Algebraic change-point detection

Michel Fliess, Cédric Join, Mamadou Mboup

Applicable Algebra in Engineering, Communication and Computing manuscript No.
(will be inserted by the editor)

Michel Fliess · Cédric Join ·
Mamadou Mboup

Algebraic change-point detection

Received: date / Revised: date

Abstract Elementary techniques from operational calculus, differential algebra, and noncommutative algebra lead to a new approach for change-point detection, which is an important field of investigation in various areas of applied sciences and engineering. Several successful numerical experiments are presented.

Keywords Change-point detection · Identifiability · Operational calculus · Differential algebra · Noncommutative algebra · Holonomic functions

1 Introduction

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a piecewise smooth function with discontinuities at $t_1, t_2, \ldots$. Its pointwise derivative $f^{(1)}$ which exists and is continuous except at $t_1, t_2, \ldots$, and its distribution derivative f' in Schwartz's sense are, as well known, related by

$$f'(t) = f^{(1)}(t) + (f(t_1+) - f(t_1-))\,\delta(t-t_1) + (f(t_2+) - f(t_2-))\,\delta(t-t_2) + \ldots \tag{1}$$

M. Fliess
INRIA-ALIEN & LIX (CNRS, UMR 7161), École polytechnique, 91128 Palaiseau, France
E-mail: Michel.Fliess@polytechnique.edu

C. Join
INRIA-ALIEN & CRAN (CNRS, UMR 7039), Nancy-Université, BP 239, 54506 Vandœuvre-lès-Nancy, France
E-mail: Cedric.Join@cran.uhp-nancy.fr

M. Mboup
INRIA-ALIEN & CReSTIC, UFR Sciences Exactes et Naturelles, Université de Reims Champagne-Ardenne, Moulin de la Housse, B.P. 1039, 51687 Reims cedex 2, France
E-mail: Mamadou.Mboup@univ-reims.fr

where

- $f(\tau+) = \lim_{t\downarrow\tau} f(t)$, $f(\tau-) = \lim_{t\uparrow\tau} f(t)$,
- δ is the Dirac delta function.

A huge literature[1] has been devoted to the detection of $t_1, t_2, \ldots$, which is a crucial question in signal processing, in diagnosis, and in many other fields of engineering and applied sciences, where those discontinuities are often called *change-points* or *abrupt changes*.[2] Difficulties are stemming from

- corrupting noises which are blurring the discontinuities,
- the combined need of
  - fast online calculations,
  - a feasible software and/or hardware implementation.

Most of the existing literature is based on statistical tools (see, for instance, [1,4,5,8] and the references therein).

The origin of our algebraic viewpoint lies in the references [18,19] which are devoted to the parametric identification of linear systems in automatic control.[3] We employ elementary techniques stemming from operational calculus,[4] differential algebra and noncommutative algebra. We are replacing Eq. (1) by its operational analogue which is easier to handle. Restricting ourselves to solutions of operational linear differential equations with rational coefficients lead to noncommutative rings of linear differential operators. By representing a change-point by a *delay operator*, i.e., an operational exponential, Sect. 2 concludes with the *identifiability* of change-points, i.e., the possibility of expressing them via measured data.[5] Higher order change-points, i.e., discontinuities of derivatives of various orders are briefly discussed in Sect. 3. Sect. 4 presents several successful numerical experiments,[6] which

- exhibit good robustness properties with respect to several types of additive and multiplicative corrupting noises;
- indicate that our approach is still valid outside of its full mathematical justification.[7]

Preliminary results may be found in [17] and [16,27].

---

[1] See the excellent account due to Basseville and Nikiforov [1] for more details.

[2] The most popular terminology in French is *ruptures*.

[3] Change-point detection has also been studied in [2] via tools stemming from [18,19], but in a quite different manner when compared to us.

[4] Mikusinski's foundation [29,30] of operational calculus, which is not based on the usual Laplace transform, is a better choice for the connection with the other algebraic tools. Mikusinski's work, which is a superb example of *algebraic analysis*, is too much neglected in spite of some advertisements like the nice book by Yosida [39].

[5] In the context of constant linear control systems with delays, which bears some similarity with what is done here, the identification of delays has also been tackled in [3,31,36] via techniques from [18,19].

[6] Let us emphasize that our techniques have already been applied in some concrete case-studies, where the signals to be processed are stemming from either biology [26,37] or finance [13,14].

[7] It goes without saying that this Section, which is mainly descriptive, is not intended to be fully rigorous.

Acknowledgement. The authors would like to thank anonymous referees for several most helpful comments.

2 Algebra via operational calculus

2.1 Differential equations

Take a commutative field k_0 of characteristic zero. The field $k_0(s)$ of rational functions over k_0 in the indeterminate s is obviously a *differential field* with respect to the derivation $\frac{d}{ds}$ and its subfield of constants is k_0 (cf. [6,34]). Write $k_0(s)[\frac{d}{ds}]$ the noncommutative ring of linear differential operators of the form

$$\sum_{\text{finite}} \varrho_\alpha \frac{d^\alpha}{ds^\alpha} \in k_0(s)\left[\frac{d}{ds}\right], \quad \varrho_\alpha \in k_0(s) \tag{2}$$

We know that $k_0(s)[\frac{d}{ds}]$ is a left and right principal ideal ring (cf. [28,34][8]). Any signal x is assumed[9] here to be *operationally holonomic*, i.e., to satisfy a linear differential equation with coefficients in $k_0(s)$: there exists a linear differential operator $\varpi \in k_0(s)[\frac{d}{ds}]$, $\varpi \neq 0$, such that $\varpi x = 0$.

Remark 1 Let us explain briefly this assumption. We consider only *holonomic* time functions $z(t)$, i.e., time functions which satisfy linear differential equations with polynomial coefficients:

$$\left(\sum_{\iota=0}^{N} p_\iota(t)\frac{d^\iota}{dt^\iota}\right) z = 0, \quad p_\iota \in \mathbb{C}[t], \quad t \geq 0$$

The corresponding operational linear differential equation reads (cf. [39])

$$\left(\sum_{\iota=0}^{N} p_\iota\left(-\frac{d}{ds}\right) s^\iota\right) \hat{z} = I(s)$$

where $I \in \mathbb{C}[s]$ depends on the initial conditions. A homogeneous linear differential equation is obtained by differentiating both sides of the previous equation enough times with respect to s.

Let $\bar{K}$ be the algebraic closure of $k_0(s)$: $\bar{K}$ is again a differential field with respect to $\frac{d}{ds}$ and its subfield of constants is the algebraic closure $\bar{k}_0$ of k_0. It is known that x belongs to a *Picard-Vessiot extension* of $\bar{K}$ (cf. [6,34]).

Remark 2 Holonomic functions play an important rôle in many parts of mathematics like, for instance, combinatorics (see, e.g., [10]).

[8] Note that [34] is not employing, contrarily to [28], the usual terminology of ring and module theory.
[9] See also [12,17,25].

2.2 Annihilators

Consider now the left $k_0(s)[\frac{d}{ds}]$-module $\mathcal{M}$ spanned by a finite set $\{x_\iota | \iota \in I\}$ of such signals. Any x_ι is a *torsion* element (cf. [28]) and therefore $\mathcal{M}$ is a *torsion module*.[10] The *annihilator* A_I of $\{x_\iota | \iota \in I\}$ is the set of linear differential operators $\varpi_I \in k_0(s)[\frac{d}{ds}]$ such that, $\forall \ \iota \in I$, $\varpi_I x_\iota = 0$. It is a left ideal of $k_0(s)[\frac{d}{ds}]$ and it is therefore generated by a single element $\Delta \in k(s)[\frac{d}{ds}]$, $\Delta \neq 0$, which is called a *minimal* annihilator of $\{x_\iota | \iota \in I\}$. It is obvious that Δ is annihilating any element belonging to the k_0-vector space $\mathrm{span}_{k_0}(x_\iota | \iota \in I)$. The next result is straightforward:

Lemma 1 *Let Δ_1, Δ_2, $\Delta_1 \neq \Delta_2$, be two minimal annihilators. There exists $\rho \in k_0(s)$, $\rho \neq 0$, such that $\Delta_1 = \rho\Delta_2$.*

We will say that the minimal annihilator is unique up to left multiplications by nonzero rational functions.

A rational function $\frac{p}{q}$, $p, q \in k_0[s]$, $q \neq 0$, is said to be *proper* (resp. *strictly proper*) if, and only if, the $\mathrm{d}^\circ p \leq \mathrm{d}^\circ q$ (resp. $\mathrm{d}^\circ p < \mathrm{d}^\circ q$). A differential operator (2) is said to be *proper* (resp. *strictly proper*) if, and only if, any ϱ_α is proper (resp. strictly proper). The next result is an obvious corollary of Lemma 1:

Corollary 1 *It is possible to choose an annihilator, which is minimal or not, in such a way that it is proper (resp. strictly proper).*

A rational function $\frac{p}{q}$, $p, q \in k_0[s]$, $q \neq 0$, is said to be in a *finite integral form* (resp. *strictly finite integral form*) if, and only if, it belongs to $k_0[\frac{1}{s}]$ (resp. $\frac{1}{s}k[\frac{1}{s}]$). A differential operator (2) is said to be in a *finite integral form* (resp. *strictly finite integral form*) if, and only if, any ϱ_α is in a finite integral form (resp. strictly finite integral form). Consider a common multiple $m \in k[s]$ of the denominators of the ϱ_α's. The operator $s^{-N}m\varpi$ is in a (strictly) finite integral form for large enough values of the integer $N \geq 0$.

Corollary 2 *It is possible to choose an annihilator, which is minimal or not, in such a way that it is in a finite integral form (resp. strictly finite integral form).*

2.3 Delay operators

Let k/k_0 be a transcendental field extension. The field $k(s)$ of rational functions over k in the indeterminate s is again a differential field with respect to $\frac{d}{ds}$ and its subfield of constants is k. The noncommutative ring $k(s)[\frac{d}{ds}]$ of linear differential operators is defined as in Sect. 2.1. Pick up an element $t_r \in k$, called *delay*, which is transcendental over k_0. Write the *delay operator* with its classic exponential notation $e^{-t_r s}$ (cf. [33]), as it satisfies the differential equation $\left(\frac{d}{ds} + t_r\right) e^{-t_r s} = 0$. According to Sect. 2.2, the differential operator $\frac{d}{ds} + t_r \in k(s)[\frac{d}{ds}]$ is a minimal annihilator of $e^{-t_r s}$.

[10] Such a module is called a *differential module* in [34].

2.4 Identifiability of the delay

2.4.1 Main result

Let $\varpi_1, \varpi_2 \in k_0(s)[\frac{d}{ds}]$ be minimal annihilators of two signals x_1, x_2, $x_1 x_2 \neq 0$. Introduce the quantity

$$X = x_1 + x_2 e^{-t_r s} \tag{3}$$

Multiplying on the left both sides of Eq. (3) by ϖ_1 yields $\varpi_1 X = \varpi_1 x_2 e^{-t_r s}$. Thus

$$\varpi_1 X e^{t_r s} = \varpi_1 x_2$$

and

$$\varpi_2' \varpi_1 X e^{t_r s} = 0 \tag{4}$$

where $\varpi_2' \in k_0(s)[\frac{d}{ds}]$ is a minimal annihilator of $\varpi_1 x_2$. The next proposition follows at once:

Proposition 1 *Eq.* (4) *is equivalent to*

$$\sum_{\text{finite}} t_r^{\nu} (\pi_\nu X) = 0, \quad \pi_\nu \in k_0(s)\left[\frac{d}{ds}\right]$$

where at least one π_ν *is not equal to* 0*.*

Write $k_0(s)\langle X \rangle$ the differential overfield of $k_0(s)$ generated by X.

Corollary 3 t_r *in Eq.* (3) *is algebraic over the differential field* $k_0(s)\langle X \rangle$*.*

Remark 3 Assume that the quantity X is *measured*, i.e., there exists a *sensor* which gives at each time instant its numerical value in the time domain. Then, according to the terminology in [15], Corollary 3 may be rephrased by saying that t_r is *algebraically identifiable.*

2.4.2 First example

Set $x_i = \sum_{\nu_i=0}^{N_i} \frac{\gamma_{\nu_i}}{s^{\nu_i+1}}$, $i = 1, 2$, $\gamma_{\nu_i} \in k_0$, where N_i is a known non-negative integer.[11] Then $\frac{d^{N_i+2}}{ds^{N_i+2}} s^{N_i+1}$ is a minimal annihilator of x_i. It follows at once that Proposition 1 and Corollary 3 apply to this case.

Straightforward calculations demonstrate that t_r is the unique solution of an equation of the form

$$\mathfrak{p}\left(\frac{d}{ds} + t_r\right)^{\varrho} X = 0 \tag{5}$$

where $\mathfrak{p} \in k_0(s)[\frac{d}{ds}]$, $1 \leq \varrho \leq N_2$.

[11] The coefficients γ_{ν_i} are not necessarily known.

2.4.3 Second example

Assume that $x_2 = \frac{\bar{\gamma}_2}{s^{N_2+1}}$, $\bar{\gamma}_2 \in k_0$, in Sect. 2.4.2. Multiply both sides of Eq. (3) by $(\frac{d}{ds} + t_r)s^{N_2+1}$ yields

$$(\frac{d}{ds} + t_r)s^{N_2+1}X = (\frac{d}{ds} + t_r)s^{N_2+1}x_1$$

Eq. (5) becomes

$$\pi_1(\frac{d}{ds} + t_r)s^{N_2+1}X = 0 \tag{6}$$

where $\pi_1 \in k_0(s)[\frac{d}{ds}]$ is a minimal annihilator of $(\frac{d}{ds} + t_r)s^{N_2+1}x_1$.

Proposition 2 *t_r satisfies an algebraic Equation (6) of degree 1.*

Remark 4 If X is measured as in Remark 3, then, according to the terminology in [18,19], Proposition 2 may be rephrased by saying that t_r is *linearly identifiable.*

2.4.4 Third example

Assume in Eq. (3) that $x_2 = \frac{a}{b} \in k_0(s)$, $a, b \in k_0[s]$, $(a, b) = 1$, is a known rational function, i.e.,

$$X = x_1 + \frac{a}{b}e^{-t_r s} \tag{7}$$

Multiplying both sides by $(\frac{d}{ds} + t_r)\frac{b}{a}$ yields $(\frac{d}{ds} + t_r)\frac{b}{a}X = (\frac{d}{ds} + t_r)\frac{b}{a}x_1$. Since t_r is constant, there exists an annihilator $\pi \in k_0(s)[\frac{d}{ds}]$ of $(\frac{d}{ds} + t_r)\frac{b}{a}x_1$, i.e.,

$$\left(\pi\frac{b}{a}X\right)t_r + \pi\frac{d}{ds}\left(\frac{b}{a}X\right) = 0 \tag{8}$$

Proposition 3 *t_r in Eq. (7) satisfies an algebraic Equation (8) of degree 1.*

Remark 5 If X is measured as in Remark 3, then, according to Remark 4, Proposition 3 may be rephrased by saying that t_r is linearly identifiable.

3 Higher order change-points

Take again as in the introduction a piecewise smooth function f, which is now assumed to be C^n, $n \geq 0$, i.e., f and its pointwise derivatives up to order n are continuous. We might be interested in the discontinuities of its $(n+1)^{th}$ order pointwise derivative, which are called *change-points*, or *abrupt changes, of order* $n + 1$.

By replacing Eq. (3) by

$$s^{(n+1)}X = x_1 + x_2e^{-t_r s}$$

it is straightforward to extend all the results of Sect. 2.4 to higher order change-points.

4 Some numerical experiments

4.1 General principles

From now on k_0 is a subfield of $\mathbb{R}$, $\mathbb{Q}$ for instance. We utilize the calculations of Sect. 2.4.3 like follows:

- Multiplying both sides of Eq. (6) by s^{-N}, where $N > 0$ is large enough, yields

$$s^{-N}\pi_1\left(\frac{d}{ds}+t_r\right)s^{N_2+1}X = 0 \tag{9}$$

  where $s^{-N}\pi_1(\frac{d}{ds}+t_r)s^{N_2+1}$ is a strictly integral operator.
- Going back to the time domain is achieved via the classic rules of operational calculus [29,30,39], where $\frac{d^\nu}{ds^\nu}$ corresponds to the multiplication by $(-t)^\nu$.
- $x_1 = \sum_{\nu_1=0}^{N_1} \frac{\gamma_{\nu_1}}{s^{\nu_1+1}}$ and $x_2 = \frac{\bar{\gamma}_2}{s^{N_2+1}}$ correspond in the time domain to the polynomial functions $\sum_{\nu_1=0}^{N_1} \frac{\gamma_{\nu_1} t^{\nu_1}}{\nu_1 !}$ and $\frac{\bar{\gamma}_2 t^{N_2}}{N_2 !}$.
- Those time functions are assumed to approximate on a "short" time interval the signal where change-points have to be detected.
- Consider the numerical value v taken by the time analogue of the left side of Eq. (9) when the value given to t_r is the middle of a given "short" time window. If v is "close" to 0, then we say that the middle of the time window is a change-point.
- This time window is sliding in order to capture the various change-points, which are assumed to be not too "close", i.e., the distance between two consecutive change-points is larger than the time window.
- The corrupting noises are attenuated by the iterated time integrals which corresponds in the time domain to the negative power of s in the left side of Eq. (9).[12]

4.2 Examples[13]

The following academic examples are investigated:

- piecewise constant and polynomial real-valued functions,
- a real-valued sinusoid plus a piecewise constant real-valued function.

[12] Noises in [11] are viewed, via *nonstandard analysis*, as quickly fluctuating phenomena (see also [23] for an introductory presentation). The noises are attenuated by the iterated time integrals, which are simple examples of low-pass filters (we may also choose, according to Lemma 2, more involved low-pass filters (see, e.g., [7])). No statistical tools are required and we are by no means restricted to Gaussian white noises, like too often in the engineering studies. Moreover the corrupting noises need not to be additive. They might also be multiplicative.

[13] Interested readers may ask C. Join for the corresponding computer programs (`Cedric.Join@cran.uhp-nancy.fr`).

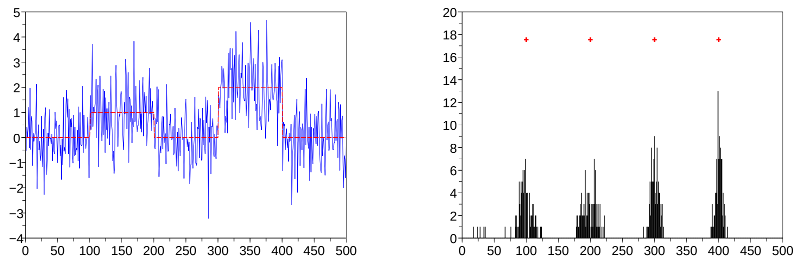
The robustness with respect to corrupting noises, which is reported in Table 1, is tested thanks to several noises, of various powers,[14] which are of the following types:

1. additive, zero mean, and either normal, uniform or Perlin,[15]
2. multiplicative, of mean 1, and uniform.

We finally note that

– piecewise polynomial functions were difficult to analyze even via recent techniques like wavelets (see, e.g., [9,24]);
– we do not need any *a priori* knowledge of the upper bound of the number of change-points (see, e.g., [21,35]);
– we are not limited to a given type of noises and we are able to handle multiplicative noises as well (see, e.g., [20,22,38]);
– the results remain satisfactory even with a very high noise level (see Figures 1 and 2).

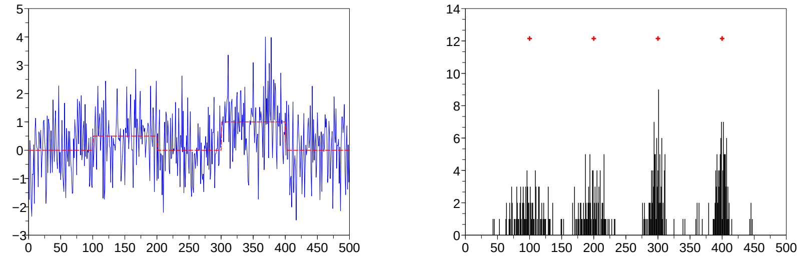
Remark 6 The so-called Perlin noises, which are not familiar in signal processing and in automatic control, contain components which are obviously not quickly fluctuating. It is all the more remarkable that our computer simulations are still good, in spite of the fact that this example goes beyond the theoretical justifications provided in Sect. 4.1.



(a) Noise-free signal (- -), signal (–) (b) Change-point detection – Exact (+)

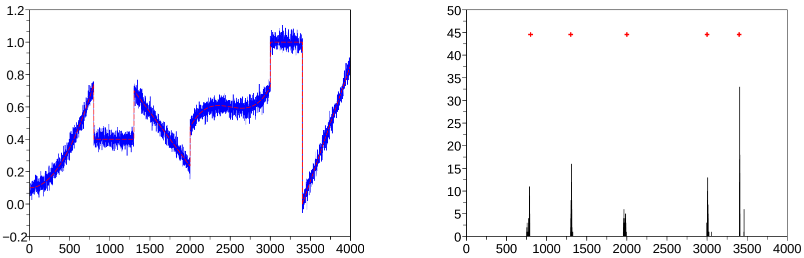
Fig. 1 Piecewise constant signal – Normal additive noise – SNR: 0 db

[14] We are utilizing the notion of *signal-to-noise ratio*, or *SNR*, which is familiar in signal processing (see *Wikipedia*, for instance).
[15] *Perlin's noises* [32] are quite popular in computer graphics.



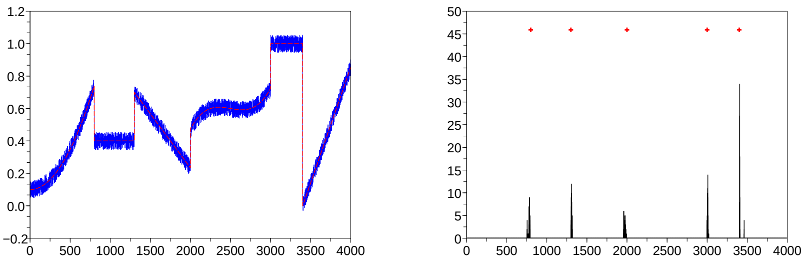
(a) Noise-free signal (- -), signal (–) (b) Change-point detection – Exact (+)

Fig. 2 Piecewise constant signal – Normal additive noise – SNR: -6 db



(a) Noise-free signal (- -), signal (–) (b) Change-point detection – Exact (+)

Fig. 3 Piecewise polynomial signal – Normal Additive noise – SNR: 25 db

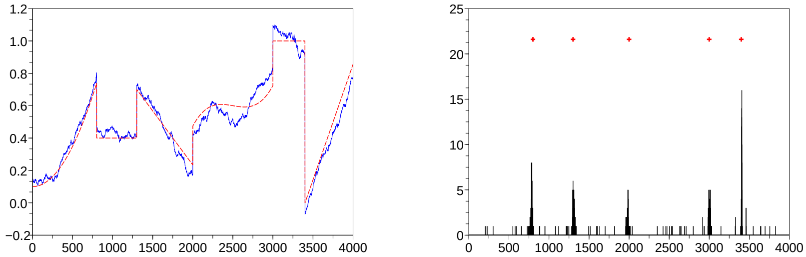


(a) Noise-free signal (- -), signal (–) (b) Change-point detection – Exact (+)

Fig. 4 Piecewise polynomial signal – Uniform additive noise – SNR: 25 db

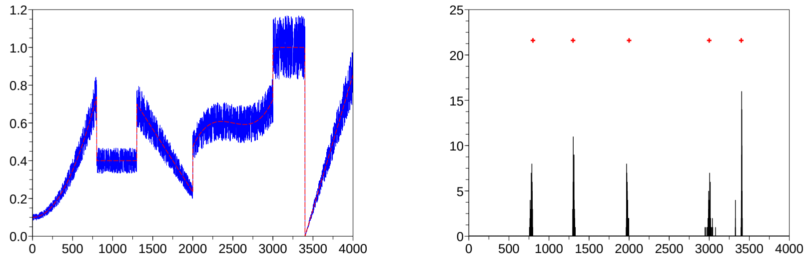
References

1. Basseville, M., Nikiforov, I.V.: Detection of Abrupt Changes: Theory and Application. Prentice-Hall (1993). Available online at `http://www.irisa.fr/sisthem/kniga/`.
2. Belkoura L.: Change point detection with application to the identification of a switching process. In: El Jai A., Afifi L., Zerrik E. (eds) Systems Theory: Modelling, Analysis and Control, Internat. Conf. Fes (Morocco),



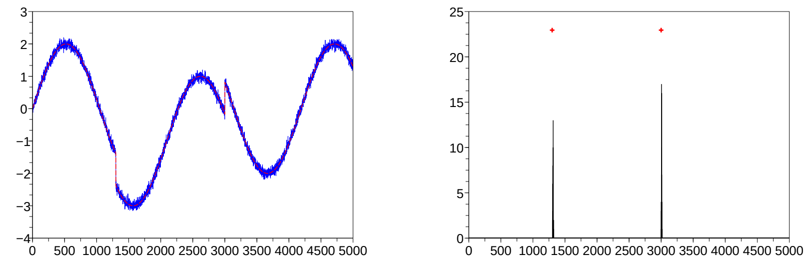
(a) Noisy free signal (- -), signal (–) (b) Change-point detection – Exact (+)

Fig. 5 Piecewise polynomial signal – Additive Perlin noise – SNR: 20 db



(a) Noise-free signal (- -), signal (–) (b) Distribution of change-point detection, exact places (+)

Fig. 6 Piecewise polynomial signal – Uniform multiplicative noise – SNR: 20 db

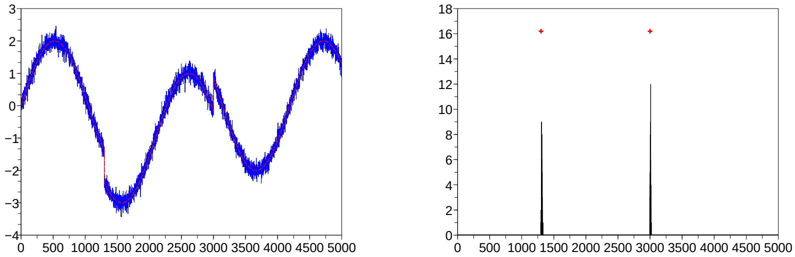


(a) Noise-free signal (- -), signal (–) (b) Change-point detection – Exact (+)

Fig. 7 Sinusoidal signal – Normal additive noise – SNR: 25 db

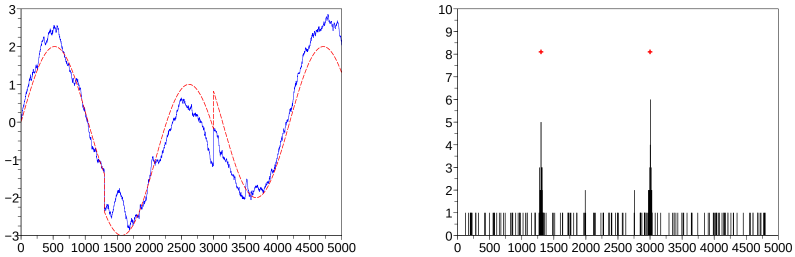
pp. 409-415, Presses Universitaires de Perpignan (2009). Available online at `http://hal.inria.fr/inria-00363679/en/`.

3. Belkoura L., Richard J.-P., Fliess M.: Parameters estimation of systems with delayed and structered entries. Automatica **45**, pp. 1117-1125 (2009).
4. Brodsky, B.E., Darkhovsky, B.S.: Nonparametric Methods in Change-Point Problems. Kluwer (1993).



(a) Noise-free signal (- -), signal (–) (b) Change-point detection – Exact (+)

Fig. 8 Sinusoidal signal – Normal additive noise – SNR: 20 db



(a) Noise-free signal (- -), signal (–) (b) Change-point detection – Exact (+)

Fig. 9 Sinusoidal signal – Additive Perlin noise – SNR: 10 db

Noise type	SNR in DB	Estimated number of segments **True value in bold font**									Figure references
		1	2	3	4	5	6	7	8	≥9	
Normal (+)	0	0	0	0	1	**98**	1	0	0	0	figure 1
Normal (+)	-6	0	0	1	8	**79**	12	0	0	0	figure 2
Normal (+)	25	0	0	0	0	9	**83**	8	0	0	figure 3
Uniform (+)	25	0	0	0	0	12	**81**	7	0	0	figure 4
Perlin (+)	20	0	0	0	2	25	**40**	16	6	11	figure 5
Uniform (×)	20	0	0	0	1	11	**74**	14	0	0	figure 6
Normal (+)	25	0	0	**100**	0	0	0	0	0	0	figure 7
Normal (+)	20	0	4	**96**	0	0	0	0	0	0	figure 8
Perlin (+)	10	0	18	**42**	16	8	3	3	2	8	figure 9

Table 1 Summary of simulation results • +: additive noise • ×: multiplicative noise

5. Brodsky, B.E., Darkhovsky, B.S.: Non-Parametric Statistical Diagnosis: Problems and Methods. Kluwer (2000).
6. Chambert-Loir, A.: Algèbre corporelle, Éditions École Polytechnique (2005). English translation: A Field Guide to Algebra. Springer (2005).
7. Chen, W.K.: Passive and Active Filters: Theory and Implementations. Wiley (1986).
8. Csörgö, M., Horváth, L.: Limit Theorems in Change-Point Analysis. Wiley (1997).
9. Dragotti, P.L., Vitterli, M.: Wavelets footprints: theory, algorithms, and applications. IEEE Trans. Signal Proc. **51**, pp. 1306-1323 (2003).
10. Flajolet P., Sedgewick R.: Analytic Combinatorics. Cambridge University Press (2009).

11. Fliess, M.: Analyse non standard du bruit. C.R. Acad. Sci. Paris Ser. I **342**, pp. 797-802 (2006).
12. Fliess, M.: Critique du rapport signal à bruit en communications numériques. ARIMA **9**, pp. 419-429 (2008). Available online at http://hal.inria.fr/inria-00311719/en/.
13. Fliess M., Join C.: Towards new technical indicators for trading systems and risk management. *15th IFAC Symp. System Identif.*, Saint-Malo (2009). Available online at http://hal.inria.fr/inria-00370168/en/.
14. Fliess M., Join C.: Systematic risk analysis: first steps towards a new definition of beta. *COGIS'09*, Paris (2009). Available online at http://hal.inria.fr/inria-00425077/en/.
15. Fliess, M., Join, C., Sira-Ramírez, H.: Non-linear estimation is easy. Int. J. Modelling Identification Control. **4**, pp. 12-27 (2008).
16. Fliess, M., Join, C., Mboup, M., Sira-Ramírez, H.: Analyse et représentation de signaux transitoires : application à la compression, au débruitage et à la détection de ruptures. 20e *Coll. GRETSI*, Louvain-la-Neuve (2005). Available online at http://hal.inria.fr/inria-00001115/en/.
17. Fliess, M., Mboup, M., Mounier, H., Sira-Ramírez, H.: Questioning some paradigms of signal processing via concrete examples. In: Sira-Ramírez H., Silva-Navarro G. (eds.) Algebraic Methods in Flatness, Signal Processing and State Estimation, pp. 1-21, Editiorial Lagares (2003). Available online at http://hal.inria.fr/inria-00001059/en/.
18. Fliess, M., Sira-Ramírez, H.: An algebraic framework for linear identification, ESAIM Control Optim. Calc. Variat. **9**, pp. 151-168 (2003).
19. Fliess, M., Sira-Ramírez, H.: Closed-loop parametric identification for continuous-time linear systems via new algebraic techniques. In: Garnier, H., Wang, L. (eds) Identification of Continuous-Time Model Identification from Sampled Data, pp. 363-391, Springer (2008).
20. Gijbels, I., Hall, P., Kneip, A.: On the estimation of jump points in smooth curves. Ann. Instit. Statistical Math. **51**, pp. 231-251 (1999).
21. Lavielle, M.: Using penalized contrasts for change-point problem. Signal Processing **85**, pp. 1501-1510 (2005).
22. Lebarbier, E.: Detecting mutiple change-points in the mean of a Gaussian process by model selection. Signal Processing **85**, pp. 717-736 (2005).
23. Lobry, C., Sari, T.: Nonstandard analysis and representation of reality. Int. J. Control **81**, pp. 517-534 (2008).
24. Mallat, S.: A Wavelet Tour of Signal Processing (2^{nd} ed.). Academic Press (1999).
25. Mboup M.: Parameter estimation for signals described by differential equations. Applicable Anal. **88**, pp. 29-52 (2009).
26. Mboup M.: A Volterra filter for neuronal spike detection. Preprint (2008). Available online at http://hal.inria.fr/inria-00347048/en/.
27. Mboup M., Join C., Fliess M.: A delay estimation approach to change-point detection. *16th Medit. Conf. Control Automat.*, Ajaccio (2008). Available online at http://hal.inria.fr/inria-00179775/en/.
28. McConnell, J., Robson, J.: Noncommutative Noetherian Rings. Amer. Math. Soc. (2000).
29. Mikusinski, J.: Operational Calculus (2^{nd} ed.), Vol. 1. PWN & Pergamon (1983).
30. Mikusinski, J., Boehme, T.: Operational Calculus (2^{nd} ed.), Vol. 2. PWN & Pergamon (1987).
31. Ollivier F., Moutaouakil S., Sadik B.: Une méthode d'identification pour un système linéaire à retards. C.R. Acad. Sci. Paris Ser. I **344**, pp.709-714 (2007).
32. Perlin, K.: An image synthetizer. ACM SIGGRAPH Comput. Graphics **19**, pp. 287-296 (1985).
33. van der Pol, B., Bremmer, H.: Operational Calculus Based on the Two-Sided Laplace Integral (2^{nd} ed.). Cambridge University Press (1955).
34. van der Put, M., Singer, M.F.: Galois Theory of Linear Differential Equations, Springer (2003).

35. Raimondo, M., Tajvidi, N.: A peaks over threshold model for change-points detection by wavelets. Statistica Sinica **14**, pp. 395-412 (2004).
36. Rudolph J., Woittennek F.: Ein algebraischer Zugang zur Parameteridentifkation in linearen unendlichdimensionalen Systemen. at–Automatisierungstechnik **55**, pp. 457-467 (2007).
37. Tiganj Z., Mboup M.: Spike detection and sorting: combining algebraic differentiations with ICA. *8*th *Int. Conf. Indep. Component Anal. Signal Separat.*, Paraty, Brazil (2009). Available online at `http://hal.inria.fr/inria-00430438/en/`.
38. Tourneret, J.-Y., Doisy, M., Lavielle, M.: Bayesian off-line detection of multiple change-points corrupted by multiplicative noise: application to SAR image edge detection. Signal Processing **83**, pp. 1871-1887 (2003).
39. Yosida, K.: Operational Calculus: A Theory of Hyperfunctions (*translated from the Japanese*). Springer (1984).