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Huaibin Tang and Qinghua Zhang
INRIA, Campus de Beaulieu, 35042 Rennes Cedex, France
E-mail: hutang@irisa.fr, zhang@irisa.fr

**Abstract**

The inverse scattering problem for coupled wave equations has various applications such as waveguide filter design and electric transmission line fault diagnosis. In this paper, an efficient numerical algorithm is presented for solving the inverse scattering problem related to generalized Zakharov-Shabat equations with two potential functions. Inspired by the work of Xiao and Yashiro on Zakharov-Shabat equations with a single potential function, this new algorithm considerably improves the numerical efficiency of an algorithm proposed by Frangos and Jaggard, by transforming the original iterative algorithm to a one-shot algorithm.

## 1 Introduction

The inverse scattering problem for coupled wave equations [1] has various applications, such as waveguide filter design [2, 3] and electric transmission line fault diagnosis [4]. The solution to such problems can be computed, in principle, by solving the Gel'fand-Levitan-Marchenko (GLM) linear integral equations. In practice, numerical algorithms are developed to solve the GLM equations. After discretization, the GLM equations lead to a set of linear algebraic equations, typically with a large number of equations and unknowns. Though in principle such a large set of linear algebraic equations can be solved by standard algorithms with digital computers, in most practical situations, due to the large size of the equations, it is important to develop efficient numerical algorithms by taking into account the particularities of the inverse scattering problem. Such a numerical algorithm was proposed in [5] for the inverse scattering problem related to the Zakharov-Shabat equations with a single potential function, then extended in [6] to the generalized Zakharov-Shabat equations with two potential functions. In this approach, by combining the GLM equations with some second order partial differential equations derived from Zakharov-Shabat equations, iterative algorithms are designed for solving discretized GLM equations. These iterative algorithms can be initialized by approximative solutions of discretized GLM equations. The result of [5] is later improved in [7] where the second order partial differential equations combined to the GLM equations are replaced by first order partial differential equations. This modification has the advantage of triangularizing the set of linear algebraic equations resulting from the discretized differential and integral equations, hence the solution to the large set of linear algebraic equations becomes much more efficient.

The algorithm after the improvement of [7] is much faster than that of [5]. However, in [7] this improvement has only been made to the the case of single-potential Zakharov-Shabat equations. In the present paper, the method of [7] is extended to the case of generalized Zakharov-Shabat equations with two-potential functions, resulting in a similar fast algorithm. The two-potential generalized Zakharov-Shabat equations have a larger scope of applications than the single-potential Zakharov-Shabat equations. For instance, when applied to electric transmission lines, the single-potential equations correspond to lossless lines, whereas the two-potential equations cover the case of lossy lines [8].

After a brief formulation of the inverse scattering problem for the generalized Zakharov-Shabat equations with two potential functions and a short presentation of their theoretic solution through GLM equations, the new numerical algorithm, extending the results of [5, 6, 7], will be presented.

## 2 The inverse scattering problem

Consider the following generalized Zakharov-Shabat equations with two potential functions $q^+(x)$ and $q^-(x)$ which will be referred to as $\mathrm{ZS}^+$:

$$\mathrm{ZS}^+ : \begin{cases} \dfrac{d\nu_1(k,x)}{dx} + ik\nu_1(k,x) = q^+(x)\nu_2(k,x) & \text{(1a)} \\ \dfrac{d\nu_2(k,x)}{dx} - ik\nu_2(k,x) = q^-(x)\nu_1(k,x) & \text{(1b)} \end{cases}$$

where $q^{\pm}(x)$ are the unknown coupling potential functions and $k$ is the wave number. Denote with $r_l(k), r_r(k)$ and $t(k)$ respectively the left reflection coefficient, the right reflection coefficient and the transmission coefficient of $\mathrm{ZS}^+$, or of (1), the inverse scattering problem considered in this paper is to determine the coupling potential functions $q^{\pm}(x)$ from the scattering data. Assume that $\mathrm{ZS}^+$ has no bound state (square integrable solution for $x \in \mathbb{R}$), then the scattering data are composed of $r_l(k), r_r(k)$ and $t(k)$ .

For a given value of $k$, the left and right Jost solutions $\nu = f_l(k,x)$ and $\nu = f_r(k,x)$ of $\mathrm{ZS}^+$ are particular solutions satisfying the limiting conditions

$$\lim_{x\to+\infty} \begin{pmatrix} f_{l1}(k,x) \\ e^{-ikx} f_{l2}(k,x) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \text{and} \quad \lim_{x\to-\infty} \begin{pmatrix} e^{ikx} f_{r1}(k,x) \\ f_{r2}(k,x) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

The three coefficients constituting the scattering data of $\mathrm{ZS}^+$ can be expressed as

$$\begin{aligned} r_l(k) &= \lim_{x\to-\infty} \frac{f_{l1}(k,x)}{f_{l2}(k,x)} e^{2ikx} \\ r_r(k) &= \lim_{x\to+\infty} \frac{f_{r2}(k,x)}{f_{r1}(k,x)} e^{-2ikx} \\ t(k) &= \frac{\lim\limits_{x\to+\infty} f_{l2}(k,x)e^{-ikx}}{\lim\limits_{x\to-\infty} f_{l2}(k,x)e^{-ikx}} = \frac{\lim\limits_{x\to-\infty} f_{r1}(k,x)e^{ikx}}{\lim\limits_{x\to+\infty} f_{r1}(k,x)e^{ikx}} \end{aligned}$$

To solve the inverse scattering problem as formulated above, it is useful to introduce the following auxiliary system, referred to as $\mathrm{ZS}^-$:

$$\text{ZS}^- : \begin{cases} \dfrac{d\nu_1^-(k,x)}{dx} + ik\nu_1^-(k,x) = q^-(x)\nu_2^-(k,x) & \text{(3a)} \\ \dfrac{d\nu_2^-(k,x)}{dx} - ik\nu_2^-(k,x) = q^+(x)\nu_1^-(k,x). & \text{(3b)} \end{cases}$$

which is obtained by interchanging the two potential functions $q^{\pm}(x)$ of $\text{ZS}^+$. The left reflection coefficient, the right reflection coefficient and the transmission coefficient of $\text{ZS}^-$ will be respectively denoted by $r_l^-(k), r_r^-(k)$ and $t^-(k)$ in the following.

The solution to the inverse scattering problem of $\text{ZS}^+$ summarized below will retrieve the two potential functions $q^{\pm}(x)$ from the two left reflection coefficients $r_l(k)$ and $r_l^-(k)$ which are respectively related to $\text{ZS}^+$ and $\text{ZS}^-$. In most applications, only the reflection and transmission coefficients of $\text{ZS}^+$ are available, not those of the auxiliary system $\text{ZS}^-$. Fortunately, in the case of real potential functions $q^{\pm}(x)$, $r_l^-(k)$ can be computed from the reflection and transmission coefficients of $\text{ZS}^+$ based on the following lemmas.

**Lemma 1.** *If the potential functions $q^{\pm}(x)$ are real, the following equalities hold, where "$*$" represents the complex conjugate:*

$$r_r(-k) = [r_r(k)]^*, \ r_l(-k) = [r_l(k)]^*, \ t(-k) = [t(k)]^*.$$

This result can be proved by simply taking the complex conjugate at both sides of $\text{ZS}^+$.

**Lemma 2.** *The reflection and transmission coefficients of* $\text{ZS}^+$ *and* $\text{ZS}^-$ *are related by*

$$\begin{pmatrix} t(k) & r_r(k) \\ r_l(k) & t(k) \end{pmatrix}^{-1} = \begin{pmatrix} t^-(-k) & r_l^-(-k) \\ r_r^-(-k) & t^-(-k) \end{pmatrix} \tag{4}$$

See [9] for a proof of this result.

By combining the above two lemmas, we have

$$\begin{aligned} r_l^-(k) &= \frac{r_r(-k)}{r_r(-k)r_l(-k) - [t(-k)]^2} \\ &= \left[\frac{r_r(k)}{r_r(k)r_l(k) - [t(k)]^2}\right]^* \end{aligned} \tag{5}$$

Hence $r_l^-(k)$ can be computed from $r_l(k), r_r(k)$ and $t(k)$, the scattering data of $\text{ZS}^+$, which are thus sufficient for retrieving the potential functions $q^{\pm}(x)$ through the inverse scattering transform.

To avoid infinite integral intervals in the GLM equations, we assume that $q^{\pm}(x) = 0$ for all $x < 0$. This is the case if $r_l^-(k)$, and $r_l(k)$ can be extended analytically to the upper half of the complex $k$-plane without any pole. Then, as derived in [9], the inverse scattering transform consists of the following steps for computing the potential functions $q^{\pm}(x)$ from $r_l(k), r_r(k)$ and $t(k)$.

1. Compute $r_l^-(k)$ from $r_l(k), r_r(k)$ and $t(k)$ through (5).

2. Compute the Fourier transforms,

$$R_l(y) \triangleq \frac{1}{2\pi}\int_{-\infty}^{+\infty} r_l(k)e^{-iky}dk \tag{6a}$$

$$R_l^-(y) \triangleq \frac{1}{2\pi}\int_{-\infty}^{+\infty} r_l^-(k)e^{-iky}dk \tag{6b}$$

3. Solve the following GLM integral equations for the kernel functions $A_{r1}(x, y)$, $A_{r2}(x, y)$, $A_{r1}^-(x, y)$, and $A_{r2}^-(x, y)$,

$$A_{r1}^-(x, y) + \int_{-y}^{x} dsR_l(y + s)A_{r2}(x, s) = 0 \tag{7a}$$

$$A_{r2}^-(x, y) + R_l(y + x) + \int_{-y}^{x} dsR_l(y + s)A_{r1}(x, s) = 0 \tag{7b}$$

$$A_{r1}(x, y) + \int_{-y}^{x} dsR_l^-(y + s)A_{r2}^-(x, s) = 0 \tag{8a}$$

$$A_{r2}(x, y) + R_l^-(x + y) + \int_{-y}^{x} dsR_l^-(y + s)A_{r1}^-(x, s) = 0 \tag{8b}$$

4. Compute the potential functions $q^{\pm}(x)$ from

$$q^+(x) = 2A_{r2}^-(x, x) \tag{9a}$$

$$q^-(x) = 2A_{r2}(x, x) \tag{9b}$$

For more details about the inverse scattering theory, we refer the readers to [9].

## 3 Numerical algorithm

Inspired by the results of [5, 6, 7], here we present an efficient numerical algorithm to solve the inverse scattering problem formulated in the previous section.

As the computation of $r_l^-(k)$ from $r_l(k), r_r(k)$ and $t(k)$ is trivial, and the Fourier transforms $R_l(y)$ and $R_l^-(y)$ can be computed with the aid of FFT from discretized values of $r_l(k)$ and $r_l^-(k)$, the main numerical computations concern the solution of the GLM equations (7) and (8) for the kernel functions $A_{r1}(x, y)$, $A_{r2}(x, y)$, $A_{r1}^-(x, y)$ and $A_{r2}^-(x, y)$.

In addition to the GLM equations (7) and (8), the following two pairs of equations respectively derived from ZS$^+$ and ZS$^-$ will also be used:

$$\frac{\partial A_{r1}(x, y)}{\partial x} + \frac{\partial A_{r1}(x, y)}{\partial y} = q^+(x)A_{r2}(x, y) \tag{10a}$$

$$\frac{\partial A_{r2}(x, y)}{\partial x} - \frac{\partial A_{r2}(x, y)}{\partial y} = q^-(x)A_{r1}(x, y) \tag{10b}$$

$$\frac{\partial A^-_{r1}(x,y)}{\partial x} + \frac{\partial A^-_{r1}(x,y)}{\partial y} = q^-(x) A^-_{r2}(x,y) \tag{11a}$$

$$\frac{\partial A^-_{r2}(x,y)}{\partial x} - \frac{\partial A^-_{r2}(x,y)}{\partial y} = q^+(x) A^-_{r1}(x,y) \tag{11b}$$

These equations are redundant to the GLM equations (7) and (8). Their combination with the GLM equations will make easier the solution of the large set of linear algebraic equations resulting from the discretization of the GLM equations.

For the case of single-potential Zakharov-Shabat equations ($q^+(x) = \pm q^-(x)$), only the pair (10) is used in [7], where it was proposed as an alternative to the similar second order equations used in [5]. This improvement made in [7] leads to an algorithm much faster than the original algorithm of [5].

Similar to the improvement made in [7] to the algorithm of [5], the use of the two pairs of first order equations (10) and (11) in this paper is in replacement of the second order equations used in the algorithm of [6].

As in [5, 6, 7], we introduce the following coordinate change which will transform the region of $x \geq |y|$ in the $x$-$y$ plane to the first quadrant in the $\xi$-$\eta$ plane:

$$\xi = \frac{x+y}{2}, \ \eta = \frac{x-y}{2} \tag{12}$$

then the potential functions can be expressed in terms of $\xi$ and $\eta$ as

$$q^+(x) = q^+(\xi+\eta), \ q^-(x) = q^-(\xi+\eta)$$

and the $A$-kernels will be rewritten as $B$-kernels after the coordinate change:

$$\begin{aligned} A^-_{r1}(x,y) &= B^-_1(\xi,\eta), \ A^-_{r2}(x,y) = B^-_2(\xi,\eta) \\ A_{r1}(x,y) &= B_1(\xi,\eta), \ A_{r2}(x,y) = B_2(\xi,\eta). \end{aligned}$$

Then the GLM equations (7), (8) and equations (9) become

$$B^-_1(\xi,\eta) + 2\int_0^\xi B_2(s+\eta, \xi-s) R_l(2s) ds = 0 \tag{13a}$$

$$B^-_2(\xi,\eta) + R_l(2\xi) + 2\int_0^\xi B_1(s+\eta, \xi-s) R_l(2s) ds = 0 \tag{13b}$$

$$B_1(\xi,\eta) + 2\int_0^\xi B^-_2(s+\eta, \xi-s) R^-_l(2s) ds = 0 \tag{13c}$$

$$B_2(\xi,\eta) + R^-_l(2\xi) + 2\int_0^\xi B^-_1(s+\eta, \xi-s) R^-_l(2s) ds = 0 \tag{13d}$$

$$q^+(\xi+\eta) = 2B^-_2(\xi+\eta, 0) \tag{13e}$$

$$q^-(\xi+\eta) = 2B_2(\xi+\eta, 0) \tag{13f}$$

By setting $\xi = 0$ in the first four equations of (13), the following boundary conditions are derived:

$$B_1^-(0,\eta) = 0 \tag{14a}$$
$$B_2^-(0,\eta) = -R_l(0), \tag{14b}$$
$$B_1(0,\eta) = 0 \tag{14c}$$
$$B_2(0,\eta) = -R_l^-(0). \tag{14d}$$

After the coordinate change (12), equations (10) and (11) become

$$\frac{\partial B_1(\xi,\eta)}{\partial\xi} = q^+(\xi+\eta)B_2(\xi,\eta) \tag{15a}$$
$$\frac{\partial B_2(\xi,\eta)}{\partial\eta} = q^-(\xi+\eta)B_1(\xi,\eta) \tag{15b}$$
$$\frac{\partial B_1^-(\xi,\eta)}{\partial\xi} = q^-(\xi+\eta)B_2^-(\xi,\eta) \tag{15c}$$
$$\frac{\partial B_2^-(\xi,\eta)}{\partial\eta} = q^+(\xi+\eta)B_1^-(\xi,\eta) \tag{15d}$$

Now discretize all these equations over the grid on the $\xi$-$\eta$ plane defined by $\xi = id$ and $\eta = jd$, with $i, j = 0, 1, 2, \ldots$, and $d$ being the discretization step size[1]. The discrete counterparts of equations (13) and (15) then write

$$B_1^-(i,j) = -2d\sum_{k=0}^{i-1} B_2(k+j, i-k)R_l(2k) \tag{16a}$$
$$B_2^-(i,j) = -R_l(2i) - 2d\sum_{k=0}^{i-1} B_1(k+j, i-k)R_l(2k) \tag{16b}$$
$$B_1(i,j) = -2d\sum_{k=0}^{i-1} B_2^-(k+j, i-k)R_l^-(2k) \tag{16c}$$
$$B_2(i,j) = -R_l^-(2i) - 2d\sum_{k=0}^{i-1} B_1^-(k+j, i-k)R_l^-(2k) \tag{16d}$$
$$q^+(i+j) = 2B_2^-(i+j,0) \tag{16e}$$
$$q^-(i+j) = 2B_2(i+j,0) \tag{16f}$$
$$B_1(i,j) = B_1(i-1,j) + dq^+(i+j-1)B_2(i-1,j) \tag{16g}$$
$$B_2(i,j) = B_2(i,j-1) + dq^-(i+j-1)B_1(i,j-1) \tag{16h}$$
$$B_1^-(i,j) = B_1^-(i-1,j) + dq^-(i+j-1)B_2^-(i-1,j) \tag{16i}$$
$$B_2^-(i,j) = B_2^-(i,j-1) + dq^+(i+j-1)B_1^-(i,j-1) \tag{16j}$$

The numerical inverse scattering algorithm should solve for $B_1(i,j), B_2(i,j), B_1^-(i,j), B_2^-(i,j)$, and hence for $q^\pm(i+j)$, from the above discretized equations. Usually the discretization grid is defined for $i$ and $j$ ranging from 0 to a large integer value (typically hundreds or thousands). Hence these equations constitute a large system

[1] The index notations $i, j$ used here should not be confused with the imaginary unit number.

of linear algebraic equations in the unknowns $B_1(i,j), B_2(i,j), B_1^-(i,j), B_2^-(i,j)$. Fortunately, these equations can be efficiently solved by observing the following particular structure of the discretized equations.

In what follows, the $L$-th diagonal of the discretization grid will refer to the set of all the grid nodes $(i,j)$ such that $i+j=L$ and $i,j \geq 0$. For any node $(i,j)$ on diagonal $L=i+j$ of the grid, we said the kernel value $B_1(i,j)$ is on diagonal $L$, so are the other similar kernel values. Moreover, $q^+(i+j)$ and $q^-(i+j)$ are also said to be on diagonal $L=i+j$.

The kernel value $B_1(i,j)$ on diagonal $L=i+j$ can be computed from kernel and potential functions values on diagonal $L-1$, following (16g), since the indexes of each quantity appearing at the right hand side of this equation sum to $i+j-1$. If the kernel values on diagonal $L-1$ are already computed, and so is $q^+(i+j-1)$, then the kernel value $B_1(i,j)$ can be easily obtained through (16g). The same observation can be made for the other kernel values and $q^-(i+j-1)$ by examining (16h), (16i) and (16j). Therefore, the kernel and potential function values can be computed diagonal by diagonal, by repeatedly increasing the value of $L$. This reasoning does not apply to nodes at the border of the first quadrant where $i=0$ or $j=0$, for which the boundary conditions (14) or equations (16b) or (16d) provide the kernel values. Following this algorithm, at each step a single unknown kernel value is computed with a new equation where the other unknowns are already computed in previous steps, without requiring any algebraic operation for eliminating unknonws. This simplicity indicates that the set of linear algebraic equations to be solved is in a triangular form.

The complete algorithm is summarized as follows.

- At diagonal $L=0$, apply the boundary conditions $B_1^-(0,0)=0, B_2^-(0,0)=-R_l(0)$, $B_1(0,0)=0, B_2(0,0)=-R_l^-(0)$, $q^+(0)=2B_2^-(0,0)=-2R_l(0)$, $q^-(0)=2B_2(0,0)=-2R_l^-(0)$.
- For $L=1,2,3,\ldots$, the nodes $(i,j)$ on diagonal $L$ are such that $i+j=L$.
  1. Compute $B_1^-(0,L)=0, B_2^-(0,L)=-R_l(0), B_1(0,L)=0, B_2(0,L)=-R_l^-(0)$.
  2. Use (16g) and (16i) to find all the other values of $B_1(i,j)$ and $B_1^-(i,j)$, and use (16h) and (16j) to find all the other values of $B_2(i,j)$ and $B_2^-(i,j)$, except $B_2(L,0)$ and $B_2^-(L,0)$.
  3. Compute $B_2(L,0)$ and $B_2^-(L,0)$ through (16b) and (16d).
  4. Compute $q^+(L)$ and $q^-(L)$ through (16e) and (16f):

$$q^+(L)=q^+(i+j)=2B_2^-(i+j,0),\ q^-(L)=q^-(i+j)=2B_2(i+j,0).$$

## 4 Conclusion

In this paper an efficient numerical algorithm is presented for solving the inverse scattering problem related to the generalized Zakharov-Shabat equations with two potential functions. Inspired by the work of [7] on Zakharov-Shabat equations with a single potential function, the second order differential equations derived from Zakharov-Shabat equations used in the algorithm of [6] are replaced by first order differential equations. This modification has the advantage of triangularizing the large set of linear algebraic equations resulting from the discretization of the GLM equations, hence these equations can be efficiently solved by a one-shot algorithm, instead of the iterative algorithm used in [6].

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