Variational approximation for a functional governing point-like singularities

Daniele Graziani, Gilles Aubert

INRIA

1 INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET EN AUTOMATIQUE

Variational approximation of a functional governing point-like singularities.

Daniele Graziani, Gilles Aubert

N° 7095

2009

2

Thème COG

Rapport de recherche

ISSN 0249-6399 ISRN INRIA/RR--7095--FR+ENG

INRIA SOPHIA ANTIPOLIS

Variational approximation of a functional governing point-like singularities.

Daniele Graziani*, Gilles Aubert†

Thème COG — Systèmes cognitifs
Projet ARIANA

Rapport de recherche n° 7095 — 2009 — xxx pages

Abstract: The aim of this paper is to provide a rigorous variational formulation for the detection of points in 2-d images. To this purpose we introduce a new functional of the calculus of variation whose minimizers give the points we want to detect. Then we build an approximating sequence of functionals, for which we prove the Γ-convergence, with respect to a suitable convergence, to the initial one.

Key-words: points detection, divergence-measure fields, p-capacity, Γ-convergence.

* ARIANA Project-team, CNRS/INRIA/UNSA, 2004 Route des lucioles-BP93, 06902 Sophia-Antipolis Cedex, France

† LABORATOIRE J.A. DIEUDONNÉ Université de Nice SOPHIA ANTIPOLIS, parc valrose 06108 Nice CEDEX 2, FRANCE.

Unité de recherche INRIA Sophia Antipolis
2004, route des Lucioles, BP 93, 06902 Sophia Antipolis Cedex (France)
Téléphone : +33 4 92 38 77 77 — Télécopie : +33 4 92 38 77 65

Variational approximation of a functional governing point-like singularities.

Résumé : Nous proposons une nouvelle méthode variationelle pour isoler des points dans une image 2-D. Dans ce but nous introduisons une energie dont les points de minimum sont donnés par l'ensemble des points que on veut détecter. En suite on approche cette energie par une suite de fonctionelles plus régulières, pour laquelle on montre la Γ-convergence vers la fonctionelle initiale.

Mots-clés : détection de points, champs avec divergence mesure, p-capacité, Γ-convergence.

Contents

1 Introduction

The issue of detecting fine structures, like points or curves in two or three dimensional biological images, is a crucial task in image processing. In particular a point may represent a viral particle whose visibility is compromised by the presence of other structures like cell membranes or some noise. Therefore one of the main goals is detecting the spots that the biologists wish to count. This operation is made harder by the presence of other singular structures.

In some biological images the image intensity is a function that takes the value 1 on points or other structures like sets with Hausdorff dimension $0 \leq \alpha < 1$, and it is close to 0 outside. In image processing these concentration sets are called discontinuities without jump, meaning that there is no jump across the set and therefore the gradient of the image is 0.

In the literature there are few variational methods dealing with this problem. In this direction one interesting approach has been proposed in [3]. In that paper the authors consider this kind of pathology as a k-codimension object, meaning that they should be regarded as a singularity of a map $U : \mathbb{R}^{k+m} \to \mathbb{R}^k$, with $k \geq 2$ and $m \geq 0$ (see [6] for a complete survey on this subject). In particular the detecting point case corresponds to the case $k = 2$ and $m = 0$.

This point of view makes possible a variational approach based on the theory of Ginzburg-Landau systems. In their work the isolated points in 2-D images are regarded as the topological singularities of a map $U : \mathbb{R}^2 \to \mathbb{S}^1$, where $\mathbb{S}^1$ is the unit sphere of $\mathbb{R}^2$. Starting from the initial image $I : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$, this strategy makes crucial the construction of an initial vector field $U_0 : \mathbb{R}^2 \to \mathbb{S}^1$ with a topological singularity of degree 1. How to build such a vector field in a rigorous way, is a subject of a current investigation.

Our first purpose here is finding a most natural variational framework in which a rigorous definition of discontinuity without jump can be given. In our model the image I is a Radon measure. It is crucial for detecting points that this Radon measure be able of charging points. The preliminary step is finding a space whose elements are able of producing this kind of measures. This space is given by $\mathcal{DM}^p(\Omega)$: the space of L^p-vector fields whose distributional divergence is a Radon measure, with $1 < p < 2$. The restriction on p is due to the fact that when $p \geq 2$ the distributional divergence $\mathrm{Div}U$ of U cannot be a measure concentrated on points (see Section 3.1 below). Then we have to construct, from the original image I, a data vector $U_0 \in \mathcal{DM}^p(\Omega)$. Clearly there are, at least in principle, many ways to do this. The one we propose here seems to be the most natural. We consider the classical elliptic problem with measure data I:

$$\begin{cases} -\Delta u_0 = I & \text{on } \Omega \\ u_0 = 0 & \text{on } \partial\Omega. \end{cases}$$

Then by setting $U_0 = \nabla u_0$ we have $U_0 \in \mathcal{DM}^p(\Omega)$ with $\mathrm{Div}U_0 = I$. However the support of the measure $\mathrm{Div}U_0$ is too large and could contains several structures like curves or fractals, while the singularities, we are interested in, are contained in the atomic part of the measure

DivU_0 and therefore we have to isolate it. To do this the notion of p-capacity of a set plays a key role. Indeed when $p < 2$ the p-capacity of a point in Ω is zero and one can say, in this sense, that it is a discontinuity with no jump. Besides every Radon measure can be decomposed (see [14]) in two mutually singular measures: the first one is absolutely continuous with respect to the p-capacity and the second one is singular with respect to the p-capacity, meaning that it is a measure concentrated on sets with 0 p-capacity.

As it is known in dimension 2, sets with 0 p-capacity, and hence discontinuities without jump, can be isolated points, countable set of points or fractals with Hausdorff dimension $0 \leq \alpha < 1$ (see Subsection 2.3 for the definiton of p-capacity and related properties).

Our goal here is keeping nothing else but points in the image. The achievement of such a purpose makes necessary the minimization of a suitable energy that must remove all the discontinuities which are not discontinuities without jump, and remove all the discontinuities without jump which are not isolated point.

From one hand we have to force the concentration set of the divergence measure of U to contain only the points we want to catch, and on the other hand we have to regularize the initial data U_0 outside the points of singularities. To this end we introduce the auxiliary space $\mathcal{SDM}^p(\Omega)$ of vector fields belonging to $\mathcal{DM}^p(\Omega)$ whose divergence measure has no absolutely continuous part with respect to the p-capacity. Then, by taking into account that the initial vector field is a gradient of a Sobolev function, our goal is to minimize the following energy:

$$\mathcal{F}(u) = \int_\Omega |\Delta u|^2 dx + \lambda \int_\Omega |\nabla u - U_0|^p dx + \mu \mathcal{H}^0(supp(\mathrm{div}^s \nabla u)_0),$$

where $u \in W_0^{1,p}(\Omega)$ with $\nabla u \in \mathcal{SDM}^p(\Omega)$, $1 < p < 2$ and λ, μ are positive weights. The gradient of a minimizer of the energy $\mathcal{F}$ is the vector field we are looking for, that is a vector field whose divergence measure can be decomposed in an absolutely continuous (with respect to the Lebesgue's measure) term plus an atomic measure concentrated on the points we want to isolate in the image.

Even if a pointwise characterization of discontinuity without jump is not available, thanks to our definition the singular set of points can be linked to the vector field ∇u, in the spirit of the classical SBV formulation of the Mumford-Shah's functional (we refer to [1] for a complete survey on the Mumford Shah's functional).

For future computational purposes, the next task is to provide an approximation in the sense of Γ-convergence introduced in [16, 17]. Our approach is close in the spirit to the one used to approximate the Mumford Shah functional by a family of depending curvature functionals as in [9]. Indeed, as in their work (see also [8]), we replace the atomic measure $\mathcal{H}^0$ by the term

$$G_\varepsilon(D) = \frac{1}{4\pi} \int_{\partial D} (\frac{1}{\varepsilon} + \varepsilon \kappa^2) d\mathcal{H}^1;$$

where D is a proper regular set containing the atomic set P, κ is the curvature of its boundary, and the constant $\frac{1}{4\pi}$ is a normalization factor. Roughly speaking the minima of these functionals are achieved on the union of balls of small radius, so that when $\varepsilon \to 0$ the sequence G_ε shrinks to the atomic measure $\mathcal{H}^0(P)$.

This leads to an intermediate approximation given by

$$\begin{aligned} F_\varepsilon(u,D) &= \int_\Omega (1-\chi_D)|\Delta u|^2 dx + \int_\Omega |\nabla u - U_0|^p dx \\ &+ \frac{1}{4\pi}\int_{\partial D}\big(\frac{1}{\varepsilon} + \varepsilon\kappa^2\big) d\mathcal{H}^1. \end{aligned} \tag{1}$$

This strategy permits to work with the perimeter measure $\mathcal{H}^1 \lfloor \partial D$, that can be approximated, according to the Modica-Mortola's approach (see [21, 22]), by the measure:

$$\mu_\varepsilon(w,\nabla w)dx = \big(\varepsilon|\nabla w|^2 + \frac{W(w)}{\varepsilon}\big)dx,$$

where $W(w) = w^2(1-w)^2$ is a double well function.

Besides by using Sard's Theorem and coarea formula (see also [4] for a similar approach) one can formally replace the integral on ∂D by an integral computed over the level sets of w, whose curvature κ becomes $\mathrm{div}\frac{\nabla w}{|\nabla w|}$ and the integral is computed over the level sets of w. So that one can formally write the complete approximating sequence:

$$\begin{aligned} \mathcal{F}_\varepsilon(u,w) &= \int_\Omega w^2|\Delta u|^2 dx + \mu\frac{1}{8\pi C}\int_{\Omega\setminus\{\nabla w=0\}}\big(\frac{1}{\beta_\varepsilon} + \beta_\varepsilon\big(\mathrm{div}(\frac{\nabla w}{|\nabla w|})\big)^2(\varepsilon|\nabla w|^2 + \frac{1}{\varepsilon}W(w)\big)dx \\ &+ \lambda\int_\Omega |\nabla u - U_0|^p dx + \frac{1}{\gamma_\varepsilon}\int_\Omega (1-w)^2 dx, \end{aligned}$$

where, as usual, $C = \int_0^1 \sqrt{W(t)}dt$, β_ε and γ_ε are infinitesimal as $\varepsilon \to 0$. The last integral is a penalization term that forces w to tend to 1 as $\varepsilon \to 0$.

The goal of the second part of this work is then to show that the family of energies $\mathcal{F}_\varepsilon$ Γ-converges to the functional $\mathcal{F}$ when the parameters are related in a suitable way.

As in [9] we deal with a suitable convergence of functions involving the Hausdorff convergence of a sub-level sets. This strategy requires a careful statement of the Γ-convergence definitions and results, in order to have that sequences asymptotically minimizing $\mathcal{F}_\varepsilon$ converges to a minimum of $\mathcal{F}$.

Despite this approach is inspired by some ideas contained in [8, 9], we point out that in our case the regularization term involves a second order differential operator, due to the fact that our goal is to detect points and not segment curves. This deep difference requires a non trivial adaptation of the arguments used in those papers.

The paper is organized as follows. Section 2 is devoted to notations, preliminary definitions and results. In Section 3 we illustrate the new variational model and we present the functional we deal with. Section 4 and 5 are devoted to the Γ-convergence result. Finally in the last Section we conclude the paper by comparing this approach with the celebrated conjecture by De Giorgi, concerning the approximation of the curvature depending functionals.

We do not give here experimental result illustrating our approach. We refer the reader for that to [19].

2 Preliminaries

2.1 Notation

In all the paper $\Omega \subset \mathbb{R}^2$ is an open bounded set with lipschitz boundary. The Euclidean norm will be denoted by $|\cdot|$, while the symbol $\|\cdot\|$ indicates the norm of some functional spaces. The brackets $\langle , \rangle$ denotes the duality product in some distributional spaces. $\mathcal{L}^d$ or dx is the d-dimensional Lebesgue measure and $\mathcal{H}^k$ is the k-dimensional Hausdorff measure. $B_\rho(x_0)$ is the ball centered at x_0 with radius ρ. We say that a set $D \subset \Omega$ is a regular set if it can be written as $\{F < 0\}$ with $F \in C_0^\infty(\Omega)$. In the following we will denote by $R(\Omega)$ the family of all regular sets in Ω. Finally we will use the symbol $\rightharpoonup$ for denoting a weak convergence.

2.2 Distributional divergence and classical spaces

In this Subsection we recall the definition of the distributional space $L^{p,q}(\mathrm{div};\Omega)$ and $\mathcal{DM}^p(\Omega)$, $1 \leq p, q \leq +\infty$, (see [2, 12]).

Definition 2.1. *We say that* $U \in L^{p,q}(\mathrm{div};\Omega)$ *if* $U \in L^p(\Omega;\mathbb{R}^2)$ *and if its distributional divergence* $\mathrm{Div}U = \mathrm{div}U \in L^q(\Omega)$. *If* $p = q$ *the space* $L^{p,q}(\mathrm{div};\Omega)$ *will be denoted by* $L^p(\mathrm{div};\Omega)$.

We say that a function $u \in W^{1,p}(\Omega)$ *belongs to* $W^{1,p,q}(\mathrm{div};\Omega)$ *if* $\nabla u \in L^{p,q}(\mathrm{div};\Omega)$. *We say that a function* $u \in W_0^{1,p}(\Omega)$ *belongs to* $W_0^{1,p,q}(\mathrm{div};\Omega)$ *if* $\nabla u \in L^{p,q}(\mathrm{div};\Omega)$.

Definition 2.2. *For* $U \in L^p(\Omega;\mathbb{R}^2)$, $1 \leq p \leq +\infty$, *set*

$$|\mathrm{Div}U|(\Omega) := \sup\{\langle U, \nabla\varphi\rangle \ : \ \varphi \in C_0^\infty(\Omega),\ |\varphi| \leq 1\}.$$

We say that U *is an* L^p*-divergence measure field, i.e.* $U \in \mathcal{DM}^p(\Omega)$, *if*

$$\|U\|_{\mathcal{DM}^p(\Omega)} := \|U\|_{L^p(\Omega;\mathbb{R}^2)} + |\mathrm{Div}U|(\Omega) < +\infty.$$

Let us recall the following classical result (see [13] Proposition 3.1).

Theorem 2.1. *Let* $\{U_k\}_k \subset \mathcal{DM}^p(\Omega)$ *be such that*

$$U_k \rightharpoonup U \quad \text{in } L^p(\Omega;\mathbb{R}^2),\ \text{as } k \to +\infty \text{ for } 1 \leq p < +\infty. \tag{2}$$

Then

$$\|U\|_{L^p(\Omega;\mathbb{R}^2)} \leq \liminf_{k\to+\infty} \|U_k\|_{L^p(\Omega;\mathbb{R}^2)}, \quad |\mathrm{Div}U|(\Omega) \leq \liminf_{k\to+\infty} |\mathrm{Div}U_k|(\Omega).$$

2.3 p-capacity

The p-capacity will be crucial to find a convenient functional framework to deal with. If $K \subset \mathbb{R}^2$ is a compact set and χ_K denotes its characteristic function, we define:

$$Cap_p(K,\Omega) = \inf\{\int_\Omega |\nabla f|^p dx,\ f \in C_0^\infty(\Omega), f \geq \chi_k\}.$$

If $U \subset \Omega$ is an open set , the p-capacity is given by

$$Cap_p(U,\Omega) = \sup_{K \subset U} Cap_p(K,\Omega).$$

Finally if $A \subset \Omega$ is a Borel set

$$Cap_p(A,\Omega) = \inf_{A \subset U \subset \Omega} Cap_p(U,\Omega).$$

We recall the following result (see for instance [20], Theorem 2.27) that explains the relationship between p-capacity and Hausdorff measures. Such a result is crucial to have geometric informations on null p-capacity sets.

Theorem 2.2. *Assume $1 < p < 2$. If $\mathcal{H}^{2-p}(A) < \infty$ then $Cap_p(A,\Omega) = 0$.*

Another useful tool to manage sets of p-capacity 0 is provided by the following characterization.

Theorem 2.3. *Let E be a compact subset of Ω. Then $Cap_p(E,\Omega) = 0$ if and only if there exists a sequence $\{\phi_k\}_k \subset C_0^\infty(\Omega)$, converging to 0 strongly in $W_0^{1,p}(\Omega)$, such that $0 \leq \phi_k \leq 1$ and $\phi_k = 1$ on E for every k.*

For a general survey we refer the reader to [18, 20, 25].

3 The Variational Model

In this section we set the functional framework and the functional to be minimized.

Roughly speaking in biological images the image is a function that could be very high on points or other structures like sets with Hausdorff dimension $0 \leq \alpha < 1$, and it is close to 0 outside. From a mathematical point of view it seems to be much more appropriate to think of the image as a Radon measure, that is $I = \mu \in (C_0(\Omega))^*$. The next step is finding a space whose elements are able of producing this kind of discontinuities: the space $\mathcal{DM}^p(\Omega)$, with $1 < p < 2$. The restriction on p is due to the fact that when $p \geq 2$ the distributional divergence of U cannot be a measure concentrated on points. Set $p \geq 2$, according to the definition, we have

$$\langle \mathrm{Div} U, \varphi \rangle = -\int_\Omega U \cdot \nabla \varphi dx \quad \text{for all } \varphi \in C_0^\infty(\Omega).$$

Since $p \geq 2$ this distribution is well-defined for any test $\varphi \in W_0^{1,p'}(\Omega)$, where $p' \leq 2$ is the dual exponent of p. In particular $\mathrm{Div}U$ belongs to the dual space $W^{-1,p'}(\Omega)$ of the Sobolev space $W_0^{1,p}(\Omega)$. Then in this case, the distributional divergence of U cannot be an atomic measure, since $\delta_0 \notin W^{-1,p'}(\Omega)$. To see this, one can consider as Ω the disk $B_1(0)$ and the function $\tilde{\varphi}(x) = \log(\log(1 + |x|)) - \log(\log(2))$. This function is in the space $W_0^{1,p'}(\Omega)$ for every $p' \leq 2$ and therefore it is an admissible test function, however it easy to check that $\langle \delta_0, \varphi \rangle = +\infty$.

When $1 < p < 2$ we have that $\mathrm{Div}U \in W^{-1,p'}(\Omega)$, but in this case since $p < 2$, we have $p' > 2$ and hence the function $\tilde{\varphi}$ is no longer an admissible test function. One can check that the distribution $\mathrm{Div}U$ is an element of $(C_0(\Omega))^*$ able of charging the points. Take for instance the map $U(x, y) = (\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2})$.

The next step is to transform the initial image I as the divergence measure of a suitable vector field. We consider the elliptic problem with measure data I:

$$\begin{cases} -\Delta u = I & \text{on } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3}$$

Classical results (see [24]) ensures the existence of a unique weak solution $u \in W_0^{1,p}(\Omega)$ with $p < 2$. Then it easy to see that the distributional divergence of ∇u is given by I. In particular by setting $U = \nabla u$, we have $U \in \mathcal{DM}^p(\Omega)$. According to the Radon-Nikodym decomposition of the measure $\mathrm{Div}U$ we have

$$\mathrm{DivU} = \mathrm{div} U + \mathrm{div}^s U,$$

where $\mathrm{div}U \in L^1(\Omega)$ and $\mathrm{div}^s U$ is a singular measure with respect to $\mathcal{L}^2$. For our purpose the support of the singular measure $\mathrm{div}^s U$ is too large. In particular the measure $\mathrm{div}^s U$ could charge sets with Hausdorff dimension $0 \leq \alpha < 2$. So that in order to isolate the singularities we are interested in, we need a further decomposition of the measure DivU.

This can be done by using the capacitary decomposition of the Radon measure $\mathrm{div}^s U$. It is known (see [14]) that given a Radon measure μ the following decomposition holds

$$\mu = \mu_a + \mu_0, \tag{4}$$

where the measure μ_a is absolutely continuous with respect to the p-capacity and μ_0 is singular with respect to the p-capacity, that is concentrated on sets with 0 p-capacity. Besides it is also known (see [14]) that every measure which is absolutely continuous with respect to the p-capacity can be characterized as an element of $L^1 + W^{-1,p'}$, leading to the finer decomposition:

$$\mu = f - \mathrm{DivG} + \mu_0, \tag{5}$$

where $G \in L^{p'}(\Omega;\mathbb{R}^2)$ with $\frac{1}{p} + \frac{1}{p'} = 1$ and $f \in L^1(\Omega)$.

By applying this decompositon to the measure $\mathrm{div}^s U$ we obtain the following decomposition of the measure DivU

$$\mathrm{DivU} = \mathrm{div}U + f - \mathrm{Div}G + (\mathrm{div}^s U)_0, \tag{6}$$

with $G \in L^{p'}(\Omega;\mathbb{R}^2)$, $f \in L^1(\Omega)$, $\mathrm{div}U \in L^1(\Omega)$, and $(\mathrm{div}^s U)_0$ is a measure concentrated on a set with 0 p-capacity.

According to this decomposition and taking into account Theorem 2.3 we give the definition of discontinuity without and with jump.

Definition 3.1. *We say that a point $x \in \Omega \subset \mathbb{R}^2$ is a point of discontinuity without jump of U if $x \in \overline{supp(\mathrm{div}^s U)_0}$.*

Remark 3.1. *The other singularities, where there is a jump, are contained in the second term of decomposition (6). Indeed the space $W^{-1,p'}(\Omega)$ contains Hausdorff measures restricted to sub-manifolds of dimension greater than or equal to 1. (We refer to [25] Section 4.7 for a detailed discussion on the space $W^{-1,p'}(\Omega)$), like for instance Hausdorff measures concentrated on regular closed curves, which are classical examples of discontinuities with jump. More precisely a contour of a regular set D is the jump set of the characteristic function of D and its p-capacity is strictly positive. This is of course in agreement with Theorem 2.3. Indeed if there were a sequence $\{\phi_k\}_k \subset C_0^\infty(\Omega)$, converging to 0 strongly in $W_0^{1,p}(\Omega)$, such that $0 \le \phi_k \le 1$ and $\phi_k = 1$ on ∂D for every k, it would be possible to define the sequence*

$$\tilde{\phi_k} = \begin{cases} \phi_k & \text{on } D \\ 1 & \text{on } \Omega \setminus D, \end{cases}$$

which converges, in the $W^{1,p}$-norm, to the BV-function $1 - \chi_D$, which cannot be approximated by regular functions in the $W^{1,p}$-norm.

Definition 3.2. *We say that a point $x \in \Omega \subset \mathbb{R}^2$ is a point of discontinuity with jump of U if $x \in \overline{supp(f - \mathrm{Div}G)}$.*

3.1 The variational framework

We shall build an energy whose minimizers will be vector fields whose divergence measure's singular part will be given by nothing else but points.

Each minimizer must be an L^p (with $p < 2$) vector field with the following properties:

1. It must be close to the initial data U_0 which is, in general, an L^p vector field U_0 with $1 < p < 2$.

2. The absolutely continuous part with respect to the Lebesgue measure of $\mathrm{Div}U$ is an L^2 function.

3. The support of the measure $(\mathrm{dive}^s U)_0$ must be given by set of points P_U with $\mathcal{H}^0(P_U) < +\infty$.

According to these considerations it is natural to introduce the space

$$\mathcal{SDM}^p(\Omega) := \{U \in \mathcal{DM}^p(\Omega), \quad f - \mathrm{Div}G = 0\}, \tag{7}$$

so that, as a consequence, decomposition (6) yields for any $U \in \mathcal{SDM}^p(\Omega)$

$$\mathrm{DivU} = \mathrm{div}U + (\mathrm{div}^s U)_0. \tag{8}$$

For our purposes the following result concerning the features of elements of the space $\mathcal{SDM}^p(\Omega)$ will play a crucial role.

Proposition 3.1. *Let $u \in W_0^{1,p,2}(\mathrm{div}; \Omega \setminus P)$, with $1 < p < 2$. Let $P \subset \Omega$ be a set of finite number of points. Then $\nabla u \in \mathcal{SDM}^p(\Omega)$, with $(\mathrm{div}^s \nabla u)_0 = P$.*

Proof. We set $P = \{x_1, ..., x_n\}$. Let $\rho(h) \to 0$ as $h \to +\infty$ be such that $B_{\rho_h}(x_i) \cap B_{\rho_h}(x_j) = \emptyset$ for h large enough and $i \neq j$. We set $\Omega_h = \bigcup_{i=1}^n B_{\rho_h}(x_i)$ and we define the following sequence $\{U_h\} \subset L^p(\Omega; \mathbb{R}^2)$.

$$\begin{cases} U_h = \nabla u & \text{on } \Omega \setminus \Omega_h, \\ 0 & \text{on } \Omega_h. \end{cases} \tag{9}$$

Since $\Delta u \in L^2(\Omega \setminus P)$, by standard elliptic regularity we deduce that $u \in W^{2,p}_{loc}(\Omega \setminus P)$. In particular the exterior trace $\gamma_0^{ext}(u) \in W^{\frac{3}{2},p}(\partial\Omega_h)$. Therefore we infer that $u \in W^{2,p}(\Omega \setminus \Omega_h)$. For every $i = 1, ..n$ and h small enough we can find an open set A_i such that $B_{\rho_h}(x_i) \subset A_i \subset \Omega \setminus \bigcup_{j \neq i} B_{\rho_h}(x_j)$ and A_i does not depend on h. Let θ_i be a cutoff function associated to A_i such that

$$\begin{cases} \theta_i = 1 & \text{on } B_{\rho_h}(x_i) \text{ for any } i = 1, ..., n, \\ 0 \le \theta_i \le 1 & \text{for any } i = 1, ..., n, \\ \theta_i = 0 & \text{on } \Omega \setminus A_i \text{ for any } i = 1, ..., n, \\ \|\nabla \theta_h\|_\infty \le \frac{M_i}{d(\partial A_i, \partial B_{\rho_h}(x_i))} & \text{for any } i = 1, ..., n. \end{cases} \tag{10}$$

Then, if $\varphi \in C^1_0(\Omega)$ with $|\varphi| \leq 1$, by applying Gauss-Green's formula we obtain:

$$\begin{aligned}
\int_\Omega U_h \cdot \nabla\varphi dx &= \int_{\Omega\setminus\Omega_h} \nabla u \cdot \nabla\varphi dx = -\int_{\Omega\setminus\Omega_h} \Delta u\varphi dx + \int_{\partial(\Omega\setminus\Omega_h)} \nabla u \cdot \nu\varphi d\mathcal{H}^1 \\
&= -\int_{\Omega\setminus\Omega_h} \Delta u\varphi dx + \sum_{i=1}^n \int_{\partial(\Omega\setminus B_{\rho_h}(x_i))} \nabla u \cdot \nu(\varphi - \theta_i\varphi(x_i)) d\mathcal{H}^1 \\
&+ \sum_{i=1}^n \varphi(x_i) \int_{\partial(\Omega\setminus B_{\rho_h}(x_i))} \theta_i \nabla u \cdot \nu d\mathcal{H}^1 \\
&= -\int_{\Omega\setminus\Omega_h} \Delta u\varphi dx + \sum_{i=1}^n \int_{\partial\Omega} \nabla u \cdot \nu(\varphi - \theta_i\varphi(x_i)) \\
&+ \sum_{i=1}^n \int_{\partial B_{\rho_h}(x_i)} \nabla u \cdot \nu(\varphi - \varphi(x_i)) d\mathcal{H}^1 \\
&+ \sum_{i=1}^n \Big\{\varphi(x_i) \int_{A_i\setminus B_{\rho_h}(x_i)} \Delta u\theta_i dx + \int_{A_i\setminus B_{\rho_h}(x_i)} \nabla u \nabla\theta_i dx\Big\}. \qquad (11)
\end{aligned}$$

where in the last equality we have applied again the Gauss-Green's formula and the definition of θ_i.

Now for every i we have that $\{\partial B_{\rho_h}(x_i)\}$ converges in the Hausdorff metric to the singleton $\{x_i\}$. Then, since the support of the function $\psi = \varphi - \varphi(x_i)$ is contained in $\Omega \setminus \{x_i\}$, we have that $supp\psi \cap \partial\{B_h(x_i)\} = \emptyset$ for h large enough, by standard properties of the Hausdorff convergence. Therefore the third term in (11) is equal to 0. Moreover for h large enough we can find a proper open regular set A, that does not depend on h, such that $u \in W^{2,p}(\Omega \setminus A)$. Hence we infer $\frac{\partial u}{\partial \nu} \in W^{\frac{1}{2},p}(\partial\Omega)$. Therefore, from (11) it follows that

$$\begin{aligned}
|\mathrm{Div} U_h|(\Omega) &\leq \sup_{0\leq\varphi\leq 1} \int_\Omega |\nabla u \cdot \nabla\varphi| dx \leq (n+1)C_1(\Omega)\|\Delta u\|_{L^2(\Omega\setminus P)} + 2n\|\frac{\partial u}{\partial\nu}\|_{W^{\frac{1}{2},p}(\partial\Omega)} \\
&+ \|\nabla u\|_{L^p(\Omega;\mathbb{R}^2)} \sum_{i=1}^n \frac{M_i}{d(\partial A_i, \partial B_{\rho_h}(x_i))} := C(n,\Omega),
\end{aligned}$$

for h large enough. Since $U_h \rightharpoonup \nabla u$ in $L^p(\Omega;\mathbb{R}^2)$, by Theorem 2.1

$$|\mathrm{Div}\nabla u|(\Omega) \leq \liminf_{h\to\infty} |\mathrm{Div}\nabla u_h| \leq C.$$

Therefore $\nabla u \in \mathcal{DM}^p(\Omega)$. Finally we know that $u \in W^{1,p,2}(\mathrm{div};\Omega \setminus P)$ and thus the support of the measure $\mathrm{div}^s\nabla u$ is given by the set P. Since $Cap_p(P,\Omega) = 0$, according to decomposition (6) the measure $f - \mathrm{DivG}$ vanishes on sets with 0 p-capacity, and we deduce $f - \mathrm{DivG} = 0$, that is $\nabla u \in \mathcal{SDM}^p(\Omega)$, with $(\mathrm{div}^s\nabla u)_0 = P$. □

3.2 The Functional

According to our purpose the natural energy to deal with is the following $F : \mathcal{SDM}^p(\Omega) \to [0, \infty]$, $1 < p < 2$, given by

$$F(U) = \int_\Omega |\mathrm{div} U|^2 dx + \lambda \int_\Omega |U - U_0|^p dx + \mu \mathcal{H}^0(supp(\mathrm{div}^s U)_0).$$

From now on we assume without loosing generality that the weights λ and μ are equal to 1.

We note that, if $\mathrm{Div} U_0 \neq 0$ in the sense of distributions, then $\inf F(U) > 0$ on $\mathcal{SDM}^p(\Omega)$. Indeed if we had $\inf\limits_{\mathcal{SDM}^p(\Omega)} F(U) = 0$ then, it would be possible exhibiting a minimizing sequence $\{U_n\}$, such that $F(U_n) \to 0$. This would imply $U_n \to U_0$ in L^p and $\mathrm{Div} U_n \to 0$ in $\mathcal{D}'(\Omega)$. On the other hand, the L^p-distance between U_n and U_0 can be arbitrary small only if $\mathrm{Div} U_0 = 0$ as well, because the constraint $\mathrm{Div} U = 0$ is stable under L^p-convergence.

4 Γ-convergence: The intermediate approximation

By analogy with the construction of U_0 we restrict ourselves to vector fields U which are the gradient of a function $u \in W_0^{1,p}(\Omega)$.

Thus the functional $\mathcal{F}$ is finite on the class of functions whose support of the measure $(\mathrm{div}^s \nabla u)_0$ is given by a finite set. Consequently it is convenient to introduce the following spaces:

$$\Delta\mathcal{M}^p(\Omega) := \{u \in W_0^{1,p}(\Omega),\ \nabla u \in S\mathcal{DM}^p(\Omega)\}, \tag{12}$$

and

$$\Delta\mathcal{AM}^{p,2}(\Omega) = \{u \in \Delta\mathcal{M}^p(\Omega) :\ \Delta u \in L^2(\Omega),\ supp(\mathrm{div}^s \nabla u)_0 = P_{\nabla u} \text{ with } \mathcal{H}^0(P_{\nabla u}) < +\infty\}. \tag{13}$$

So that the target-limit energy $\mathcal{F} : \Delta\mathcal{AM}^{p,2}(\Omega) \to (0,\infty)$ is given by

$$\mathcal{F}(u) = \int_\Omega |\Delta u|^2 dx + \int_\Omega |\nabla u - U_0|^p dx + \mathcal{H}^0(P_{\nabla u}). \tag{14}$$

In the spirit [9] we introduce an intermediate variational approximation of the functional $\mathcal{F}$. We define a sequence of functionals where the counting measure $\mathcal{H}^0(P_{\nabla u})$ is replaced by a functional defined on regular sets D and which involves the curvature of the boundary ∂D. The approximating sequence is given by:

$$\begin{aligned} F_\varepsilon(u,D) &= \int_\Omega (1-\chi_D)|\Delta u|^2 dx + \int_\Omega |\nabla u - U_0|^p dx \\ &+ \frac{1}{4\pi}\int_{\partial D} \big(\frac{1}{\varepsilon} + \varepsilon\kappa^2\big) d\mathcal{H}^1. \end{aligned}$$

Where $u \in W_0^{1,p,2}(\mathrm{div};\Omega)$, D is a regular set, and κ denotes the curvature of its boundary.

In order to guarantee that the measure of the sets D is small we define a new functional still denoted by $F_\varepsilon(u,D)$ given by

$$F_\varepsilon(u,D) = \int_\Omega (1-\chi_D)|\Delta u|^2 dx + \int_\Omega |\nabla u - U_0|^p dx + \frac{1}{4\pi}\int_{\partial D} \big(\frac{1}{\varepsilon} + \varepsilon\kappa^2\big) d\mathcal{H}^1 + \frac{1}{\varepsilon}\mathcal{L}^2(D) \quad \text{on } Y(\Omega), \tag{15}$$

where $Y(\Omega) = \{(u,D)\ u \in W_0^{1,p,2}(\mathrm{div};\Omega),\ D \in R(\Omega)\}$. We endow the set $Y(\Omega)$ with the following convergence.

Definition 4.1. *Let $h \in \mathbb{N}$ go to $+\infty$. We say that a sequence $\{(u_h, D_h)\}_h \subset Y(\Omega)$ H-converges to $u \in \Delta\mathcal{AM}^{p,2}(\Omega)$ if the following conditions hold*

1. *$\mathcal{L}^2(D_h) \to 0$;*
2. *$\{\partial D_h\}_h \to P \subset \Omega$ in the Hausdorff metric, where P is a finite set of points;*
3. *$u_h \to u$ in $L^p(\Omega)$ and $P_{\nabla u} \subseteq P$.*

As in [9] we adopte the following ad hoc definition of Γ-convergence.

Definition 4.2. *Let $h \in \mathbb{N}$ go to $+\infty$. We say that F_ε Γ-converges to $\mathcal{F}$ if for every sequence of positive numbers $\{\varepsilon_h\} \to 0$ and for every $u \in \Delta\mathcal{AM}^{p,2}(\Omega)$ we have:*

1. *for every sequence $\{(u_h, D_h)\}_h \subset Y(\Omega)$ H-converging to $u \in \Delta\mathcal{AM}^{p,2}(\Omega)$*

$$\liminf_{h\to+\infty} F_{\varepsilon_h}(u_h, D_h) \geq \mathcal{F}(u);$$

2. *there exists a sequence $\{(u_h, D_h)\}_h \subset Y(\Omega)$ H-converging to u such that*

$$\limsup_{h\to+\infty} F_{\varepsilon_h}(u_h, D_h) \leq F(u).$$

We point out that with this approach, the fundamental theorem of the Γ-convergence cannot be applied directly, since we do not deal with a metric space (for a complete survey on Γ-convergence we refer to [7, 10]). However it is still possible to prove that a sequence $\{(u_h, D_h)\}_h$ asymptotically minimizing $F_\varepsilon(u, D)$ admits a subsequence H-converging to a minimizer of $\mathcal{F}(u)$. Indeed we will show at the end of the Section (see Theorem 4.4) that the convergence of the minimum problems can still obtained as a consequence of compactness of the minimizing sequence of F_ε, $\Gamma - \liminf$ inequality (1) and $\Gamma - \limsup$ inequality (2) .

4.1 Compactness

We state and prove the following compactness result.

Theorem 4.1. *Let $h \in \mathbb{N}$ go to $+\infty$ and $\varepsilon_h \to 0$ such that*

$$F_{\varepsilon_h}(u_h, D_h) \leq M, \tag{16}$$

then there exist a subsequence $\{(u_{h_k}, D_{h_k})\}_k \subset Y(\Omega)$, a function $u \in \Delta\mathcal{AM}^{p,2}(\Omega)$ and a set $P \subset \overline{\Omega}$ of finite number of points, such that $\{(u_{h_k}, D_{h_k})\}_k$ H-converges to u.

Proof. We adapt an argument of [9]. From (16) we have immediately $\{D_h\} \subset R(\Omega)$ with $\mathcal{L}^2(D_h) \to 0$. Then we can parametrize every $C_h = \partial D_h$ by a finite and disjoint union of Jordan curves. Let us set for every h, $C_h = \bigcup_{i=1}^{m(h)} \gamma^i$. Then we have according to the 2-dimensional version of Gauss-Bonnet's Theorem and Young's inequality

$$M \geq \frac{1}{4\pi}\int_{\partial D_h} (\frac{1}{\varepsilon_h} + \varepsilon_h {\kappa_h}^2) d\mathcal{H}^1 \geq \frac{1}{4\pi}\int_{\partial D_h} 2\kappa_h d\mathcal{H}^1 = \frac{1}{4\pi}\int_{\bigcup_h C_h} 2\kappa_h d\mathcal{H}^1 = m(h).$$

Note that the number $m(h) \leq M$, with $M \geq 0$, is independent of h. Then it is possible to extract a subsequence C_{h_k} with the number of curves in C_{h_k} equal to some n for every k. Then we set $C_{h_k} = \{\gamma^1_{h_k}, ..., \gamma^n_{h_k}\}$ for any k. From (16) we also have for any $\gamma \in C_{h_k}$ that

$\mathcal{H}^1(\gamma) \leq 4\pi M\varepsilon_{h_k}$ and consequently $\max\{\mathcal{H}^1(\gamma) : \gamma \in C_{h_k}\} \to 0$. Then there exists a finite set of point $P = \{x_1, ..., x_n\} \subset \overline{\Omega}$ such that for any radius ρ there is an index k_ρ with

$$\gamma^i_{h_k} \subset B_\rho(x_i) \text{ for all } k > k_\rho \text{ and } i \in \{1, ..., n\},$$

so that if we set $\partial D_{h_k} = \bigcup_{i=1}^n \gamma^i_{h_k} \subset \bigcup_{i=1}^n B_\rho(x_i)$, then the Hausdorff distance $d_H(\partial D_{h_k}, P) \to 0$ since $\mathcal{L}^2(D_{h_k}) \to 0$ and therefore $\rho \to 0$ as well.

Now we prove the compactness property for u_h. First of all from the estimate

$$\|\nabla u_h\|^p_{L^p(\Omega)} \leq 2^p(\|\nabla u_h - U_0\|^p_{L^p(\Omega)} + \|U_0\|^p_{L^p(\Omega)}), \tag{17}$$

and (16), we may extract a subsequence $\{u_{h_k}\} \subset W^{1,p}_0(\Omega)$ weakly convergent to $u \in W^{1,p}_0(\Omega)$.

Let Ω_j be a sequence of open sets $\Omega_j \subset\subset \Omega \backslash P$ invading $\Omega \backslash P$. We claim that it is possible to extract a sequence of D_{h_k} such that $\Omega_j \cap \partial D_{h_k} = \emptyset$. Indeed since the distance between Ω_j and P is positive for any j there exists η_j such that $\Omega_j \cap (\bigcup_i^n B_{\eta_j}(x^i)) = \emptyset$. On the other hand we know that for every ρ we can find k_ρ such that $\partial D_{h_k} = \bigcup_{i=1}^n \gamma^i_{h_k} \subset \bigcup_{i=1}^n B_\rho(x_i)$. Then in particular if $\rho = \eta_j$ there exists k_j such that for all $k \geq k_j$

$$\Omega_j \cap \partial D_{h_k} = \emptyset.$$

Therefore for any $x \in \Omega_j$ there exists $\delta > 0$ such that either $B_\delta(x) \subset D_{h_k}$ or $B_\delta(x) \subset \Omega \backslash D_{h_k}$. Finally by taking into account that $\mathcal{L}^2(D_{h_k}) \to 0$ we conclude $\Omega_j \cap \partial D_{h_k} = \emptyset$ for $k \geq k_j$.

Then for every $k \geq k_j$ we have that $u_{h_k} \in W^{1,p,2}(\text{div}; \Omega_j)$ and by (16) we get

$$\int_{\Omega_j} |\Delta u_h|^2 dx \leq \int_{\Omega \backslash D_{h_k}} |\Delta u_{h_k}|^2 dx \leq M. \tag{18}$$

Then we can extract a further subsequence still denoted by $\{u_{h_k}\} \subset W^{1,p,2}(\text{div}; \Omega_j)$ such that

$$\begin{cases} u_{h_k} \to u & \text{in } L^p(\Omega_j; \mathbb{R}^2) \text{ and a.e.} \\ \nabla u_{h_k} \rightharpoonup \nabla u & \text{in } L^p(\Omega_j; \mathbb{R}^2) \\ \Delta u_{h_k} \rightharpoonup \Delta u & \text{in } L^2(\Omega_j). \end{cases}$$

Let now $x \in \Omega^{'} \subset\subset \Omega \setminus P$. Then there exists a sequence $x_j \to x$ with $j \in \mathbb{N}$. By applying the diagonal argument to the sequence $u_{h_{k_l}}(x_j)$ we obtain a subsequence $u_l = u_{h_{k_l}}(x_l)$ such that Δu_l converges weakly in $L^2(\Omega')$ to Δu for any $\Omega' \subset\subset \Omega$. Then by the semicontinuity of the L^2-norm we have

$$\sup_j \int_{\Omega_j} |\Delta u|^2 dx \leq \sup_j \liminf_{l \to +\infty} \int_{\Omega_j} |\Delta u_l|^2 dx \leq M.$$

If we set $\tilde{P} = P \backslash \partial\Omega$, then we deduce $u \in W^{1,p,2}_0(\text{div}; \Omega \backslash \tilde{P})$ and therefore $\nabla u \in \mathcal{SDM}^p(\Omega)$ with $P_{\nabla u} \subseteq P$, by Proposition 3.1. So we conclude that $u \in \Delta\mathcal{AM}^{p,2}(\Omega)$. $\square$

4.2 Lower bound

We provide the lower bound (1) in Definition 4.2.

Theorem 4.2. *Let $h \in \mathbb{N}$ go to $+\infty$. Let $\{\varepsilon_h\}_h$ be a sequence of positive numbers converging to zero. For every sequence $\{(u_h, D_h)\}_h \subset Y(\Omega)$, H-converging to $u \in \Delta\mathcal{AM}^{p,2}(\Omega)$, we have*

$$\liminf_{h\to\infty} F_{\varepsilon_h}(u_h, D_h) \geq \mathcal{F}(u).$$

Proof. Up to a subsequence we may assume that the lim inf is a actually a limit. As in the proof of Theorem 4.1, by setting for every h, $C_h = \bigcup_{i=1}^{m(h)} \gamma^i$, we get

$$M \geq \frac{1}{4\pi}\int_{\partial D_h} (\frac{1}{\varepsilon_h} + \varepsilon_h k^2) d\mathcal{H}^1 = m(h).$$

Up to subsequences we have $m(h) = n$ for some natural number n. Hence there exists a set P_1 of n points such that ∂D_h converges in the Hausdorff metric to P_1. On the other hand we have that ∂D_h converges in the Hausdorff metric to P with $P_{\nabla u} \subseteq P$. Then, since the limit is unique, we have $P = P_1$.

Let now $\{\Omega_j\}_j$ be a sequence of open sets $\Omega_j \subset\subset \Omega \setminus P_1$ invading $\Omega \setminus P_1$. As in the proof of Theorem 4.1 we may assume up to a subsequence, that $\Delta u_h \rightharpoonup \Delta u$ in $L^2(\Omega_j)$. Furthermore, since in this case all the sequence D_h converges to the set P_1 we have, by the same argument used in the proof of Theorem 4.1, $\Omega_j \subset \Omega \setminus D_h$ for h large and for any j. Consequently

$$\liminf_{h\to+\infty} \int_{\Omega\setminus D_h} |\Delta u_h|^2 dx \geq \liminf_{h\to+\infty} \int_{\Omega_j} |\Delta u_h|^2 dx \geq \int_{\Omega_j} |\Delta u|^2 dx.$$

On the other hand, arguing as in Theorem 4.1, we infer that the limit u of the subsequence u_h belongs to $\Delta\mathcal{AM}^{p,2}(\Omega)$, with $\Delta u \in L^2(\Omega \setminus P_1)$ and $P_{\nabla u} \subseteq P_1$. So that by monotone convergence

$$\liminf_{h\to+\infty} \int_{\Omega\setminus D_h} |\Delta u_h|^2 dx \geq \int_{\Omega\setminus P_1} |\Delta u|^2 dx = \int_{\Omega} |\Delta u|^2 dx. \tag{19}$$

As in the proof of Theorem 4.1, inequality (17) holds. Then we easily get

$$\lim_{h\to\infty} \int_{\Omega} |\nabla u_h - U_0|^p dx \geq \int_{\Omega} |\nabla u - U_0|^p dx. \tag{20}$$

Finally we have

$$\frac{1}{4\pi}\int_{\partial D_h} (\frac{1}{\varepsilon_h} + \varepsilon_h k^2) d\mathcal{H}^1 \geq n = \mathcal{H}^0(P_1) \geq \mathcal{H}^0(P_{\nabla u}). \tag{21}$$

Eventually by (19),(20) (21) and by the superlinearity property of the lim inf operator we achieve the result. □

4.3 Upper bound

In [9] for the construction of the optimal sequence it is crucial to use a result due to Chambolle and Doveri (see [11]). This result states that it is possible to approximate, in the H^1-norm, a function $u \in W^{1,2}(\Omega \setminus C)$ (where C is a closed set), by means of a sequence of functions $u_h \in W^{1,2}(\Omega \setminus C_h)$ with C_h convergent to C in the Hausdorff metric. In our case this argument does not apply due to presence of a second order differential operator. Nevertheless since we work only with set of points it is possible to build an optimal sequence in a more direct way.

Theorem 4.3. *Let $h \in \mathbb{N}$ go to $+\infty$. Let ε_h be a sequence of positive converging to 0. For every $u \in \Delta\mathcal{AM}^{p,2}(\Omega)$ there exists a sequence $\{(u_h, D_h)\}_h \subset Y(\Omega)$ H-converging to u such that*

$$\limsup_{h\to+\infty} F_{\varepsilon_h}(u_h, D_h) \le \mathcal{F}(u). \tag{22}$$

Proof. We start by the construction of the sequence D_h. Let n be the number of points x_i in $P_{\nabla u}$. Then we take $D_h = \bigcup_{i=1}^n B_{\varepsilon_h}(x_i)$. So that $\mathcal{L}^2(D_h) \to 0$, $\frac{1}{\varepsilon_h}\mathcal{L}^2(D_h) \to 0$ and ∂D_h converges with respect to the Hausdorff distance to $P_{\nabla u}$. Moreover for h large enough we may assume $B_{\varepsilon_h}(x_i) \cap B_{\varepsilon_h}(x_j) = \emptyset$ for $i \neq j$. Now we build u_h. Let $\{\rho_h\} \subset \mathbb{R}$ be such that $\rho_h \ge 0$ and $\rho_h \to 0$ when $h \to \infty$. Let $\theta_h \in C^\infty(\Omega)$ with the following property:

$$\begin{cases} \theta_h = 1 & \text{on } B_{\frac{\rho_h}{2}}(x_i) \text{ for any } i = 1, ..., n \\ 0 \le \theta_h \le 1 & \text{on } B_{\rho_h}(x_i) \setminus B_{\frac{\rho_h}{2}}(x_i) \text{ for any } i = 1, ..., n \\ \theta = 0 & \text{on } \Omega \setminus B_{\rho_h}(x_i) \text{ for any } i = 1, ..., n \\ \|\nabla\theta_h\|_\infty \le \frac{1}{\rho_h}. \end{cases} \tag{23}$$

We set $u_h = (1-\theta_h)u$. It is not difficult to check that $\{(u_h, D_h)\}_h \subset Y(\Omega)$ and H-converges to u. We claim that the pair (u_h, D_h) realizes the inequality (22) for a suitable choice of the sequence ρ_h. By making the computation we have

$$\nabla u_h = (1-\theta_h)\nabla u - u\nabla\theta_h.$$

Then

$$\int_\Omega |\nabla u_h - U_0|^p dx = \int_\Omega |\nabla u - U_0 - \theta_h \nabla u - u\nabla\theta_h|^p dx,$$

so that

$$\limsup_{h\to+\infty} \int_\Omega |\nabla u_h - U_0|^p dx \le \limsup_{h\to+\infty} \Big((\int_\Omega |\nabla u - U_0|^p dx)^{\frac{1}{p}} + (\int_\Omega |\theta_h \nabla u|^p dx)^{\frac{1}{p}} + (\int_\Omega |\nabla\theta_h u|^p dx)^{\frac{1}{p}} \Big)^p. \tag{24}$$

Since $|\nabla u|^p \in L^1(\Omega)$, we have by applying the dominated convergence theorem $\int_\Omega |\theta_h \nabla u|^p dx \to 0$. Let us focus on the term $\int_\Omega |\nabla\theta_h u|^p$. By the Sobolev embedding we have $u \in L^{p^*}(\Omega)$ with $p^* = \frac{2p}{2-p}$ and hence $|u|^p \in L^{\frac{p^*}{p}}(\Omega)$, with $\frac{p^*}{p} = \frac{2}{2-p}$.

By (23), using Holder's inequality with dual exponents $\frac{2}{2-p}$ and $\frac{2}{p}$, and taking into account that $p<2$

$$
\begin{aligned}
\int_\Omega |\nabla\theta_h u|^p dx &\leq \sum_{i=1}^n \int_{B_{\rho_h}(x_i)\setminus B_{\frac{\rho_h}{2}}(x_i)} |\nabla\theta_h u|^p dx = \sum_{i=1}^n \Big(\int_{B_{\rho_h}(x_i)} |\nabla\theta_h u|^p dx - \int_{B_{\frac{\rho_h}{2}}(x_i)} |\nabla\theta_h u|^p dx \Big) \\
&\leq \sum_{i=1}^n (\int_{B_{\rho_h}(x_i)} |\nabla\theta_h|^2 dx)^{\frac{p}{2}} \|u\|_{L^{\frac{2}{2-p}}(\Omega)} \leq \sum_{i=1}^n \|u\|_{L^{p^*}(\Omega)} (\frac{\pi^2\rho_h^2}{\rho_h^p}) \to 0. \qquad (25)
\end{aligned}
$$

From (24) it follows that

$$
\begin{aligned}
\limsup_{h\to+\infty} \int_\Omega |\nabla u_h - U_0|^p dx &\leq \lim_{h\to+\infty} \Big((\int_\Omega |\nabla u - U_0|^p dx)^{\frac{1}{p}} | + (\int_\Omega |\theta_h \nabla u|^p dx)^{\frac{1}{p}} + (\int_\Omega |\nabla\theta_h u|^p dx)^{\frac{1}{p}} \Big)^p \\
&= \Big((\int_\Omega |\nabla u - U_0|^p dx)^{\frac{1}{p}} \Big)^p = \int_\Omega |\nabla u - U_0|^p dx. \qquad (26)
\end{aligned}
$$

Now we compute Δu_h. The identity $\mathrm{div}(fA) = f\mathrm{div}A + \nabla f \cdot A$ yields

$$
\Delta u_h = (1-\theta_h)\Delta u - 2\nabla\theta_h \nabla u - \Delta\theta_h u.
$$

Then by choosing ρ_h small enough we have from (23)

$$
\limsup_{h\to+\infty} \int_{\Omega\setminus D_h} |\Delta u_h|^2 dx \leq \lim_{h\to+\infty} \int_{\Omega\setminus D_h} |\Delta u|^2 dx \to \int_\Omega |\Delta u|^2 dx. \qquad (27)
$$

Finally since for h large we have $B_{\varepsilon_h}(x_i) \cap B_{\varepsilon_h}(x_j) = \emptyset$ for $i \neq j$ we get

$$
\lim_h \frac{1}{4\pi} \int_{\partial D_h} (\frac{1}{\varepsilon_h} + \varepsilon_h k^2) d\mathcal{H}^1 = \lim_h \sum_{i=1}^n \frac{1}{4\pi} \int_{\partial B_{\varepsilon_h}} (\varepsilon_h \frac{1}{\varepsilon_h} k^2) d\mathcal{H}^1 = n = \mathcal{H}^0(P_{\nabla u}). \qquad (28)
$$

By recalling that the $\limsup$ is sublinear operation and by (26),(27),(28), we achieve the result. $\square$

4.4 Variational property

We conclude this section by properly stating and proving the particular version of fundamental Theorem, which is, in this case, a direct consequence of Theorems 4.1, 4.2, 4.3. The proof can be achieved by a classical argument (see [7], Section 1.5). However we prefer to give the proof in order to make clear that the classical variational setting is not directly available, and therefore the variational property has to be proven.

Theorem 4.4. *Let $h \in \mathbb{N}$ go to $+\infty$. Let F_ε and $\mathcal{F}$ be given respectively by (15) and (14). If $\{\varepsilon_h\}$ is a sequence of positive numbers converging to zero and $\{(u_h, D_h)\} \subset Y(\Omega)$ such that*

$$
\lim_{h\to+\infty} (F_{\varepsilon_h}(u_h, D_h) - \inf_{Y(\Omega)} F_{\varepsilon_h}(u, D)) = 0,
$$

then there exists a subsequence $\{(u_{h_k}, D_{h_k})\} \subset Y(\Omega)$ *and a minimizer* $\overline{u}$ *of* $\mathcal{F}(u)$ *with* $\overline{u} \in \Delta\mathcal{AM}^{p,2}(\Omega)$*, such that* $\{(u_{h_k}, D_{h_k})\}$ *H-converges to* $\overline{u}$*.*

Proof. We know from Theorems 4.2 and 4.3 that F_ε Γ-converges to $\mathcal{F}$. Let $u \in \Delta\mathcal{AM}^{p,2}(\Omega)$ be such that

$$\mathcal{F}(u) \leq \inf_{\Delta\mathcal{AM}^{p,2}(\Omega)} \mathcal{F}(u) + \delta.$$

From Theorem 4.3 there exists a sequence $\{(\S\tilde{u_h}, \tilde{D_h})\} \subset Y(\Omega)$, such that

$$\inf_{\Delta\mathcal{AM}^{p,2}(\Omega)} \mathcal{F} + \delta \geq \mathcal{F}(u) \geq \limsup_{h\to+\infty} F_{\varepsilon_h}(\tilde{u_h}, \tilde{D_h}).$$

Then since δ is arbitrary it follows that

$$\limsup_{h\to+\infty} \inf_{Y(\Omega)} F_{\varepsilon_h} \leq \limsup_{h\to+\infty} F_{\varepsilon_h}(\tilde{u_h}, \tilde{D_h}) \leq \inf_{\Delta\mathcal{AM}^{p,2}(\Omega)} \mathcal{F}. \tag{29}$$

Let now $\{(u_h, D_h)\} \subset Y(\Omega)$ be such that $\lim_{h\to+\infty}(F_{\varepsilon_h}(u_h, D_h) - \inf_{Y(\Omega)} F_{\varepsilon_h}(u, D)) = 0$. Then from Theorem 4.1, up to subsequences, the sequence $\{(u_h, D_h)\}_h$ H-converges to some $\overline{u} \in \Delta\mathcal{AM}^{p,2}(\Omega)$. Then by Theorem 4.2 and taking into account (29) we deduce

$$\inf_{\Delta\mathcal{AM}^{p,2}(\Omega)} \mathcal{F} \leq \mathcal{F}(\overline{u}) \leq \liminf_{h\to+\infty} \inf_{Y(\Omega)} F_{\varepsilon_h} \leq \limsup_{h\to+\infty} \inf_{Y(\Omega)} F_{\varepsilon_h} \leq \inf_{\Delta\mathcal{AM}^{p,2}(\Omega)} \mathcal{F}.$$

Then we easily get the thesis. $\square$

5 Approximation by smooth function

By following the Braides-March's approach in [9] we approximate the measure $\mathcal{H}^1\lfloor\partial D$ by the Modica-Mortola's energy density given by $(\varepsilon|\nabla w|^2+\frac{1}{\varepsilon}W(w))dx$ where $W(w)=w^2(1-w)^2$ and $w\in C^\infty(\Omega)$. The next step is to replace the regular set D with the level set of w. Let us set $Z=\{\nabla w(x)=0\}$. By Sard's Lemma we have that $\mathcal{L}^1(w(Z))=0$. In particular, if w takes values into the interval $[0,1]$, we infer that for almost every $t\in(0,1)$ the set $Z\cap w^{-1}(t)$ is empty. Consequently for almost every $t\in(0,1)$ the t-level set $\{w<t\}$ is a regular set with boundary $\{w=t\}$. Now, since we want to replace the set D, we need that $\{w<t\}\subset\subset\Omega$. Then we require $1-w\in C_0^\infty(\Omega;[0,1])$. Furthermore for almost every t, we have $k(\{w=t\})=\mathrm{div}(\frac{\nabla w}{|\nabla w|})$. From all of this we are led to define the following space:

$$S(\Omega)=\{(u,w);\ u\in W_0^{1,p,2}(\mathrm{div};\Omega);\ 1-w\in C_0^\infty(\Omega;[0,1])\} \tag{30}$$

and having in mind the coarea formula, the following sequence of functionals defined on $S(\Omega)$

$$\begin{aligned}\mathcal{G}_\varepsilon(u,w) &= \int_\Omega w^2|\Delta u|^2dx+\frac{1}{8\pi C}\int_{\Omega\setminus\{\nabla w=0\}}\Big(\frac{1}{\beta_\varepsilon}+\beta_\varepsilon\big(\mathrm{div}(\frac{\nabla w}{|\nabla w|})\big)^2(\varepsilon|\nabla w|^2+\frac{1}{\varepsilon}W(w)\Big)dx\\ &+\int_\Omega|\nabla u-U_0|^pdx+\frac{1}{\gamma_\varepsilon}\int_\Omega(1-w)^2dx,\end{aligned} \tag{31}$$

with $C=\int_0^1\sqrt{W(t)}dt$. The last term forces w_ε be 1 almost everywhere in the limit. From now on the parameters ε,β_ε,γ_ε will be related as follows

$$\lim_{\varepsilon\to 0^+}\frac{\beta_\varepsilon}{\gamma_\varepsilon}=0, \tag{32}$$

$$\lim_{\varepsilon\to 0^+}\frac{\varepsilon|\log(\varepsilon)|}{\beta_\varepsilon}=0. \tag{33}$$

The convergence that plays the role of the H-convergence is the following. With a slight abuse of notation this convergence will be still denoted by H.

Definition 5.1. *Let $h\in\mathbb{N}$ goto $+\infty$ and $\{(u_h,w_h)\}_h$ be a sequence $S(\Omega)$. Set $D_h^t=\{w_h<t\}$. We say that $\{(u_h,w_h)\}_h$ H-converges to $u\in\Delta\mathcal{AM}^{p,2}(\Omega)$, if for every $t\in(0,1)$ the sequence $\{(u_h,D_h^t)\}_h$ in $Y(\Omega)$ H-converges to u.*

As in the previous Section, we adopte the ad hoc definition of Γ-convergence with respect to the convergence above.

Definition 5.2. *Let $h\in\mathbb{N}$ go to $+\infty$. We say that $\mathcal{G}_\varepsilon$ Γ-converges to $\mathcal{F}$ if, for every sequence of positive numbers $\varepsilon_h\to 0$ and for every $u\in\Delta\mathcal{AM}^{p,2}(\Omega)$, we have:*

1. *for every sequence* $\{(u_h, w_h)\}_h \subset S(\Omega)$ *H-converging to* u

$$\liminf_{h\to+\infty} \mathcal{G}_{\varepsilon_h}(u_h, w_h) \geq \mathcal{F}(u);$$

2. *there exists a sequence* $\{(u_h, w_h)\}_h \subset S(\Omega)$ *H-converging to* u *such that*

$$\limsup_{h\to+\infty} \mathcal{G}_{\varepsilon_h}(u_h, w_h) \leq F(u).$$

As in the previous Section, we remark that the convergence of the minimum problems must be proved, since we cannot apply the fundamental Theorem of Γ-convergence. We will state the analogous of Theorem 4.4 at the end of the Section.

5.1 Compactness

The compactness result goes as follows.

Theorem 5.1. *Let* $h \in \mathbb{N}$ *goes to* $+\infty$ *and* $\varepsilon_h \to 0$ *such that*

$$F_{\varepsilon_h}(u_h, w_h) \leq M. \tag{34}$$

Then there exists a subsequence $\{(u_{h_k}, w_{h_k})\}_k \subset S(\Omega)$, $u \in \Delta\mathcal{AM}^{p,2}(\Omega)$ *such that* $\{(u_{h_k}, w_{h_k})\}_k$ *H-converges to* u.

Proof. The first part of proof is as in [9]. For the convenience of the reader we give the complete proof.

By Young's inequality and by (34) we get

$$M \geq 2\int_{\Omega\setminus\{|\nabla w_h|=0\}} |\nabla w_h|\sqrt{W(w_h)}\Big(\big(\frac{1}{\beta_{\varepsilon_h}} + \beta_{\varepsilon_h}\big(\mathrm{div}(\frac{\nabla w_h}{|\nabla w_h|})\big)^2\Big)dx.$$

Now by coarea formula, we obtain

$$M \geq 2\int_0^1 \sqrt{W(t)}\int_{\{w_h=t\}\cap\{|\nabla w_h|\neq 0\}} \Big(\frac{1}{\beta_{\varepsilon_h}} + \beta_{\varepsilon_h}\big(\mathrm{div}(\frac{\nabla w_h}{|\nabla w_h|})\big)^2\Big)d\mathcal{H}^1 dt. \tag{35}$$

Thanks to Sard's Lemma, for any h there exists a $\mathcal{L}^1$-negligible set $\mathcal{N}_{w_h} \subseteq (0,1)$ such that

$$\{w_h = t\} = \partial\{w_h < t\},\ \{w_h < t\} \in R(\Omega),\ \text{for } t \in (0,1)\setminus\mathcal{N}_{w_h}.$$

On $\{w_h = t\}$ for $t \in (0,1)\setminus\mathcal{N}_{w_h}$ we have

$$|\nabla w_h| \neq 0 \text{ and } \kappa(\{w_h = t\}) = \mathrm{div}(\frac{\nabla w_h}{|\nabla w_h|}).$$

Now since the union $\bigcup_h \mathcal{N}_{w_h h}$ of the sets $\mathcal{N}_{w_h}$ is $\mathcal{L}^2$-negligible (almost countable) from (35) we have

$$M \geq 2 \int_{(0,1)\setminus \bigcup_h \mathcal{N}_{w_h}} \sqrt{W(t)} \int_{\partial\{w_h<t\}} \Big(\frac{1}{\beta_{\varepsilon_h}} + \beta_{\varepsilon_h}\kappa^2\Big) d\mathcal{H}^1 dt.$$

By applying Fatou's Lemma and taking into account that the set $\bigcup_h \mathcal{N}_{w_h}$ does not depend on h we get

$$M \geq 2 \int_{(0,1)\setminus \bigcup_h \mathcal{N}_{w_h}} \sqrt{W(t)} \liminf_{h\to+\infty} \int_{\partial\{w_h<t\}} (\frac{1}{\beta_{\varepsilon_h}} + \beta_{\varepsilon_h}\kappa^2) d\mathcal{H}^1 dt. \tag{36}$$

Hence we deduce the existence of a $\mathcal{L}^2$-negligible set Q, with $\bigcup_h \mathcal{N}_{w_h} \subseteq Q$, such that

$$\liminf_{h\to+\infty} \int_{\partial\{w_h<t\}} (\frac{1}{\beta_{\varepsilon_h}} + \beta_{\varepsilon_h}\kappa^2) d\mathcal{H}^1 \leq M_t, \tag{37}$$

where the constant M_t does not depend on h.

Then for every $t \in (0,1) \setminus Q$ we can extract a sequence $\{w_h^t\}_h$ such that $\partial\{w_h^t < t\}$ converges with respect to the Hausdorff metric to a set $P^t \subset \overline{\Omega}$. Let $\mathcal{N} = \{t_i\}$ in $(0,1)$ be a dense countable set. Up to a diagonal argument we can find a subsequence $\{w_{h_k}\}_k$ such that, for every $t_i \in \mathcal{N}$, $\partial\{w_{h_k} < t_i\}$ converges to P^{t_i}. Let $t_i \in \mathcal{N}$ such that $t_i > t$ and consequently $\{w_{h_k} < t\} \subseteq \{w_{h_k} < t_i\}$. From the definition of Hausdorff convergence it follows that for every $\rho > 0$ there exists $k_0(\rho)$ such that for any $k > k_0$ we have $\{w_{h_k} < t_i\} \cap B_\rho(x) \neq \emptyset$ for every $x \in P^{t_i}$. Since the $t-$-level set is open for every ρ and for every $x \in P^{t_i}$ such that $\{w_{h_k} < t\} \cap B_\rho(x) \neq \emptyset$, we may choose $t_n \in \mathcal{N}$ with $t_n < t$ and so obtain for k large enough $\{w_{h_k} < t_n\} \cap B_\rho(x) \neq \emptyset$. By choosing $t_{max} = \max\limits_{x\in P^{t_i}} t_n(x)$ for every $x \in P^{t_i}$, the inclusion $\{w_{h_k} < t_{max}\} \subset \{w_{h_k} < t\}$ gives

$$\{w_{h_k} < t_{max}\} \cap B_\rho(x) \subseteq \{w_{h_k} < t\} \cap B_\rho(x) \subseteq \{w_{h_k} < t_n\} \cap B_\rho(x),$$

with $t_{max}, t_i \in \mathcal{N}$. Then by taking the limit $\rho \to 0^+$ we infer $\partial\{w_{h_k} < t\}$ converges with respect to the Hausdorff metric to a set $P^t \subset \overline{\Omega}$ for every $t \in (0,1)$.

Finally for any $t \in (0,1)$ since $0 \leq w_h \leq 1$, we have $\mathcal{L}^2(\{w_h < t\}) = \mathcal{L}^2(\{1 - w_h > 1 - t\}) \leq \mathcal{L}^2(\{1 - w_h > (1-t)^2\})$, then by Chebyshev's inequality and by (34)

$$\mathcal{L}^2(\{w_h < t\}) \leq \frac{M\gamma_{\varepsilon_h}}{(1-t)^2}. \tag{38}$$

Therefore, as in the proof of Theorem 4.1, we can extract a subsequence $\{u_{h_k}\}_k$ which converges strongly in $L^p(\Omega)$ to a function $u \in \Delta\mathcal{AM}^{p,2}(\Omega)$ with $P_{\nabla u} \subseteq P^t$ for every $t \in (0,1)$. Hence we have that for every $t \in (0,1)$ the sequence $\{(u_{h_k}, D^t_{h_k})\}_k$ H-converges to u and the proof is achieved. $\square$

5.2 Lower bound

We give the proof of the lower bound (1) in Definition 5.2. In the proof it will be crucial having the convergence of the t-level set for every $t \in (0,1)$.

Theorem 5.2. *Let $h \in \mathbb{N}$ go to $+\infty$. Let $\{\varepsilon_h\}_h$ be a sequence of positive numbers converging to zero. For every sequence $\{(u_h, w_h)\}_h \subset S(\Omega)$ H-converging to $u \in \Delta\mathcal{AM}^{p,2}(\Omega)$, we have*

$$\liminf_{h\to+\infty} F_{\varepsilon_h}(u_h, w_h) \geq \mathcal{F}(u).$$

Proof. Without loss of generality we assume, up to subsequences,

$$+\infty > \liminf_{h\to+\infty} \mathcal{F}_{\varepsilon_h} = \lim_{h\to+\infty} \mathcal{F}_{\varepsilon_h}.$$

As in the proof of Theorem 5.1 we get that for every $t \in (0,1)$ $\mathcal{L}^2(\{w_h < t\}) \to 0$ and $\partial\{w_h < t\} \to P^t$ in the Hausdorff distance. For any $t \in (0,1)$ we have (see also [9] for a similar argument)

$$\int_\Omega {w_h}^2|\Delta u_h|^2 dx = \int_{\{w_h<t\}} {w_h}^2|\Delta u_h|^2 dx + \int_{\{w_h\geq t\}} {w_h}^2|\Delta u_h|^2 dx \geq t^2 \int_\Omega (1-\chi_{\{w_h<t\}})|\Delta u_h|^2 dx. \tag{39}$$

Let $\{\Omega_j\}_j$ be a sequence of open sets $\Omega_j \subset\subset \Omega \setminus P^t$ invading $\Omega \setminus P^t$. Then we may assume that $u_h \rightharpoonup$ weakly in $W_0^{1,p}(\Omega)$ and Δu_h converges weakly in $L^2(\Omega_j)$ to Δu. Therefore as in the proof of Theorem 4.2 we get

$$\lim_{h\to+\infty} \int_\Omega |\nabla u_h - U_0|^p dx \geq \int_\Omega |\nabla u - U_0|^p dx, \tag{40}$$

and

$$\liminf_{h\to+\infty} t^2 \int_\Omega (1-\chi_{\{w_h<t\}})|\Delta u_h|^2 dx \geq t^2 \int_{\Omega_j} |\Delta u|^2 dx,$$

for any j.

Then by (39) and, by taking into account that $|\Delta u|$ is in $L^2(\Omega \setminus P^t)$ with $P_{\nabla u} \subseteq P_t$, it follows that

$$\liminf_{h\to+\infty} \int_\Omega {w_h}^2|\Delta u_h|^2 dx \geq t^2 \int_\Omega |\Delta u|^2 dx.$$

And eventually by taking the limit $t \to 1$

$$\liminf_{h\to+\infty} \int_\Omega {w_h}^2|\Delta u_h|^2 dx \geq \int_\Omega |\Delta u|^2 dx. \tag{41}$$

Finally, as in the proof of Theorem 4.2 (inequality 21) we have

$$\liminf_{h\to+\infty} \frac{1}{4\pi} \int_{\partial\{w_h<t\}} \Big(\frac{1}{\beta_{\varepsilon_h}} + \beta_{\varepsilon_h}\ k^2\Big) d\mathcal{H}^1 \geq \mathcal{H}^0(P^t) \geq \mathcal{H}^0(P_{\nabla u}). \tag{42}$$

Now arguing as in the proof of Theorem 5.1 and by taking into account (42), we get

$$\begin{aligned}
&\liminf_{h\to+\infty}\int_{\Omega\setminus\{\nabla w_h=0\}}\Big(\frac{1}{\beta_{\varepsilon_h}}+\beta_{\varepsilon_h}\big(\mathrm{div}(\frac{\nabla w_h}{|\nabla w_h|})\big)^2(\varepsilon_h|\nabla w_h|^2+\frac{1}{\varepsilon_h}W(w_h))dx\\
\geq\;& 2\liminf_{h\to+\infty}\int_{(0,1)\setminus\bigcup_h\mathcal{N}_{w_h}}\sqrt{W(t)}\liminf_{h\to+\infty}\int_{\partial\{w_h<t\}}(\frac{1}{\beta_{\varepsilon_h}}+\beta_{\varepsilon_h}k^2)d\mathcal{H}^1dt\\
\geq\;& 2\int_{(0,1)}\mathcal{H}^0(P^t)\sqrt{W(t)}dt\geq 8\pi C\mathcal{H}^0(P_{\nabla u}). \qquad (43)
\end{aligned}$$

By collecting (40) (41) and (43) we achieve the thesis. □

5.3 Upper bound

As in [9] to build w_h we use the construction given in [4], while the optimal sequence u_k is chosen as in Theorem 4.3.

Theorem 5.3. *Let $h\in\mathbb{N}$ go to $+\infty$. Let ε_h be a sequence of positive numbers with $\varepsilon_h\to 0$. For every $u\in\Delta\mathcal{AM}^{p,2}$ there exists a sequence $\{(u_h,w_h)\}_h\subset S(\Omega)$, H-converging to u, such that*

$$\limsup_{h\to\infty}F_{\varepsilon_h}(u_h,w_h)\leq\mathcal{F}(u). \qquad (44)$$

Proof. If $A\subset\mathbb{R}^2$ we set

$$\delta_A(x)=d(x,A)-d(x,\mathbb{R}^2\setminus A).$$

We start with the construction of w_h.

As in the proof of Theorem 4.4 we set $P_{\nabla u}=\{x_1,...,x_n\}$ and we define

$$D_h=\bigcup_{i=1}^n B_{\beta_{\varepsilon_h}}(x_i).$$

Since D_h is a regular set by taking into account the condition (33) for h large enough we have

$$\{x\in\Omega:\ d(x,D_h)<2\varepsilon_h|\log\varepsilon_h|\}\subset\subset\Omega. \qquad (45)$$

Let η be the optimal profile for Modica-Mortola's energy, that is the solution of the ODE

$$\begin{cases}\eta'(t)=\sqrt{W(t)} & \text{on }\mathbb{R}\\ \eta(-\infty)=0,\\ \eta(+\infty)=1,\end{cases}$$

given by $\eta(t)=\frac{1}{2}(1+\tanh\frac{t}{2})$.

For every h let $\psi_h : [0, +\infty) \to [0, 1]$ be a C^∞-function such that

$$\begin{cases} \psi_h = 1 & \text{on } [0, |\log \varepsilon_h|] \\ \psi_h = 0 & \text{on } [2|\log \varepsilon_h|, +\infty] \\ {\psi_h}' < 0 & \text{on } [|\log \varepsilon_h|, 2|\log \varepsilon_h|] \\ \|{\psi_h}'\|_{L^\infty(|\log \varepsilon_h|, 2|\log \varepsilon_h|)} = O(\frac{1}{|\log \varepsilon_h|}). \end{cases}$$

As in [4] and in [9] we define

$$\eta_h(t) = \begin{cases} \eta(\frac{t}{\varepsilon_h})\psi(\frac{t}{\varepsilon_h}) + 1 - \psi(\frac{t}{\varepsilon_h}) & \text{if } t \geq 0 \\ \psi(\frac{t}{\varepsilon_h}) - \eta(\frac{t}{\varepsilon_h})\psi(\frac{t}{\varepsilon_h}) & \text{if } t < 0. \end{cases}$$

Then we set $w_h(x) = \eta_h(\delta_{D_h}(x))$. We claim that $1 - w_h(x) \in C_0^\infty(\Omega; [0, 1])$ for h large enough. Indeed for any $x \in \Omega \setminus D_h$ we have $\delta_{D_h}(x) \geq 0$, hence

$$1 - w_h(x) = \psi_h(\frac{\delta_{D_h}(x)}{\varepsilon_h})(1 - \eta(\frac{\delta_{D_h}(x)}{\varepsilon_h})).$$

Then $0 \leq 1 - w_h \leq 1$ by the properties of ψ_h and η. The case $x \in D_h$ is similar. Let now $x \in \partial\Omega$ then $\delta_{D_h}(x) \geq 0$ and $1 - w_h(x) = \psi_h(\frac{\delta_{D_h}(x)}{\varepsilon_h})(1 - \eta(\frac{\delta_{D_h}(x)}{\varepsilon_h}))$. From (45), it follows $\delta_{D_h}(x) \geq 2|\log \varepsilon_h|$ for h large enough, hence the claim follows. Then we take $\{(u_h, w_h)\}_h$ as optimal sequence, where u_h is given as in Theorem 4.4

First of all we have to check that $\{(u_h, w_h)\}_h$ H-converges to u. For any $x \in \Omega \setminus P_{\nabla u}$ we have that for h large enough $\delta_{D_h}(x) \geq 0$ and one can check that $w_h(x) \to 1$ for every $x \in \Omega \setminus P_{\nabla u}$. This implies that $\mathcal{L}^2(\{w_h < t\}) \to 0$ for every $t \in (0, 1)$. Now for every $t \in (0, 1)$ we write

$$\{w_h = t\} = \big(\{w_h = t\} \cap D_h\big) \cup \big(\{w_h = t\} \cap \Omega \setminus D_h\big). \tag{46}$$

Hence, since $w_h(x) \to 1$ for $x \in \Omega \setminus D_h$, for any $t \in (0, 1)$ there exists $h(t)$ such that $\{w_h = t\} \cap \Omega \setminus D_h = \emptyset$ for every $h \geq h(t)$. So that from (46) it follows that for every $t \in (0, 1)$, $\{w_h = t\} \to P_{\nabla u}$ when $h \to +\infty$. So we can conclude that (u_h, w_h) H-converges to u.

As in [4] we set

$$D_h^1 = \{x \in \Omega : |\delta_{D_h}(x)| < \varepsilon_h|\log \varepsilon_h|\},\ D_h^2 = \{x \in \Omega : \varepsilon_h|\log \varepsilon_h| < |\delta_{D_h}(x)| < 2|\varepsilon_h|\log \varepsilon_h|\}.$$

Therefore we can write

$$\begin{aligned} &\int_{\Omega\setminus\{\nabla w_h = 0\}} (\frac{1}{\beta_{\varepsilon_h}} + \beta_{\varepsilon_h}\big(\mathrm{div}(\frac{\nabla w_h}{|\nabla w_h|})\big)^2(\varepsilon_h|\nabla w_h|^2 + \frac{1}{\varepsilon_h}W(w_h))dx \\ &= \int_{D_h^1} (\frac{1}{\beta_{\varepsilon_h}} + \beta_{\varepsilon_h}\big(\mathrm{div}(\frac{\nabla w_h}{|\nabla w_h|})\big)^2(\varepsilon_h|\nabla w_h|^2 + \frac{1}{\varepsilon_h}W(w_h))dx + \\ &\quad \int_{D_h^2} (\frac{1}{\beta_{\varepsilon_h}} + \beta_{\varepsilon_h}\big(\mathrm{div}(\frac{\nabla w_h}{|\nabla w_h|})\big)^2(\varepsilon_h|\nabla w_h|^2 + \frac{1}{\varepsilon_h}W(w_h))dx = \\ &= I_h + II_h. \end{aligned} \tag{47}$$

For $x \in D_h^1$, we have $\frac{|\delta_{D_h}|}{\varepsilon_h} < \log \varepsilon_h$ therefore $w_h(x) = \eta(\frac{|\delta_{D_h}|}{\varepsilon_h})$. By taking into account the definition of η we have $\frac{\varepsilon_h}{1+\varepsilon_h} \leq w_h \leq \frac{1}{1+\varepsilon_h}$. Moreover it easy to check that

$$\eta_h'(t) = \frac{1}{\varepsilon_h}\eta'(t) = \frac{1}{\varepsilon_h}\sqrt{W(t)}; \quad |\nabla w_h| = |\eta_k'(\delta_{D_h})|.$$

This, together with the coarea formula yields

$$I_h = 2\int_{\frac{\varepsilon_h}{1+\varepsilon_h}}^{\frac{1}{1+\varepsilon_h}} \sqrt{W(t)} \int_{\{w_h = t\}} (\frac{1}{\beta_h} + \beta_h k^2) d\mathcal{H}^1 dt.$$

Now we have that $\partial D_h = \{w_h = \frac{1}{2}\}$ then

$$I_h = 2\int_{\frac{\varepsilon_h}{1+\varepsilon_h}}^{\frac{1}{1+\varepsilon_h}} \sqrt{W(t)} \int_{\partial D_h} (\frac{1}{\beta_h} + \beta_h k^2) d\mathcal{H}^1 dt + O(\varepsilon_h \log(\varepsilon_h)) \int_{\frac{\varepsilon_h}{1+\varepsilon_h}}^{\frac{1}{1+\varepsilon_h}} \sqrt{W(t)} dt.$$

Then

$$\lim_{h\to+\infty} I_h = 8\pi\mathcal{H}^0(P_{\nabla u}) \int_0^1 \sqrt{W(t)} dt. \tag{48}$$

Moreover with the same argument and by using the definition of w_h one can check that

$$\lim_{h\to\infty} II_h = 0 \tag{49}$$

By (48) and (49) we have

$$\lim_{h\to\infty} \frac{1}{8\pi C} \int_{\Omega\setminus\{\nabla w_h = 0\}} (\frac{1}{\beta_{\varepsilon_h}} + \beta_{\varepsilon_h}\big(\mathrm{div}(\frac{\nabla w_h}{|\nabla w_h|})\big)^2 (\varepsilon_h |\nabla w_{\varepsilon_h}|^2 + \frac{1}{\varepsilon_h} W(w_{\varepsilon_h})) dx = \mathcal{H}^0(P_{\nabla u}). \tag{50}$$

Now let us examine the terms involving u_h. As in the proof of Theorem 4.3 we have

$$\limsup_{h\to+\infty} \int_\Omega |\nabla u_h - U_0|^p dx \leq \int_\Omega |\nabla u - U_0|^p dx. \tag{51}$$

Furthermore, by taking into account that $w_h(x) = 1$ if $\delta_{D_h}(x) \geq 2\varepsilon_h |\log(\varepsilon_h)|$ and $w_h(x) = 0$ if $\delta_{D_h}(x) < -2\varepsilon_h |\log(\varepsilon_h)|$, we have

$$\int_\Omega w_h^2 |\Delta u_h|^2 dx = \int_{\Omega\setminus D_h^0} |\Delta u_h|^2 dx,$$

where $D_h^0 = \{x \in \Omega : \ \delta_{D_h}(x) < -2\varepsilon_h |\log(\varepsilon_h)|\}$. By choosing ρ_h in such a way that $\rho_h \leq 2\varepsilon_h |\log(\varepsilon_h)|$, we obtain

$$\limsup_{h\to+\infty} \int_\Omega w_h^2 |\Delta u_h|^2 dx = \lim_{h\to+\infty} \int_{\Omega\setminus D_h^0} |\Delta u|^2 dx = \int_{\Omega\setminus P_{\nabla u}} |\Delta u|^2 dx = \int_\Omega |\Delta u|^2 dx. \tag{52}$$

Finally from the definition of w_h, it follows that $w_h = 1$ outside the disk $(D_h)_{2\log\varepsilon_h}$ and hence thanks to (32) and (33)

$$\limsup_{h\to+\infty} \frac{1}{\gamma_h}\int_\Omega (1-w_h)^2 dx \leq \lim_{h\to+\infty} \mathcal{L}(D_{h\,2\log\varepsilon_h})\frac{1}{\gamma_h} = 0. \tag{53}$$

The thesis follows by collecting (50), (51), (52) and (53) $\square$

5.4 Variational property

Also in this case we obtain, as a direct consequence of Theorems 5.1, 5.2 and 5.3, the convergence of the minimum problems. The proof is as in Theorem 4.4.

Theorem 5.4. *Let $h \in \mathbb{N}$ go to $+\infty$. Let $\mathcal{G}_\varepsilon$ and $\mathcal{F}$ be given respectively by (31) and (14). If $\{\varepsilon_h\}$ is a sequence of positive numbers converging to zero and $\{(u_h, w_h)\} \subset S(\Omega)$ such that*

$$\lim_{h\to+\infty}(\mathcal{G}_{\varepsilon_h}(u_{\varepsilon_h}, w_{\varepsilon_h}) - \inf_{S(\Omega)} \mathcal{F}_{\varepsilon_h}(u, w)) = 0,$$

then there exists a subsequence $\{(u_{h_k}, w_{h_k})\} \subset S(\Omega)$ and a minimizer $\overline{u}$ of $\mathcal{F}(u)$, with $\overline{u} \in \Delta\mathcal{AM}^{p,2}(\Omega)$, such that $\{(u_{h_k}, w_{h_k})\}_k$ H-converges to $\overline{u}$.

6 De Giorgi's Conjecture

The aim of De Giorgi was finding a variational approximation of a curvature depending functional of the type:

$$F^2(D) = \int_{\partial D} (1+\kappa^2) d\mathcal{H}^1;$$

where D is a regular set and κ is a curvature of its boundary ∂D.

Since ∂D can be represented as the discontinuity set of the function $w_0 = 1 - \chi_D$, by Modica-Mortola's Theorem it follows that there is a sequence of non constant local minimizers such that $w_\varepsilon \to w_0$ with respect to the L^1-convergence such that

$$\lim_{\varepsilon\to 0} F^1_\varepsilon(w_\varepsilon) := C_V \mathcal{H}^1(\partial D).$$

Furthermore looking at the Euler-Lagrange equation associated to a contour length term, yields a contour curvature term κ, while the Euler-Lagrange equations for the functional $F^1_\varepsilon(w)$ contains a term $2\varepsilon\Delta w - \frac{W'(w)}{\varepsilon}$.

Then De Giorgi suggested to approximate by Γ-convergence the functional F^2 by adding to the Modica-Mortola approximating functionals the following term

$$F^2_\varepsilon(w) = \int_\Omega (2\varepsilon\Delta w - \frac{W'(w)}{\varepsilon})^2 dx.$$

In ([5]) Bellettini and Paolini have proven the lim sup inequality, while the validity of lim inf inequality for a modified version of the original conjecture has been proven by Röger and Shätzle (see [23]).

Inspired by the De Giorgi's conjecture (see [15] for the original statement) it appears natural to investigate, in the spirit of [9], the possibility of approximating the functional $\mathcal{F}$ by means of a sequence $\mathcal{F}_\varepsilon$ much more convenient from a numerical point view (see [19]):

$$\begin{aligned}\mathcal{F}_\varepsilon(u,w) &= \int_\Omega w^2|\Delta u|^2 dx + \frac{1}{8\pi C}\Big(\frac{\beta_\varepsilon}{2\varepsilon}\int_\Omega (2\varepsilon\Delta w - \frac{W'(w)}{\varepsilon})^2 dx + \frac{1}{\beta_\varepsilon}\int_\Omega (\varepsilon|\nabla w|^2 + \frac{1}{\varepsilon}W(w))dx\Big)\\ &+ \int_\Omega |\nabla u - U_0|^p dx + \int_\Omega \frac{1}{\gamma_\varepsilon}(1-w^2)dx.\end{aligned}$$

The presence of the term $\frac{1}{2\varepsilon}$ will be clear in the proof. By the way we are able to prove only the Γ-limsup inequality.

Theorem 6.1. *Let $h \in \mathbb{N}$ go to $+\infty$. Let ε_h be a sequence of positive numbers with $\varepsilon_h \to 0$. For every $u \in \Delta\mathcal{AM}^{p,2}(\Omega)$, there exists a sequence $\{(u_h, w_h)\}_h \subset S(\Omega)$ H-converging to u such that*

$$\limsup_{\to+\infty} \mathcal{F}_{\varepsilon_h}(u_h, w_h) \le \mathcal{F}(u). \tag{54}$$

Proof. Let $\{(u_h, w_h)\}_h$ be the optimal sequence of Theorem 5.3. It is not difficult to see that for every $x \in D_h^1$ the function $\delta_h(x)$ is regular and using the definition of w_h and taking into account that $\eta' = \sqrt{W(\eta)}$ the following identity holds

$$2\varepsilon_h \Delta w_h - \frac{W'(w_h)}{\varepsilon_h} = 2\varepsilon_h \eta_h' \Delta \delta_{D_h}(x) + 2\varepsilon \eta_h'' - \frac{W'(w_h)}{\varepsilon_h} = 2\varepsilon_h \eta_h'(\delta_{D_h}(x)).$$

For h large enough we also have $\Delta\delta_{D_h}(x) = \kappa(\{\delta_{D_h}(x) = t\})$. Besides on D_h^1 we have $w_h(x) = \eta(\frac{\delta_{D_h}(x)}{\varepsilon_h})$ and hence the level set $\{\delta_{D_h}(x) = t\}$ corresponds to the level set $\{w_h(x) = \eta(\frac{t}{\varepsilon_h}\}$ with $0 \leq \eta \leq 1$, so that we infer

$$\kappa(\{\delta_{D_h}(x) = t\}) = \operatorname{div}(\frac{\nabla w_h}{|\nabla w_h|}).$$

By proceeding as in the proof of Theorem 5.3 and taking into account the equality $2\varepsilon_h |\eta_h'(\delta_{D_h}(x))| = 2\varepsilon_h |\nabla w_h|$ we have

$$\begin{aligned} I_h &= \int_{D_h^1} \frac{\beta_{\varepsilon_h}}{2\varepsilon_h} (2\varepsilon_h \Delta w_h - \frac{W'(w_h)}{\varepsilon_h})^2 + \frac{1}{\beta_{\varepsilon_h}} (\varepsilon_h |\nabla w_h|^2 + \frac{1}{\varepsilon_h} W(w_h)) dx \\ &= 2 \int_{D_h^1} (\beta_\varepsilon (\operatorname{div}(\frac{\nabla w_h}{|\nabla w_h|})^2 + \frac{1}{\beta_{\varepsilon_h}}) \sqrt{W(w_h)} |\nabla w_h| dx. \end{aligned}$$

Then as in the proof of Theorem 5.3 we conclude

$$\lim_{h \to +\infty} I_h = 8\pi \mathcal{H}^0(P_{\nabla u}) \int_0^1 \sqrt{W(t)} dt.$$

By the same calculation on D_h^2 one can check that the integral over D_h^2 vanishes as in the proof of Theorem 5.3.

The other terms can be estimated exactly as in the proof of Theorem 5.3 and therefore the thesis is achieved. $\square$

References

[1] L.Ambrosio, N.Fusco, D.Pallara. *Functions of bounded variation and free discontinuity problems.* Oxford University Press (2000).

[2] G.Anzellotti. *Pairings between measures and bounded functions and compensated compactness.* Ann. Mat. Pura Appl. **135** (1983), 293-318.

[3] G.Aubert, J. Aujol, L.Blanc-Feraud. *Detecting Codimension-Two Objects in an image with Ginzurbg-Landau Models.* International Journal of Computer Vision **65** (2005), 29-42.

[4] G.Bellettini. *Variational approximation of functionals with curvatures and related properties.* J.Conv. Anal. **4** (1997)., 91-108.

[5] G.Bellettini, M.Paolini. *Approssimazione variazionale di funzionali con curvatura.* Seminario di Analisi Matematica, Dipartimento di Matematica dell'Università di Bologna, 1993.

[6] F.Bethuel, H.Brezis and F.Hélein. *G*inzburg-Landau Vortices. Birkäuser, Boston (1994).

[7] A.Braides. Γ*-convergence for beginners.* Oxford University Press, New york (2000).

[8] A.Braides, A.Malchiodi. *Curvature Theory of Boundary phases: the two dimensional case.* Interfaces Free. Bound. **4** (2002), 345-370.

[9] A.Braides, R.March. *Approximation by* Γ*-convergence of a curvature-depending functional in Visual Reconstruction.* Comm. Pure Appl. Math. **59** (2006), 71-121.

[10] G.Dal Maso. *Introduction to* Γ*-convergence.* Birkhäuser, Boston(1993).

[11] A.Chambolle, F.Doveri. *Continuity of Neumann linear elliptic problems on varying two-dimensionals bounded open sets.* Comm.Partial Diff.Eq. **22** (1997), 811-840.

[12] G.Q.Chen, H.Fried *Divergence-measure fields and conservation laws.* Arch. Rational Mech. Anal. **147** (1999), 35-51.

[13] G.Q.Chen, H.Fried. *On the theory of divergence-measure fields and its applications.*. Bol. Soc. Bras. Math. **32** (2001), 1-33.

[14] G.Dal Maso, F.Murat, L.Orsina, A.Prignet. *Renormalized solutions of elliptic equations with general measure data*, Ann. Scuola Norm. Sup. Pisa Cl. Sci, **28** (1999), 741-808.

[15] E.De Giorgi. *Some Remarks on* Γ*-convergence and least square methods, in* Composite Media and Homogenization Theory (G. Dal Maso and G.F.Dell'Antonio eds.), Birkhauser, Boston, 1991, 135-142.

[16] E.De Giorgi, T.Franzoni. *Su un tipo di convergenza variazionale.* Atti Accad. Naz. Lincei Rend. Cl. Sci. Mat. Natur. **58** (1975), 842-850.

[17] E.De Giorgi, T.Franzoni. *Su un tipo di convergenza variazionale.* Rend. Sem. Mat. Brescia **3** (1979), 63-101.

[18] L.C.Evans, R.F. Gariepy. *Measure Theory and Fine Properties of Functions.* CRC Press (1992).

[19] D.Graziani, L.Blanc-Feraud, G.Aubert. *A formal* Γ*-convergence approach for the detection of points in* 2*-D images.* Research report Inria (2008).

[20] J.Heinonen, T.Kilpeläinen, O.Martio. *Nonlinear Potential Theory of Degenerate Elliptic Equations*, Oxford University Press, (1993).

[21] L.Modica, S.Mortola. *Un esempio di* Γ*-convergenza.* Boll. Un. Mat. Ital. 14-B(1977), $285-299$.

[22] L.Modica. *The gradient theory of phase transitions and the minimal interface criterion.* Arch. Rational Mech. Anal. **98** (1987), 123-142.

[23] M.Röger, R.Shätzle. *On a modified conjecture of De Giorgi.* Math. Zeitschrift. **254** (2006), 675-714

[24] G.Stampacchia. *Le problème de Dirichlet pour les equations elliptiques du seconde ordre à coefficients discontinuous.* Ann. Inst. Fourier (Grenoble), **15** (1965), 180-258.

[25] W.Ziemer. *Weakly Differentiable Functions.* Springer-Verlag, New york (1989).

Unité de recherche INRIA Sophia Antipolis
2004, route des Lucioles - BP 93 - 06902 Sophia Antipolis Cedex (France)

Unité de recherche INRIA Futurs : Parc Club Orsay Université - ZAC des Vignes
4, rue Jacques Monod - 91893 ORSAY Cedex (France)
Unité de recherche INRIA Lorraine : LORIA, Technopôle de Nancy-Brabois - Campus scientifique
615, rue du Jardin Botanique - BP 101 - 54602 Villers-lès-Nancy Cedex (France)
Unité de recherche INRIA Rennes : IRISA, Campus universitaire de Beaulieu - 35042 Rennes Cedex (France)
Unité de recherche INRIA Rhône-Alpes : 655, avenue de l'Europe - 38334 Montbonnot Saint-Ismier (France)
Unité de recherche INRIA Rocquencourt : Domaine de Voluceau - Rocquencourt - BP 105 - 78153 Le Chesnay Cedex (France)

Éditeur
INRIA - Domaine de Voluceau - Rocquencourt, BP 105 - 78153 Le Chesnay Cedex (France)
http://www.inria.fr
ISSN 0249-6399