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Dynamic Geometric Graph Processes: Adjacency Operator Approach

René Schott and G. Stacey Staples

Abstract. The d-dimensional unit cube $[0, 1]^d$ is discretized to create a collection V of vertices used to define geometric graphs. Each subset of V is uniquely associated with a geometric graph. Defining a dynamic random walk on the subsets of V induces a walk on the collection of geometric graphs in the discretized cube. These walks naturally model addition-deletion networks and can be visualized as walks on hypercubes with loops. Adjacency operators are constructed using subalgebras of Clifford algebras and are used to recover information about the cycle structure and connected components of the n graph of a sequence.

1. Introduction

Consider n points distributed uniformly and independently in the unit cube $[0, 1]^d$. Given fixed real number $r > 0$, connect two points by an edge if their Euclidean distance is at most r. Graphs of this type are called random geometric graphs, commonly denoted $G^{(d)}(\chi_n, ; r)$ [9].

Random geometric graphs are of particular interest as models of wireless networks [3], [6], [7], [1]. The vertices (or nodes) of the graph typically represent wireless devices that can communicate with each other when their physical distance is less than some prescribed range. Of particular interest is the graph's connectivity. Ad hoc networks are modeled by addition-deletion processes.

Considering a geometric graph on n vertices, Xue and Kumar [20] have recently shown that the number of neighbors of each vertex needs to grow like $\Theta(\log n)$ if the graph is to be connected.

The philosophy presented in this paper is to first discretize the unit cube by partitioning it into sub-cubes whose center points serve as the vertices of a geometric graph. For fixed radius r, graphs are then uniquely determined by their vertex sets. A geometric graph process is then associated with a random walk on a hypercube of appropriate dimension induced by an algebraic process [17]. Other

algebraic methods (cf. [12], [14], [16], [17]) are then applied to determine the cycle structure and connectivity properties of the graph on the discretized cube.

2. Notational Preliminaries

A *graph* $G = (V, E)$ is a collection of vertices V and a set E of unordered pairs of vertices called *edges*. Two vertices $v_i, v_j \in V$ are *adjacent* if there exists an edge $\{v_i, v_j\} \in E$. Given a graph G, it will sometimes be convenient to use the notation V_G and E_G to denote, respectively, the vertices and the edges of G.

A *k-walk* $(v_0, \dots, v_k)$ in a graph G is a sequence of vertices in G with *initial vertex* v_0 and *terminal vertex* v_k such that there exists an edge $\{v_j, v_{j+1}\} \in E$ for each $0 \le j \le k - 1$. Note that a k-walk contains k edges. A *k-path* is a k-walk in which no vertex appears more than once. A *closed k-walk* is a k-walk whose initial vertex is also its terminal vertex. A *k-cycle* ($k \ge 3$) is a closed k-path with $v_0 = v_k$. A *Hamiltonian cycle* is an n-cycle in a graph on n vertices; i.e., it contains V. An edge from a vertex to itself is called a *loop*.

The *circumference* of a graph is the length of the longest cycle contained in the graph. The *girth* of a graph is defined as the length of the shortest cycle contained in the graph.

A graph G is said to be *connected* if for every pair $u \ne v \in V_G$, there exists a $k - walk$ from u to v for some positive integer k. A *component* of G is a connected subgraph of maximal size contained in G.

A *tree* is a connected graph that contains no k-cycles for $k \ge 3$. A *spanning tree* in a graph G is a subgraph of G that forms a tree and contains all vertices of G.

2.1. Random geometric graphs

A *random graph* $G(n, p)$ is a graph with n vertices in which each possible edge is independently included with probability p.

Let r be an arbitrary positive real number, and let $V = v_1, \dots, v_n$ be a set of points in a metric space with norm $\| \cdot \|$. A *geometric graph* $G(V, r)$ is defined as the graph with vertex set V and edge set E defined by

$$\{v_i, v_j\} \in E \Leftrightarrow 0 < \|v_i - v_j\| < r. \tag{2.1}$$

Definition 2.1. A *random geometric graph* $G(n, r)$ is a geometric graph in which the n vertices are independently and uniformly distributed in a metric space. It is a random graph in which the edge existence probability p between two vertices is defined by

$$p = \begin{cases} 1 & \text{if } 0 < \|v_i - v_j\| \le r, \\ 0 & \text{otherwise.} \end{cases} \tag{2.2}$$

Consider first the unit d-cube $[0, 1]^d$. Dividing the sides into N equal subintervals yields N^d sub-cubes. Center points of the sub d-cubes will serve as vertices of a geometric graph.

The set of vertices V is defined by

$$V = \left\{ \left(\frac{2j_1 - 1}{2N}, \ldots, \frac{2j_d - 1}{2N} \right) : 1 \le j_1, \ldots, j_d \le N \right\}. \tag{2.3}$$

The partitioned d-cube just described will be said to have *mesh* $1/N^d$.

Given any subset $U \subseteq V$, the topology of the geometric graph on vertex set U is uniquely determined by

$$v_1 \sim v_2 \Leftrightarrow 0 < \|v_1 - v_2\| \le r. \tag{2.4}$$

The geometric graph with vertex set U will be denoted by G_U.

2.2. Three useful algebras from Clifford algebras

Given a collection of commuting null-square elements $\{\zeta_j\}$ in one-to-one correspondence with the vertex set V, let $\mathcal{C}\ell_V{}^{\mathrm{nil}}$ denote the associative algebra generated by $\{\zeta_j\}$ and the unit scalar $1 = \zeta_\emptyset$. In particular, $\zeta_i\,\zeta_j = \zeta_j\,\zeta_i$ when $i \ne j$ and $\zeta_i{}^2 = 0$ for each i.

For convenience, generators of $\mathcal{C}\ell_V{}^{\mathrm{nil}}$ will be labeled with elements of V. The basis of $\mathcal{C}\ell_V{}^{\mathrm{nil}}$ is then in one-to-one correspondence with the power set of V. For any subset $U \subseteq V$, define the blade $\zeta_U = \prod_{v \in U} \zeta_v$. An arbitrary element $z \in \mathcal{C}\ell_V{}^{\mathrm{nil}}$ then has canonical expansion of the form

$$z = \sum_{U \subseteq V} \alpha_U\,\zeta_U, \tag{2.5}$$

where $\alpha_U \in \mathbb{R}$.

By convention a *blade* will refer to any basis monomial in an algebra.

The algebra $\mathcal{C}\ell_V{}^{\mathrm{nil}}$ is constructed within $\mathcal{C}\ell_{2|V|,2|V|}$ as follows: define $f_i = (\mathbf{e}_i - \mathbf{e}_{n+i}) \in \mathcal{C}\ell_{2|V|,2|V|}$ for each $1 \le i \le 2|V|$. Then letting $\zeta_i = f_{2i-1}\,f_{2i}$ for $1 \le i \le |V|$ completes the construction.

Remark 2.2. The algebra $\mathcal{C}\ell_V{}^{\mathrm{nil}}$ is referred to as a *zeon* algebra by Feinsilver [2]. It is the algebra referred to as $\mathcal{N}_V$ in Schott and Staples [12].

Assuming a fixed enumeration of elements of V, a probability mapping φ is induced on the generators of $\mathcal{C}\ell_V{}^{\mathrm{nil}}$ by

$$\varphi(\zeta_{v_j}) = \mu(v_j). \tag{2.6}$$

Denote by $\{e_i\}$ the collection of orthonormal basis vectors of $\mathbb{R}^{|V|2^{|V|}}$. The Dirac notation $\langle e_i|$ will represent a row vector, while the conjugate transpose, $|e_i\rangle$ represents a column vector. In this way,

$$\langle e_i | e_j \rangle = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases} \tag{2.7}$$

Moreover, $|e_i\rangle\langle e_i|$ is the rank-one orthogonal projector onto the linear subspace $\mathrm{span}(e_i)$.

Fix an enumeration $f : 2^V \to \{1, \ldots, 2^{|V|}\}$ of the power set 2^V. Notation of the form $|e_U\rangle$ and $\langle e_U|$ should be understood to use the fixed enumeration of 2^V for subsets $U \subseteq V$.

Define an enumeration of $2^V \times V$ by

$$(U, \{v_j\}) \mapsto (f(U) - 1)|V| + j. \tag{2.8}$$

The enumeration of $2^V \times V$ is then used as a double-index for the unit basis vectors of $\mathbb{R}^{|V|2^{|V|}}$. Notation of the form $|e_{U,v_i}\rangle$ and $\langle e_{U,e_i}|$ should be viewed in this context.

For each subset of vertices $U \subseteq V$, denote the nilpotent adjacency operator of the corresponding subgraph G_U by $\Phi_r^{(U)}$. In particular,

$$\Phi_r^{(U)} = \sum_{\substack{v_i, v_j \in U \\ 0 < \|v_1 - v_2\| < Nr}} \zeta_{\{v_2\}} |e_{U,v_1}\rangle \langle e_{U,v_2}| . \tag{2.9}$$

Given a collection of commuting idempotent elements $\{\gamma_j\}$ in one-to-one correspondence with the vertex set V, let $\mathcal{C}\ell_V{}^{\text{idem}}$ denote the associative algebra generated by $\{\gamma_j\}$ and the unit scalar $1 = \gamma_\emptyset$. In particular, $\gamma_i \gamma_j = \gamma_j \gamma_i$ when $i \neq j$, and $\gamma_i^2 = \gamma_i$ for each i.

The algebra $\mathcal{C}\ell_V{}^{\text{idem}}$ is constructed within the Clifford algebra $\mathcal{C}\ell_{|V|,|V|}$ by letting $\gamma_i = \frac{1}{2}\left(1 + \mathbf{e}_i \mathbf{e}_{|V|+i}\right) \in \mathcal{C}\ell_{|V|,|V|}$ for each $1 \leq i \leq |V|$.

For convenience, generators of $\mathcal{C}\ell_V{}^{\text{idem}}$ will be labeled with elements of V. The basis of $\mathcal{C}\ell_V{}^{\text{idem}}$ is then in one-to-one correspondence with the power set of V. For any subset $U \subseteq V$, define the notation $\gamma_U = \prod_{v \in U} \gamma_v$. An arbitrary element $z \in \mathcal{C}\ell_V{}^{\text{idem}}$ then has canonical expansion of the form

$$z = \sum_{U \subseteq V} \alpha_U \gamma_U, \tag{2.10}$$

where $\alpha_U \in \mathbb{R}$.

Define the *degree* mapping $\delta : \mathcal{C}\ell_V{}^{\text{idem}} \to \mathbb{N}_0$ by

$$\delta\left(\sum_{U \in 2^V} \alpha_U \gamma_U\right) = \max_{\alpha_U \neq 0} \{|U|\}. \tag{2.11}$$

In other word, $\delta(z)$ is the size of the maximal multi-index in the canonical expansion of $z \in \mathcal{C}\ell_V{}^{\text{idem}}$.

For each subset U of the collection of vertices V, denote the corresponding idempotent adjacency operator by

$$\Xi_r^{(U)} = \sum_{\substack{v_1, v_2 \in U \\ \|v_1 - v_2\| \leq r}} \gamma_{v_2} |e_{U,v_1}\rangle \langle e_{U,v_2}| . \tag{2.12}$$
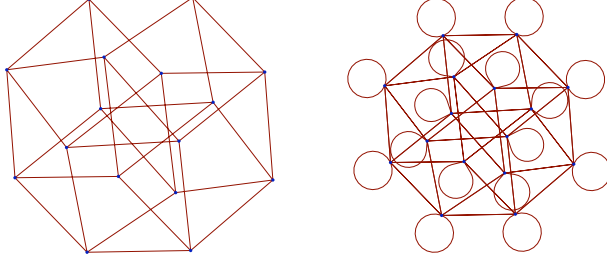
FIGURE 1. Graphs $\mathcal{Q}_4$ and $\tilde{\mathcal{Q}}_4$ associated with walk (Υ_n).

Remark 2.3. By using the inequality $\|v_1 - v_2\| \leq r$ in place of $0 < \|v_1 - v_2\| \leq r$, "loops" are placed at each vertex of the graph. This allows every pair of vertices in a given component to be joined by a closed walk of length $2|V| - 1$.

Given a collection of commuting unipotent elements $\{\varsigma_j\}$ in one-to-one correspondence with the vertex set V, let $\mathcal{C}\ell_V{}^{\mathrm{sym}}$ denote the associative algebra generated by $\{\varsigma_j\}$ and the unit scalar $1 = \varsigma_\emptyset$. In particular, $\varsigma_i \varsigma_j = \varsigma_j \varsigma_i$ when $i \neq j$, and $\varsigma_i{}^2 = 1$ for each i.

The algebra $\mathcal{C}\ell_V{}^{\mathrm{sym}}$ is constructed within the Clifford algebra $\mathcal{C}\ell_{|V|,|V|}$ by letting $\varsigma_i = \mathbf{e}_i \, \mathbf{e}_{|V|+i} \in \mathcal{C}\ell_{|V|,|V|}$ for each $1 \leq i \leq |V|$.

For convenience, generators of $\mathcal{C}\ell_V{}^{\mathrm{sym}}$ will be labeled with elements of V. The basis of $\mathcal{C}\ell_V{}^{\mathrm{sym}}$ is then in one-to-one correspondence with the power set of V. For any subset $U \subseteq V$, define the notation $\varsigma_U = \displaystyle\prod_{v \in U} \varsigma_{\{v\}}$. An arbitrary element $z \in \mathcal{C}\ell_V{}^{\mathrm{sym}}$ then has canonical expansion of the form

$$z = \sum_{U \subseteq V} \alpha_U \, \varsigma_U, \tag{2.13}$$

where $\alpha_U \in \mathbb{R}$.

Note that each blade ς_U in $\mathcal{C}\ell_V{}^{\mathrm{sym}}$ is associated with a unique geometric graph G_U.

Each vertex of the hypercube $\mathcal{Q}_{|V|}$ is uniquely identified with a geometric graph. Hence, any graph process corresponds to a random walk on the hypercube $\mathcal{Q}_{|V|}$ or a hypercube with loops at each vertex $\tilde{\mathcal{Q}}_{|V|}$, depending on whether or not the walk is allowed to revisit the same graph at consecutive steps.

3. Graph Processes

A *geometric graph process* is a sequence of random geometric graphs $(\mathcal{G}_n)_{n \geq 0}$ such that $|V_{n-1} \triangle V_n| \leq 1$ for each $n \geq 1$. Here, $\triangle$ is the set symmetric difference operator. An example is the sequence in Figure 3.

The strategy is now to define "second quantization" adjacency operators that encode information about the graphs in a graph process.

Use the fixed enumeration of 2^V to define the *second quantization nilpotent adjacency operator* by

$$\Phi_r = \sum_{U \in 2^V} \left(\Phi_r^{(U)} \otimes |e_U\rangle \langle e_U| \right). \tag{3.1}$$

By construction, Φ_r is an operator on the $N^d 2^{N^d}$-dimensional product space $\mathcal{C}\ell_V^{\mathrm{nil}|V|2^{|V|}}$. In particular, Φ_r is defined by

$$\langle e_{U,v_i}| \Phi_r |e_{U,v_j}\rangle = \begin{cases} \zeta_{v_j} & \text{if } v_i \sim v_j \text{ in } G_U \\ 0 & \text{otherwise.} \end{cases} \tag{3.2}$$

Recalling the canonical expansion $x = \sum\limits_{U \subseteq V} x_U \zeta_U \in \mathcal{C}\ell_V^{\mathrm{nil}}$, let $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$ and define the function $\psi : \mathcal{C}\ell_V^{\mathrm{nil}} \to \mathbb{N}_0$ by

$$\psi(x) = \sum_{U \subseteq V} \langle x, \zeta_U\rangle = \sum_{U \subseteq V} x_u. \tag{3.3}$$

In other words, $\psi(x)$ is the sum of the scalar coefficients in the canonical expansion of x.

For convenience, define the notation $\vec{e}_V = e_1 + e_2 + \cdots + e_{|V|!}$, and for any $U \subseteq V$, define the *U-trace* of Φ_r by

$$\mathrm{Tr}_U(\Phi_r) = \sum_{j=1}^{|V|} \langle e_{U,j}| \Phi_r |e_{U,j}\rangle. \tag{3.4}$$

The *second quantization idempotent adjacency operator* is defined by

$$\Xi_r = \sum_{U \in 2^V} \left(\Xi_r^{(U)} \otimes |e_U\rangle \langle e_U| \right). \tag{3.5}$$

By construction, Ξ_r is an operator on the $N^d 2^{N^d}$-dimensional product space $\mathcal{C}\ell_V^{\mathrm{idem}|V|2^{|V|}}$. In particular, Ξ_r is defined by

$$\langle e_{U,v_i}| \Xi_r |e_{U,v_j}\rangle = \begin{cases} \gamma_{v_j} & \text{if } v_i \sim v_j \text{ in } G_U \\ 0 & \text{otherwise.} \end{cases} \tag{3.6}$$

It should be clear that because a geometric graph is uniquely determined by its vertex set, every graph process $(\mathcal{G}_n)$ is uniquely determined by a sequence (U_n) of subsets of V satisfying $|U_n| = n$ for each $n \geq 1$. Moreover, a graph process on

the partitioned unit cube of mesh $1/N^d$ is a random walk on a subgraph of the N^d-dimensional hypercube $\mathcal{Q}_{|V|}$ restricted to strictly increasing vertex weights.

Given a finite set V, vertices of the $|V|$-dimensional hypercube $\mathcal{Q}_{|V|}$ are represented by binary strings of length $|V|$. A one-to-one correspondence between the blades of $\mathcal{Cl}_V{}^{\mathrm{sym}}$ and vertices of $\mathcal{Q}_{|V|}$ is obtained through the binary string representation of subsets of V. Representing a binary string by $(b_{|V|} \ldots b_2\, b_1)$, where each b_i takes values in $\{0,1\}$, such a correspondence takes the form

$$(b_{|V|} \ldots b_2\, b_1) \mapsto \prod_{i:b_i=1} \varsigma_{\{v_i\}}. \tag{3.7}$$

For example, the blade representation of vertex $(0\,1\,0\,1)$ in the hypercube $\mathcal{Q}_4$ is $\varsigma_{\{v_1,v_3\}} \in \mathcal{Cl}_V{}^{\mathrm{sym}}$.

To construct a random walk on the $|V|$-dimensional hypercube, one can think of a binary string of length $|V|$ and consider the effect of "flipping" a single digit, either from 0 to 1 or vice-versa, at each discrete time step. This can be accomplished within $\mathcal{Cl}_V{}^{\mathrm{sym}}$ simply by multiplying the blade representation of a vertex by the blade representation of the digit being flipped.

Given a random variable ξ, the *expectation* of ξ will be denoted by either $\langle \xi \rangle$ or $\mathbb{E}\,(\xi)$. Given a sequence of random variables $\{\xi_N\}$, the notation $\xi_N \xrightarrow{\mathcal{D}} \psi$ denotes *convergence in distribution* to the random variable ψ. The notation $\xi_N \xrightarrow{\mathcal{P}} u$ denotes *convergence in probability* to u.

Fix nonnegative integers p and q, and let $n = p + q$. Following the approach of Guillotin-Plantard and Schott [4], let $\Sigma = (E, \mathcal{A}, \mu, T)$ be a dynamical system where $(E, \mathcal{A}, \mu)$ is a probability space and T is a transformation on E.

Let $x \in E$, and define the collection $\{p_j\}_{1 \le j \le N^d}$ such that $0 \le p_j < \dfrac{1}{N^d}$ for each $j = 1, 2, \ldots, N^d$ and $\displaystyle\sum_{j=1}^{N^d} p_j \le 1$. Further, define the collection $\{f_j(x)\}_{1 \le j \le N^d}$ such that for each $i \ge 1$

$$0 \le f_j(T^i x) \le p_j, \tag{3.8}$$

and define

$$f_0(T^i x) = 1 - \sum_{j=1}^{N^d} f_j(T^i x). \tag{3.9}$$

For $n \ge 1$, let $\{Y_1, \ldots, Y_n\}$ be i.i.d. random variables taking values in $\{0, \ldots, N^d\}$ with probability

$$\mathbb{P}\,(Y_i = \ell) = f_\ell(T^i x). \tag{3.10}$$

We are interested in the *dynamic random walk* on $\mathcal{Cl}_V{}^{\mathrm{sym}}$ defined by

$$\Upsilon_n = \prod_{k=1}^{n} \varsigma_{\{Y_k\}}. \tag{3.11}$$

Note that when $\sum_{j=1}^{N^d} p_j = 1$, the resulting walk (Υ_n) is represented by a walk on the N^d-dimensional hypercube $\mathcal{Q}_{N^d}$, while $\sum_{j=1}^{N^d} p_j < 1$ results in a walk on a hypercube with loops, denoted $\tilde{\mathcal{Q}}_{N^d}$. Figure 1 illustrates these two cases.

Using the one-to-one correspondence between multivectors ς_U and geometric graphs G_U, the walk (Υ_n) induces a dynamic random walk $(\mathcal{G}_n)$ on geometric graphs. Independence of the collection $\{f_j(T^i x)\}$ then gives

$$\mathbb{P}\left(\mathcal{G}_n = G_U\right) = \left\langle \prod_{i=1}^{n} \left(f_0(T^i x) + \sum_{j=1}^{N^d} f_j(T^i x)\varsigma_{\{j\}} \right) , \varsigma_U \right\rangle \tag{3.12}$$

For convenience, define the notation

$$\tau_n = \prod_{i=1}^{n} \left(f_0(T^i x) + \sum_{j=1}^{N^d} f_j(T^i x)\varsigma_{\{j\}} \right) \tag{3.13}$$

so that

$$\langle \tau_n, \varsigma_U \rangle = \mathbb{P}\left(\mathcal{G}_n = G_U\right). \tag{3.14}$$

Theorem 3.1. *Let $k \geq 3$ be fixed. Let Φ_r denote the second quantization nilpotent adjacency operator. Let $X_k(n)$ denote the number of k-cycles in the nth geometric graph of the sequence in the partitioned d-cube with mesh $1/N^d$. Then,*

$$\mathbb{E}(X_k(n)) = \frac{1}{2k} \sum_{U \in 2^V} \langle \tau_n, \varsigma_U \rangle \; \psi\left(\text{Tr}_U(\Phi_r{}^k)\right). \tag{3.15}$$

Proof. For fixed $U \subseteq V$, a straightforward proof by induction shows that $\psi\left(\text{Tr}_U(\Phi_r{}^k)\right)$ is a sum of products of k generators of $\mathcal{C}\ell_V{}^{\text{nil}}$ corresponding to closed k-walks contained in G_U. Because $\zeta_{\{v\}}{}^2 = 0$ for all $v \in V$, the only nonzero terms correspond to the k-cycles of G_U. Since each cycle has k possible choices of basepoint and two orientations, it is evident that

$$\psi\left(\text{Tr}_U(\Phi_r{}^k)\right) = 2k\, X_k(U), \tag{3.16}$$

where $X_k(U)$ denotes the number of k-cycles contained in G_U.

Now,

$$\frac{1}{2k} \sum_{U \in 2^V} \langle \tau_n, \varsigma_U \rangle \ \psi\left(\mathrm{Tr}_U({\Phi_r}^k)\right)$$

$$= \frac{1}{2k} \sum_{U \in 2^V} \mathbb{P}\left(\mathcal{G}_n = G_U\right) \ \psi\left(\mathrm{Tr}_U({\Phi_r}^k)\right)$$

$$= \sum_{U \in 2^V} \mathbb{P}\left(\mathcal{G}_n = G_U\right) X_k(U) = \mathbb{E}\left(X_k(n)\right). \quad (3.17)$$

$\square$

Example 3.2. To compute the expected number of 5-cycles in the 7th graph of a dynamic geometric process in the partitioned unit square with mesh $1/9$, we begin by defining the probability functions $\{f_j(x)\}$. The ergodic mapping T acts on S^1 according to $Tx = e^i x$. Additional Mathematica procedures for computing products in $\mathcal{C}\ell_n{}^{\mathrm{nil}}$ and $\mathcal{C}\ell_n{}^{\mathrm{sym}}$ can be found in the appendix.

```
In[312]:= (* Define functions f_j(x) taking values in [0, 1/n²] for j=1,..., n², *)

       (* and let f_0(x) =1 - Σ_j f_j(x).  The probability that the ith graph of the process  *)
       (* G_n on the unit square partitioned into n² subsquares is the graph indexed by integer *)
       (* B is given by <τ_i, e_Int2Mult[B,n]> *)

       x = 1;
       n = 3;
       f_j_[s_] := 1/n² Abs[-1 - s] / 2 * √2
           Norm[{1 / 2, 1 / 2} - pnts[[IntegerPart[(j - 1) / n] + 1]][[Mod[j, n] + 1]]] /; (j ≥ 1)
       f_j_[s_] := 1 - Sum[1/n² Abs[-1 - s] / 2 * k / n², {k, 1, n²}] /; (j == 0)
       T_i_[r_] := If[i == 0, 1, If[i == 1, Return[Expand[e^i r]], Return[Expand[e^i T_{i-1}[r]]]]];
       τ_m_ := If[m == 1, Return[N[ClSymExpand[f_0[T_1[x]] + Sum[f_j[T_1[x]] e_{j}, {j, 1, n²}]]]],
           Return[N[ClSymExpand[ClSymExpand[f_0[T_m[x]] + Sum[f_j[T_m[x]] e_{j}, {j, 1, n²}]] ⊕ τ_{m-1}]]]]
```

For simplicity, subsets of vertices are associated with binary representations of integers.

```
In[4]:= (* Convert binary string to coordinates of subsquare centers
        in [0,1]² with mesh 1/n².  B is an integer in the range 0 to 2^{n²}-1 *)

    Bin2Coord[B_, n_] := (
      d = PadLeft[IntegerDigits[B, 2], n²];
      pnts = Table[{(2 j - 1)/(2 n), (2 k - 1)/(2 n)}, {j, 1, n}, {k, 1, n}];
      Y = Table[d[[j]] pnts[[IntegerPart[(j - 1) / n] + 1]][[Mod[j, n] + 1]], {j, 1, n²}];
      Return[DeleteCases[Y, {0, 0}]];
    )
```

The following Mathematica procedure generates the nilpotent adjacency matrix associated with a subset of vertices.

In[8]:= `(* Build nilpotent adjacency matrix of G_U in [0,1]^2 (mesh 1/n^2) *)`

```
BuildMatrix[B_, n_, r_] := (
  d = PadLeft[IntegerDigits[B, 2], n^2];
  pnts = Table[{(2 j - 1)/(2 n), (2 k - 1)/(2 n)}, {j, 1, n}, {k, 1, n}];
  Y = Table[d[[j]] pnts[[IntegerPart[(j - 1)/n] + 1]][[Mod[j, n] + 1]], {j, 1, n^2}];
  U = DeleteCases[Y, {0, 0}];
  edgs = {};
  For[k = 1, k ≤ Length[U], k++,
   edgs = Append[edgs, {U[[k]]}];
   For[j = k + 1, j ≤ Length[U], j++,
    edgs = If[Norm[U[[k]] - U[[j]]] ≤ r, Append[edgs, {U[[k]], U[[j]]}], edgs]]];
  (* Build the adjacency matrix *)
  A = Table[If[MemberQ[edgs, {U[[i]], U[[j]]}] || MemberQ[edgs, {U[[j]], U[[i]]}], 1, 0],
    {i, Length[U]}, {j, Length[U]}];
  Return[A.DiagonalMatrix[Table[e_{i}, {i, 1, Length[A]}]]];
  )
```

We now generate nilpotent adjacency matrices for all subsets.

In[11]:= `(* Build all possible adjacency matrices in [0,1]^2 with mesh 1/9 and r=√2/3 *)`

```
Λ_0 = {{0}};
For[m = 1, m ≤ 511, m++,
 Λ_m = BuildMatrix[m, 3, √2/3];
]
```

The next procedure converts an integer into its appropriate multiindex.

In[226]:= `Int2Mult[B_, n_] := DeleteCases[Table[If[BitAnd[B, 2^j] == 2^j, j + 1, 0], {j, 0, n^2 - 1}], 0]`

Finally, we compute the expected number of 5-cycles in $\mathcal{G}_7$.

In[318]:= `(* Calculate expected number of 5-`
`cycles in 7th random geometric graph on partition of mesh 1/9 in [0,1]^2. *)`

$$\frac{1}{10} \text{Sum}[\text{Coefficient}[\tau_7, e_{\text{Int2Mult[gnum,3]}}]$$
`ScalarSum[ClNilExpand[Tr[QClNilMatrixPower[Λ_gnum, 5]]]], {gnum, 0, 511}]`

Out[318]= `0.00202193`

Corollary 3.3. *The probability that the nth random geometric graph of a sequence contains exactly ℓ cycles of length k is given by*

$$\mathbb{P}(X_k(n) = \ell) = \sum_{U \in 2^V} \left\langle e_{\psi\left(\text{Tr}_U(\Phi_r{}^k)\right)}, e_{2k\ell} \right\rangle \langle \tau_n, \varsigma_U \rangle . \qquad (3.18)$$

Proof. Note that the geometric graph G_U contains ℓ k-cycles if and only if $\psi\left(\text{Tr}_U(\Phi_r{}^k)\right) = 2k\ell$. Hence, G_U contains ℓ k-cycles if and only if

$$\left\langle e_{\psi\left(\text{Tr}_U(\Phi_r{}^k)\right)}, e_{2k\ell} \right\rangle = 1. \qquad (3.19)$$

Summing the probabilities $\mathbb{P}\left(\mathcal{G}_n = G_U\right)$ over all graphs G_U gives the stated result.
$\square$

Example 3.4. The probabilities of ℓ 5-cycles in the seventh graph of the sequence considered in Example 3.2 are displayed for $0 \le \ell \le 10$.

```
In[77]:= (* Probability that mᵗʰ graph is indexed by gnum *)
        μ[gnum_, m_] := Coefficient[τₘ, eInt2Mult[gnum,3]]

        (* Initialize variables for probabilities *)
        For[j = 0, j ≤ 511, j++,
          sⱼ = 0];

        (* Compute probability of j 5-
         cycles in 7th graph of sequence for any j by considering all possible graphs *)
        s₀ = μ[0, 7];
        For[gnum = 1, gnum ≤ 511, gnum++,

          tr = 1/10 ScalarSum[ClNilExpand[Tr[ClNilMatrixPower[Λgnum, 5]]]];

          sₜᵣ = sₜᵣ + μ[gnum, 7];]
```

```
In[66]:= (* Print probabilities of ell 5-cycles for some values of ell *)
        For[ell = 0, ell ≤ 6, ell++,
         Print["Probability of ", ell, " 5-cycles: ", sₑₗₗ]]
```

```
Probability of 0 5-cycles: 0.720566

Probability of 1 5-cycles: 0.000344756

Probability of 0 5-cycles: 0.720566

Probability of 1 5-cycles: 0.000344756

Probability of 2 5-cycles: 0.000671417

Probability of 3 5-cycles: 0

Probability of 4 5-cycles: 0.0000165376

Probability of 5 5-cycles: 0.0000333949

Probability of 6 5-cycles: 0
```

Recall that the *circumference* of a graph G is the length of the longest cycle contained in G. Circumference will be denoted by $\mathrm{Circ}(G)$.

Lemma 3.5. *Fix integer ℓ such that $3 \le \ell \le |V|$. Then, the geometric graph G_U has circumference ℓ if and only if*

$$\left\langle e_{\psi\left(\mathrm{Tr}_U(\Phi_r{}^\ell)\right)}, \vec{e}_V \right\rangle \prod_{k=\ell+1}^{|V|} \left\langle e_{\psi\left(\mathrm{Tr}_U(\Phi_r{}^k)\right)}, e_0 \right\rangle = 1. \tag{3.20}$$

Proof. Note that $\left\langle e_{\psi\left(\mathrm{Tr}_U(\Phi_r{}^\ell)\right)}, \vec{e}_V \right\rangle = 1$ if and only if G_U contains an ℓ cycle, while $\displaystyle\prod_{k=\ell+1}^{|V|} \left\langle e_{\psi\left(\mathrm{Tr}_U(\Phi_r{}^k)\right)}, e_0 \right\rangle = 1$ if and only if G_U does not contain a k cycle for any $\ell < k \le |V|$. $\square$

The preceding lemma can be applied to compute the expected circumference of the nth graph of a process.

Proposition 3.6. *The expected circumference of the nth geometric graph of the graph process is given by*

$$\mathbb{E}(\mathrm{Circ}(\mathcal{G}_n)) = \sum_{\ell=3}^{|V|} \ell \left(\sum_{U \in 2^V} \left\langle e_{\psi\left(\mathrm{Tr}_U(\Phi_r{}^\ell)\right)}, \vec{e}_V \right\rangle \prod_{k=\ell+1}^{|V|} \left\langle e_{\psi\left(\mathrm{Tr}_U(\Phi_r{}^k)\right)}, e_0 \right\rangle \langle \tau_n, \varsigma_U \rangle \right). \tag{3.21}$$

Proof. In light of Lemma 3.5,

$$\sum_{\ell=3}^{|V|} \ell \left(\sum_{U \in 2^V} \left\langle e_{\psi\left(\mathrm{Tr}_U(\Phi_r{}^\ell)\right)}, \vec{e}_V \right\rangle \prod_{k=\ell+1}^{|V|} \left\langle e_{\psi\left(\mathrm{Tr}_U(\Phi_r{}^k)\right)}, e_0 \right\rangle \langle \tau_n, \varsigma_U \rangle \right)$$

$$= \sum_{\ell=3}^{|V|} \ell \sum_{\{U : \mathrm{Circ}(G_U) = \ell\}} \mathbb{P}\left(\mathcal{G}_n = G_U\right)$$

$$= \sum_{\ell=3}^{|V|} \ell\, \mathbb{P}\left(\mathrm{Circ}(\mathcal{G}_n) = \ell\right). \tag{3.22}$$

$\square$

A concept "dual" to graph circumference is graph girth. Recall that the *girth* of a graph G is the length of the shortest cycle contained in G. Girth will be denoted by $\mathrm{Girth}(G)$.

Lemma 3.7. *Fix integer ℓ such that $3 \leq \ell \leq |V|$. Then, the geometric graph G_U has girth ℓ if and only if*

$$\left\langle e_{\psi\left(\mathrm{Tr}_U(\Phi_r{}^\ell)\right)}, \vec{e}_V \right\rangle \prod_{k=3}^{\ell-1} \left\langle e_{\psi\left(\mathrm{Tr}_U(\Phi_r{}^k)\right)}, e_0 \right\rangle = 1. \tag{3.23}$$

Proof. The proof is analogous to that of Lemma 3.5. $\square$

This necessary and sufficient condition can now be used to compute the expected girth of the nth graph of a graph process.

Proposition 3.8. *The expected girth of the nth random geometric graph $\mathcal{G}_n$ is given by*

$$\mathbb{E}(\mathrm{Girth}(\mathcal{G}_n)) = \sum_{\ell=3}^{|V|} \ell \left(\sum_{U \in 2^V} \left\langle e_{\psi\left(\mathrm{Tr}_U(\Phi_r{}^\ell)\right)}, \vec{e}_V \right\rangle \prod_{k=2}^{\ell-1} \left\langle e_{\psi\left(\mathrm{Tr}_U(\Phi_r{}^k)\right)}, e_0 \right\rangle \langle \tau_n, \varsigma_U \rangle \right). \tag{3.24}$$

Proof. In light of Lemma 3.7,

$$\sum_{\ell=3}^{|V|} \ell \left(\sum_{U \in 2^V} \left\langle e_{\psi\left(\mathrm{Tr}_U(\Phi_r{}^\ell)\right)}, \vec{e}_V \right\rangle \prod_{k=2}^{\ell-1} \left\langle e_{\psi\left(\mathrm{Tr}_U(\Phi_r{}^k)\right)}, e_0 \right\rangle \langle \tau_n, \varsigma_U \rangle \right)$$

$$= \sum_{\ell=3}^{|V|} \ell \sum_{\{U : \mathrm{Girth}(G_U) = \ell\}} \mathbb{P}\left(\mathcal{G}_n = G_U\right)$$

$$= \sum_{\ell=3}^{|V|} \ell\, \mathbb{P}\left(\mathrm{Girth}(\mathcal{G}_n) = \ell\right). \quad (3.25)$$

$\square$

Of particular interest in models of wireless networks is the size of a graph's largest connected component. The size is expressed as the number of vertices contained in the component.

Proposition 3.9. *The size of the largest component $C_{\max}$ in G_U is given by*

$$|C_{\max}| = \delta\left(\mathrm{Tr}_U\left(\Xi_r{}^{2|U|-1}\right)\right). \quad (3.26)$$

Proof. An inductive argument shows that the terms of $\mathrm{Tr}_U\left(\Xi_r{}^{2n-1}\right)$ are sums of idempotents representing closed walks of length $2n-1$ on the graph. Because the graph contains $|U|$ vertices, the maximal connected component of G will be covered by a closed walk of length $2|U|-1$ or less. All components can be covered by closed walks of length equal to $2|U|-1$ by the inclusion of a loop based at each vertex in the definition of the idempotent-adjacency operator. $\square$

Clearly, a connected graph consists of a single component containing all of the graph's vertices. The following corollary is an immediate consequence of Proposition 3.9.

Corollary 3.10. *The graph G_U on vertices $U \in 2^V$ is connected* if and only if

$$\delta\left(\langle e_{U,j} | \Xi_r{}^{2|V|-1} | e_{U,j}\rangle\right) = |U|, \quad (3.27)$$

for any $v_j \in U$.

The expected maximal component size in a graph of a process can also be computed.

Proposition 3.11. *The expected size of a maximal component in the nth geometric graph of the process is given by*

$$\mathbb{E}(|C_{\max}|) = \sum_{U \in 2^V} \langle \tau_n, \varsigma_U \rangle\, \delta\left(\mathrm{Tr}_U\left(\Xi_r{}^{2|V|-1}\right)\right). \quad (3.28)$$

Proof. In light of Proposition 3.9,

$$\sum_{U \in 2^V} \langle \tau_n, \varsigma_U \rangle \, \delta \left(\mathrm{Tr}_U \left(\Xi_r^{2|V|-1} \right) \right) = \sum_{\{U : G_U \text{ is connected}\}} |U| \cdot \mathbb{P}\left(\mathcal{G}_n = G_U \right). \quad (3.29)$$

$\square$

Another question of interest in the wireless network model context is whether or not a connected graph contains small cycles. Ideally, a network should be connected without too much redundancy (or too many unnecessary connections). The next result gives the probability that a graph of the process is connected and contains no cycles of length less than or equal to k_0.

Proposition 3.12. *Let $\mathcal{G}_n$ be the nth graph of a geometric graph process, and let $k_0 \geq 3$ be an integer. Then,*

$$\mathbb{P}(\mathcal{G}_n \text{ connected and contains no } k\text{-cycles for all } k \leq k_0)$$
$$= \sum_{U \in 2^V} \langle \tau_n, \varsigma_U \rangle \left\langle e_{\delta\left(\mathrm{Tr}_U(\Xi_r^{2|V|-1})\right)}, e_{|U|} \right\rangle \left\langle e_{\sum_{k=3}^{k_0} \psi\left(\mathrm{Tr}_U(\Phi_r^k)\right)}, e_0 \right\rangle. \quad (3.30)$$

Proof. Note that for fixed $U \subseteq V$, G_U contains no k-cycles for all $3 \leq k \leq k_0$ if and only if

$$\left\langle e_{\sum_{k=3}^{k_0} \psi\left(\mathrm{Tr}_U(\Phi_r^k)\right)}, e_0 \right\rangle = 1. \quad (3.31)$$

Moreover, G_U is connected if and only if

$$\left\langle e_{\delta\left(\mathrm{Tr}_U(\Xi_r^{2|V|-1})\right)}, e_{|U|} \right\rangle = 1. \quad (3.32)$$

Hence,

$$\sum_{U : \in 2^V} \langle \tau_n, \varsigma_U \rangle \left\langle e_{\delta\left(\mathrm{Tr}_U(\Xi_r^{2|V|-1})\right)}, e_{|U|} \right\rangle \left\langle e_{\sum_{k=3}^{k_0} \psi\left(\mathrm{Tr}_U(\Phi_r^k)\right)}, e_0 \right\rangle$$

$$= \sum_{\substack{\{U : G_U \text{ connected and} \\ \text{contains no } k\text{-cycles } \forall k \leq k_0\}}} \mathbb{P}\left(\mathcal{G}_n = G_U \right). \quad (3.33)$$

$\square$

The following corollary deals with spanning trees, i.e. cycle-free connected graphs.

Corollary 3.13. *The probability that the nth geometric graph of the process is a spanning tree is given by*

$$\sum_{U \in 2^V} \langle \tau_n, \varsigma_U \rangle \left\langle e_{\delta\left(\mathrm{Tr}_U(\Xi_r^{2|V|-1})\right)}, e_{|U|} \right\rangle \left\langle e_{\sum_{k=3}^{|U|} \psi\left(\mathrm{Tr}_U(\Phi_r^k)\right)}, e_0 \right\rangle. \quad (3.34)$$

One final goal is to enumerate the connected components in a geometric graph G_U. To this end, let $x \in \mathcal{Cl}_V{}^{\text{idem}}$ and define the mapping $\eta : \mathcal{Cl}_V{}^{\text{idem}} \to \mathcal{Cl}_V{}^{\text{idem}}$ by

$$\eta(x) = \eta \left(\sum_{U \in V} \alpha_U \, \gamma_U \right) = \sum_{|U| = \delta(x)} \alpha_U \, \gamma_U. \tag{3.35}$$

Define the function $\rho : \mathcal{Cl}_V{}^{\text{idem}} \to \mathbb{N}_0$ by

$$\rho(x) = \min_{W \ni x} \{ \dim(W) \}. \tag{3.36}$$

In other words, $\rho(x)$ is the dimension of the smallest linear subspace of $\mathcal{Cl}_V{}^{\text{idem}}$ containing x. Now in a manner similar to the enumeration of cycles, it is possible to enumerate components.

Proposition 3.14. *Let Ξ_r denote the second quantization nilpotent adjacency operator. Let X denote the number of connected components in the nth graph in the partitioned d-cube with mesh $1/N^d$. Then,*

$$\mathbb{E}(X) = \sum_{U \in 2^V} \langle \tau_n, \varsigma_U \rangle \, \rho \left(\sum_{v_j \in U} \eta \left(\langle e_{U,j} | \, \Xi_r{}^{2|V|-1} \, | e_{U,j} \rangle \right) \right). \tag{3.37}$$

Proof. Note that as in the proof of Proposition 3.9, the quantity $\langle e_{U,j} | \, \Xi_r{}^{2|V|-1} \, | e_{U,j} \rangle$ is a sum of products of idempotents representing connected components containing vertex v_j in the graph G_U. Hence, $\eta \left(\langle e_{U,j} | \, \Xi_r{}^{2|V|-1} \, | e_{U,j} \rangle \right)$ is a sum of terms representing maximal components containing vertex v_j in graph G_U.

After summing over all vertices, the number of maximal components in the graph G_U is given by the quantity $\rho \left(\sum_{v_j \in U} \eta \left(\langle e_{U,j} | \, \Xi_r{}^{2|V|-1} \, | e_{U,j} \rangle \right) \right)$ by construction of ρ.

As seen before, $\langle \tau_n, \varsigma_U \rangle = \mathbb{P}(\mathcal{G}_n = G_U)$. Summing over all graphs G_U completes the proof. $\square$

In addition to computing the expected number of components, we can calculate the probability that a graph in the process contains exactly ℓ components for any nonnegative integer ℓ.

Proposition 3.15. *Let X_n denote the number of components in the nth geometric graph of the process. The probability that $X_n = \ell$ is given by*

$$\mathbb{P}(X_n = \ell) = \sum_{U \in 2^V} \left\langle e_{\rho \left(\sum_{v_j \in U} \eta \left(\langle e_{U,j} | \Xi_r{}^{2|V|-1} | e_{U,j} \rangle \right) \right)}, e_\ell \right\rangle \langle \tau_n, \varsigma_U \rangle. \tag{3.38}$$

Proof. By construction, $\left\langle e_{\rho \left(\sum_{v_j \in U} \eta \left(\langle e_{U,j} | \Xi_r{}^{2|V|-1} | e_{U,j} \rangle \right) \right)}, e_\ell \right\rangle = 1$ if and only if G_U contains ℓ components. Summing the probabilities of graphs G_U containing ℓ components gives (3.38). $\square$
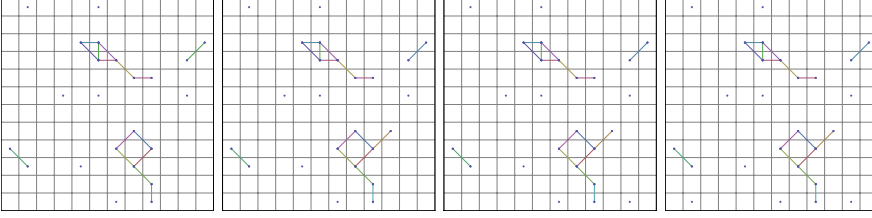
FIGURE 2. Four graphs of a dynamic geometric graph process.

The graphs on the partitioned cube can be partitioned into two sets: those that are connected, and those that are not.

Define $\vartheta \in \mathcal{C}\ell_V{}^{\text{sym}}$ by

$$\vartheta = \sum_{U \in 2^V} \left\langle e_{\delta\left(\text{Tr}_U\left(\Xi_{r^{2|V|-1}}\right)\right)}, e_{|U|} \right\rangle \varsigma_U. \tag{3.39}$$

Note that ϑ is a sum of blades indexed by vertex sets of connected graphs in the partitioned cube $[0,1]^d$. Hence,

$$\langle \vartheta, \varsigma_U \rangle = \begin{cases} 1 & \text{if } G_U \text{ is connected,} \\ 0 & \text{otherwise.} \end{cases} \tag{3.40}$$

Similarly, define

$$\tilde{\vartheta} = \sum_{U \in 2^V} \left(1 - \left\langle e_{\delta\left(\text{Tr}_U\left(\Xi_{r^{2|V|-1}}\right)\right)}, e_{|U|} \right\rangle\right) \varsigma_U, \tag{3.41}$$

and note that $\tilde{\vartheta}$ is a sum of blades indexed by vertex sets of disconnected graphs in the partitioned cube $[0,1]^d$.

Often one wants to know the probability that a graph in the process is connected. The next result gives an expression for this probability.

Proposition 3.16. *The probability that the nth graph of the process is connected is given by*

$$\mathbb{P}\left(\mathcal{G}_n \text{ is connected}\right) = \langle \tau_n, \vartheta \rangle. \tag{3.42}$$

Proof.

$$\mathbb{P}\left(\mathcal{G}_n \text{ is connected}\right) = \sum_{U \in 2^V} \langle \tau_n, \varsigma_U \rangle \langle \varsigma_U, \vartheta \rangle = \langle \tau_n, \vartheta \rangle. \tag{3.43}$$

$\square$

Define the projection $\pi_{\tilde{\vartheta}}$ by

$$\pi_{\tilde{\vartheta}}(x) = \sum_{\{U : \langle \varsigma_U, \vartheta \rangle = 0\}} x_U \, \varsigma_U. \tag{3.44}$$

Similarly, define π_ϑ by

$$\pi_\vartheta(x) = \sum_{\{U : \langle \varsigma_U, \vartheta \rangle = 1\}} x_U \, \varsigma_U. \tag{3.45}$$

For $n \geq 0$, define the mappings $\nu_n : \mathcal{C}\ell_V{}^{\mathrm{sym}} \to \mathcal{C}\ell_V{}^{\mathrm{sym}}$ by

$$\nu_0(x) = x \tag{3.46}$$
$$\nu_1(x) = \pi_\vartheta \left(x \, \tau_1 \right), \tag{3.47}$$
$$\nu_n(x) = \pi_\vartheta \left(\nu_{n-1}(x) \, \tau_n \right) \ (n \geq 2). \tag{3.48}$$

Similarly, define the mappings $\tilde{\nu}_n : \mathcal{C}\ell_V{}^{\mathrm{sym}} \to \mathcal{C}\ell_V{}^{\mathrm{sym}}$ by

$$\tilde{\nu}_0(x) = x \tag{3.49}$$
$$\tilde{\nu}_1(x) = \pi_{\tilde{\vartheta}} \left(x \, \tau_1 \right), \tag{3.50}$$
$$\tilde{\nu}_n(x) = \pi_{\tilde{\vartheta}} \left(\tilde{\nu}_{n-1}(x) \, \tau_n \right) \ (n \geq 2). \tag{3.51}$$

Proposition 3.17. *Let $\mathcal{G}_0 = G_U$ be a connected geometric graph, and let n be a positive integer. The probability that $\mathcal{G}_k$ is connected for all $0 \leq k \leq n$ is*

$$\mathbb{P} \left(\mathcal{G}_0, \ldots, \mathcal{G}_n \text{ connected} \right) = \langle \nu_n(\varsigma_U), \vartheta \rangle . \tag{3.52}$$

Similarly, when $\mathcal{G}_0 = G_U$ is a disconnected geometric graph, the probability that $\mathcal{G}_k$ is disconnected for all $0 \leq k \leq n$ is

$$\mathbb{P} \left(\mathcal{G}_0, \ldots, \mathcal{G}_n \text{ disconnected} \right) = \left\langle \tilde{\nu}_n(\varsigma_U), \tilde{\vartheta} \right\rangle . \tag{3.53}$$

Proof. Let $\mathcal{G}_0 = G_U$ be a connected geometric graph. Proof of the first part of the proposition is by induction on $n \geq 1$. When $n = 1$, definition of ν_1 gives

$$\nu_1(\varsigma_U) = \pi_\vartheta(\varsigma_U \, \tau_1) = \pi_\vartheta \left(\sum_{v \in V \cup \{0\}} \mathbb{P} \left(\mathcal{G}_1 = G_{U \triangle \{j\}} \right) \varsigma_{U \triangle \{v\}} \right)$$

$$= \sum_{\{U : G_U \text{ connected}\}} \mathbb{P} \left(\mathcal{G}_1 = G_U \right) \varsigma_U. \tag{3.54}$$

Hence, $\langle \nu_1(\varsigma_U), \vartheta \rangle = \mathbb{P} \left(\mathcal{G}_0, \mathcal{G}_1 \text{ connected} \right)$. Now assume true for some positive integer n and proceed by induction. For convenience, denote by U_n the vertex set of $\mathcal{G}_n$; i.e., $\mathcal{G}_n = G_{U_n}$.

$$\nu_{n+1}(\varsigma_U) = \pi_\vartheta(\nu_n(\varsigma_U) \, \tau_{n+1})$$

$$= \pi_\vartheta \left(\sum_{v \in V \cup \{0\}} \mathbb{P} \left(\mathcal{G}_0, \ldots, \mathcal{G}_n \text{ connected and } \mathcal{G}_{n+1} = G_{U_n \triangle \{v\}} \right) \varsigma_{U_n \triangle \{v\}} \right)$$

$$= \sum_{\{U : G_U \text{ connected}\}} \mathbb{P} \left(\mathcal{G}_0, \ldots, \mathcal{G}_n \text{ connected and } \mathcal{G}_{n+1} = G_U \right) \varsigma_U. \tag{3.55}$$

Hence,

$$\langle \nu_{n+1}(\varsigma_U), \vartheta \rangle = \sum_{\{U:\, G_U \text{ connected}\}} \mathbb{P}\left(\mathcal{G}_0, \ldots, \mathcal{G}_n \text{ connected and } \mathcal{G}_{n+1} = G_U\right)$$

$$= \mathbb{P}\left(\mathcal{G}_0, \ldots, \mathcal{G}_{n+1} \text{ connected}\right). \quad (3.56)$$

The proof of the second part of the proposition follows from the same logic. $\qquad \square$

Given an initial graph that is not connected, the next two results deal with the emergence of a connected graph in the process.

Corollary 3.18. *Let $\mathcal{G}_0 = G_U$ be a disconnected geometric graph. For $n \geq 1$, the probability that $\mathcal{G}_n$ is the first connected graph of the process is given by*

$$\mathbb{P}\left(\mathcal{G}_n \text{ first connected graph}\right) = \langle \tilde{\nu}_{n-1}(\varsigma_U)\tau_n, \vartheta \rangle. \quad (3.57)$$

Corollary 3.19. *Let $\mathcal{G}_0 = G_U$ be a disconnected geometric graph. The expected time n when the geometric graph $\mathcal{G}_n$ is the first connected graph of the process is given by*

$$\mathbb{E}\left(n : \mathcal{G}_n \text{ first connected graph}\right) = \sum_{n \geq 1} n \langle \tilde{\nu}_{n-1}(\varsigma_U)\tau_n, \vartheta \rangle. \quad (3.58)$$

Similarly, given an initial graph that is connected, the following results deal with the occurrence of the first disconnected graph in the process.

Corollary 3.20. *Let $\mathcal{G}_0 = G_U$ be a connected geometric graph. For $n \geq 1$, the probability that $\mathcal{G}_n$ is the first disconnected graph of the process is given by*

$$\mathbb{P}\left(\mathcal{G}_n \text{ first disconnected graph}\right) = \left\langle \nu_{n-1}(\varsigma_U)\tau_n, \tilde{\vartheta} \right\rangle. \quad (3.59)$$

Corollary 3.21. *Let $\mathcal{G}_0 = G_U$ be a connected geometric graph. The expected time n when the geometric graph $\mathcal{G}_n$ is the first disconnected graph of the process is given by*

$$\mathbb{E}\left(n : \mathcal{G}_n \text{ first disconnected graph}\right) = \sum_{n \geq 1} n \left\langle \nu_{n-1}(\varsigma_U)\tau_n, \tilde{\vartheta} \right\rangle. \quad (3.60)$$

3.1. Vertex degrees in $\mathcal{G}_n$

Given a graph $G = (V, E)$, the *degree of a vertex* $v \in V$ is defined by

$$\deg(v) = |\{\text{edges incident with } v \text{ in } G\}|. \quad (3.61)$$

The *degree of a graph* G is defined by

$$\deg(G) = \inf_{v \in G} \deg(v). \quad (3.62)$$

In the context of wireless networks, the vertex degree represents the number of nodes within a node's transmission range. The degree of a graph then represents the minimum number of nodes within transmission range of any node in the network.

The lemmas and propositions below follow immediately from definitions and results acquired thus far.

Lemma 3.22. *Let the geometric graph G_U be given. By construction of the second quantization nilpotent adjacency operator Φ_r, the degree of vertex $v_j \in U$ s given by*

$$\deg(v_j) = \psi\left(\langle e_{U,j}|\Phi_r{}^2|e_{U,j}\rangle\right). \tag{3.63}$$

It follows naturally that the degree of G_U is given by

$$\deg(G_U) = \inf_{j \in U} \psi\left(\langle e_{U,j}|\Phi_r{}^2|e_{U,j}\rangle\right). \tag{3.64}$$

The preceding lemma can now be used to compute the probability that a graph in the process has degree k for some nonnegative integer k.

Proposition 3.23. *Let $(\mathcal{G}_n)$ be a geometric graph process associated with Φ_r. Then, for nonnegative integer k, the probability that $\deg(\mathcal{G}_n) = k$ is given by*

$$\mathbb{P}\left(\deg(\mathcal{G}_n) = k\right) = \sum_{U \subseteq V} \langle \tau_n, \varsigma_U \rangle \left\langle e_{\inf_{j \in U} \psi(\langle e_{U,j}|\Phi_r{}^2|e_{U,j}\rangle)}, e_k \right\rangle. \tag{3.65}$$

In the next pair of propositions, the expected degree of a graph in a process and expected degree of a vertex in the nth graph of a process are calculated.

Proposition 3.24. *Let $(\mathcal{G}_n)$ be a geometric graph process associated with Φ_r. Then, the expected value of $\deg(\mathcal{G}_n)$ is given by*

$$\mathbb{E}\left(\deg(\mathcal{G}_n)\right) = \sum_{U \subseteq V} \langle \tau_n, \varsigma_U \rangle \inf_{j \in U} \psi\left(\langle e_{U,j}|\Phi_r{}^2|e_{U,j}\rangle\right). \tag{3.66}$$

Proposition 3.25. *Let $(\mathcal{G}_n)$ be a geometric graph process associated with Φ_r. Then randomly choosing a vertex v from the vertices of $\mathcal{G}_n$ with all vertices equiprobable,*

$$\mathbb{E}\left(\deg(v)\right) = \sum_{U \subseteq V} \langle \tau_n, \varsigma_U \rangle \psi\left(\text{Tr}\left(\Phi_r{}^2\right)\right). \tag{3.67}$$

A number of results involving conditional probabilities and conditional expectations are now possible.

Proposition 3.26. *Let $(\mathcal{G}_n)$ be a geometric graph process associated with Φ_r, let k be a nonnegative integer, and let ϑ be defined as in (3.39). Then the conditional probability that $\mathcal{G}_n$ has minimum vertex degree k given that $\mathcal{G}_n$ is connected is given by*

$$\mathbb{P}\left(\deg(\mathcal{G}_n) = k \mid \mathcal{G}_n \text{ is connected}\right) = \frac{\sum_{U \subseteq V} \langle \vartheta, \varsigma_U \rangle \langle \tau_n, \varsigma_U \rangle \left\langle e_{\deg(G_U)}, e_k \right\rangle}{\sum_{U \subseteq V} \langle \vartheta, \varsigma_U \rangle \langle \tau_n, \varsigma_U \rangle}. \tag{3.68}$$

Corollary 3.27. *Let $(\mathcal{G}_n)$ be a geometric graph process associated with Φ_r, and let ϑ be defined as in (3.39). Then the conditional expected value of $\deg(\mathcal{G}_n)$ given that $\mathcal{G}_n$ is connected is given by*

$$\mathbb{E}\left(\deg(\mathcal{G}_n) \mid \mathcal{G}_n \text{ is connected}\right) = \frac{\sum_{U \subseteq V} \deg(G_U) \langle \vartheta, \varsigma_U \rangle \langle \tau_n, \varsigma_U \rangle}{\sum_{U \subseteq V} \langle \vartheta, \varsigma_U \rangle \langle \tau_n, \varsigma_U \rangle}. \tag{3.69}$$

Proposition 3.28. *Let* $(\mathcal{G}_n)$ *be a geometric graph process associated with* Φ_r, *let* k *be a nonnegative integer, and let* ϑ *be defined as in* (3.39). *Then the conditional probability that* $\mathcal{G}_n$ *is connected given that it has minimum vertex degree* k *is given by,*

$$\mathbb{P}\left(\mathcal{G}_n \text{ is connected} \,|\, \deg(\mathcal{G}_n) = k\right) = \frac{\sum_{U \subseteq V} \langle \vartheta, \varsigma_U \rangle \langle \tau_n, \varsigma_U \rangle \langle e_{\deg(G_U)}, e_k \rangle}{\sum_{U \subseteq V} \langle e_{\deg(G_U)}, e_k \rangle \langle \tau_n, \varsigma_U \rangle}. \quad (3.70)$$

Proposition 3.29. *Let* $(\mathcal{G}_n)$ *be a geometric graph process associated with* Ξ_r. *Let* k *be a nonnegative integer, and let* X_n *denote the number of connected components in the* n^{th} *graph of the process. Then,*

$$\mathbb{E}(X_n \,|\, \deg(\mathcal{G}_n) = k)$$

$$= \frac{\sum_{U \in 2^V} \langle \tau_n, \varsigma_U \rangle \, \rho \left(\sum_{v_j \in U} \eta \left(\langle e_{U,j} | \, \Xi_r^{2|V|-1} \, | e_{U,j} \rangle \right) \right) \langle e_{\deg(G_u)}, e_k \rangle}{\sum_{U \in 2^V} \langle \tau_n, \varsigma_U \rangle \langle e_{\deg(G_u)}, e_k \rangle}. \quad (3.71)$$

3.2. Energy and Laplacian Energy of Geometric Graphs

Given the adjacency matrix A associated with a graph G on n vertices, the *energy of* G is defined as the sum of the singular values of A:

$$\mathcal{E}(G) = \sum_{i=1}^{n} |\lambda_i|, \quad (3.72)$$

where each λ_i is an eigenvalue of A. It is evident that

$$\text{Tr}(A^2) = \sum_{i=1}^{|V|} \lambda_i^2 = 2|E|. \quad (3.73)$$

Let $\lambda_1, \ldots, \lambda_{|V|}$ denote the spectrum of A. Since A is symmetric, A has a spectral decomposition $A\mathcal{U} = \mathcal{U}\Lambda$ where $\mathcal{U}$ is an orthogonal matrix and Λ is a diagonal matrix with the eigenvalues of A along the main diagonal. It follows that $\Lambda = \mathcal{U}^{-1}A\mathcal{U}$ so that the energy of G is given by

$$\mathcal{E}(G) = \text{Tr}\left(\left((\mathcal{U}^{-1}A\mathcal{U})^2\right)^{1/2}\right). \quad (3.74)$$

For each subset of vertices $U \subseteq V$, denote the ordinary adjacency operator of the corresponding subgraph G_U by $A^{(U)}$. In particular,

$$A^{(U)} = \sum_{\substack{v_i, v_j \in U \\ 0 < \|v_1 - v_2\| < Nr}} |e_{U,v_1}\rangle \langle e_{U,v_2}|. \quad (3.75)$$

This adjacency operator is related to the corresponding nilpotent adjacency operator by

$$A^{(U)} Z^{(U)} = \Phi_r^{(U)}, \quad (3.76)$$

where $Z^{(U)}$ is the diagonal matrix of generators of $\mathcal{C}\ell_V^{\text{nil}}$ corresponding to the subset $U \subseteq V$. As before, each $A^{(U)}$ is symmetric and therefore has spectral decomposition of the form $A^{(U)}\mathcal{U}^{(U)} = \mathcal{U}^{(U)}\Lambda^{(U)}$.

Given the second quantization nilpotent adjacency operator Φ_r associated with a graph process $(\mathcal{G}_n)$, define the ordinary second quantization adjacency operator A_r associated with Φ_r by

$$A_r = \sum_{U \in 2^V} \left(A^{(U)} \otimes |e_U\rangle \langle e_U| \right). \tag{3.77}$$

Similarly, the unitary second quantization operator $\mathcal{U}_r$ is defined by

$$\mathcal{U}_r = \sum_{U \in 2^V} \left(\mathcal{U}^{(U)} \otimes |e_U\rangle \langle e_U| \right). \tag{3.78}$$

The following result is an immediate consequence of definitions.

Lemma 3.30. *For subset $U \subset V$, the energy of the geometric graph G_U is given by*

$$\mathcal{E}(G_U) = \mathrm{Tr}_U \left(((\mathcal{U}_r^{-1} A_r \mathcal{U}_r)^2)^{1/2} \right). \tag{3.79}$$

As a corollary of previous results, we obtain the next result.

Proposition 3.31. *Given geometric graph process $(\mathcal{G}_n)$ with associated ordinary second quantization adjacency operator A_r, the expected energy of the nth graph of the process is given by*

$$\mathbb{E}\left(\mathcal{E}(\mathcal{G}_n)\right) = \sum_{U \subseteq V} \langle \tau_n, \varsigma_U \rangle \, \mathrm{Tr}_U \left(((\mathcal{U}_r^{-1} A_r \mathcal{U}_r)^2)^{1/2} \right). \tag{3.80}$$

Recall that by Lemma 3.22, the degree of vertex $v_j \in U \subseteq V$ is given by

$$\deg(v_j) = \psi \left(\langle e_{U,j} | \Phi_r^2 | e_{U,j} \rangle \right). \tag{3.81}$$

Let $A^{(U)}$ denote the ordinary adjacency matrix of the graph G_U, and define the diagonal matrix of vertex degrees $D^{(U)}$ by

$$(D^{(U)})_{jj} = \psi \left(\langle e_{U,j} | \Phi_r^2 | e_{U,j} \rangle \right). \tag{3.82}$$

For each subset of vertices $U \subseteq V$, denote the combinatorial Laplacian of the corresponding subgraph G_U by $L^{(U)}$. In particular,

$$L^{(U)} = D^{(U)} - A^{(U)}. \tag{3.83}$$

Given a graph G on n vertices and m edges, the Laplacian spectrum of G is the spectrum of its Laplacian matrix. The Laplacian spectrum consists of the numbers $\mu_1 \geq \mu_2 \geq \cdots \mu_n \geq 0$. The *Laplacian energy* of G is defined by

$$\mathcal{E}_{\mathcal{L}}(G) = \sum_{i=1}^{n} \left| \mu_i - \frac{2m}{n} \right|. \tag{3.84}$$

The Laplacian also has spectral decomposition of the form $L^{(U)} \mathcal{O}_U = \mathcal{O}_U \Lambda_U$ for orthogonal $\mathcal{O}_U$ and diagonal matrix Λ_U.

Lemma 3.32. *For subset $U \subset V$, the* Laplacian energy *of the geometric graph G_U is given by*

$$\mathcal{L}_{\mathcal{E}}(G_U) = \mathrm{Tr}\left(\left(\left(\mathcal{O}_U^{-1} L^{(U)} \mathcal{O}_U - \frac{\mathrm{Tr}(L^{(U)})}{|U|} I \right)^2 \right)^{1/2} \right). \qquad (3.85)$$

Proof. By the handshaking theorem, the sum of vertex degrees is twice the number of edges. $\qquad \square$

Proposition 3.33. *Given geometric graph process $(\mathcal{G}_n)$ with associated second quantization combinatorial Laplacian L_r, the expected Laplacian energy of the nth graph of the process is given by*

$$\mathbb{E}\left(\mathcal{E}_{\mathcal{L}}(\mathcal{G}_n) \right) = \sum_{U \subseteq V} \langle \tau_n, \varsigma_U \rangle \, \mathrm{Tr}\left(\left(\left(\mathcal{O}_U^{-1} L^{(U)} \mathcal{O}_U - \frac{\mathrm{Tr}(L^{(U)})}{|U|} I \right)^2 \right)^{1/2} \right). \quad (3.86)$$

4. Convergence Conditions and Limit Theorems

In order to establish limit theorems for the results of the preceding section, it is necessary to establish convergence of the sequence (τ_n) in $\mathcal{C}\ell_V{}^{\mathrm{sym}}$.

The norm on $\mathcal{C}\ell_V{}^{\mathrm{sym}}$ defined by

$$\left\| \sum_{U \subseteq V} \alpha_U \, \varsigma_U \right\|^2 = \sum_{U \subseteq V} \alpha_U{}^2 \qquad (4.1)$$

corresponds to the Euclidean norm when $\mathcal{C}\ell_V{}^{\mathrm{sym}}$ is viewed as a real $2^{|V|}$-dimensional vector space. A sequence (ξ_n) in $\mathcal{C}\ell_V{}^{\mathrm{sym}}$ will be said to *converge* to $\xi \in \mathcal{C}\ell_V{}^{\mathrm{sym}}$ if for every $\epsilon > 0$, $\exists k_\epsilon \in \mathbb{N}$ such that $\|\xi_n - \xi\| < \epsilon$ whenever $n \geq k_\epsilon$.

By definition of τ_n,

$$\tau_n = \prod_{i=1}^{n} \left(\sum_{j=0}^{N^d} f_j(T^i x) \varsigma_{\{j\}} \right) = \sum_{U \in 2^V} \mathbb{P}\left(\Upsilon_n = \varsigma_U \right) \varsigma_U. \qquad (4.2)$$

Theorem 4.1. *Define the set*

$$J := \{ k \in V : \ f_k(x) > 0 \ \forall x \in E \}. \qquad (4.3)$$

If $f_0(x) > 0 \ \forall x \in E$, then

$$\lim_{n \to \infty} \tau_n = \frac{1}{2^{|J|}} + \sum_{\{U \in 2^V : U \cap J \neq \emptyset\}} \frac{1}{2^{|J|}} \varsigma_U. \qquad (4.4)$$

Proof. Let $U \subseteq V$ and let $k \in V$. Note that if $k \notin U$ and $f_k(x) = 0$ for all $x \in E$, then $\langle \tau_n, \varsigma_U \rangle = 0$ for all n.

Fix $i \in V$, and let (Υ_n) be the dynamic walk corresponding to the collection $\{f_v(x) : v \in V\}$. Define the real-valued sequence (x_n) by

$$x_n := \mathbb{P}\left(i \in L \mid \Upsilon_n = \varsigma_L\right), \ n \geq 0. \tag{4.5}$$

Then

$$x_0 = 0, \tag{4.6}$$

$$x_1 = f_i(Tx), \tag{4.7}$$

$$x_{n+1} = x_n(1 - f_i(T^{n+1}x)) + (1 - x_n)f_i(T^{n+1}x), \ \ \forall n \geq 1. \tag{4.8}$$

It will be shown by induction that $x_n < 1/2$ for all $n \geq 1$. By definition, $f_i(x) < \frac{1}{2}$ for all $x \in E$. Write $f_i(T^n x) = 1/2 - \kappa(n)$ where $\kappa(n) > 0$, $\forall n$. In the basis step, $x_1 = f_i(Tx) < 1/2$.

It is now assumed that $x_n < 1/2$ for some $n \geq 1$. Then,

$$x_{n+1} = x_n(1 - f_i(T^n x)) + (1 - x_n)f_i(T^n x)$$

$$= x_n\left(1 - \frac{1}{2} + \kappa(n)\right) + (1 - x_n)\left(\frac{1}{2} - \kappa(n)\right)$$

$$= \frac{x_n}{2} + x_n\kappa(n) + \frac{1}{2} - \frac{x_n}{2} - \kappa(n) + x_n\kappa(n)$$

$$= \frac{1}{2} - \kappa(n) + 2x_n\kappa(n)$$

$$< \frac{1}{2} - \kappa(n) + \kappa(n) = \frac{1}{2}. \tag{4.9}$$

Hence, $x_n < \frac{1}{2}$ for all $n \geq 1$. It will now be shown that $x_{n+1} > x_n$ for all $n \geq 1$. Using the recurrence (5.12), $x_n < \frac{1}{2}$ implies

$$x_{n+1} = x_n + f_i(T^n x) - 2f_i(T^n x)x_n > x_n + f_i(T^n x) - f_i(T^n x) = x_n. \tag{4.10}$$

Now the sequence (x_n) converges to $\xi \in (0, \frac{1}{2})$ by the Monotone Convergence Theorem. In particular, for any $n \in \mathbb{N}$, ξ must satisfy

$$\xi = \xi(1 - f_i(T^n x)) + (1 - \xi)f_i(T^n x) = \xi + f_i(T^n x) - 2\xi f_i(T^n x). \tag{4.11}$$

Hence, ξ satisfies $0 = f_i(T^n x) - 2\xi f_i(T^n x) = f_i(T^n x)(1 - 2\xi)$, which implies $\xi = \frac{1}{2}$.

It is evident that ξ represents the limit of the probability that vertex i is present in the random geometric graph $\mathcal{G}_n$ as $n \to \infty$. Fixing $U \subseteq V$ and letting J be the set described in the statement of the theorem, it is evident that the limiting distribution is uniform on the vertices of J; i.e.,

$$\lim_{n \to \infty} \mathbb{P}\left(\Upsilon_n = \varsigma_U\right) = \left(\prod_{j \in J \cap U} \frac{1}{2}\right)\left(\prod_{j \in J \cap U'} \left(1 - \frac{1}{2}\right)\right) = \frac{1}{2^{|J|}}. \tag{4.12}$$

$\square$

The following theorem is a corollary of the previous section's results.

Theorem 4.2. *Assume the collection $\{f_j(x)\}_{j \in V}$ satisfies the hypotheses of Theorem 4.1, and let $(\mathcal{G}_n)$ be the corresponding dynamic geometric graph process. Let J be the set defined in the statement of Theorem 4.1. Let Φ_r denote the second quantization nilpotent adjacency operator, and let Ξ_r denote the second quantization nilpotent adjacency operator associated with $(\mathcal{G}_n)$, respectively.*

Let $k \geq 3$ be fixed, and let $z_k(n)$ denote the number of k-cycles in the nth geometric graph of the sequence in the partitioned d-cube with mesh $1/N^d$. Let $|C_{\max}(n)|$ denote the size of a maximal component in the nth geometric graph of the process. Let $K(n)$ denote the number of connected components in the nth graph in the partitioned d-cube with mesh $1/N^d$.

Then,

$$\lim_{n \to \infty} \mathbb{E}(z_k(n)) = \frac{1}{k 2^{|J|+1}} \sum_{U \in 2^V} \psi\left(\mathrm{Tr}_U(\Phi_r{}^k)\right), \tag{4.13}$$

$$\lim_{n \to \infty} \mathbb{E}(\mathrm{Cr}(\mathcal{G}_n)) =$$

$$\frac{1}{2^{|J|}} \sum_{\ell=3}^{|V|} \ell \left(\sum_{U \in 2^V} \left\langle e_{\psi\left(\mathrm{Tr}_U(\Phi_r{}^\ell)\right)}, \vec{e}_V \right\rangle \prod_{k=\ell+1}^{|V|} \left\langle e_{\psi\left(\mathrm{Tr}_U(\Phi_r{}^k)\right)}, e_0 \right\rangle \right), \tag{4.14}$$

$$\lim_{n \to \infty} \mathbb{E}(\mathrm{Girth}(\mathcal{G}_n)) =$$

$$\frac{1}{2^{|J|}} \sum_{\ell=3}^{|V|} \ell \left(\sum_{U \in 2^V} \left\langle e_{\psi\left(\mathrm{Tr}_U(\Phi_r{}^\ell)\right)}, \vec{e}_V \right\rangle \prod_{k=2}^{\ell-1} \left\langle e_{\psi\left(\mathrm{Tr}_U(\Phi_r{}^k)\right)}, e_0 \right\rangle \right), \tag{4.15}$$

$$\lim_{n \to \infty} \mathbb{E}(|C_{\max}(n)|) = \frac{1}{2^{|J|}} \sum_{U \in 2^V} \delta\left(\mathrm{Tr}_U\left(\Xi_r{}^{2^{|V|}-1} \right) \right), \tag{4.16}$$

and

$$\lim_{n \to \infty} \mathbb{E}(K(n)) = \frac{1}{2^{|J|}} \sum_{U \in 2^V} \rho\left(\sum_{v_j \in U} \eta\left(\left\langle e_{U,j} \,|\, \Xi_r{}^{2^{|V|}-1} \,|\, e_{U,j} \right\rangle \right) \right). \tag{4.17}$$

5. Time-homogeneous walks on random geometric graphs

In this section results are obtained for graph processes in which the vertex addition/deletion probabilities do not vary with time.

Proposition 5.1. *Let Y be a random variable taking values in $[|V|] \cup \{0\}$ with probabilities $p_i = \mathrm{Pr}\{Y = i\}$ for each $0 \leq i \leq |V|$, and let $\{Y_k\}_{k>0}$ be the sequence of independent random variables obtained from repeated observations of Y. Let $1 = \varsigma_\emptyset \in \mathcal{C}\ell_V{}^{\mathrm{sym}}$ represent the initial probability density on the vertices of $\mathcal{Q}_{|V|}$; i.e., the*

initial vertex corresponds to the empty graph. Let $\tau = \sum_{i=0}^{|V|} p_i \varsigma_i$. *For convenience,* $\varsigma_{\{0\}}$ *is identified with* $\varsigma_{\emptyset}$. *Then for* $k > 0$, *the probability density function* f_k *on the power set* 2^V *at time* k *is defined by*

$$f_k(U) = \left\langle \tau^k, \varsigma_U \right\rangle. \tag{5.1}$$

Proof. For $k \geq 0$, let $\xi_k \in \mathcal{Cl}_V{}^{\mathrm{sym}}$ be defined by

$$\xi_k = \sum_{U \in 2^V} f_k(U)\varsigma_U, \tag{5.2}$$

and note that $\xi_0 = \varsigma_{\emptyset} = 1$. We show that $\xi_k = \tau^k$ for all $k \geq 1$.

We see that the probability density at time $k = 1$ is given by

$$\xi_1 = \sum_{U \in 2^V} f_1(U)\varsigma_U = \sum_{U \in 2^V} \langle \tau, \varsigma_U \rangle \varsigma_U = \tau. \tag{5.3}$$

Assuming true for k, using independence of $\{Y_k\}$, and proceeding by induction, we find

$$\xi_{k+1} = \sum_{U \in 2^V} f_{k+1}(U)\varsigma_U = \sum_{U \in 2^V} \sum_{j=1}^{|V|} f_{k+1}(U \triangle \{j\}) \varsigma_{U \triangle \{j\}}$$

$$= \sum_{U \in 2^V} f_k(U)\varsigma_U \sum_{j=1}^{|V|} p_j \varsigma_{\{j\}} = \xi_k \left(\sum_{j=1}^{|V|} p_j \varsigma_{\{j\}} \right) = \left(\sum_{j=1}^{|V|} p_j \varsigma_{\{j\}} \right)^k \left(\sum_{j=1}^{|V|} p_j \varsigma_{\{j\}} \right)$$

$$= \tau^k \, \tau = \tau^{k+1}. \tag{5.4}$$

$\square$

When $p_0 = 0$, the resulting sequence (Υ_n) is a walk on the hypercube $\mathcal{Q}_{|V|}$. When $p_0 > 0$, (Υ_n) is a walk on a hypercube with loops.

In the homogeneous case, the sequence (τ_n) defined by (3.13) is replaced by (τ^n). All of the previous results hold as stated with this modification.

Example 5.2. Let $\{p_0, p_1, \ldots, p_4\} = \{.15, .1, .2, .3, .25\}$. The probability density on the collection of geometric graphs G_U at time $t = 32$ is determined by setting $\tau := \sum_{i=0}^{4} p_i \varsigma_{\{i\}}$ and computing τ^{32} with Mathematica in Figure 3.

In[320]:= τ = 0.15 + 0.1 e_{{1}} + 0.2 e_{{2}} + 0.3 e_{{3}} + 0.25 e_{{4}}

Out[320]= 0.15 + 0.1 e_{{1}} + 0.2 e_{{2}} + 0.3 e_{{3}} + 0.25 e_{{4}}

In[321]:= ClSymExpand[ClSymPower[τ, 32]]

Out[321]= 0.0625502 + 0.0624498 e_{{1}} + 0.0625488 e_{{2}} + 0.0625488 e_{{3}} + 0.0625488 e_{{4}} + 0.0624512 e_{{1,2}} + 0.0624512 e_{{1,3}} + 0.0624512 e_{{1,4}} + 0.0625502 e_{{2,3}} + 0.0625502 e_{{2,4}} + 0.0625502 e_{{3,4}} + 0.0624498 e_{{1,2,3}} + 0.0624498 e_{{1,2,4}} + 0.0624498 e_{{1,3,4}} + 0.0625488 e_{{2,3,4}} + 0.0624512 e_{{1,2,3,4}}

FIGURE 3. Probability that $\mathcal{G}_{32} = G_U$ for each $U \subseteq \{v_1, \ldots, v_4\}$.

Now the probability that $\mathcal{G}_{32} = G_{\{v_1,v_2,v_3\}}$ is the coefficient of $\sigma_{\{1,2,3\}}$, seen to be .0624498.

Lemma 5.3. *Let $(\mathcal{G}_n)$ be the homogeneous geometric graph process defined in Proposition 5.1 with $p_0 > 0$, and let $i \in V$. Then, given $\mathcal{G}_n = G_{U_n}$,*

$$\mathbb{P}\left(i \in U_n\right) = \frac{1 - (1 - 2p_i)^n}{2} \tag{5.5}$$

Proof. Fix $i \in V$, and let (Υ_n) be the time-homogeneous walk corresponding to the collection $\{p_i : v \in V\}$. The subset $U_n \subseteq V$ is defined by $\Upsilon_n = \varsigma_{U_n}$. Now define the sequence (x_n) by $x_n = \mathbb{P}\left(i \in U_n\right)$ for $n \geq 0$. Then

$$x_0 = 0, \tag{5.6}$$

$$x_n = x_{n-1}(1 - p_i) + (1 - x_{n-1})p_i, \quad \forall n \geq 1. \tag{5.7}$$

By back-substitution, the recurrence relation has solution given by

$$\begin{aligned}
x_n &= (1 - 2p_i)x_{n-1} + p_i \\
&= (1 - 2p_i)^2 x_{n-2} + (1 - 2p_i)p_i + p_i \\
&\qquad \vdots \\
&= (1 - 2p_i)^n x_0 + p_i \sum_{j=0}^{n-1} (1 - 2p_i)^j \\
&= p_i \sum_{j=0}^{n-1} (1 - 2p_i)^j = \frac{1 - (1 - 2p_i)^n}{2}. \tag{5.8}
\end{aligned}$$

$\square$

Theorem 5.4. *Let $J = \{k \in V : p_k > 0\}$. If $p_0 > 0$, then*

$$\lim_{n \to \infty} \tau^n = \frac{1}{2^{|J|}} + \sum_{\{U \in 2^V : U \cap J \neq \emptyset\}} \frac{1}{2^{|J|}} \varsigma_U. \tag{5.9}$$

Proof. Let $U \subseteq V$ and let $k \in V$. Note that if $p_k = 0$ and $k \notin U$, then $\langle \tau^n, \varsigma_U \rangle = 0$ for all n.

Fix $i \in V$, and let (Υ_n) be the time-homogeneous walk corresponding to the collection $\{p_i : v \in V\} \cup \{p_0\}$. The subset $L \subseteq V$ is defined by $\Upsilon_n = \varsigma_L$.

Now define the sequence (x_n) by $x_n = \mathbb{P}\left(i \in L\right)$ for $n \geq 0$. Then

$$x_0 = 0, \tag{5.10}$$

$$x_1 = p_i, \tag{5.11}$$

$$x_{n+1} = x_n(1 - p_i) + (1 - x_n)p_i, \quad \forall n \geq 1. \tag{5.12}$$

Note that $p_0 > 0$ implies $p_i < 1$. By Lemma 5.3,

$$\lim_{n \to \infty} x_n = \lim_{n \to \infty} \frac{1 - (1 - 2p_i)^n}{2} = \frac{1}{2}. \tag{5.13}$$

It is evident that $\frac{1}{2}$ represents the limit of the probability that vertex i is present in the random geometric graph $\mathcal{G}_n$ as $n \to \infty$. Fixing $U \subseteq V$ and letting J be the set described in the statement of the theorem, it is evident that the limiting distribution is uniform on the vertices of J; i.e.,

$$\lim_{n \to \infty} \mathbb{P}\left(\Upsilon_n = \varsigma_U\right) = \left(\prod_{j \in J \cap U} \frac{1}{2}\right)\left(\prod_{j \in J \cap U'} \left(1 - \frac{1}{2}\right)\right) = \frac{1}{2^{|J|}}. \qquad (5.14)$$

$\square$

6. Mathematica Procedures

This section contains general purpose Mathematica code for performing computations in the algebras $\mathcal{Cl}_n{}^{\mathrm{nil}}$ and $\mathcal{Cl}_n{}^{\mathrm{sym}}$. This code underlies the examples computed in the paper.

```
Unprotect[CirclePlus];
ClearAll[CirclePlus];
SetAttributes[CirclePlus, {Flat, OneIdentity, Listable}];
CirclePlus[x_?NumericQ , y_?NumericQ] := x y ;
CirclePlus[arg1_, y_?NumericQ] := y arg1;
CirclePlus[x_?NumericQ, arg2_] := x arg2;
CirclePlus[x_?NumericQ arg1_, y_?NumericQ] := x y arg1;
CirclePlus[x_?NumericQ, y_?NumericQ arg2_] := x y arg2;
CirclePlus[x_?NumericQ arg1_, y_?NumericQ arg2_] := x y CirclePlus[arg1, arg2];
CirclePlus[x_?NumericQ arg1_,  arg2_] := x CirclePlus[arg1, arg2];
CirclePlus[arg1_, y_?NumericQ arg2_] := y CirclePlus[arg1, arg2];

CirclePlus[e_a_ , e_b_] :=
   (-1)^Length[Complement[a⋂b,pSet]] * e_((a⋃b)⋂(Complement[Union[a,b],Intersection[a,b]])) /. {e_{} → 1};

CirclePlus = Symbol["CirclePlus"];
Protect[CirclePlus];


Unprotect[ClSymExpand];
ClearAll[ClSymExpand];
SetAttributes[ClSymExpand, Listable];
ClSymExpand[x_] := x /; FreeQ[x, _⊕_];
ClSymExpand[x_ + y_] := Simplify[ClSymExpand[x] + ClSymExpand[y]];
ClSymExpand[x_?NumericQ arg1_] := x ClSymExpand[arg1];
ClSymExpand[arg1_] := Module[{arg2},
   arg2 = Distribute[ExpandAll[arg1], Plus, CirclePlus];
   If[! FreeQ[arg2, _⊕_] && arg2 =!= arg1,
    arg2 = ClSymExpand[arg2];
   ];
   Return[arg2];
  ];
Protect[ClSymExpand];
```

```
(* Compute powers of Cl^sym elements *)
ClSymPower[x_, n_Integer] := Module[{y}, y = ClSymExpand[x]; Switch[EvenQ[n],

    True, If[n == 0, Return[1], Return[ClSymPower[ClSymExpand[y⊕y], n/2]]],

    False, If[n == 1, Return[y], Return[ClSymExpand[y⊕ClSymPower[ClSymExpand[y⊕y], (n-1)/2]]]]]];


(* Define product for commutative algebra
 Cl_n^nil.  Generators satisfy the following conditions: *)
(*e_{j} ⊖ e_{k} = e_{k} ⊖ e_{j}     k≠j *)
(* e_{j} ⊖ e_{j}=0                  *)

Unprotect[CircleMinus];
Unprotect[CircleMinus];
ClearAll[CircleMinus];
SetAttributes[CircleMinus, {Flat, OneIdentity, Listable}];
CircleMinus[x_?NumericQ , y_?NumericQ] := x y ;
(* One 'no arguments' case *)
CircleMinus[arg1_, y_?NumericQ] := y arg1;
(* Four 'one argument' cases *)
CircleMinus[x_?NumericQ, arg2_] := x arg2;
CircleMinus[x_?NumericQ arg1_, y_?NumericQ] := x y arg1;
CircleMinus[x_?NumericQ, y_?NumericQ arg2_] := x y arg2;
CircleMinus[x_?NumericQ arg1_, y_?NumericQ arg2_] := x y CircleMinus[arg1, arg2];
(* Three 'two arguments' cases *)
CircleMinus[x_?NumericQ arg1_, arg2_] := x CircleMinus[arg1, arg2];
CircleMinus[arg1_, y_?NumericQ arg2_] := y CircleMinus[arg1, arg2];
CircleMinus[e_a_, e_b_] := If[Length[a⋂b] > 0, 0, e_(a⋃b)];
CircleMinus = Symbol["CircleMinus"];
Protect[CircleMinus];


Unprotect[ClNilExpand];
ClearAll[ClNilExpand];
SetAttributes[ClNilExpand, Listable];
ClNilExpand[x_] := x /; FreeQ[x, _⊖_];
ClNilExpand[x_ + y_] := Simplify[ClNilExpand[x] + ClNilExpand[y]];
ClNilExpand[x_?NumericQ arg1_] := x ClNilExpand[arg1];
ClNilExpand[arg1_] := Module[{arg2},
   arg2 = Distribute[ExpandAll[arg1], Plus, CircleMinus];
   If[! FreeQ[arg2, _⊖_] && arg2 =!= arg1,
    arg2 = ClNilExpand[arg2];
   ];
   Return[arg2];
  ];
Protect[ClNilExpand];


(* Compute powers of Cl^nil elements *)
ClNilPower[x_, n_Integer] := Module[{y}, y = ClNilExpand[x]; Switch[EvenQ[n],

    True, If[n == 0, Return[1], Return[ClNilPower[ClNilExpand[y⊖y], n/2]]],

    False,

    If[n == 1, Return[y], Return[ClNilExpand[y⊖ClNilPower[ClNilExpand[y⊖y], (n-1)/2]]]]]];

(* Procedure to multiply matrices with Cl^nil entries *)
ClNilMatrixProduct[A_, B_] := If[Dimensions[A][[2]] ≠ Dimensions[B][[1]], Abort[];,
   Table[Total[ClNilExpand[CircleMinus[A[[i]], Transpose[B][[j]]]]],
    {i, 1, Length[A]}, {j, 1, Dimensions[B][[2]]}]];
```

```
(* A procedure for computing powers of Cl^nil matrices. In this method,
A^m is computed by recursive squaring ((A^2)^2...)A *)
ClNilMatrixPower[A_, m_] := Module[{y}, y = ClNilExpand[A]; Switch[EvenQ[m],

    True, If[m == 0, Return[IdentityMatrix[Length[y]]],

     Return[ClNilExpand[ClNilMatrixPower[ClNilExpand[ClNilMatrixProduct[y, y]], m/2]]]],

    False, If[m == 1, Return[y], Return[ClNilExpand[ClNilMatrixProduct[

       ClNilMatrixPower[ClNilExpand[ClNilMatrixProduct[y, y]], (m-1)/2], y]]]]]];
```

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