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# *Planar graphs with maximum degree $\Delta \geq 9$ are $(\Delta + 1)$-edge-choosable – short proof*

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**Abstract:** We give a short proof of the following theorem due to Borodin [2]. Every planar graph with maximum degree $\Delta \geq 9$ is $(\Delta + 1)$-edge-choosable.

**Key-words:** edge-colouring, list colouring, List Colouring Conjecture, planar graphs

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# Les graphes planaires de degré maximum $\Delta \geq 9$ sont $(\Delta + 1)$-arête-choisissables – une preuve courte

**Résumé :** Nous présentons une preuve courte d'un résultat de Borodin [2] : tout graphe planaire de degré maximum $\Delta \geq 9$ est $(\Delta + 1)$-arête-choisissable.

**Mots-clés :** arête-coloration, coloration par liste, Conjecture de la coloration par liste, graphes planaires

# Planar graphs with maximum degree $\Delta \geq 9$ are $(\Delta + 1)$-edge-choosable – a short proof

Nathann Cohen*, Frédéric Havet

We give a short proof of the following theorem due to Borodin [2]. Every planar graph with maximum degree $\Delta \geq 9$ is $(\Delta + 1)$-edge-choosable.

## 1 Introduction

All graphs considered in this paper are simple and finite. An *edge-colouring* of a graph $G$ is a mapping $f$ from $E(G)$ into a set $S$ of *colours* such that incident edges have different colours. If $|S| = k$ then $f$ is a *k-edge-colouring*. A graph is *k-edge-colourable* if it has a $k$-edge-colouring. The *chromatic index* $\chi'(G)$ of a graph $G$ is the least $k$ such that $G$ is $k$-edge-colourable.

Since edges sharing an end-vertex need different colours, $\chi'(G) \geq \Delta(G)$ where $\Delta(G)$ denotes the maximum degree of $G$. The celebrated Vizing's Theorem [13] (also shown independently by Gupta [5]) states that $\chi'(G) \in \{\Delta(G), \Delta(G) + 1\}$.

**Theorem 1 (Vizing [13])** *Let $G$ be a graph. Then $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$.*

An *edge-list-assignment* of a graph $G$ is an application $L$ which assigns to each edge $e \in E(G)$ a prescribed list of colours $L(e)$. An edge-list-assignment is a *k-edge-list-assignment* if each list is of size at least $k$. An *L-edge-colouring* of $G$ is an edge-colouring such that $\forall v \in V(G), v \in L(v)$. A graph $G$ is *L-edge-colourable* if there exists an edge-colouring of $G$. It is *k-edge-choosable* if it is $L$-colourable for every $k$-list-assignment $L$. The *choice index* or *list chromatic index* $ch'(G)$ is the least $k$ such that $G$ is $k$-edge-choosable.

One of the most celebrated conjecture on graph colouring is the List Colouring Conjecture asserting that the chromatic index is always equal to the list chromatic index.

**Conjecture 2 (List Colouring Conjecture)** *For every graph $G$, $\chi'(G) = ch'(G)$.*

Bollobás and Harris [1] proved that $ch'(G) < c\Delta(G)$ when $c > 11/6$ for sufficiently large $\Delta$. Using probabilistic methods, Kahn [9] proved Conjecture 2 asymptotically: $ch'(G) \leq (1 + o(1))\Delta(G)$. The error term was sharpened by Häggkvist and Janssen [7]: $ch'(G) \leq \Delta(G) + O(\Delta(G)^{2/3}\sqrt{\log \Delta(G)}$ and further on by Molloy and Reed [10]: $ch'(G) \leq \Delta(G) +$

$O(\Delta(G)^{1/2}(\log \Delta(G))^4)$. Galvin [6] proved the List Colouring Conjecture for bipartite graphs. (See also Slivnik [12]).

The List-Colouring Conjecture and Vizing's Theorem imply the following conjecture :

**Conjecture 3** *For any graph $G$, $ch'(G) \leq \Delta(G) + 1$.*

Borodin [2] settled this conjecture for planar graphs of maximum degree at least 9.

**Theorem 4 (Borodin [2])** *Let $\Delta \geq 9$. Every planar graph of maximum degree at most $\Delta$ is $(\Delta + 1)$-edge-choosable.*

This theorem does not imply the List Colouring Conjecture for planar graphs of large maximum degree. Indeed, Sanders and Zhao [11] showed that planar graphs with maximum degree $\Delta \geq 7$ are $\Delta$-edge-colourable. Vizing Edge-Colouring Conjecture [14] asserts that it remains true for $\Delta = 6$. This would be best possible as for any $\Delta \in \{2, 3, 4, 5\}$, there are some planar graphs with maximum degree $\Delta$ and chromatic index equal to $\Delta + 1$ [14].

Borodin, Kostochka and Woodall [3] showed that if $G$ is planar and $\Delta(G) \geq 12$ then $ch'(G) \leq \Delta(G)$, thus proving the List Colouring Conjecture for such planar graphs of maximum degree at least 12. Another proof has been given by Cole, Kowalik and Škrekovski [4] which yields a linear time algorithm to $L$-edge-colour a planar graph $G$ for any $\max\{\Delta(G), 12\}$-list edge-assignment. Conjecture 3 is still open for planar graphs of maximum degree between 5 and 8 and it is still unknown if planar graphs of maximum degree $\Delta$ are $\Delta$-edge-choosable for $6 \leq \Delta \leq 11$.

In this paper, we give a short proof of Theorem 4.

## 2 Proof of Theorem 4

Our proof uses the discharging method.

A vertex of degree $d$ (respectively at least $d$, respectively at most $d$) is said to be a *$d$-vertex* (respectively a *$(\geq d)$-vertex*, respectively a *$(\leq d)$-vertex*). The notion of a *$d$-face* (respectively a *$(\leq d)$-face*, respectively a *$(\geq d)$-face*) is defined analogously regarding the size of a face.

Consider a minimal counter-example $G$ to the theorem. Let $L$ be a $(\Delta + 1)$-list edge-assignment so that $G$ is not $L$-edge-colourable. $G$ has no edge $uv$ such that $d(u) + d(v) \leq \Delta + 2$, otherwise any $L$-colouring of $G \setminus uv$ could be extended to one of $G$ by giving to $e$ a colour distinct from the ones of its $\Delta$ adjacent edges. In particular, $\delta(G) \geq 3$ and for any $i \geq 3$ the neighbours of a $i$-vertex have degree at least $\Delta + 3 - i$.

Let $V_3$ be the set of 3-vertices and $V_\Delta$ the set of vertices of degree $\Delta$.

**Claim 4.1** $|V_\Delta| > 2|V_3|$.

**Proof**. Let $F$ the set of edges with an end-vertex of degree 3 (and so the other end-vertex of degree $\Delta$) and $H$ the bipartite subgraph $(V_3 \cup V_\Delta, F)$ of $G$.

Let us first show that $H$ is a forest. Suppose by way of contradiction that $H$ has cycle $C$. Then $C$ is even because $H$ is bipartite. By minimality of $G$, $G \setminus E(C)$ has an $L$-edge-colouring. Now every edge of $C$ has at least two available colours since it is adjacent to $\Delta+1$ edges and $\Delta-1$ coloured ones. Since the even cycles are 2-edge-choosable, one can extend the $L$-edge-colouring to $G$, which is a contradiction. Then, as any $v \in V_3$ is of degree 3 in $H$ (implying $|E(H)| = 3|V_3|$), we can write $|V_\Delta| + |V_3| > 3|V_3|$. □

Let us assign a charge of its degree to every vertex and face. It follows easily from Euler's Formula that $\Sigma = \sum_{v \in V(G)}(d(v)-4) + \sum_{f \in F(G)}(d(f)-4) = -8$. Let us now discharge along the following rules:

(R1) Every $\Delta$-vertex gives $1/2$ to a common pot from which each 3-vertex receives 1;

(R2) Every $(\geq 8)$-vertex gives $1/2$ to each of its incidents 3-faces;

(R3) Every $d$-vertex with $d \in \{5,6,7\}$ gives $\frac{d-4}{d}$ to each of its incident 3-faces.

Let us show that after the final charge $f$ of every vertex or face is non-negative as well as the charge of the common pot which contradicts $\Sigma < 0$.

• As $|V_\Delta| > 2|V_3|$ by Claim 4.1, the charge of the common pot is positive.

• Let $x$ be a $d$-vertex.

If $d = 3$ then $x$ receives at least $1/3$ from each of its neighbours (they must have degree $\Delta$), so $f(x) \geq 0$. If $d = 4$, the charge of $x$ does not change so $f(x) = d \geq 0$. If $d \in \{5,6,7\}$, then $x$ sends at most $\frac{d-4}{d}$ to each of its incident face so $f(x) \geq d(1-\frac{d-4}{d})-4 \geq 0$. If $8 \leq d \leq \Delta-1$, then $x$ sends at most $1/2$ to each of its incident faces so $f(x) \geq d-4-d/2 \geq 0$. If $d = \Delta$, then the most $x$ can send is $d \times 1/2 + d/2 \times 1/3$ since a 3-face contains at most one 3-vertex. So $f(x) \geq d-4-d/2-d/6 \geq 4$ because $d \geq 12$.

• Let $x$ be a $d$-face.

If $d \geq 4$ then its charge does not change so $f(x) = d(x)-4 \geq 0$. Suppose now that $d = 3$. If $x$ contains a $(\leq 4)$-vertex then the two other neighbours have degree at least $\Delta-1 \geq 8$ so it receives $1/2$ from each of those two. So $f(x) = 3-4+2\times 1/2 = 0$. If $x$ contains a 5-vertex then its two other vertices have degree at least $\Delta-2 \geq 7$. So it receives at $\frac{1}{5}$ from its 5-vertex and at least $\frac{3}{7}$ from the other two vertices. So $f(x) \geq 3-4+1/5+2\times 3/7 > 0$. Otherwise, all the vertices incident to $x$ are $(\geq 6)$-vertices. Hence $f(x) \geq 3-4+3\times 1/3 = 0$.

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