Lipschitz classification of almost-Riemannian distances on compact oriented surfaces

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Lipschitz Classification of Almost-Riemannian Distances on Compact Oriented Surfaces

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March 17, 2010

Abstract

Two-dimensional almost-Riemannian structures are generalized Riemannian structures on surfaces for which a local orthonormal frame is given by a Lie bracket generating pair of vector fields that can become collinear. We consider the Carnot–Caratheodory distance canonically associated with an almost-Riemannian structure and study the problem of Lipschitz equivalence between two such distances on the same compact oriented surface. We analyse the generic case, allowing in particular for the presence of tangency points, i.e., points where two generators of the distribution and their Lie bracket are linearly dependent. The main result of the paper provides a characterization of the Lipschitz equivalence class of an almost-Riemannian distance in terms of a labelled graph associated with it.

1 Introduction

Consider a pair of smooth vector fields X and Y on a two-dimensional smooth manifold M. If the pair (X,Y) is Lie bracket generating, i.e., if $\mathrm{span}\{X(q),Y(q),[X,Y](q),[X,[X,Y]](q),\ldots\}$ is full-dimensional at every $q\in M$, then the control system

$$\dot q = uX(q)+vY(q)\,,\quad u^2+v^2\le 1\,,\quad q\in M\,, \tag{1}$$

is completely controllable and the minimum-time function defines a continuous distance d on M. When X and Y are everywhere linear independent (the only possibility for this to happen is that M is parallelizable), such distance is Riemannian and it corresponds to the metric for which (X,Y) is an orthonormal frame. Our aim is to study the geometry obtained starting from a pair of vector fields which may become collinear. Under generic hypotheses, the set $\mathcal{Z}$ (called *singular locus*) of points of M at which X and Y are parallel is a one-dimensional embedded submanifold of M (possibly disconnected).

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Metric structures that can be defined *locally* by a pair of vector fields (X, Y) through (1) are called almost-Riemannian structures.

Equivalently, an almost-Riemannian structure $\mathcal{S}$ can be defined as an Euclidean bundle E of rank two over M (i.e. a vector bundle whose fibre is equipped with a smoothly-varying scalar product $\langle\cdot,\cdot\rangle_q$) and a morphism of vector bundles $f : E \to TM$ such that the evaluation at q of the Lie algebra generated by the submodule

$$\Delta := \{f \circ \sigma \mid \sigma \,\text{section of}\, E\} \tag{2}$$

of the algebra of vector fields on M is equal to T_qM for every $q \in M$.

If E is orientable, we say that $\mathcal{S}$ is *orientable*. The singular locus $\mathcal{Z}$ is the set of points q of M at which $f(E_q)$ is one-dimensional. An almost-Riemannian structure is Riemannian if and only if $\mathcal{Z} = \emptyset$, i.e. f is an isomorphism of vector bundles.

The first example of genuinely almost-Riemannian structure is provided by the Grushin plane, which is the almost-Riemannian structure on $M = \mathbb{R}^2$ with $E = \mathbb{R}^2 \times \mathbb{R}^2$, $f((x, y), (a, b)) = ((x, y), (a, bx))$ and $\langle\cdot,\cdot\rangle$ the canonical Euclidean structure on $\mathbb{R}^2$. The model was originally introduced in the context of hypoelliptic operator theory [13, 15] (see also [4, 10]). Notice that the singular locus is indeed nonempty, being equal to the y-axis. Another example of almost-Riemannian structure appeared in problems of control of quantum mechanical systems (see [8, 9]).

Almost-Riemannian structures present very interesting phenomena. For instance, even in the case where the Gaussian curvature is everywhere negative (where it is defined, i.e., on $M \setminus \mathcal{Z}$), geodesics may have conjugate points. This happens for instance on the Grushin plane (see [2] and also [6, 5] in the case of surfaces of revolution). The structure of the cut and conjugate loci is described in [7] under generic assumptions.

In [1], we provided an extension of the Gauss–Bonnet theorem to almost-Riemannian structures, linking the Euler number of the vector bundle E to a suitable principal part of the integral of the curvature on M. For generalizations of the Gauss-Bonnet formula in related context see also [18].

The results in [1] have been obtained under a set of generic hypotheses called **(H0)**. To introduce it, let us define the *flag* of the submodule Δ defined in (2) as the sequence of submodules $\Delta = \Delta_1 \subset \Delta_2 \subset \cdots \subset \Delta_m \subset \cdots$ defined through the recursive formula

$$\Delta_{k+1} = \Delta_k + [\Delta, \Delta_k].$$

Under generic assumptions, the singular locus $\mathcal{Z}$ has the following properties: **(i)** $\mathcal{Z}$ is an embedded one-dimensional submanifold of M; **(ii)** the points $q \in M$ at which $\Delta_2(q)$ is one-dimensional are isolated; **(iii)** $\Delta_3(q) = T_qM$ for every $q \in M$. We say that $\mathcal{S}$ satisfies **(H0)** if properties **(i)**,**(ii)**,**(iii)** hold true. If this is the case, a point q of M is called *ordinary* if $\Delta(q) = T_qM$, *Grushin point* if $\Delta(q)$ is one-dimensional and $\Delta_2(q) = T_qM$, i.e. the distribution is transversal to $\mathcal{Z}$, and *tangency point* if $\Delta_2(q)$ is one-dimensional, i.e. the distribution is tangent to $\mathcal{Z}$. Local normal forms around ordinary, Grushin and tangency points have been provided in [2]. When an ARS $\mathcal{S} = (E, f, \langle\cdot,\cdot\rangle)$ satisfying **(H0)** is oriented and the surface itself is oriented, M is split into two open sets M^+, M^- such that $\mathcal{Z} = \partial M^+ = \partial M^-$, $f : E|_{M^+} \to TM^+$ is an orientation preserving isomorphism and $f : E|_{M^-} \to TM^-$ is an orientation reversing isomorphism. Moreover, in this case it is possible to associate with each tangency point q an integer τ_q in the following way. Choosing on $\mathcal{Z}$ the orientation induced by M^+, $\tau_q = 1$ if walking along the oriented curve $\mathcal{Z}$ in a neighborhood of q the angle between the distribution and the tangent space to $\mathcal{Z}$ increases, $\tau_q = -1$ if the angle decreases.

In this paper we provide a classification of orientable two-dimensional almost-Riemannian structures in terms of graphs. With an oriented almost-Riemannian structure, we associate a graph whose vertices correspond to connected components of $M \setminus \mathcal{Z}$ and whose edges correspond to connected components of $\mathcal{Z}$. The edge corresponding to a connected component W of $\mathcal{Z}$ joins the two vertices corresponding to the connected components of $M \setminus \mathcal{Z}$ adjacent to W. Every vertex is labelled with its orientation (± 1 if it a subset of $M^{\pm}$) and its Euler characteristic. Every edge is labelled with the ordered sequence of signs (modulo cyclic permutations) given by the contributions at the tangency points belonging to W. See Figure 1 for an example of almost-Riemannian structure and its corresponding graph. We say that two labelled graphs are equivalent if they are equal or they can be obtained by the same almost-Riemannian structure reversing the orientation of the vector bundle.

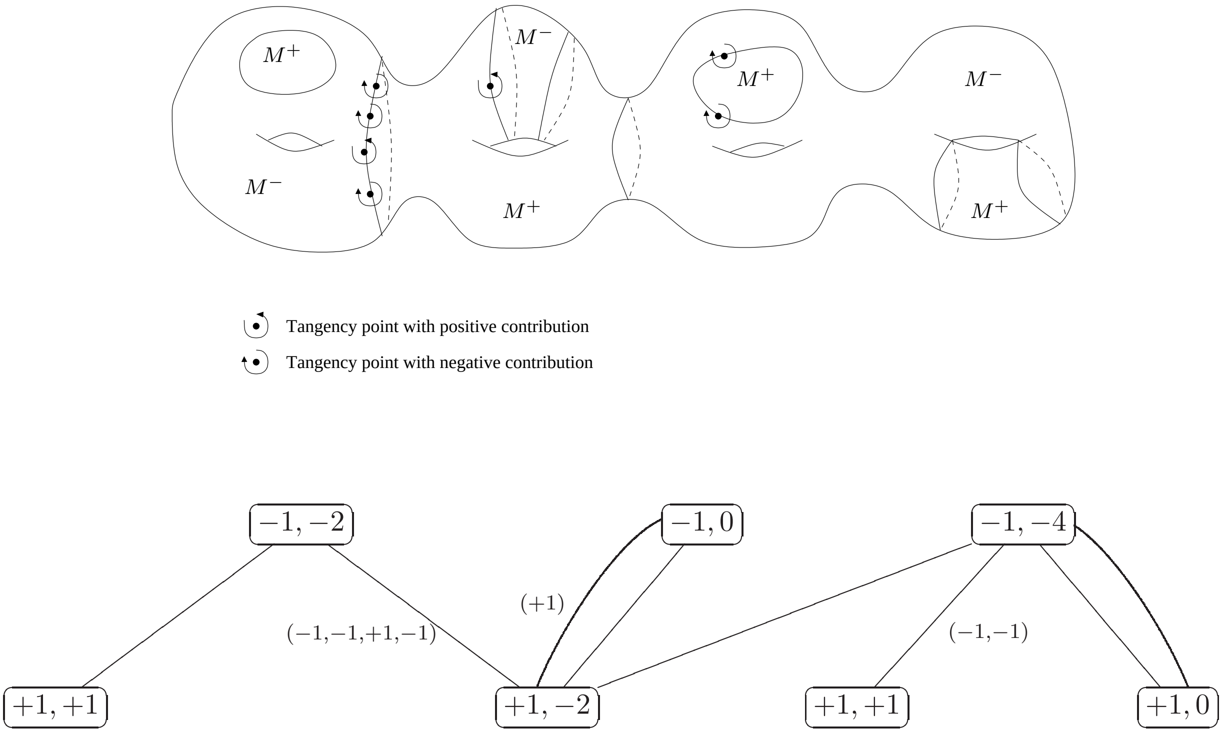

Figure 1: Example of ARS on a surface of genus 4 and corresponding labelled graph

The main result of the paper is the following.

Theorem 1 *Two oriented almost-Riemannian structures, defined on compact oriented surfaces and satisfying* **(H0)***, are Lipschitz equivalent if and only if they have equivalent graphs.*

In the statement above, two almost-Riemannian structures are said to be Lipschitz equivalent if there exists a diffeomorphism between their base surfaces which is bi-Lipschitz with respect to the two almost-Riemannian distances.

This theorem shows another interesting difference between Riemannian manifolds and almost-Riemannian ones: in the Riemannian context, Lipschitz equivalence coincides with the equivalence as differentiable manifolds; in the almost-Riemannian context, Lipschitz equivalence is a stronger condition. Notice, however, that in general Liptschitz equivalence does not imply isometry. Indeed, the Lipschitz equivalence between two structures does not depend on the metric structure but only on the submodule Δ. This is highlightened by the fact that the graph itself depends only on Δ.

The structure of the paper is the following. In section 2, we recall some basic notion of sub-Riemannian geometry. Section 3 introduces the definitions of the number of revolution of a one-dimensional distribution along a closed oriented curve and of the graph associated with an almost-Riemannian structure. In section 4 we demonstrate Theorem 1. Section 4.1 provides the proof of the fact that having equivalent graphs is a necessary condition for Lipschitz equivalent structures. Finally, in section 4.2 we show this condition to be sufficient.

2 Preliminaries

This section is devoted to recall some basic definitions in the framework of sub-Riemannian geometry following [1, 2], see also [4, 17].

Let M be a n-dimensional manifold. Throughout the paper, unless specified, manifolds are smooth (i.e., $\mathcal{C}^\infty$) and without boundary; vector fields and differential forms are smooth. Given a vector bundle E over M, the $\mathcal{C}^\infty(M)$-module of smooth sections of E is denoted by $\Gamma(E)$. For the particular case $E = TM$, the set of smooth vector fields on M is denoted by $\mathrm{Vec}(M)$.

Definition 2 *A (n,k)-rank-varying distribution on a n-dimensional manifold M is a pair (E,f) where E is a vector bundle of rank k over M and $f : E \to TM$ is a morphism of vector bundles, i.e.,* **(i)** *the diagram*

$$\begin{array}{ccc} E & \xrightarrow{f} & TM \\ & \searrow^{\pi_E} & \downarrow{\scriptstyle \pi} \\ & & M \end{array}$$

commutes, where $\pi : TM \to M$ and $\pi_E : E \to M$ denote the canonical projections and **(ii)** *f is linear on fibers. Moreover, we require the map $\sigma \mapsto f \circ \sigma$ from $\Gamma(E)$ to $\mathrm{Vec}(M)$ to be injective.*

Given a (n,k)-rank-varying distribution, we denote by $f_* : \Gamma(E) \to \mathrm{Vec}(M)$ the morphism of $\mathcal{C}^\infty(M)$-modules that maps $\sigma \in \Gamma(E)$ to $f \circ \sigma \in \mathrm{Vec}(M)$. The following proposition shows that all the information about a rank-varying distribution is carried by the submodule $f_*(\Gamma(E))$.

Proposition 3 *Given two (n,k)-rank-varying distributions $(E_i, f_i), i = 1,2$, assume that they define the same submodule of $\mathrm{Vec}(M)$, i.e., $(f_1)_*(\Gamma(E_1)) = (f_2)_*(\Gamma(E_2)) = \Delta \subseteq \mathrm{Vec}(M)$. Then, there exists an isomorphism of vector bundles $h : E_1 \to E_2$ such that $f_2 \circ h = f_1$.*

Proof. Since $(f_i)_* : \Gamma(E_i) \to \Delta, i = 1,2$, are isomorphisms of $\mathcal{C}^\infty(M)$-modules, then $(f_2)_*^{-1} \circ (f_1)_* : \Gamma(E_1) \to \Gamma(E_2)$ is an isomorphism. A classical result given in [14, Proposition XIII p.78] states that the map $f \mapsto f_*$ is an isomorphism of $\mathcal{C}^\infty(M)$-modules from the set of morphisms from E_1 to E_2 to the set of morphisms from $\Gamma(E_1)$ to $\Gamma(E_2)$. Applying this result, there exists a unique isomorphism $h : E_1 \to E_2$ such that $h_* = (f_2)_*^{-1} \circ (f_1)_*$. By construction, $(f_2)_* \circ h_* = (f_1)_*$ and applying again [14, Proposition XIII p.78] we get $f_2 \circ h = f_1$. ■

Let (E,f) be a (n,k)-rank-varying distribution, $\Delta = f_*(\Gamma(E)) = \{f \circ \sigma \mid \sigma \in \Gamma(E)\}$ be its associated submodule and denote by $\Delta(q)$ the linear subspace $\{V(q) \mid V \in \Delta\} = f(E_q) \subseteq T_qM$. Let $\mathrm{Lie}(\Delta)$ be the smallest Lie subalgebra of $\mathrm{Vec}(M)$ containing Δ and, for every $q \in M$, let

$\mathrm{Lie}_q(\Delta)$ be the linear subspace of T_qM whose elements are evaluation at q of elements belonging to $\mathrm{Lie}(\Delta)$. We say that (E, f) satisfies the *Lie bracket generating condition* if $\mathrm{Lie}_q(\Delta) = T_qM$ for every $q \in M$.

A property (P) defined for (n, k)-rank-varying distributions is said to be *generic* if for every vector bundle E of rank k over M, (P) holds for every f in an open and dense subset of the set of morphisms of vector bundles from E to TM, endowed with the $\mathcal{C}^\infty$-Whitney topology. E.g., the Lie bracket generating condition is a generic property among (n, k)-rank-varying distributions satisfying $k > 1$.

We say that a (n, k)-rank-varying distribution (E, f) is *orientable* if E is orientable as a vector bundle.

A rank-varying sub-Riemannian structure is defined by requiring that E is an Euclidean bundle.

Definition 4 *A (n, k)-rank-varying sub-Riemannian structure is a triple $\mathcal{S} = (E, f, \langle\cdot,\cdot\rangle)$ where (E, f) is a Lie bracket generating (n, k)-rank-varying distribution on a manifold M and $\langle\cdot,\cdot\rangle_q$ is a scalar product on E_q smoothly depending on q.*

Several classical structures can be seen as particular cases of rank-varying sub-Riemannian structures, e.g., Riemannian structures and classical (constant-rank) sub-Riemannian structures (see [3, 17]). An (n, n)-rank-varying sub-Riemannian structure is called *n-dimensional almost-Riemannian structure*. In this paper, we focus on 2-dimensional almost-Riemannian structures (2-ARSs for short).

Let $\mathcal{S} = (E, f, \langle\cdot,\cdot\rangle)$ be a (n, k)-rank-varying sub-Riemannian structure. The Euclidean structure on E and the injectivity of the morphism f_* allow to define a symmetric positive definite $\mathcal{C}^\infty(M)$-bilinear form on the submodule Δ by

$$\begin{array}{rcl} G : \Delta \times \Delta & \to & \mathcal{C}^\infty(M) \\ (V, W) & \mapsto & \langle \sigma_V, \sigma_W \rangle, \end{array}$$

where σ_V, σ_W are the unique sections of E such that $f \circ \sigma_V = V, f \circ \sigma_W = W$.

If $\sigma_1, \ldots, \sigma_k$ is an orthonormal frame for $\langle\cdot,\cdot\rangle$ on an open subset Ω of M, an *orthonormal frame for G* on Ω is given by $f \circ \sigma_1, \ldots, f \circ \sigma_k$. Orthonormal frames are systems of local generators of Δ.

For every $q \in M$ and every $v \in \Delta(q)$ define

$$\mathbf{G}_q(v) = \inf\{\langle u, u\rangle_q \mid u \in E_q, f(u) = v\}.$$

In this paper, a curve $\gamma : [0, T] \to M$ absolutely continuous with respect to the differential structure is said to be *admissible* for $\mathcal{S}$ if there exists a measurable essentially bounded function

$$[0, T] \ni t \mapsto u(t) \in E_{\gamma(t)}$$

called *control function*, such that $\dot\gamma(t) = f(u(t))$ for almost every $t \in [0, T]$. Given an admissible curve $\gamma : [0, T] \to M$, the *length of* γ is

$$\ell(\gamma) = \int_0^T \sqrt{\mathbf{G}_{\gamma(t)}(\dot\gamma(t))}\, dt.$$

The *Carnot-Caratheodory distance* (or sub-Riemannian distance) on M associated with $\mathcal{S}$ is defined as

$$d(q_0, q_1) = \inf\{\ell(\gamma) \mid \gamma(0) = q_0, \gamma(T) = q_1, \gamma \text{ admissible}\}.$$

The finiteness and the continuity of $d(\cdot, \cdot)$ with respect to the topology of M are guaranteed by the Lie bracket generating assumption on the rank-varying sub-Riemannian structure (see [3]). The Carnot-Caratheodory distance associated with $\mathcal{S}$ endows M with the structure of metric space compatible with the topology of M as differential manifold.

We give now a characterization of admissible curves.

Proposition 5 *Let $(E, f, \langle\cdot,\cdot\rangle)$ be a rank-varying sub-Riemannian structure on a manfold M. Let $\gamma : [0, T] \to M$ be an absolutely continuous curve. Then γ is admissible if and only if it is Lipschitz continuous with respect to the sub-Riemannian distance.*

Proof. First we prove that if the curve is admissible then it is Lipschitz with respect to d (*d-Lipschitz* for short). This is a direct consequence of the definition of the sub-Riemannian distance. Indeed, let

$$[0, T] \ni t \mapsto u(t) \in E_{\gamma(t)}$$

be a control function for γ and let $L > 0$ be the essential supremum of $\sqrt{\langle u, u\rangle}$. Then, for every subinterval $[t_0, t_1] \subset [0, T]$ one has

$$d(\gamma(t_0), \gamma(t_1)) \leq \int_{t_0}^{t_1} \sqrt{\mathbf{G}_{\gamma(t)}(\dot\gamma(t))}dt \leq \int_{t_0}^{t_1} \sqrt{\langle u(t), u(t)\rangle}dt \leq L(t_1 - t_0).$$

Hence γ is d-Lipschitz.

Viceversa, assume that γ is d-Lipschitz with Lipschitz constant L. Since γ is absolutely continuous, it is differentiable almost everywhere on $[0, T]$. Thanks to the Ball-Box Theorem (see [4]), for every $t \in [0, T]$ such that the tangent vector $\dot\gamma(t)$ exists, $\dot\gamma(t)$ belongs to the distribution $\Delta(\gamma(t))$ (if not, the curve would fail to be d-Lipschitz). Hence for almost every $t \in [0, T]$ there exists $u_t \in E_{\gamma(t)}$ such that $\dot\gamma(t) = f(u_t)$. Moreover, since the curve is d-Lipschitz, one has that $\mathbf{G}_{\gamma(t)}(\dot\gamma(t)) \leq L^2$ for almost every $t \in [0, T]$. This can be seen computing lengths in privileged coordinates (see [4] for the definition of this system of coordinates). Hence, we can assume that $\langle u_t, u_t\rangle \leq L^2$ almost everywhere. Finally, we apply Filippov Theorem (see [12, Theorem 3.1.1 p.36]) to the differential inclusion

$$\dot\gamma(t) \in \{f(u) \mid \pi_E(u) = \gamma(t) \text{ and } \langle u, u\rangle \leq L^2\}.$$

that assures the existence of a measurable choice of the control function corresponding to γ. Thus γ is admissible. ■

Given a 2-ARS $\mathcal{S}$, we define its *singular locus* as the set

$$\mathcal{Z} = \{q \in M \mid \Delta(q) \subsetneq T_qM\}.$$

Since Δ is bracket generating, the subspace $\Delta(q)$ is nontrivial for every q and $\mathcal{Z}$ coincides with the set of points q where $\Delta(q)$ is one-dimensional.

We say that $\mathcal{S}$ *satisfies condition* **(H0)** if the following properties hold: **(i)** $\mathcal{Z}$ is an embedded one-dimensional submanifold of M; **(ii)** the points $q \in M$ at which $\Delta_2(q)$ is one-dimensional are isolated; **(iii)** $\Delta_3(q) = T_qM$ for every $q \in M$, where $\Delta_1 = \Delta$ and $\Delta_{k+1} = \Delta_k + [\Delta, \Delta_k]$. It is not difficult to prove that property **(H0)** is generic among 2-ARSs (see [2]). This hypothesis was essential to show Gauss–Bonnet type results for ARSs in [1, 2, 11]. The following theorem recalls the local normal forms for ARSs satisfying hypothesis **(H0)**.

Theorem 6 ([2]) *Given a 2-ARS $\mathcal{S}$ satisfiyng* **(H0)**, *for every point $q \in M$ there exist a neighborhood U of q, an orthonormal frame (X, Y) for G on U and smooth coordinates defined on U such that $q = (0, 0)$ and (X, Y) has one of the forms*

$$\begin{array}{lll} \text{(F1)} & X(x,y) = (1,0), & Y(x,y) = (0, e^{\phi(x,y)}), \\ \text{(F2)} & X(x,y) = (1,0), & Y(x,y) = (0, xe^{\phi(x,y)}), \\ \text{(F3)} & X(x,y) = (1,0), & Y(x,y) = (0, (y - x^2\psi(x))e^{\xi(x,y)}), \end{array}$$

where ϕ, ξ and ψ are smooth real-valued functions such that $\phi(0, y) = 0$ and $\psi(0) > 0$.

Let $\mathcal{S}$ be a 2-ARS satisfying **(H0)**. A point $q \in M$ is said to be an *ordinary point* if $\Delta(q) = T_qM$, hence, if $\mathcal{S}$ is locally described by (F1). We call q a *Grushin point* if $\Delta(q)$ is one-dimensional and $\Delta_2(q) = T_qM$, i.e., if the local description (F2) applies. Finally, if $\Delta(q) = \Delta_2(q)$ has dimension one and $\Delta_3(q) = T_qM$ then we say that q is a *tangency point* and $\mathcal{S}$ can be described near q by the normal form (F3). We define

$$\mathcal{T} = \{q \in \mathcal{Z} \mid q \text{ tangency point of } \mathcal{S}\}.$$

Assume $\mathcal{S}$ and M to be oriented. Thanks to the hypothesis **(H0)**, $M \setminus \mathcal{Z}$ splits into two open sets M^+ and M^- such that $f : E|_{M^+} \to TM^+$ is an orientation-preserving isomorphism and $f : E|_{M^-} \to TM^-$ is an orientation-reversing isomorphism.

3 Number of revolutions and graph of a 2-ARS

From now on M is a compact oriented surface and $\mathcal{S} = (E, f, \langle\cdot,\cdot\rangle)$ is an oriented ARS on M satisfying **(H0)**.

Fix on $\mathcal{Z}$ the orientation induced by M^+ and consider a connected component W of $\mathcal{Z}$. Let $V \in \Gamma(TW)$ be a never-vanishing vector field whose duality product with the fixed orientation on W is positive. Since M is oriented, $TM|_W$ is isomorphic to the trivial bundle of rank 2 over W. We choose an isomorphism $t : TM|_W \to W \times \mathbb{R}^2$ such that t is orientation-preserving and for every $q \in W$, $t \circ V(q) = (q, (1, 0))$. This trivialization induces an orientation-preserving isomorphism between the projectivization of $TM|_W$ and $W \times S^1$. For the sake of readability, in what follows we omit the isomorphism t and identify $TM|_W$ (respectively, its projectivization) with $W \times \mathbb{R}^2$ (respectively, $W \times S^1$).

Since $\Delta|_W$ is a subbundle of rank one of $TM|_W$, $\Delta|_W$ can be seen as a section of the projectivization of $TM|_W$, i.e., a smooth map (still denoted by Δ) $\Delta : W \to W \times S^1$ such that $\pi_1 \circ \Delta = \mathrm{Id}_W$, where $\pi_1 : W \times S^1 \to W$ denotes the projection on the first component. We define $\tau(\Delta, W)$, the *number of revolutions* of Δ along W, to be the degree of the map $\pi_2 \circ \Delta : W \to S^1$, where $\pi_2 : W \times S^1 \to S^1$ is the projection on the second component. Notice that $\tau(\Delta, W)$ changes sign if we reverse the orientation of W.

Let us show how to compute $\tau(\Delta, W)$. By construction, $\pi_2 \circ V : W \to S^1$ is constant. Let $\pi_2 \circ V(q) \equiv \theta_0$. Since $\Delta_3(q) = T_q M$ for every $q \in M$, θ_0 is a regular value of $\pi_2 \circ \Delta$. By definition,

$$\tau(\Delta, W) = \sum_{q | \pi_2 \circ \Delta(q) = \theta_0} \operatorname{sign}(d_q(\pi_2 \circ \Delta)) = \sum_{q \in W \cap \mathcal{T}} \operatorname{sign}(d_q(\pi_2 \circ \Delta)), \tag{3}$$

where d_q denotes the differential at q of a smooth map and $\operatorname{sign}(d_q(\pi_2 \circ \Delta)) = 1$, resp. -1, if $d_q(\pi_2 \circ \Delta)$ preserves, resp. reverses, the orientation. The equality in (3) follows from the fact that a point q satisfies $\pi_2 \circ \Delta(q) = \theta_0$ if and only if $\Delta(q)$ is tangent to W at q, i.e., $q \in \mathcal{T}$.

Define the *contribution at a tangency point* q as $\tau_q = \operatorname{sign}(d_q(\pi_2 \circ \Delta))$ (see Figure 2). Moreover, we define

$$\tau(\mathcal{S}) = \sum_{W \in \mathfrak{C}(\mathcal{Z})} \tau(\Delta, W),$$

where $\mathfrak{C}(\mathcal{Z}) = \{W \mid W \text{ connected component of } \mathcal{Z}\}$. Clearly, $\tau(\mathcal{S}) = \sum_{q \in \mathcal{T}} \tau_q$.

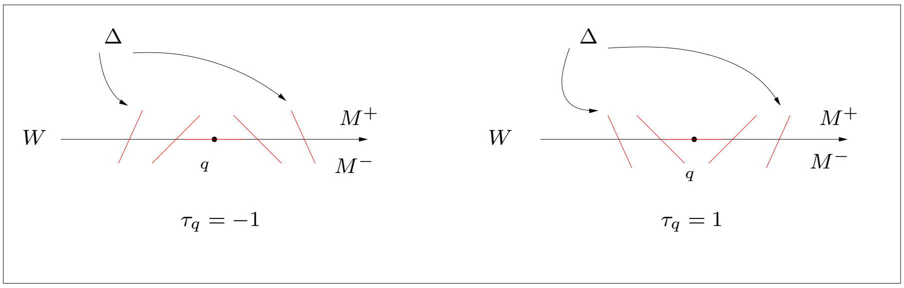

Figure 2: Tangency points with opposite contributions

Let us associated with the 2-ARS $\mathcal{S}$ the graph $\mathcal{G} = (\mathcal{V}(\mathcal{G}), \mathcal{E}(\mathcal{G}))$ where

- each vertex in $\mathcal{V}(\mathcal{G})$ represents a connected component of $M \setminus \mathcal{Z}$;
- each edge in $\mathcal{E}(\mathcal{G})$ represents a connected component of $\mathcal{Z}$;
- the edge corresponding to a connected component W connects the two vertices corresponding to the connected components M_1 and M_2 of $M \setminus \mathcal{Z}$ such that $W \subset \partial M_1 \cap \partial M_2$.

Thanks to the hypothesis **(H0)**, every connected component of $\mathcal{Z}$ joints a connected component of M^+ and one of M^-. Thus the graph $\mathcal{G}$ turns out to be *bipartite*, i.e., there exists a partition of the set of vertices into two subsets V^+ and V^- such that each edge of $\mathcal{G}$ joins a vertex of V^+ to a vertex of V^-. Conversely, it is not difficult to see that every finite bipartite graph can be obtained from an oriented 2-ARS (satisfying **(H0)**) on a compact oriented surface.

Using the bipartite nature of $\mathcal{G}$ we introduce an orientation on $\mathcal{G}$ given by two functions $\alpha, \omega : \mathcal{E}(\mathcal{G}) \to \mathcal{V}(\mathcal{G})$ defined as follows. If e corresponds to W then $\alpha(e) = v$ and $\omega(e) = w$, where v and w correspond respectively to the connected components $M_v \subset M^-$ and $M_w \subset M^+$ such that $W \subseteq \partial M_v \cap \partial M_w$.

We label each vertex v corresponding to a connected component $\hat{M}$ of $M \setminus \mathcal{Z}$ with a pair $(\operatorname{sign}(v), \chi(v))$ where $\operatorname{sign}(v) = \pm 1$ if $\hat{M} \subset M^{\pm}$ and $\chi(v)$ is the Euler characteristic of $\hat{M}$. We define for every $e \in E(\mathcal{G})$ the number $\tau(e) = \sum_{q \in W \cap \mathcal{T}} \tau_q$, where W is the connected component of $\mathcal{Z}$ corresponding to e.

Finally, we define a label for each edge e corresponding to a connected component W of $\mathcal{Z}$ containing tangency points. Let $s \geq 1$ be the cardinality of the set $W \cap \mathcal{T}$. The label of e is an equivalence class of s-uples with entries in $\{\pm 1\}$ defined as follows. Fix on W the orientation induced by M^+ and choose a point $q \in W \cap \mathcal{T}$. Let $q_1 = q$ and for every $i = 1, \ldots, s-1$ let q_{i+1} be the first element in $W \cap \mathcal{T}$ that we meet after q_i walking along W in the fixed orientation. We associate with e the equivalence class of $(\tau_{q_1}, \tau_{q_2}, \ldots, \tau_{q_s})$ in the set of s-uples with entries in $\{\pm 1\}$ modulo cyclic permutations. In figure 3 an ARS on a surface of genus 4 and its labelled graph (figure 3(a)) are portrayed. According to our definition of labels on edges, figures 3(a) and 3(b) represent equal graphs associated with the same ARS. On the other hand, the graph in figure 3(c) is not the graph associated to the ARS of figure 3. In figure 4 two steps in the construction of the labelled graph associated with the ARS in figure 1 are shown.

Remark 7 *Once an orientation on E is fixed the labelled graph associated with $\mathcal{S}$ is unique.*

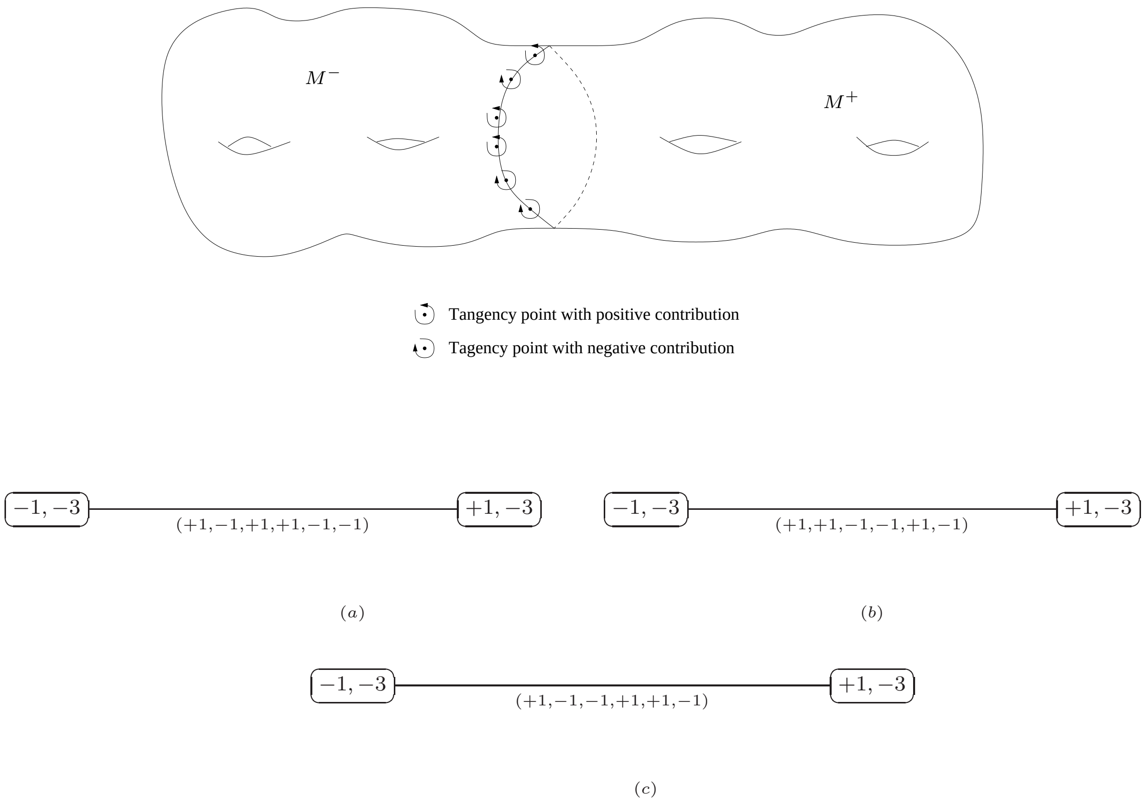

Figure 3: Example of ARS on a surface of genus 4. Figures (a) and (b) illustrate equal labelled graphs associated with the ARS. Figure (c) gives an example of labelled graph different from the graph in figure (a)

We define an equivalence relation on the set of graphs associated with oriented ARS on M satisfying hypothesis **(H0)**.

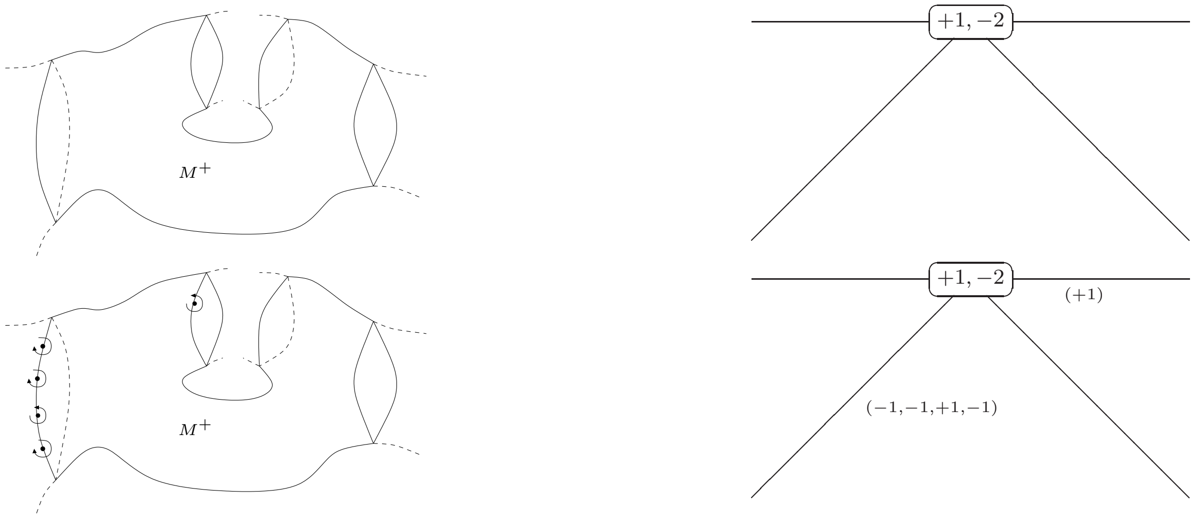

Figure 4: Algorythm to build the graph

Definition 8 *Let $\mathcal{S}_i = (E_i, f_i, \langle\cdot,\cdot\rangle_i)$, $i = 1, 2$, be two oriented almost-Riemannian structures on a compact oriented surface M satisfying hypothesis* **(H0)**. *Let $\mathcal{G}_i$ be the labelled graph associated with $\mathcal{S}_i$ and denote by $\alpha_i, \omega_i : \mathcal{E}(\mathcal{G}_i) \to \mathcal{V}(\mathcal{G}_i)$ the functions defined as above. We say that $\mathcal{S}_1$ and $\mathcal{S}_2$ have* equivalent graphs *if , after possibly changing the orientation on E_2, they have the same labelled graph.*

In other words, after possibly changing the orientation on E_2 and still denoting by $\mathcal{G}_2$ the associated graph, there exist bijections $u : \mathcal{V}(\mathcal{G}_1) \to \mathcal{V}(\mathcal{G}_2)$, $k : \mathcal{E}(\mathcal{G}_1) \to \mathcal{E}(\mathcal{G}_2)$ such that the diagram

$$\begin{array}{ccc} \mathcal{V}(\mathcal{G}_1) & \xrightarrow[u]{} & \mathcal{V}(\mathcal{G}_2) \\ \alpha_1 \uparrow & & \uparrow \alpha_2 \\ \mathcal{E}(\mathcal{G}_1) & \xrightarrow{k} & \mathcal{E}(\mathcal{G}_2) \end{array} \tag{4}$$

commutes and u and k preserve labels.

Figure 5 illustrates the graph associated with the ARS obtained by reversing the orientation of the ARS in figure 1.

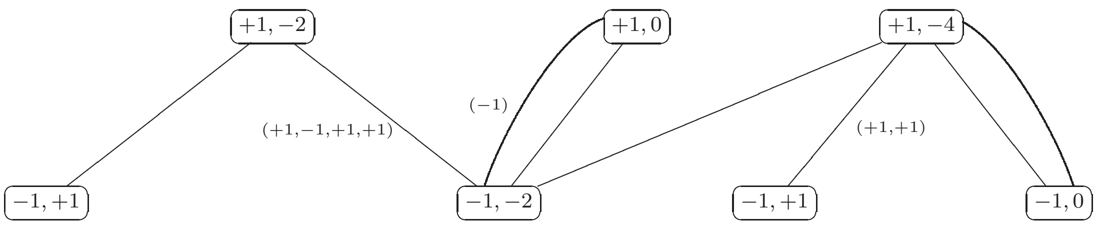

Figure 5: Equivalent graph to the one in figure 1

4 Lipschitz equivalence

This section is devoted to the proof of Theorem 1 which is a generalization to ARSs of the well-known fact that all Riemannian structures on a compact oriented surface are Lipschitz equivalent.

Let M_1, M_2 be two manifolds. For $i = 1, 2$, let $\mathcal{S}_i = (E_i, f_i, \langle\cdot,\cdot\rangle_i)$ be a sub-Riemannian structure on M_i. Denote by d_i the Carnot–Caratheodory distance on M_i associated with $\mathcal{S}_i$.

Definition 9 *We say that a diffeomorphism $\varphi : M_1 \to M_2$ is a* Lipschitz equivalence *if it is bi-Lipschitz as a map from (M_1, d_1) to (M_2, d_2).*

Notice that in Theorem 1 we can assume $M_1 = M_2 = M$. Indeed, if two ARSs are Lipschitz equivalent, then by definition there exists a diffeomorphism $\varphi : M_1 \to M_2$. On the other hand, if the associated graphs are equivalent, by [1, Theorem 1] it follows that E_1 and E_2 are isomorphic vector bundles. Hence the underlying surfaces are diffeomorphic.

4.1 Necessity

Denote by M_i^+, respectively M_i^-, the set where f_i is an orientation-preserving, respectively orientation-reversing, isomorphism of vector bundles, and by Δ^i the submodule $\{f_i \circ \sigma \mid \sigma \in \Gamma(E_i)\}$. Let $\mathcal{Z}_i$ be the singular locus of $\mathcal{S}_i$ and $\mathcal{T}_i$ the set of tangency points of $\mathcal{S}_i$. Finally, for every $q \in \mathcal{T}_i$, denote by τ_q^i the contribution at the tangency point defined in Section 3 with $\Delta = \Delta^i$.

In this section we assume $\varphi : (M, d_1) \to (M, d_2)$ to be a Lipschitz equivalence and we show that $\mathcal{S}_1$ and $\mathcal{S}_2$ have equivalent graphs. As a consequence of the Ball-Box Theorem (see, for instance, [4]) one can prove the following result.

Lemma 10 *If p is an ordinary, Grushin or tangency point for $\mathcal{S}_1$, then $\varphi(p)$ is an ordinary, Grushin or tangency point for $\mathcal{S}_2$, respectively.*

Thanks to Lemma 10, for every connected component $\hat{M}$ of $M \setminus \mathcal{Z}_1$, $\varphi(\hat{M})$ is a connected component of $M \setminus \mathcal{Z}_2$ and for every connected component W of $\mathcal{Z}_1 \cap \partial\hat{M}$, $\varphi(W)$ is a connected component of $\mathcal{Z}_2 \cap \partial\varphi(\hat{M})$. Moreover, since $\varphi|_{\overline{\hat{M}}}$ is a diffeomorphism, it follows that $\chi(\hat{M}) = \chi(\varphi(\hat{M}))$. After possibly changing the orientation on E_2, we may assume $\varphi(M_1^{\pm}) = M_2^{\pm}$. We will prove that, in this case, the labelled graphs are equal. Indeed, if $v \in \mathcal{V}(\mathcal{G}_1)$ corresponds to $\hat{M}$, define $u(v) \in \mathcal{V}(\mathcal{G}_2)$ as the vertex corresponding to $\varphi(\hat{M})$. If $e \in \mathcal{E}(\mathcal{G}_1)$ corresponds to W define $k(e) \in \mathcal{E}(\mathcal{G}_2)$ as the edge corresponding to $\varphi(W)$. Then $\chi(u(v)) = \chi(v)$, $\mathrm{sign}(u(v)) = \mathrm{sign}(v)$, and, by construction, the diagram (4) commutes.

Let us compute the contribution at a tangency point q of an ARS $(E, f, \langle\cdot,\cdot\rangle)$ using the corresponding normal form given in Theorem 6.

Lemma 11 *Let $\gamma : [0, T] \to M$ be a smooth curve such that $\gamma(0) = q \in \mathcal{T}$ and $\dot\gamma(0) \in \Delta(q) \setminus \{0\}$. Assume moreover that γ is d-Lipschitz, where d is the almost-Riemannian distance, and that $\gamma((0, T))$ is contained in one of the two connected components of $M \setminus \mathcal{Z}$. Let (x, y) be a coordinate system centered at q such that the form (F3) of Theorem 6 applies. Then $\gamma((0, T)) \subset \{(x, y) \mid y - x^2\psi(x) < 0\}$. Moreover, if $\{(x, y) \mid y - x^2\psi(x) < 0\} \subseteq M^+$, resp. M^-, then $\tau_q = 1$, resp. -1.*

Proof. Since $\gamma(0) = (0, 0)$ and $\dot\gamma(0) \in \mathrm{span}\{(1, 0)\} \setminus \{0\}$, there exist two smooth functions $\overline{x}(t), \overline{y}(t)$ such that $\gamma(t) = (t\overline{x}(t), t^2\overline{y}(t))$ and $\overline{x}(0) \neq 0$. Assume by contradiction that $\gamma((0, T)) \subset \{(x, y) \mid y - x^2\psi(x) > 0\}$, i.e., for $t \in (0, T)$, $\overline{y}(t) > \psi(t\overline{x}(t))\overline{x}(t)^2$. Since $\psi(0) > 0$, for t sufficiently small

$\psi(t\overline{x}(t)) > 0$ and $\overline{y}(t)^{1/3} > \psi(t\overline{x}(t))^{1/3}|\overline{x}(t)|^{2/3}$. By the Ball-Box Theorem (see [4]) there exist c_1, c_2 positive constants such that, for t sufficiently small we have

$$c_1(|t\overline{x}(t)| + |t^2\overline{y}(t)|^{1/3}) \leq d(\gamma(t),(0,0)) \leq c_2(|t\overline{x}(t)| + |t^2\overline{y}(t)|^{1/3}).$$

On the other hand, for t sufficiently small,

$$|t\overline{x}(t)| + |t^2\overline{y}(t)|^{1/3} > t^{2/3}|\overline{x}(t)|^{2/3}\psi(t\overline{x}(t))^{1/3}.$$

Hence, for t sufficiently small, $d(\gamma(t),(0,0)) > c_3 t^{2/3}$, with $c_3 > 0$. This implies that γ is not Lipschitz with respect to the almost-Riemannian distance. Finally, a direct computation shows the assertion concerning τ_q, see Figure 2. ∎

Next lemma, jointly with Lemma 10, guarantees that the two bijections u and k preserve labels.

Lemma 12 *Let $q \in \mathcal{T}_1$. Then $\tau^1_q = \tau^2_{\varphi(q)}$.*

Proof. Apply Theorem 6 to $\mathcal{S}_1$ and find a neighborhood U of q and a coordinate system (x, y) on U such that $q = (0,0)$ and $\mathcal{Z}_1 \cap U = \{(x, y) \mid y = x^2\psi(x)\}$. Let $\sigma, \rho \in \Gamma(E|_U)$ be the local orthonormal frame such that $f_1 \circ \sigma = X$ and $f_1 \circ \rho = Y$. Assume that $U_1^+ = M_1^+ \cap U = \{(x, y) \mid y - x^2\psi(x) > 0\}$. Fix $T > 0$ and consider the smooth curve $\gamma : [0,T] \to U$ defined by $\gamma(t) = (t, 0)$. Then γ is admissible for $\mathcal{S}_1$ with control function $u(t) = \sigma(t, 0)$. By definition, for T sufficiently small $\gamma((0,T))$ lies in a single connected component of $U \setminus \mathcal{Z}_1$. Moreover, by Proposition 5, γ is a d_1-Lipschitz map with Lipschitz constant less or equal to 1. Hence, according to Lemma 11, $\tau^1_q = -1$.

Consider the curve $\tilde{\gamma} = \varphi \circ \gamma : [0,T] \to \varphi(U)$. Since φ is Lipschitz, $\tilde{\gamma}$ is d_2-Lipschitz as a map from the interval $[0,T]$ to the metric space $(\varphi(U), d_2)$. Moreover, $\tilde{\gamma}$ is smooth and $\dot{\tilde{\gamma}}(0) \in \Delta^2(\varphi(q)) \setminus \{0\}$, φ being a diffeomorphism mapping $\mathcal{Z}_1$ to $\mathcal{Z}_2$. Finally, since $\varphi(M_1^-) = M_2^-$, then $\tilde{\gamma}((0,T)) \subset U_2^- = \varphi(U) \cap M_2^-$. Thus, by Lemma 11, $\tau^2_{\varphi(q)} = -1$. Analogously, one can prove the statement in the case $U_1^+ = \{(x, y) \mid y - x^2\psi(x) < 0\}$ (for which $\tau^1_q = \tau^2_q = 1$). ∎

Lemma 12 implies that $\mathcal{S}_1$ and $\mathcal{S}_2$ have equal labelled graphs. This concludes the proof that having equivalent graphs is a necessary condition for two ARSs being Lipschitz equivalent.

4.2 Sufficiency

In this section we prove that if $\mathcal{S}_1$ and $\mathcal{S}_2$ have equivalent graphs then there exists a Lipschitz equivalence between (M, d_1) and (M, d_2). After possibly changing the orientation on E_2, we assume the associated labelled graphs to be equal, i.e., there exist two bijections u, k as in Definition 8 such that diagram (4) commutes.

The proof is in five steps. The first step consists in proving that we may assume $E_1 = E_2$. The second step shows that we can restrict to the case $\mathcal{Z}_1 = \mathcal{Z}_2$ and $\mathcal{T}_1 = \mathcal{T}_2$. In the third step we prove that we can assume that $\Delta^1(q) = \Delta^2(q)$ at each point $q \in M$. As fourth step, we demonstrate that the submodules Δ^1 and Δ^2 coincide. In the fifth and final step we remark that we can assume $f_1 = f_2$ and conclude. The Lipschitz equivalence between the two structures will be the composition of the diffeomorphisms singled out in steps 1, 2, 3, 5.

By construction, the push-forward of $\mathcal{S}_1$ along a diffeomorphism ψ of M, denoted by $\psi_*\mathcal{S}_1$, is Lipschitz equivalent to $\mathcal{S}_1$ and has the same labelled graph of $\mathcal{S}_1$. Notice, moreover, that the singular locus of $\psi_*\mathcal{S}_1$ coincides with $\psi(\mathcal{Z}_1)$ and the set of tangency points coincides with $\psi(\mathcal{T}_1)$.

Step 1. Having the same labelled graph implies

$$\sum_{v\in\mathcal{V}(\mathcal{G}_1)} \mathrm{sign}(v)\chi(v) + \sum_{e\in\mathcal{E}(\mathcal{G}_1)} \tau(e) = \sum_{v\in\mathcal{V}(\mathcal{G}_2)} \mathrm{sign}(v)\chi(v) + \sum_{e\in\mathcal{E}(\mathcal{G}_2)} \tau(e).$$

By [1, Theorem 1], this is equivalent to say that the Euler numbers of E_1 and E_2 are equal. Since E_1 and E_2 are oriented vector bundles of rank 2, with the same Euler number, over a compact oriented surface, then they are isomorphic. Hence, we assume $E_1 = E_2 = E$.

Step 2. Using the bijections u, k and the classification of compact oriented surfaces with boundary (see, for instance, [16]), one can prove the following lemma.

Lemma 13 *There exists a diffeomorphism $\tilde{\varphi} : M \to M$ such that $\tilde{\varphi}(M_1^+) = M_2^+$, $\tilde{\varphi}(M_1^-) = M_2^-$, $\tilde{\varphi}|_{\mathcal{Z}_1} : \mathcal{Z}_1 \to \mathcal{Z}_2$ is a diffeomorphism that maps $\mathcal{T}_1$ into $\mathcal{T}_2$, and, for every $q \in \mathcal{T}_1$ $\tau^2_{\tilde{\varphi}(q)} = \tau^1_q$. Moreover, if $v \in \mathcal{V}(\mathcal{G}_1)$ corresponds to $\hat{M} \subset M \setminus \mathcal{Z}_1$, then $\tilde{\varphi}(\hat{M})$ is the connected component of $M \setminus \mathcal{Z}_2$ corresponding to $u(v) \in \mathcal{V}(\mathcal{G}_2)$; if $e \in \mathcal{E}(\mathcal{G}_1)$ corresponds to $W \subset \mathcal{Z}_1$, then $\tilde{\varphi}(W)$ is the connected component of $\mathcal{Z}_2$ corresponding to $k(e) \in \mathcal{E}(\mathcal{G}_2)$.*

The lemma implies that the singular locus of $\tilde{\varphi}_*\mathcal{S}_1$ coincides with $\mathcal{Z}_2$ and the set of tangency points coincides with $\mathcal{T}_2$. For the sake of readability, in the following we rename $\tilde{\varphi}_*\mathcal{S}_1$ simply by $\mathcal{S}_1$ and we will denote by $\mathcal{Z}$ the singular locus of the two structures, by $\mathcal{T}$ the set of their tangency points, and by $M^\pm$ the set $M_i^\pm$.

Step 3. Remark that the subspaces $\Delta^1(q)$ and $\Delta^2(q)$ coincide at every ordinary and tangency point q. We are going to show that there exists a diffeomorphism of M that carries $\Delta^1(q)$ into $\Delta^2(q)$ at every point q of the manifold.

Lemma 14 *Let W be a connected component of $\mathcal{Z}$. There exist a tubular neighborhood $\mathbf{W}$ of W and a diffeomorphism $\varphi_W : \mathbf{W} \to \varphi_W(\mathbf{W})$ such that $d_q\varphi_W(\Delta^1(q)) = \Delta^2(\varphi_W(q))$ for every $q \in \mathbf{W}$, $\varphi_W|_W = Id|_W$ and $\varphi(\mathbf{W} \cap M^\pm) \subset M^\pm$.*

Proof. The idea of the proof is first to consider a smooth section A of $\mathrm{Hom}(TM|_W; TM|_W)$ such that for every $q \in W$, $A_q : T_qM \to T_qM$ is an isomorphism and $A_q(\Delta^1(q)) = \Delta^2(q)$. Secondly, we build a diffeomorphism φ_W of a tubular neighborhood of W such that $d_q\varphi_W = A_q$ for every point $q \in W$.

Choose on a tubular neighborhood $\mathbf{W}$ of W a parameterization (θ, t) such that $W = \{(\theta, t) \mid t = 0\}$, $M^+ \cap \mathbf{W} = \{(\theta, t) \mid t > 0\}$ and $\frac{\partial}{\partial\theta}\big|_{(\theta,0)}$ induces on W the same orientation as M^+. We are going to show the existence of two smooth functions $a, b : W \to \mathbb{R}$ such that b is positive and for every $(\theta, 0) \in W$,

$$\begin{pmatrix} 1 & a(\theta) \\ 0 & b(\theta) \end{pmatrix} (\Delta^1(\theta, 0)) = \Delta^2(\theta, 0). \tag{5}$$

Then, for every $q = (\theta, 0) \in W$ defining $A_q : T_qM \to T_qM$ by

$$A_{(\theta,0)} = \begin{pmatrix} 1 & a(\theta) \\ 0 & b(\theta) \end{pmatrix}, \tag{6}$$

we will get an isomorphism smoothly depending on the point q and carrying $\Delta^1(q)$ into $\Delta^2(q)$.

Let $W \cap \mathcal{T} = \{(\theta_1, 0), \ldots, (\theta_s, 0)\}$, with $s \geq 0$. Using the chosen parametrization, there exist two smooth functions $\beta_1, \beta_2 : W \setminus \{(\theta_1, 0), \ldots, (\theta_s, 0)\} \to \mathbb{R}$ such that $\Delta^i(\theta, 0) = \mathrm{span}\{(\beta_i(\theta), 1)\}$.

For every $j = 1, \dots s$, there exists a smooth function g_j^i defined on a neighborhood of $(\theta_j, 0)$ in W such that $g_j^i(\theta_j) \neq 0$, $\tau^i_{(\theta_j,0)} = \text{sign}(g_j^i(\theta_j))$ and

$$\beta_i(\theta) = \frac{1}{(\theta - \theta_j)g_j^i(\theta)}, \quad \theta \sim \theta_j.$$

Since the graphs associated with $\mathcal{S}_1, \mathcal{S}_2$ are equivalent, for every $j = 1 \dots s$ we have $\tau^1_{(\theta_j,0)} = \tau^2_{(\theta_j,0)}$. Hence $\frac{g_j^2(\theta_j)}{g_j^1(\theta_j)} > 0$ for every j. Let $b : W \to \mathbb{R}$ be a positive smooth function such that for each $j \in \{1, \dots s\}$, $b(\theta_j) = \frac{g_j^2(\theta_j)}{g_j^1(\theta_j)}$. Define $a : W \to \mathbb{R}$ by

$$a(\theta) = b(\theta)\beta_2(\theta) - \beta_1(\theta).$$

Clearly a is smooth on $W \setminus \{(\theta_1, 0), \dots, (\theta_s, 0)\}$. Moreover, thanks to our choice of b, a is smooth at θ_j, and, by construction, we have (5). The existence of a, b is established.

Define A_q as in (6). Let us extend the isomorphism A_q defined for $q \in W$ to a tubular neighborhood. Define $\varphi_W : \mathbf{W} \to \mathbf{W}$ by

$$\varphi_W(\theta, t) = (a(\theta)t + \theta, b(\theta)t).$$

By construction, $d_{(\theta,0)}\varphi_W$ is an isomorphism. Hence, reducing $\mathbf{W}$ if necessary, $\varphi_W : \mathbf{W} \to \varphi_W(\mathbf{W})$ turns out to be a diffeomorphism. Finally, by definition, $\varphi_W(\theta, 0) = (\theta, 0)$ and, since b is positive, $\varphi(\mathbf{W} \cap M^\pm) \subset M^\pm$. ■

We apply Lemma 14 to every connected component W of $\mathcal{Z}$. We reduce, if necessary, the tubular neighborhood $\mathbf{W}$ of W in such a way that every pair of distinct connected component of $\mathcal{Z}$ have disjoint corresponding tubular neighborhoods built as in Lemma 14. We claim that there exists a diffeomorphism $\varphi : M \to M$ such that $\varphi|_{\mathbf{W}} = \varphi_W$ for every connected component W of $\mathcal{Z}$. This is a direct consequence of the fact that the labels on vertices of $\mathcal{G}_1$ and $\mathcal{G}_2$ are equal and of the classification of compact oriented surfaces with boundary (see [16]). By construction, the push-forward of $\mathcal{S}_1$ along φ is Lipschitz equivalent to $\mathcal{S}_1$ and has the same labelled graph as $\mathcal{S}_1$. To simplify notations, we denote $\varphi_*\mathcal{S}_1$ by $\mathcal{S}_1$. By Lemma 14, $\Delta^1(q) = \Delta^2(q)$ at every point q.

Step 4. The next point is to prove that Δ^1 and Δ^2 coincide as $\mathcal{C}^\infty(M)$-submodules.

Lemma 15 *The submodules Δ^1 and Δ^2 associated with $\mathcal{S}_1$ and $\mathcal{S}_2$ coincide.*

Proof. It is sufficient to show that for every $p \in M$ there exist a neighborhood U of p such that $\Delta^1|_U$ and $\Delta^2|_U$ are generated as $\mathcal{C}^\infty(M)$-submodules by the same pair of vector fields.

If p is an ordinary point, then taking $U = M \setminus \mathcal{Z}$, we have $\Delta^1|_U = \Delta^2|_U = \text{Vec}(U)$.

Let p be a Grushin point and apply Theorem 6 to $\mathcal{S}_1$ to find a neighborhood U of p such that

$$\Delta^1|_U = \text{span}_{\mathcal{C}^\infty(M)}\{F_1, F_2\}, \text{ where } F_1(x, y) = (1, 0),\ F_2(x, y) = (0, xe^{\phi(x,y)}).$$

Up to reducing U we assume the existence of a frame

$$G_1(x, y) = (a_1(x, y), a_2(x, y)), \quad G_2(x, y) = (b_1(x, y), b_2(x, y))$$

such that $\Delta^2|_U = \text{span}_{\mathcal{C}^\infty(M)}\{G_1, G_2\}$. Since $\Delta^1(q) = \Delta^2(q)$ at every point $q \in M$, $a_2(0, y) \equiv 0$ and $b_2(0, y) \equiv 0$. Since $\Delta^2(0, y)$ is one-dimensional, let us assume $a_1(0, y) \neq 0$ for every y.

Moreover, after possibly further reducing U, $\Delta^2|_U = \mathrm{span}_{\mathcal{C}^\infty(M)}\{(1/a_1)G_1, G_2 - (b_1/a_1)G_1\}$ hence we may assume $a_1(x,y) \equiv 1$ and $b_1(x,y) \equiv 0$. The conditions $a_2(0,y) \equiv 0$ and $b_2(0,y) \equiv 0$ imply $a_2(x,y) = x\overline{a}_2(x,y)$ and $b_2(x,y) = x\overline{b}_2(x,y)$ respectively, with $\overline{a}_2, \overline{b}_2$ smooth functions. Since $[G_1, G_2]|_{(0,y)} = (0, \overline{b}_2(0,y))$, thanks to hypothesis **(H0)** on $\mathcal{S}_2$, we have $\overline{b}_2(0,y) \neq 0$. Hence, reducing U if necessary,

$$\begin{aligned}
\Delta^2|_U &= \mathrm{span}_{\mathcal{C}^\infty(M)}\{G_1 - (\overline{a}_2(x,y)/\overline{b}_2(x,y))G_2, (e^{\phi(x,y)}/\overline{b}_2(x,y))G_2\} \\
&= \mathrm{span}_{\mathcal{C}^\infty(M)}\{F_1, F_2\} = \Delta^1|_U.
\end{aligned}$$

Finally, let p be a tangency point. Apply Theorem 6 to $\mathcal{S}_1$, i.e., choose a neighborhood U of p and a system of coordinates (x,y) such that $p = (0,0)$,

$$\Delta^1|_U = \mathrm{span}_{\mathcal{C}^\infty(M)}\{F_1, F_2\}, \text{ where } F_1(x,y) = (1,0), \; F_2(x,y) = (0, (y - x^2\psi(x))e^{\xi(x,y)}),$$

and ψ, ξ are smooth functions such that $\psi(0) > 0$. Consider the change of coordinates

$$\tilde{x} = x, \quad \tilde{y} = y - x^2\psi(x).$$

Then

$$F_1(\tilde{x}, \tilde{y}) = (1, \tilde{x}a(\tilde{x})), \quad F_2(\tilde{x}, \tilde{y}) = (0, \tilde{y}e^{\xi(\tilde{x}, \tilde{y} + \tilde{x}^2\psi(\tilde{x}))}),$$

where $a(\tilde{x}) = -2\psi(\tilde{x}) - \tilde{x}\psi'(\tilde{x})$. To simplify notations, in the following we rename $\tilde{x}, \tilde{y}$ by x, y respectively and we still denote by $\xi(x,y)$ the function $\xi(x, y + x^2\psi(x))$. In the new coordinate system we have $p = (0,0)$, $\mathcal{Z} \cap U = \{(x,y) \mid y = 0\}$, $F_1(x,y) = (1, xa(x))$ and $F_2(x,y) = (0, ye^{\xi(x,y)})$. Reducing U, if necessary, let $G_1(x,y) = (a_1(x,y), a_2(x,y)), G_2(x,y) = (b_1(x,y), b_2(x,y))$ be a frame for $\Delta^2|_U$. Since $\Delta^1(q) = \Delta^2(q)$ at every point, we have $a_2(0,0) = b_2(0,0) = 0$. Since $\Delta^2(0,0)$ is one-dimensional, we may assume $a_1(0,0) \neq 0$. After possibly further reducing U, $\Delta^2|_U = \mathrm{span}_{\mathcal{C}^\infty(M)}\{(1/a_1)G_1, G_2 - (b_1/a_1)G_1\}$ and we can assume $a_1(x,y) \equiv 1$ and $b_1(x,y) \equiv 0$. Moreover, by $\Delta^1(x,0) = \Delta^2(x,0)$ we get $a_2(x,0) = xa(x)$ and $b_2(x,0) \equiv 0$, whence $a_2(x,y) = xa(x) + y\overline{a}_2(x,y)$ and $b_2(x,y) = y\overline{b}_2(x,y)$, with $\overline{a}_2, \overline{b}_2$ smooth functions. Computing the Lie brackets we get

$$[G_1, G_2]|_{(x,0)} = (0, xa\overline{b}_2)|_{(x,0)}, \quad [G_1, [G_1, G_2]]|_{(0,0)} = (0, a\overline{b}_2))|_{(0,0)}.$$

Applying hypothesis **(H0)** to $\mathcal{S}_2$ we have $\overline{b}_2(x,0) \neq 0$ for all x in a neighborhood of 0. Hence, up to reducing U,

$$\begin{aligned}
\Delta^2|_U &= \mathrm{span}_{\mathcal{C}^\infty(M)}\{G_1 - (\overline{a}_2(x,y)/\overline{b}_2(x,y))G_2, (e^{\xi(x,y)}/\overline{b}_2(x,y))G_2\} \\
&= \mathrm{span}_{\mathcal{C}^\infty(M)}\{F_1, F_2\} = \Delta^1|_U.
\end{aligned}$$

■

Step 5. Thanks to Lemma 15 and Proposition 3 we can assume $f_1 = f_2 = f$. In other words, we reduce to the case $\mathcal{S}_1 = (E, f, \langle\cdot,\cdot\rangle_1)$ and $\mathcal{S}_2 = (E, f, \langle\cdot,\cdot\rangle_2)$. By compactness of M, there exists a constant $k \geq 1$ such that

$$\frac{1}{k}\langle u, u\rangle_2 \leq \langle u, u\rangle_1 \leq k\langle u, u\rangle_2, \; \forall\, u \in E. \tag{7}$$

For every $q \in M$ and $v \in \Delta(q)$ let $\mathbf{G}_q^i(v) = \inf\{\langle u, u\rangle_i \mid u \in E_q, f(u) = v\}$ (see section 2). Clearly,

$$\frac{1}{k}\mathbf{G}_q^2(v) \leq \mathbf{G}_q^1(v) \leq k\mathbf{G}_q^2(v), \; \forall\, v \in f(E_q). \tag{8}$$

By (7), admissible curves for $\mathcal{S}_1$ and $\mathcal{S}_2$ coincide. Moreover, given an admissible curve $\gamma : [0,T] \to M$, we can compare its length with respect to $\mathcal{S}_1$ and $\mathcal{S}_2$ using (8). Namely,

$$\frac{1}{\sqrt{k}} \int_0^T \sqrt{\mathbf{G}^2_{\gamma(s)}(\dot\gamma(s))}ds \leq \int_0^T \sqrt{\mathbf{G}^1_{\gamma(s)}(\dot\gamma(s))}ds \leq \sqrt{k} \int_0^T \sqrt{\mathbf{G}^2_{\gamma(s)}(\dot\gamma(s))}ds.$$

Since the Carnot-Caratheodory distance between two points is defined as the infimum of the lengths of the admissible curves joining them, we get

$$\frac{1}{\sqrt{k}} d_2(p,q) \leq d_1(p,q) \leq \sqrt{k} d_2(p,q), \ \forall\, p,q \in M.$$

This is equivalent to say that the identity map is a Lipschitz equivalence between $\mathcal{S}_1$ and $\mathcal{S}_2$. ■

Acnowledgements. The authors are grateful to Andrei Agrachev for very helpful discussions.

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