_3)$ and $r^2 = \frac{1}{a_0^2}(a_1^2 + a_2^2 + a_3^2 - a_0 a_4), \quad a_0 \neq 0.$

We are thus led to the *complex projective subspace*: $P_4^{(s)} \subset P_9 = P(Sym_C(4))$ consisting of all quadrics of the above form Q, with $a = (a_0 : ...a_4) \in P_4$. For $a_0 \neq 0$, we shall continue to designate the expressions given for c and r^2 as the "center and squared radius" of Q.

Using $(x_1 : x_2 : x_3 : x_4)$ as homogeneous coordinates in P_3, the family $P_4^{(s)}$ can also be described as the family of all quadrics in P_3 passing through the "imaginary conic at infinity":

$$x_4 = 0, \quad < x, x >= x_1^2 + x_2^2 + x_3^2 = 0$$

Lemma 8.1 *The subgroup of transformations in $PGL_R(4)$ preserving the family $P_4^{(s)}$ and its real structure consists of all similarities of $R^3 = \{x = (x_1 : x_2 : x_3 : 1) \in P_3\}$ with respect to the above inner product $< \, , \, >$, that is: compositions of isometries and rescalings. Its complexification consists of all transformations in $PGL_C(4)$ which take the "imaginary conic at infinity" to itself.* $\square$

Lemma 8.2 *The formula: $det(Q) = -a_0^2(a_1^2 + a_2^2 + a_3^2 - a_0 a_4)$ shows that the locus $P_4^{(s)} \cap \mathcal{R}_8^3$ decomposes into a quadric and the double-hyperplane $a_0^2 = 0$.*

In P_9, the three-space $a_0 = 0$ is tangent to the rank one locus at the point given by $a = (0 : 0 : 0 : 0 : 1)$, which is the only rank one point in $P_4^{(s)}$. All rank two points in $P_4^{(s)}$ are on $a_0 = 0$. $\square$

Proposition 8.3 *Let $q_i, \ i = 1, 2, 3$ be three distinct quadrics of rank at least three in $P_4^{(s)}$. Suppose there's a common basket for $q_i, \ i = 1, 2, 3$. Then:*
(i) the span $[q_1, q_2, q_3]$ contains the rank one point $T = (0 : 0 : 0 : 0 : 1)$;
or, equivalently:
(ii) the centers c_i of $q_i, \ i = 1, 2, 3$ are collinear.

A generic triple satisfying these conditions has a common basket.

Proof: We have to interpret Proposition 3.3 for $P_4^{(s)}$.

Considering that the rank two locus in $P_4^{(s)}$ is $a_0 = 0$, our triple must satisfy the generic condition in (C_{19}^3) for a tangent, or condition (H_{14}^3). Both imply (i).

The equivalence of (i) and (ii) is a simple computation, and the generic converse follows from section 3. $\square$

Proposition 8.4 *Let* $q_i,\ i = 1, 2, 3, 4$ *be four distinct quadrics of rank at least three in* $P_4^{(s)}$. *Suppose there's a common basket for* $q_i,\ i = 1, 2, 3, 4$.

Then, the four quadrics q_i *lie on a conic tangent to the rank two locus* $a_0 = 0$ *at the rank one point* $T = (0 : 0 : 0 : 0 : 1) \in P_4^{(s)}$, *and the four centers are collinear.*

A generic quadruple (q_i) *satisfying this property has a common basket.*

Proof: We have to interpret Proposition 4.6 for $P_4^{(s)}$.

We see that we must be in case (G_{20}^4) for a tangent, or (J_{15}^4). This yields the condition, with the conic degenerating to a double line through T in case (J_{15}^4). The fact that the four centers must be collinear is obvious from the previous result on triples.

The generic converse is covered by constructions in section 4. $\square$

Remark: For the generic case above, we have $[q_1, ..., q_4] = P_2$. Using the triangle T, q_1, q_2 as simplex of reference in this plane, and with:

$$q_3 = \alpha_0 T + \alpha_1 q_1 + \alpha_2 q_2 \quad \text{and} \quad q_4 = \beta_0 T + \beta_1 q_1 + \beta_2 q_2$$

the existence of the conic amounts to:

$$\alpha_0\left(\frac{1}{\alpha_1} + \frac{1}{\alpha_2}\right) = \beta_0\left(\frac{1}{\beta_1} + \frac{1}{\beta_2}\right)$$

We turn now to the problem of understanding the variety of common tangents to four spheres in R^3.

At the *complex projective level*, the corresponding four complex quadrics in P_3 have a common curve "at infinity" i.e. in $x_4 = 0$, namely the "imaginary conic": $< x, x >= 0$. Tangents to this conic are common tangents and define a conic in the Grassmannian $G(2, 4)$. Thus, what has to be identified is the remaining part of the variety of common (complex) tangents.

At the *real level*, we have to consider *only the real points of this residual complex piece*, because there's no real tangent at infinity.

Thus, one is led to coordinates particularly adapted to lines in the affine part $R^3 \subset P_3(R)$, respectively $C^3 \subset P_3$.

A line ℓ in R^3 is completely characterized by the pair $(p, v) \in R^3 \times P(R^3)$, where p is the orthogonal projection of the origin in R^3 on the given line, and v the projective point determined by a direction vector along the line. (One may represent $P(R^3)$ as the plane at infinity for R^3, and then v is simply the point where the (completed) line meets infinity.) Clearly:

$$< p, v >= \sum_{k=1}^{3} p_k v_k = 0$$

Over C, the same description works generically. The resulting relation with $G(2,4)$ is expressed in:

Proposition 8.5 *There's a natural birational equivalence:*

$$G(2,4) \approx I_4 \subset P_3 \times P_2, \quad \ell \mapsto (p, v)$$

,
where I_4 is the P_2-bundle over P_2 defined by:

$$I_4 = \{(p, v) \in P_3 \times P_2 \mid \ < p, v >= \sum_{k=1}^{3} p_k v_k = 0\}$$

Let Γ_4 denote the closed graph of this birational map:

$$\Gamma_4 = \{(\ell, p, v) \in G(2,4) \times P_3 \times P_2 \mid p \in \ell, v \in \ell, < p, v >= 0\}$$

(Here $v \in \ell$ is to be understood via the identification of directions with points at infinity: $P_3 = C^3 \cup P_2$.)

The projection $\Gamma_4 \to G(2,4)$ is a modification over lines at infinity (i.e. in $x_4 = 0$) and lines through the origin of $C^3 \subset P_3$ with null direction (i.e. $< v, v >= 0$).

The projection $\Gamma_4 \to I_4$ is a blow-up of the rational curve $\{(p, v) \in I_4 \mid p_4 = 0,\ p = v$ as points at infinity $\}$.

Remark: The fibers of $\Gamma_4 \to G(2,4)$ over tangents to the imaginary conic at infinity are unions of two rational curves with a common point, while elsewhere one-dimensinal fibers are rational curves. This eventually relates to the contribution of this conic in $G(2,4)$ in counting the isolated common tangents to four spheres by other techniques (cf. [Ful]).

For our approach, the relevant fact in the above set-up is that the composition $G(2,4) \cdot\cdot \to I_4 \to P_2$ is induced by a *linear projection* $P_5 \cdot\cdot \to P_2$, and lifting to Γ_4 resolves the indeterminacies of the map to I_4, and hence to P_2 as well. $\square$

We consider now four real quadrics of rank at least three, and belonging to the family $P_4^{(s)}$. 'Centers' and 'squared radii' maintain a formal sense and, after a translation, we may assume the centers are at $0, c_1, c_2, c_3 \in R^3$, with corresponding squared radii r^2, r_1^2, r_2^2, r_3^2.

A way to set aside the component given by tangents 'at infinity', is to write the equations for common tangents in (p, v) coordinates, with $p \in C^3$. As in [ST], the equations are:

$$< p, v >= 0$$

$$< p, p >= r^2$$

$$< c_i, p >= \frac{1}{2 < v, v >}[- < c_i, v >^2 + < v, v > (< c_i, c_i > + < p, p > -r_i^2)]$$

Proposition 8.6 *Suppose the four centers are affinely independent (i.e. the real span of c_i, $i = 1, 2, 3$ is R^3). Then, counting multiplicity, there are twelve complex common tangents 'away from infinity' for the four quadrics.*

Proof: With centers understood as column vectors, we put $M = [c_1\ c_2\ c_3]^t$. It is a *real invertible matrix* and the last three equations take the form:

$$Mp = \frac{1}{2 < v, v >}[\Phi_2(v)+ < v, v > \Phi_0]$$

where $\Phi_2(v)$ is the column vector with entries $- < c_i, v >^2$, and Φ_0 is the column vector with entries $< c_i, c_i > +r^2 - r_i^2$.

Thus $v \in P_2$ determines p, and must satisfy:

$$< M^{-1}(\Phi_2(v)+ < v, v > \Phi_0), v >= 0$$

$$< M^{-1}(\Phi_2(v)+ < v, v > \Phi_0), M^{-1}(\Phi_2(v)+ < v, v > \Phi_0) >= 4r^2 < v, v >^2$$

We prove that there can be no one-dimensional component in the intersection of the above cubic and quartic curves by showing that the further intersection with the conic $< v, v >= 0$ is empty. Indeed, the equations yield the system:

$$< v, v >= 0$$

$$< M^{-1}\Phi_2(v), v >= 0$$

$$< M^{-1}\Phi_2(v), M^{-1}\Phi_2(v) >= 0$$

The first two equations say that $M^{-1}\Phi_2(v)$ is on the tangent at v to the smooth conic $< v, v >= 0$, and the last that $M^{-1}\Phi_2(v)$ is itself on the same conic. This means:

$$M^{-1}\Phi_2(v) = \mu v \quad \text{that is:} \quad \Phi_2(v) = \mu M v$$

But this gives:

$$(< c_1, v >^2 : < c_2, v >^2 : < c_3, v >^2) = (< c_1, v > : < c_2, v > : < c_3, v >)$$

which has only *real solutions*, namely: $(1:1:1)$ or $(1:1:0)$ or $(1:0:0)$, up to permutation. In all cases, with M real, the resulting v is *real* and we cannot have $< v, v >= 0$.

The cubic and quartic curves have therefore zero-dimensional intersection, that is, counting multiplicity, they meet in twelve points. The twelve solutions determine twelve common tangents 'away from infinity'. □

Corollary 8.7 *Four spheres in R^3 with affinely independent centers have at most twelve common real tangents.*

Remark: Configurations of four spheres with twelve distinct real common tangents are constructed in [MPT]. See also [ST].

The next case to consider is when *the four centers are coplanar but no three of them are collinear.* It requires more detailed computations for ruling out the possibility of infinitely many common tangents in the real case.

Proposition 8.8 *Four quadrics of rank at least three from $P_4^{(s)}(R)$, with coplanar centers but no three of them collinear, have only isolated common tangents 'away from infinity'. Their number is at most twelve.*

Proof: By Lemma 8.1, we may assume that the centers span $x_3 = 0$. Thus:

$$M = \begin{pmatrix} c_{11} & c_{12} & 0 \\ c_{21} & c_{22} & 0 \\ c_{31} & c_{32} & 0 \end{pmatrix}$$

We let M_{12} stand for the 2×2 upper left corner.

Eliminating p from the equations yields in this case *a sextic and a conic* in $v \in P_2$, and our aim is to show that their intersection has to be zero-dimensional. The *conic* E_2 is obtained by using a non-zero vector k in the kernel of M^t:

$$\sum_{i=1}^{3} k_i c_i = 0, \qquad k \neq 0$$

$$0 = 2 < v,v >< Mp,k >=< \Phi_2(v),k > + < v,v >< \Phi_0,k >$$

The sextic is obtained by writing $p = p_{12} + p_3 e_3$, with $p_{12} \perp e_3$. Then, with similar notations for $v = (v_{12}, v_3)$, $\Phi_2(v)$ etc. :

$$p_3 = -\frac{< p_{12}, v_{12} >}{v_3} \quad \text{from} \quad < p,v >= 0$$

$$p_{12} = \frac{1}{2 < v,v >} M_{12}^{-1} [\Phi_2(v)_{12} + < v,v > \Phi_{0,12}]$$

With $\Psi_2(v) = \Psi_2(v_{12}) = M_{12}^{-1}\Phi_2(v)_{12}$, $\Psi_0 = M_{12}^{-1}\Phi_{0,12}$, and $||x||^2 =< x,x >\in C$ this gives the *sextic* E_6:

$$v_3^2||\Psi_2(v) + < v,v > \Psi_0||^2 + < \Psi_2(v) + < v,v > \Psi_0, v_{12} >^2 -4 < v,v >^2 v_3^2 r^2 = 0$$

Considering that our centers $c_i,\ i = 1,2,3$ are in the plane of the first two coordinates, we shall envisage them as two-dimensional vectors when this simplifies formulae. Thus, from:

$$\frac{1}{k_3} k_{12} = -(M_{12}^t)^{-1} c_3$$

we obtain an equivalent expression for the conic E_2:

$$< \Psi_2(v), c_3 > + < c_3, v_{12} >^2 - < v,v >< \Phi_0, k >= 0$$

¿From here on, our proof relies on various computational consequences of the above equations for the sextic E_6 and conic E_2, which we present in a sequence of lemmas.

Lemma 8.9 *E_6, E_2 and $< v,v >= 0$ have no common solution $v \in P_2$, unless $v_{12} = c_i^{\perp} = (c_{i2} : -ci1)$ as points in P_1, for some $i \in \{1,2,3\}$, and $< c_i^{\perp}, c_j - c_k >= 0$. (The last condition means that the four centers are the vertices of a trapeze.)*

Proof: With $< v,v >= 0$, E_6 and E_2 become equations in $v_{12} \in P_1$:

$$||v_{12}||^2||\Psi_2(v_{12})||^2 - < \Psi_2(v_{12}), v_{12} >^2 = 0 \qquad (E_6^{12})$$

$$< \Psi_2(v_{12}), c_3 > + < c_3, v_{12} >^2 = 0 \qquad (E_2^{12})$$

The first equation requires:

$$\Psi_2(v_{12}) = \lambda v_{12} \qquad i.e. \qquad \Phi_2(v)_{12} = \lambda M_{12} v_{12}$$

which gives with E_2^{12}:

$$< c_i, v >^2 = -\lambda < c_i, v > \quad \text{for} \ \ i = 1, 2, 3$$

Any two centers being independent, v_{12} must be orthogonal to some c_i and then $< c_i^{\perp}, c_j - c_k >= 0$. $\square$

Lemma 8.10 *For E_2 and E_6 to have a common one-dimensional component, it is necessary that E_2^{12} and E_6^{12} have two common solutions.*

Proof: Suppose there's a single solution $v_{12} = c_i^{\perp}$. By relabelling, if necessary, we may assume $i = 3$. Then the common component of E_6 and E_2 must be the line in P_2 through $(c_3^{\perp} : \pm i < c_3, c_3 >^{1/2})$, or, for $< c_3, c_3 >= 0$, the tangent $< c_3, v >= 0$ to $< v, v >= 0$. In either case it's the line through $(c_3^{\perp} : 0)$ and $(0 : 0 : 1)$.

This means that, when we rewrite E_6 as an equation in v_3 (actually v_3^2), with coefficients depending on v_{12}, all these coefficients must vanish identically for $v_{12} = c_3^{\perp}$. In other words, we put $< v, v >=< v_{12}, v_{12} > +v_3^2$ in E_6 and obtain a cubic in v_3^2. The vanishing of its four coefficients for $v_{12} = c_3^{\perp}$ yields:

$$||\Psi_0||^2 = 4r^2$$

$$2 < \Psi_2(c_3^{\perp}), \Psi_0 > + < \Psi_0, c_3^{\perp} >^2 = 0$$

$$||\Psi_2(c_3^{\perp})||^2 + 2||c_3||^2 < \Psi_2(c_3^{\perp}), \Psi_0 >= 0$$

$$< \Psi_2(c_3^{\perp}), c_3^{\perp} > +||c_3||^2 < \Psi_0, c_3^{\perp} >= 0$$

¿From the previous lemma we know that:

$$\Psi_2(c_3^{\perp}) = \mu c_3^{\perp} \quad \text{with} \quad \mu = - < c_1, c_3^{\perp} >= - < c_2, c_3^{\perp} >$$

and the last three equations become:

$$< \Psi_0, c_3^{\perp} > (< \Psi_0, c_3^{\perp} > +2\mu) = 0$$

$$||c_3||^2(\mu + 2 < \Psi_0, c_3^{\perp} >) = 0$$

$$||c_3||^2(< \Psi_0, c_3^{\perp} > +\mu) = 0$$

With real quadrics, we have $||c_3||^2 \neq 0$, and the last two equations already provide a contradiction, since $\mu \neq 0$. $\square$

Again, by relabelling the centers, we may assume that the two common solutions of E_2^{12} and E_6^{12} are $v_{12} = c_1^\perp$ and $v_{12} = c_2^\perp$. Thus, $c_1 + c_2 = c_3$ i.e. the centers form a parallelogram.

This suggests using a *translation* which brings the origin at the center of the parallelogram. We assume therefore that the four centers are now at $a = (a_1, a_2, 0)^t, b = (b_1, b_2, 0)^t, -a$, and $-b$, with squared radii $r_i^2,\ i = 1, ..., 4$.

The original system becomes:

$$< p, v >= 0$$

$$< a, p >= \frac{1}{2 < v, v >}[- < a, v >^2 + < v, v > (< a, a > + < p, p > -r_1^2)]$$

$$- < a, p >= \frac{1}{2 < v, v >}[- < a, v >^2 + < v, v > (< a, a > + < p, p > -r_3^2)]$$

$$< b, p >= \frac{1}{2 < v, v >}[- < b, v >^2 + < v, v > (< b, b > + < p, p > -r_2^2)]$$

$$- < b, p >= \frac{1}{2 < v, v >}[- < b, v >^2 + < v, v > (< b, b > + < p, p > -r_4^2)]$$

Subtraction in the last two pairs of equations gives:

$$2 < a, p >= r_3^2 - r_1^2$$

$$2 < b, p >= r_4^2 - r_2^2$$

This shows that the first two coordinates p_{12} of p are determined by centers and squared radii alone, and remain constant. But this means that *all common tangents to our four real quadrics meet the perpendicular drawn from p_{12} to the plane of the centers.*

Remark: A theorem in [MS] already addresses a situation of this nature, and shows that the common tangents to three spheres which meet at the same time a fixed line cannot be infinitely many *unless the three spheres have collinear* *centers* (and the fixed line adequate position). However, it is not necessary to rely on this result in order to prove our proposition, as we show next.

Completing to an equivalent system, we have:

$$p = p_{12} - \frac{< p_{12}, v_{12} >}{v_3} e_3$$

$$< a, v >^2 = < v, v > [< a, a > + < p, p > - \frac{1}{2}(r_3^2 + r_1^2)]$$

$$< b, v >^2 = < v, v > [< b, b > + < p, p > - \frac{1}{2}(r_4^2 + r_2^2)]$$

With: $\alpha = < a, a > - \frac{1}{2}(r_3^2 + r_1^2)$ and $\beta = < b, b > - \frac{1}{2}(r_4^2 + r_2^2)$, we obtain:

$$\frac{< a, v >^2}{< b, v >^2} = \frac{\alpha + < p, p >}{\beta + < p, p >}$$

For $v \in P_2$ the system amounts now to intersecting a conic and a quartic:

$$< a + b, v >< a - b, v > = < v, v > [< a + b, a - b > + \frac{1}{2}(r_4^2 + r_2^2 - r_3^2 - r_1^2)]$$

$$\frac{< a, v_{12} >^2}{< b, v_{12} >^2} = \frac{(\alpha + < p_{12}, p_{12} >)v_3^2 + < p_{12}, v_{12} >^2}{(\beta + < p_{12}, p_{12} >)v_3^2 + < p_{12}, v_{12} >^2}$$

With: $A = \alpha + < p_{12}, p_{12} >$, $B = \beta + < p_{12}, p_{12} >$ and $C = A - B = \alpha - \beta$, the equations say:

$$v_3^2 = \frac{1}{C} < a + b, v >< a - b, v > - < v_{12}, v_{12} >$$

$$v_3^2 = \frac{< p_{12}, v_{12} >^2 < a + b, v_{12} >< a - b, v_{12} >}{A < b, v_{12} >^2 - B < a, v_{12} >^2}$$

Thus, for the conic and quartic to have a common one-dimensional component it is necessary that:

$$< a + b, v_{12} >< a - b, v_{12} > [C < p_{12}, v_{12} >^2 + B < a, v_{12} >^2 - A < b, v_{12} >^2] =$$

$$= -C < v_{12}, v_{12} > [A < b, v_{12} >^2 - B < a, v_{12} >^2]$$

identically in $v_{12} \in P_1$.

Now, evaluating at $v_{12} = (a + b)^{\perp}$ and $(a - b)^{\perp}$, we find: $A = B = C = 0$

Returning these conditions into the system gives:

$$< a + b, v_{12} >< a - b, v_{12} >= 0$$

$$< a, v_{12} >^2 =< b, v_{12} >^2 = (< v_{12}, v_{12} > + v_3^2) p_3^2$$

The first equation requires: $v_{12} = (a + b)^{\perp}$ or $(a - b)^{\perp}$, and then the second determines v_3, since we cannot have $p_3 = 0$ with a and b linealy independent. Thus, there's no one-dimensional family of solutions. □

Corollary 8.11 *Four spheres with coplanar centers but no three of them collinear have at most twelve common real tangents.*

Finally, when *three of the centers are collinear*, we have rotational symmetry around this axis for the common tangents to the corresponding three quadrics. Thus, either (i) the three quadrics have a common conic in the affine part C^3, or (ii) the three quadrics have a common basket (and only one by Proposition 8.3 and Lemma 3.2).

Accordingly, the fourth quadric cannot have a curve of common tangents with the other three in the affine part unless it passes through the same common conic, in case (i), or has the same common basket in case (ii).

Both cases require the four centers to be collinear, and, restricting to the case of spheres and real tangents, we obtain the result described in the introduction:

Theorem 8.12 *Four distinct spheres in R^3 have infinitely many common real tangents if and only if they have collinear centers and at least one common real tangent.*

This means that either all four spheres intersect in a circle, possibly degenerating to a common tangency point, or each sphere has a curve of tangency with one and the same real quadric of revolution with symmetry axis determined by the line passing through all centers. This quadric can be a cone, a cylinder, or a one-sheeted hyperboloid.

Remark: Our argument has made effective use of reality assumptions. It will be observed that the complex case allows more possibilities for degenerate configurations.

References

[BPVdV] Barth, W., Peters, C., and Van de Venn, A.: *Compact Complex Surfaces,* Springer-Verlag (1984).

[Bor1] Borcea, C.S.: *Point configurations and Cayley-Menger varieties,* arXiv:math.AG/0207110.

[Bor2] Borcea, C.S.: *Tangents to quadrics and nested Calabi-Yau varieties,* in preparation.

[Dol] Dolgachev, I.: *Abstract configurations in Algebraic Geometry,* arXiv:math.AG/0304258.

[Ful] Fulton, W.: *Intersection Theory,* Springer-Verlag (1984).

[GKZ] Gelfand, I.M., Kapranov, M.M. and Zelevinsky, A.V.: *Discriminants, resultants and multidimensional determinants,* Birkhäuser (1994).

[GH] Griffiths, Ph. and Harris, J.: *Principles of Algebraic Geometry,* Wiley Classics Library (1994).

[Har] Harris, J.: *Algebraic Geometry,* Graduate Texts in Math. 133, Springer-Verlag (1993).

[HC-V] Hilbert, D. and Cohn-Vossen, S.: *Geometry and the Imagination,* Chelsea Publishing Comp., New York (1952).

[Hud] Hudson, R.W.H.T.: *Kummer's quartic surface,* Cambridge University Press (1905), reissued in Cambridge Mathematical Library series (1990).

[Jes] Jessop, C.M.: *A Treatise on the Line Complex,* Cambridge University Press (1903)

[MPT] Macdonald, I.G., Pach, J. and Theobald, T.: *Common tangents to four unit balls in R^3,* Discrete and Comput. Geom. 26 (2001), 1-17.

[MS] Megyesi, G. and Sottile, F.: *The envelope of lines meeting a fixed line and tangent to two spheres,* arXiv:math.AG/0304346.

[Nik] Nikulin, V.V.: *Kummer surfaces,* Math. USSR Izv.9 (1975), no.2, 261-275 (1996).

[ST] Sottile, F. and Theobald, T.: *Lines tangent to $2n-2$ spheres in R^n,* Trans. Amer. Math. Soc. 354 (2002), no.12, 4815-4829.

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