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Research Article

Local existence and uniqueness of the mild solution to the 1D Vlasov–Poisson system with an initial condition of bounded variation

Simon Labrunie*, Sandrine Marchal, Jean-Rodolphe Roche

We propose a result of local existence and uniqueness of a mild solution to the one-dimensional Vlasov–Poisson system. We establish the result for an initial condition lying in the space $W^{1,1}(\mathbb{R}^2)$, then we extend it to initial conditions lying in the space $BV(\mathbb{R}^2)$, without any assumption of continuity, boundedness or compact support. Copyright © 2010 John Wiley & Sons, Ltd.

Keywords: Vlasov-Poisson system, bounded variation functions, Banach contraction mapping principle

1. Introduction

1.1. Position of the problem

In this paper we study the one-dimensional Vlasov–Poisson system:

$$\forall (t, x, v) \in [0, T] \times \mathbb{R}^2, \qquad \frac{\partial f}{\partial t}(t, x, v) + v \frac{\partial f}{\partial x}(t, x, v) + E(t, x) \frac{\partial f}{\partial v}(t, x, v) = 0, \tag{1}$$

$$\forall (t, x) \in [0, T] \times \mathbb{R}, \qquad \frac{\partial E}{\partial x}(t, x) = \int_{\mathbb{R}} f(t, x, v)\, dv - n_{\mathrm{b}}(t, x), \tag{2}$$

$$\forall (x, v) \in \mathbb{R}^2, \qquad f(0, x, v) = f_0(x, v). \tag{3}$$

This system models the behaviour of a gas of charged particles in an electrostatic field, created by the particles and by a so-called *neutralising background*, whose density n_{b} is assumed to be given. Collisions between particles are neglected. In [7], Cooper and Klimas show the existence and uniqueness of a global mild solution to this system, i.e. a solution defined by characteristics, for a continuous and bounded initial condition f_0 which has its first two moments in v uniformly bounded in x, and a constant background density n_{b}. This was extended by Bostan [5] to the initial-boundary value problem, with slightly more general hypotheses on the initial and boundary conditions, namely, that they are bounded but not necessarily continuous, and have one moment in v uniformly bounded in x. In [12], Guo showed that there exists a unique local weak solution to (1–3) in the space $L^\infty([0, T], BV(\mathbb{R}^2))$ for initial and boundary conditions with compact support and in the space $L^\infty(\mathbb{R}^2) \cap BV(\mathbb{R}^2)$. (The latter two works do not consider a neutralising background, but can be easily extended to this situation.)

In higher dimensions, the known existence and uniqueness results usually rest upon restrictive assumptions. To get strong or mild solutions, the standard hypotheses on f_0 are either explicit decay bounds in v (of the form $C(1 + |v|)^{-\alpha}$) for f_0 and its first derivatives [13], or local boundedness and Lipschitz-style conditions together with a finite energy [17], or even a bounded (at least in v) support [2, 19]. As for weak or renormalised solutions, their existence is generally proven under assumptions including $f_0 \in L^1(\mathbb{R}^{2d}) \cap L^p(\mathbb{R}^{2d})$, with p large enough, and a certain number of global moments in v finite (at least $\iint |v|^2 f_0\, dx\, dv$, corresponding to kinetic energy); see Horst–Hunze [14], DiPerna–Lions [8], Lions–Perthame [15]. Uniqueness results generally assume, once more, a compact support [18], or the boundedness of the spatial density [16], sometimes complemented by

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Lipschitz-style conditions [15]. Let us finally mention the existence and uniqueness result of compactly supported solutions in $L^1(\mathbb{R}^6) \cap L^\infty(\mathbb{R}^6)$ by Zhidkov [21].

In this article we extend the results of Guo to the initial value problem with intial data in the space $BV(\mathbb{R}^2)$, thus not necessarily compactly supported, bounded, or continuous. As far as Lebesgue regularity is concerned, this implies $f_0 \in L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$. Our proof is based on the contraction mapping principle of Banach, and consists of two steps: first we establish the local existence and uniqueness of a mild solution for an initial data f_0 in $W^{1,1}(\mathbb{R}^2)$, then we extend the result to $f_0 \in BV(\mathbb{R}^2)$.

1.2. Notations and main results

We introduce the following notations (see [12]). Given $T > 0$, we denote

$$U_T = (0, T) \times \mathbb{R} \quad \text{and} \quad V_T = (0, T) \times \mathbb{R}^2.$$

For $s \in [0, T]$, we denote $\Pi_s = \{s\} \times \mathbb{R}^2$ the slice $t = s$ of $\overline{V_T}$. Then we introduce the following functional spaces:

$$L(T) = L^\infty(0, T; W^{1,1}(\mathbb{R}^2)), \quad X(T) = L^\infty(0, T; W^{1,\infty}(\mathbb{R})).$$

The space $L(T)$ will be that of the solutions f to the Vlasov equation with initial data $f_0 \in W^{1,1}(\mathbb{R}^2)$; we equip it with its natural norm. As for $X(T)$, it is a space of electrostatic fields E for which the characteristic curves are globally well defined and Lipschitz-continuous in all their variables [5]. This can be shown by adapting the proof of the Cauchy–Lipschitz theorem: the only difference is that we integrate L^∞ functions instead of C^0 functions and so we get continuous solutions differentiable almost everywhere in the time variable and with bounded derivative. We equip it with the following norm:

$$\forall E \in X(T), \quad \|E\|_{X(T)} = \max(\|E\|_{L^\infty(U_T)}, \|\partial_x E\|_{L^\infty(U_T)}).$$

Morover, for any $E \in X(T)$, we set

$$C(E) = \max(\|\partial_x E\|_{L^\infty(U_T)}, 1), \tag{4}$$

and we denote by Y_E the Vlasov differential operator:

$$Y_E = \frac{\partial}{\partial t} + v\frac{\partial}{\partial t} + E(t, x)\frac{\partial}{\partial v}. \tag{5}$$

We recall the definition of the total variation of a function $f \in L^1(\mathbb{R}^2)$ (see for example [9, p. 39]):

$$\forall f \in L^1(\mathbb{R}^2), \ TV[f] = TV_x[f] + TV_v[f], \tag{6}$$

where:

$$TV_x[f] = \limsup_{\epsilon \longrightarrow 0} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \left|\frac{f(x+\epsilon, v) - f(x, v)}{\epsilon}\right| dx dv \tag{7}$$

$$TV_v[f] = \limsup_{\epsilon \longrightarrow 0} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \left|\frac{f(x, v+\epsilon) - f(x, v)}{\epsilon}\right| dx dv \tag{8}$$

The space of functions of bounded variation is defined as:

$$BV(\mathbb{R}^2) = \{f \in L^1(\mathbb{R}^2) : TV[f] < +\infty\}, \tag{9}$$

and equipped with the norm $\|f\|_{BV(\mathbb{R}^2)} = \|f\|_{L^1(\mathbb{R}^2)} + TV[f]$.

Finally, we denote by $L^{bv}(T)$ the space $L^\infty(0, T; BV(\mathbb{R}^2))$ equipped with its natural norm. We shall establish the following two theorems:

Theorem 1 (Local existence and uniqueness in $W^{1,1}$) *Let f_0 and n_b be non-negative given functions in $W^{1,1}(\mathbb{R}^2)$ and $L^\infty_{loc}(\mathbb{R}^+; L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}))$, respectively. For any (R, T) such that:*

$$R \ \geq \ \max\left(\tfrac{1}{2}\left[\|f_0\|_{L^1(\mathbb{R}^2)} + \|n_b\|_{L^\infty(0,T;L^1(\mathbb{R}))}\right], \|n_b\|_{L^\infty(U_T)}, |f_0|_{W^{1,1}(\mathbb{R}^2)}, 1\right)$$

$$T \ \in \ \left[0, \frac{1}{R}\ln\left(\frac{R}{|f_0|_{W^{1,1}(\mathbb{R}^2)}}\right)\right],$$

there exists a unique mild solution $(f, E) \in L(T) \times X(T)$ to (1–3).

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Moreover, we have a lower bound on the existence time T_{ex} of the maximal solution to (1–3). Let $T_* > 0$ be chosen arbitrarily to bound the various norms of n_b, and set $R_0 := \max(\frac{1}{2}(\|f_0\|_{L^1(\mathbb{R}^2)} + \|n_b\|_{L^\infty(0,T_*;L^1(\mathbb{R}))}), \|n_b\|_{L^\infty(U_{T_*})}, 1)$. Then:

$$T_{ex} \geq \min\left(T_*, \frac{1}{(e-1)|f_0|_{W^{1,1}}}\right) \quad if \quad |f_0|_{W^{1,1}} \geq \frac{R_0}{e}, \tag{10}$$

$$T_{ex} \geq \min\left(T_*, \frac{1}{R_0}\left[\ln\frac{R_0}{|f_0|_{W^{1,1}}} + \frac{1}{(e-1)}\right]\right) \quad if \quad |f_0|_{W^{1,1}} \leq \frac{R_0}{e}. \tag{11}$$

If the $W^{1,1}$ norm of $f(t)$, or equivalently the L^∞ norm of the density $n(t) := \int f(t)\,dv$, remains bounded for $t \in (0, T_{ex})$, then $T_{ex} = +\infty$, i.e. the solution is global. Otherwise, the following bounds hold:

$$|f(t)|_{W^{1,1}} \geq \frac{1}{(e-1)(T_{ex}-t)}, \quad \|n(t)\|_{L^\infty} \geq C\,|\ln(T_{ex}-t)|, \tag{12}$$

for some constant C and $T_{ex} - t$ small enough.

Theorem 2 (Local existence and uniqueness in BV) *Assume now that $f_0 \in BV(\mathbb{R}^2)$, while keeping all other hypotheses of Theorem 1. If (R, T) are such that:*

$$R \geq \max\left(\tfrac{1}{2}[\|f_0\|_{L^1(\mathbb{R}^2)} + \|n_b\|_{L^\infty(0,T;L^1(\mathbb{R}))}], \|n_b\|_{L^\infty(U_T)}, TV[f_0], 1\right)$$

$$T \in \left[0, \frac{1}{R}\ln\left(\frac{R}{TV[f_0]}\right)\right],$$

there exists a unique mild solution $(f, E) \in L^{bv}(T) \times X(T)$ to (1–3). The bounds (10), (11), (12) hold, with the semi-norms $|f_0|_{W^{1,1}}, |f(t)|_{W^{1,1}}$ replaced with total variations $TV[f_0]$, $TV[f(t)]$.

The proof is organised as follows. In §2, we recall the definitions of weak and mild solutions to the linear Vlasov equation (i.e. (1) and (3) with E a known function of (t, x)) and to the Vlasov–Poisson system (1–3). Then, in §3, we estimate the mild solutions to the linear Vlasov equation with initial data in $W^{1,1}(\mathbb{R}^2)$, and use these results to construct a contraction mapping on a suitable set, whose fixed point gives a mild solution to the Vlasov–Poisson system. Finally, we extend these results to initial conditions lying in $BV(\mathbb{R}^2)$ in §4.

2. Weak and mild solutions

2.1. Definition of a weak solution

We recall the definition of a weak solution to (1–3) by using the spaces of test functions and the functionals introduced by Guo in [11]. We define two spaces of test functions, one for the Vlasov equation and the other for the Poisson equation:

$$\mathcal{V} = C_c^\infty([0, T) \times \mathbb{R}^2), \quad \mathcal{M} = C_c^\infty([0, T) \times \mathbb{R}).$$

We define for $(E, f, f_0) \in L_{loc}^\infty(U_T) \times L_{loc}^1(V_T) \times L_{loc}^1(\mathbb{R}^2)$ and $\alpha \in \mathcal{V}$ (still like in [11]) the following functional:

$$A(f, E, f_0, \alpha) = \int_{\mathbb{R}^2} f_0(x, v)\,\alpha(0, x, v)\,dxdv + \int_0^T \int_{\mathbb{R}^2} [(Y_E\alpha)\,f](t, x, v)\,dxdv\,dt.$$

We define for $(E, f, n_b) \in L_{loc}^\infty(U_T) \times L_{loc}^1((0, T) \times \mathbb{R}_x; L^1(\mathbb{R}_v)) \times L_{loc}^\infty(U_T)$ and $\psi \in \mathcal{M}$ the following functional:

$$C(f, E, n_b, \psi) = \int_0^T \int_{\mathbb{R}} \left\{ E(t, x)\,\partial_x\psi(t, x) + \psi(t, x)\left[\int_{\mathbb{R}} f(t, x, v)\,dv - n_b(t, x)\right] \right\} dx\,dt.$$

These functionals are well-defined.

A weak solution to the linear Vlasov equation associated to $E \in L_{loc}^\infty(U_T)$ with initial condition $f_0 \in L_{loc}^1(\mathbb{R}^2)$ is a function $f \in L_{loc}^1(V_T)$ which satisfies:

$$\forall \alpha \in \mathcal{V}, \quad A(f, E, f_0, \alpha) = 0.$$

A weak solution to the one-dimensional Vlasov–Poisson system with intial condition $f_0 \in L_{loc}^1(\mathbb{R}^2)$ and neutralising background density $n_b \in L_{loc}^\infty(U_T)$ is a pair $(E, f) \in L_{loc}^\infty(U_T) \times L_{loc}^1((0, T) \times \mathbb{R}_x; L^1(\mathbb{R}_v))$ which verifies:

$$\forall(\alpha, \psi) \in \mathcal{V} \times \mathcal{M}, \quad A(f, E, f_0, \alpha) = 0 \quad \text{and} \quad C(f, E, n_b, \psi) = 0.$$

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2.2. Characteristic curves associated to $E \in X(T)$

We recall the following results on the characteristic curves of a transport equation, see for example [7] or [20]. Given $E \in X(T)$ and $(t, x, v) \in \overline{V_T}$, we consider the differential system :

$$\frac{dX}{ds}(s) = V(s),$$
$$\frac{dV}{ds}(s) = E(s, X(s)),$$
$$(X(t), V(t)) = (x, v). \tag{13}$$

As remarked above, this system admits a unique solution for all $(t, x, v) \in \overline{V_T}$, which we denote $\Gamma(s; t, x, v) = (X(s; t, x, v), V(s; t, x, v))$ and is called the *characteristic curve* passing by (t, x, v).

As E is bounded on $[0, T] \times \mathbb{R}$, every characteristic curve is defined from $s = 0$ to $s = T$; moreover, the characteristic curves form a partition of V_T. Thus for every characteristic $\Gamma(s; t, x, v)$, we can define an origin on Π_0: $\Gamma(0; t, x, v) = (X(0; t, x, v), V(0; t, x, v))$.

Let $(t, s) \in [0, T]$. We denote by $\phi_{t,s}$ the *characteristic flow* of E, namely the function:

$$\phi_{t,s} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$
$$(x, v) \longmapsto \Gamma(s; t, x, v). \tag{14}$$

$\phi_{t,s}$ transports a point (t, x, v) of the slice Π_t to a point (s, x', v') of the slice Π_s by following the characteristic curve passing by (t, x, v). It is well-known that $\phi_{t,s}$ is a bijection (one-to-one and onto mapping) of $\mathbb{R}^2$, which admits bounded partial derivatives and whose Jacobian is identically equal to 1.

2.3. Definition of a mild solution

Let $E \in X(T)$ and (X, V) be the associated characteristic curves. A mild solution to the linear Vlasov equation associated to E with initial condition $f_0 \in L^1_{loc}(\mathbb{R}^2)$ is a function $f \in L^1_{loc}(V_T)$ which satisfies:

$$f(t, x, v) = f_0(X(0; t, x, v), V(0; t, x, v)) \quad \text{for a.e. } (t, x, v) \in \overline{V_T}.$$

We recall the following result (see for example [3]):

Proposition 3 *Let $E \in X(T)$ and $f_0 \in L^1(\mathbb{R}^2)$. Then $f \in L^1(V_T)$ is a weak solution to the linear Vlasov equation associated to E with initial condition f_0 if and only if it is a mild solution.*

This can be shown by using the characteristic change of variables: $(t, x, v) \mapsto (t, x_0, v_0) = (t, \phi_{t,0}(x, v))$, as e.g. in Guo [12]. We deduce the existence and uniqueness of a solution $f \in L^1(V_T)$ to the linear Vlasov equation associated to a field $E \in X(T)$:

Corollary 4 *Let $E \in X(T)$ and $f_0 \in L^1(\mathbb{R}^2)$. The linear Vlasov equation associated to E with initial condition f_0 admits a unique weak solution in $L^1(V_T)$ defined as: $\forall (t, x, v) \in V_T$, $f(t, x, v) = f_0(X(0; t, x, v), V(0; t, x, v))$.*

Finally, a mild solution to the Vlasov–Poisson system with intial condition $f_0 \in L^1(\mathbb{R}^2)$ and neutralising background density $n_b \in L^\infty_{loc}(U_T)$ is defined as a weak solution (E, f), which belongs to $X(T) \times L^1(V_T)$, and such that f coincides a.e. with the mild solution to the linear Vlasov equation associated to E with initial condition f_0.

3. Proof of Theorem 1

3.1. A priori *estimates*

The proof of Theorem 1 relies on the following two theorems whose version for a half space is given by Guo in [12].

Theorem 5 *Let $E \in X(T)$ and $p \in [1, +\infty)$. We suppose that $u \in L^p(V_T)$ and $Y_E u \in L^p(V_T)$. Then:*

1. *There exists $u_0 \in L^1_{loc}(\Pi_0) \simeq L^1_{loc}(\mathbb{R}^2)$, called the trace of u on Π_0, such that $\forall \alpha \in C^\infty_c([0, T[\times\mathbb{R}^2)$,*

$$\int_{V_T} (Y_E u \, \alpha + u \, Y_E \alpha)(t, x, v) \, dx dv \, dt = -\int_{\mathbb{R}^2} u_0(x, v) \, \alpha(0, x, v) \, dx dv.$$

2. *If $u_0 \in L^p(\mathbb{R}^2)$, then $\forall s \in [0, T]$, $u(s) \in L^p(\mathbb{R}^2)$ and*

$$\int_{\mathbb{R}^2} |u(s)|^p \, dx dv = \int_{\mathbb{R}^2} |u_0|^p \, dx dv + p \int_0^s \int_{\mathbb{R}^2} (\text{sgn } u \, |u|^{p-1} \, Y_E u)(\tau) \, dx dv \, d\tau.$$

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Theorem 6 *Let $E \in X(T)$, and $\phi_{t,s}$ be its characteristic flow. We suppose that $u \in L^1(V_T)$ and $Y_E u \in L^1(V_T)$. Let u_0 be the trace of u on Π_0 defined in Theorem 5. If K is a measurable set of $\mathbb{R}^2$ with non-vanishing Lebesgue measure, then:*

$$\int_{\phi_{0,s}(K)} |u(s)| \, dx dv = \int_K |u_0| \, dx dv + \int_0^s \int_{\phi_{0,\tau}(K)} (\operatorname{sgn} u \, Y_E u)(\tau) \, dx dv \, d\tau.$$

The proofs rely on the characteristic change of variables and are entirely similar to those of [12].

With these results, we can prove the fundamental estimate on the solutions to the linear Vlasov equation. We introduce the semi-norm $| \cdot |_{W^{1,1}}$ defined by

$$\forall f \in W^{1,1}(\mathbb{R}^2), \quad |f|_{W^{1,1}} = \|\partial_x f\|_1 + \|\partial_v f\|_1.$$

Theorem 7 *Let $E \in X(T)$ and $f_0 \in W^{1,1}(\mathbb{R}^2)$. Let f be the unique mild solution in $L^1(V_T)$ of the linear Vlasov equation associated to E with initial condition f_0. Then $\forall s \in [0, T]$, $f(s) \in W^{1,1}(\mathbb{R}^2)$ and*

$$\|f(s)\|_{L^1(\mathbb{R}^2)} = \|f_0\|_{L^1(\mathbb{R}^2)} ; \tag{15}$$

$$|f(s)|_{W^{1,1}(\mathbb{R}^2)} \leq |f_0|_{W^{1,1}(\mathbb{R}^2)} \exp(C(E)s). \tag{16}$$

Thus, integrating from 0 to T:

$$\int_0^T |f(\tau)|_{W^{1,1}(\mathbb{R}^2)} \, d\tau \leq |f_0|_{W^{1,1}(\mathbb{R}^2)} \frac{\exp(C(E)T) - 1}{C(E)}. \tag{17}$$

Proof. Equation (15) is an immediate consequence of point 2 of Theorem 5 (with $p = 1$), or of Theorem 6 (with $K = \mathbb{R}^2$), given that $Y_E f = 0$.

We now establish the estimate (16) on derivatives. The set of the indefinitely differentiable functions with compact support on $\mathbb{R}^2$ is dense in $W^{1,1}(\Pi_0)$ [1, p. 54]. Thus there exists a sequence $(f_0^n)_n$ of elements of $C_c^\infty(\mathbb{R}^2)$, such that $\|f_0^n - f_0\|_{W^{1,1}(\mathbb{R}^2)} \longrightarrow 0$ when $n \longrightarrow +\infty$.

Similarly, we regularise $E \in L^\infty(0, T; , W^{1,\infty}(\mathbb{R}))$ in the following way. We define for all $t \in [0, T]$, $E_n(t, .) = E(t, .) * \rho_n$, where $(\rho_n) \in C_c^\infty(\mathbb{R}_x)$ is a mollifying sequence. The sequence $(E_n)_n$ satisfies: $E_n \in L^\infty(0, T; W^{1,\infty} \cap C^1(\mathbb{R}_x))$; $\|E_n\|_{L^\infty(U_T)} \leq \|E\|_{L^\infty(U_T)}$, $\|\partial_x E_n\|_{L^\infty(U_T)} \leq \|\partial_x E\|_{L^\infty(U_T)}$, and $\|E - E_n\|_{L^\infty(U_T)} \longrightarrow 0$ when $n \longrightarrow +\infty$. We denote by (X^n, V^n) and $\phi_{t,s}^n$ the characteristic curves and flow associated to E_n.

Let f_n be the solution to the linear problem associated to E_n with initial condition f_0^n; we recall that this solution is given for a.e. $(t, x, v) \in \overline{V_T}$ by $f_n(t, x, v) = f_0^n(X^n(0; t, x, v), V^n(0; t, x, v))$. As f_0^n is compactly supported, so is f_n: its support is contained in the image of the compact $[0, T] \times \operatorname{Supp} f_0^n$ by the continuous mapping $(s, x_0, v_0) \mapsto (s, X(s; 0, x_0, v_0), V(s; 0, x_0, v_0))$. Moreover, the characteristics associated to E_n are Lipschitz-continuous in all their variables (s, t, x, v), therefore $f_n \in W^{1,\infty}(V_T)$.

All together, we have $\partial_x f_n$ and $\partial_v f_n \in L_c^\infty(V_T)$, thus $\partial_x f_n$ and $\partial_v f_n$ lie in $L^1(V_T)$. Moreover $Y_{E_n} \partial_x f_n = -\partial_x E_n \, \partial_v f_n$ in $\mathcal{D}'(V_T)$, thus $Y_{E_n} \partial_x f_n$ lies in $L^1(V_T)$. By an integration by parts, it can be shown that the trace of $\partial_x f_n$ on Π_0 is $\partial_x f_0^n$. If K is a measurable subset of $\mathbb{R}^2$ of non-vanishing Lebesgue measure, we get by Theorem 6:

$$\int_{\phi_{0,s}^n(K)} |\partial_x f_n(s)| = \int_K |\partial_x f_0^n| - \int_0^s \int_{\phi_{0,\tau}^n(K)} (\operatorname{sgn}(\partial_x f_n) \, \partial_x E_n \, \partial_v f_n)(\tau) \, d\tau ;$$

for the sake of brevity we have omitted the kinetic integration element $dx dv$. Thus:

$$\int_{\phi_{0,s}^n(K)} |\partial_x f_n(s)| \leq \int_K |\partial_x f_0^n| + \|\partial_x E_n\|_{L^\infty([0,s] \times \mathbb{R})} \int_0^s \int_{\phi_{0,\tau}^n(K)} |\partial_v f_n(\tau)| \, d\tau,$$

$$\int_{\phi_{0,s}^n(K)} |\partial_x f_n(s)| \leq \int_K |\partial_x f_0^n| + \|\partial_x E\|_{L^\infty(U_T)} \int_0^s \int_{\phi_{0,\tau}^n(K)} |\partial_v f_n(\tau)| \, d\tau. \tag{18}$$

In the same way, we have $\partial_v f_n \in L_1(V_T)$ and $Y_{E_n} \partial_v f_n = -\partial_x f_n \in \mathcal{D}'(V_T)$, thus $Y_{E_n} \partial_v f_n$ lies in $L^1(V_T)$; and one shows that the trace of $\partial_v f_n$ on Π_0 is $\partial_v f_0^n$. Reasoning as above, we obtain:

$$\int_{\phi_{0,s}^n(K)} |\partial_v f_n(s)| \leq \int_K |\partial_v f_0^n| + \int_0^s \int_{\phi_{0,\tau}^n(K)} |\partial_x f_n(\tau)| \, d\tau. \tag{19}$$

We add (18) and (19):

$$\int_{\phi_{0,s}^n(K)} \{|\partial_x f_n(s)| + |\partial_v f_n(s)|\} \leq \int_K \{|\partial_v f_0^n| + |\partial_x f_0^n|\}$$

$$+ \max(\|\partial_x E\|_{L^\infty(U_T)}, 1) \int_0^s \int_{\phi_{0,\tau}^n(K)} \{|\partial_x f_n(\tau)| + |\partial_v f_n(\tau)|\} \, d\tau.$$

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Then we utilize the Grönwall lemma, and we get:

$$\int_{\phi_{0,s}^n(K)} \{|\partial_x f_n(s)| + |\partial_v f_n(s)|\} \le \exp(C(E)s) \int_K \{|\partial_v f_0^n| + |\partial_x f_0^n|\}. \tag{20}$$

Therefore:

$$\int_0^T \int_{\phi_{0,s}^n(K)} |\nabla f_n(s)|\, ds \le \frac{\exp(C(E)T) - 1}{C(E)} \int_K |\nabla f_0^n|. \tag{21}$$

Now we utilize the Dunford–Pettis weak compactness criterion in L^1, that can be found for example in [6, p. 76] or [4, p. 167]:

Theorem 8 (Dunford–Pettis) *Let $(f_n)_n$ be a bounded sequence of $L^1(\Omega)$. The sequence is weakly compact if and only if $\{f_n\}_{n\in\mathbb{N}}$ is equiintegrable, that is to say:*

$$\forall \epsilon > 0,\ \exists K_\epsilon \text{ compact } \subset \Omega \text{ s.t. } \sup_n \int_{\Omega \setminus K_\epsilon} |f_n|\, d\Omega < \epsilon, \quad \text{and:}$$

$$\forall \epsilon > 0,\ \exists \eta > 0,\ \forall \mathcal{A} \subset \Omega \text{ measurable, meas}(\mathcal{A}) < \eta \implies \sup_n \int_{\mathcal{A}} |f_n|\, d\Omega < \epsilon.$$

Let $\epsilon > 0$. The sequences $(\partial_x f_0^n)_n$ and $(\partial_v f_0^n)_n$ converge in $L^1(\mathbb{R}^2)$, thus are weakly compact in $L^1(\mathbb{R}^2)$. By the Dunford–Pettis criterion, these sequences are equiintegrable. Thus, there exists a compact K_ϵ^0 of $\mathbb{R}^2$, and $\eta > 0$ such that:

$$\sup_n \int_{\mathbb{R}^2 \setminus K_\epsilon^0} \{|\partial_x f_0^n| + |\partial_v f_0^n|\} < e^{-C(E)T} \epsilon, \quad \text{and:}$$

$$\forall \mathcal{A} \subset \mathbb{R}^2 \text{ measurable, meas}(\mathcal{A}) < \eta \implies \sup_n \int_{\mathcal{A}} \{|\partial_x f_0^n| + |\partial_v f_0^n|\} < e^{-C(E)T} \epsilon.$$

Let $\mathcal{A}$ be a subset of $\mathbb{R}^2$ such that meas$(\mathcal{A}) \le \eta$. We have for all $n \in \mathbb{N}$, meas$(\phi_{s,0}^n(\mathcal{A})) = $ meas$(\mathcal{A}) \le \eta$, and we can apply the inequality (20) to get

$$\sup_n \int_{\mathcal{A}} |\nabla f_n(s)| \le \epsilon. \tag{22}$$

Thus we see that the sequences $(\partial_x f_n(s))_n$ and $(\partial_v f_n(s))_n$ verify the second part of the Dunford–Pettis criterion. For the first part of this criterion, we construct a compact K_ϵ such that all the $\phi_{0,s}^n(K_\epsilon^0) \subset K_\epsilon$. Let $(X_L(s; t, x, v), V_L(s; t, x, v))$ and $\phi_{t,s}^L$ be the characteristic curves and flow associated to free transportation $(E = 0)$. Of course, we have : $V_L(t; 0, x_0, v_0) = v_0$ and $X_L(t; 0, x_0, v_0) = x_0 + v_0 t$. We denote $L_\epsilon = \phi_{0,s}^L(K_\epsilon^0)$; this set is a compact as the continuous image of a compact. Then, using the estimate on the divergence of characteristics from [7, Lemma 1] or [5, Lemma 4.8], we obtain:

$$\forall t \in [0, T], \quad |V^n(t; 0, x_0, v_0) - V_L(t; 0, x_0, v_0)| \le t \, \|E_n\|_{L^\infty(U_t)} \, ; \tag{23}$$

$$|X^n(t; 0, x_0, v_0) - X_L(t; 0, x_0, v_0)| \le t^2 \, \|E_n\|_{L^\infty(U_t)}. \tag{24}$$

Thus we can take for K_ϵ the compact:

$$K_\epsilon = \{(x, v) \in \mathbb{R}^2 : \exists (x_1, v_1) \in L_\epsilon, |x - x_1| \le T \, \|E\|_{L^\infty(U_T)}$$
$$\text{and } |v - v_1| \le T^2 \, \|E\|_{L^\infty(U_T)}\}.$$

We have: $\forall n \in \mathbb{N},\ \phi_{0,s}^n(K_\epsilon^0) \subset K_\epsilon$. Thus,

$$\sup_{n \in \mathbb{N}} \int_{\mathbb{R}^2 \setminus K_\epsilon} |\nabla f_n(s)| \le \sup_{n \in \mathbb{N}} \int_{\mathbb{R}^2 \setminus \phi_{0,s}^n(K_\epsilon^0)} |\nabla f_n(s)| \tag{25}$$

$$\le \sup_n \int_{\mathbb{R}^2 \setminus K_\epsilon^0} \{|\partial_x f_0^n| + |\partial_v f_0^n|\}\, e^{C(E)T} \le \epsilon. \tag{26}$$

Therefore, $(\partial_x f_n(s))_n$ and $(\partial_v f_n(s))_n$ verify the Dunford-Pettis criterion and thus converge weakly (after extracting a subsequence) in $L^1(\mathbb{R}^2)$ toward some functions g and h of $L^1(\mathbb{R}^2)$.

On the other hand, we have $Y_E(f_n - f) = (E - E_n)\, \partial_v f_n$, thus $f_n - f$ and $Y_E(f_n - f)$ are in $L^1(V_T)$. Applying point 2 of Theorem 5 and then the bound (21), we find:

$$\int_{\mathbb{R}^2} |f(s) - f_n(s)| \le \int_{\mathbb{R}^2} |f_0 - f_0^n| + \int_0^s \int_{\mathbb{R}^2} |E(\tau) - E_n(\tau)|\, |\partial_v f_n(\tau)|\, d\tau$$

$$\le \int_{\mathbb{R}^2} |f_0 - f_0^n| + \|E - E_n\|_{L^\infty(U_T)} \frac{\exp(C(E)T) - 1}{C(E)} \int_{\mathbb{R}^2} |\nabla f_0^n|.$$

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Thus, $f_n(s)$ converges toward $f(s)$ in $L^1(\mathbb{R}^2)$. As a consequence, $\partial_x f_n(s)$ and $\partial_v f_n(s)$ converge toward $\partial_x f(s)$ and $\partial_v f(s)$ in $\mathcal{D}'(\mathbb{R}^2)$; therefore $g = \partial_x f(s)$ and $h = \partial_v f(s)$, i.e. $\partial_x f(s)$ and $\partial_v f(s)$ lie in $L^1(\mathbb{R}^2)$. In other words, $f(s)$ appears as the weak limit in $W^{1,1}(\mathbb{R}^2)$ of the sequence $(f_n(s))_n$. By passing to the limit in (20), we get:

$$\int_{\mathbb{R}^2} |\nabla f(s)| \leq \liminf_{n \to +\infty} \int_{\mathbb{R}^2} |\nabla f_n(s)| \leq \exp(C(E)s) \|\nabla f_0\|_{L^1(\mathbb{R}^2)},$$

which is (16), and yields (17) by integrating from 0 to T.   ■

3.2. Construction of a contraction mapping

We now study the non-linear Vlasov–Poisson problem. In the rest of §3, we choose $f_0 \in W^{1,1}(\mathbb{R}^2)$ and $n_b \in L^\infty_{loc}(\mathbb{R}^+; L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}))$, both being non-negative. In order to construct a contraction mapping from a closed subset of a Banach space to itself, we define the following mappings:

- $\rho : L^1(\mathbb{R}^2) \to L^1(\mathbb{R})$ maps $g \in L^1(\mathbb{R}^2)$ to its spatial density defined as: $\forall x \in \mathbb{R}$, $\rho[g](x) = \int_{-\infty}^{+\infty} g(x,v)\, dv$;
- $\phi_1 : X(T) \to L^\infty(0,T;L^1(\mathbb{R}^2))$ maps $E \in X(T)$ to the unique mild solution f to the linear Vlasov equation associated to E and with initial condition f_0;
- $\phi_2 : L^\infty(0,T;L^1(\mathbb{R}^2)) \to L^\infty(U_T)$ maps $f \in L^\infty(0,T;L^1(\mathbb{R}^2))$ to the function $\mathcal{E}$ defined as:

$$\mathcal{E}(t,x) = \frac{1}{2}\int_{-\infty}^x (\rho[f(t)](y) - n_b(t,y))\, dy - \frac{1}{2}\int_x^{+\infty} (\rho[f(t)](y) - n_b(t,y))\, dy.$$

  Of course, it satisfies $\partial_x \mathcal{E} = \rho[f] - n_b$, thus it is a solution to the Poisson equation (2). This particular choice corresponds to the fundamental solution $G(x) = \frac{1}{2}\,\text{sgn}(x) = \frac{1}{2}\,x/|x|$, the counterpart of those appearing in higher dimensions.

The following lemma will be crucial in our proof. It is left as an exercise to the reader.

Lemma 9 *For $R \geq 0$, let B'_R be the closed ball of center 0 and radius R of the Banach space $(X(T), \|\cdot\|_{X(T)})$. Then, B'_R is a closed subset of the Banach space $(L^\infty(U_T), \|\cdot\|_\infty)$, hence it is complete for this norm.*

Then we state and prove the following property of the mapping ρ:

Lemma 10 *Let $g \in W^{1,1}(\mathbb{R}^2)$. Then, $\rho[g] \in L^\infty(\mathbb{R})$ and $\|\rho[g]\|_\infty \leq \|\partial_x g\|_1 \leq |g|_{W^{1,1}}$.*

Proof. By Fubini's theorem, the mapping $x \mapsto g(x,v)$ is in $W^{1,1}(\mathbb{R})$, for a.e. $v \in \mathbb{R}$, hence it satisfies $\lim_{x \to -\infty} g(x,v) = 0$. We have thus:

$$\rho[g](x) = \int_{-\infty}^{+\infty} g(x,v)\, dv = \int_{-\infty}^{+\infty}\int_{-\infty}^x \partial_x g(y,v)\, dy\, dv;$$

$$|\rho[g](x)| \leq \int_{-\infty}^{+\infty}\int_{-\infty}^x |\partial_x g(y,v)|\, dy\, dv \leq \|\partial_x g\|_{L^1(\mathbb{R}^2)};$$

hence the result.   ■

3.2.1. Stability and Lipschitz continuity of $\phi_2 \circ \phi_1$
Let $E \in X(T)$. Theorem 7 gives $\phi_1(E) \in L(T)$; moreover we have:

$$\|\phi_1(E)\|_{L^\infty(0,T;L^1(\mathbb{R}^2))} = \|f_0\|_1; \quad \|\nabla\phi_1(E)\|_{L^\infty(0,T;L^1(\mathbb{R}^2))} \leq |f_0|_{W^{1,1}}\, e^{C(E)T}. \tag{27}$$

Let $f \in L(T)$. From the definition of ϕ_2, we deduce: $|\phi_2(f)(t,x)| \leq \frac{1}{2}\|\rho[f](t) - n_b(t)\|_{L^1(\mathbb{R})}$, and thus

$$\|\phi_2(f)\|_{L^\infty(U_T)} \leq \frac{1}{2}[\|f\|_{L^\infty(0,T;L^1(\mathbb{R}^2))} + \|n_b\|_{L^\infty(0,T;L^1(\mathbb{R}))}]. \tag{28}$$

Assume that f is non-negative; thus the same holds for $\rho[f(t)]$, for a.e. t. The equation $\partial_x\phi_2(f)(t,x) = \rho[f(t)](x) - n_b(t,x)$ gives $|\partial_x\phi_2(f)(t,x)| \leq \max(\rho[f(t)](x), n_b(t,x))$. By Lemma 10, we infer:

$$\|\partial_x\phi_2(f)\|_{L^\infty([0,T]\times\mathbb{R})} \leq \max\left(\|\partial_x f\|_{L^\infty([0,T],L^1(\mathbb{R}^2))}, \|n_b\|_{L^\infty([0,T]\times\mathbb{R})}\right).$$

Together with (28), we obtain the following bound

$$\|\phi_2(f)\|_{X(T)} \leq \max\left\{\frac{1}{2}[\|f\|_{L^\infty(0,T;L^1(\mathbb{R}^2))} + \|n_b\|_{L^\infty(0,T;L^1(\mathbb{R}))}], \|n_b\|_{L^\infty([0,T]\times\mathbb{R})}, \|\partial_x f\|_{L^\infty([0,T],L^1(\mathbb{R}^2))}\right\}. \tag{29}$$

Now, if $f = \phi_1(E)$, it is non-negative as soon as the initial data f_0 is. We combine (29) and (27) and find:

$$\forall E \in X(T), \quad \|\phi_2 \circ \phi_1(E)\|_{X(T)} \leq \max\{M_T, M'_T, |f_0|_{W^{1,1}(\mathbb{R}^2)}\, e^{C(E)T}\}, \tag{30}$$
$$\text{where:} \quad M_T := \frac{1}{2}[\|f_0\|_{L^1(\mathbb{R}^2)} + \|n_b\|_{L^\infty(0,T;L^1(\mathbb{R}))}], \quad M'_T := \|n_b\|_{L^\infty(U_T)}.$$

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Now we show that $\phi_2 \circ \phi_1$ is a Lipschitz-continuous mapping in the norm of $L^\infty(U_T)$. Let $E_1, E_2 \in X(T)$; we denote $f_1 = \phi_1(E_1)$ and $f_2 = \phi_1(E_2)$. There holds: $Y_{E_1}(f_1 - f_2) = (Y_{E_2} - Y_{E_1})(f_2) = (E_2 - E_1)\,\partial_v f_2$. Thus, $(f_1 - f_2) \in L^1(V_T)$ and $Y_{E_1}(f_1 - f_2) \in L^1(V_T)$; we apply Theorem 5 and find:

$$\int_{\mathbb{R}^2} |f_1(s) - f_2(s)| \leq \int_0^s \int_{\mathbb{R}^2} |E_1(\tau) - E_2(\tau)|\, |\partial_v f_2(\tau)|\, d\tau.$$

Thus $\|f_1 - f_2\|_{L^\infty(0,T;L^1(\mathbb{R}^2))} \leq \|E_1 - E_2\|_{L^\infty(U_T)}\|\partial_v f_2\|_{L^1(V_T)}$; applying the bound (17), we obtain:

$$\|\phi_1(E_1) - \phi_1(E_2)\|_{L^\infty(0,T;L^1(\mathbb{R}^2))} \leq$$
$$\|E_1 - E_2\|_{L^\infty(U_T)}\, |f_0|_{W^{1,1}(\mathbb{R}^2)}\, \frac{\exp(C(E_2)T) - 1}{C(E_2)}. \tag{31}$$

As for ϕ_2, it is an affine mapping, the associated linear mapping corresponding to the case $n_b \equiv 0$. Therefore, for $f_1,\ f_2 \in L(T)$, one can apply the bound (28) with $f = f_1 - f_2$ and $n_b \equiv 0$:

$$\|\phi_2(f_1) - \phi_2(f_2)\|_{L^\infty(U_T)} \leq \tfrac{1}{2} \|f_1 - f_2\|_{L^\infty(0,T;L^1(\mathbb{R}^2))}. \tag{32}$$

Finally we arrive at:

$$\|\phi_2 \circ \phi_1(E_1) - \phi_2 \circ \phi_1(E_2)\|_{L^\infty(U_T)} \leq$$
$$\tfrac{1}{2} \|E_1 - E_2\|_{L^\infty(U_T)}\, |f_0|_{W^{1,1}(\mathbb{R}^2)}\, \frac{\exp(C(E_2)T) - 1}{C(E_2)}. \tag{33}$$

3.2.2. Local existence and uniqueness We now give conditions on the parameters R and T in order to have: (i) the closed ball B'_R stable by $\phi_2 \circ \phi_1$, and (ii) $\phi_2 \circ \phi_1$ a contraction mapping on B'_R. In order to bound the various norms of n_b, we settle on an arbitrary finite time interval $(0, T_*)$; notice that M_T and M'_T are non-decreasing functions of T. The stability estimate (30) implies (i) provided: $|f_0|_{W^{1,1}(\mathbb{R}^2)} \exp(\max(R, 1)T) \leq R$ and $M_{T_*},\ M'_{T_*} \leq R$. Thus we choose:

$$R \geq \max(|f_0|_{W^{1,1}(\mathbb{R}^2)}, M_{T_*}, M'_{T_*}) \quad \text{and} \quad T \leq \frac{1}{\max(R, 1)}\, \ln\left(\frac{R}{|f_0|_{W^{1,1}(\mathbb{R}^2)}}\right).$$

As for the point (ii), the Lipschitz estimate (33) yields the sufficient condition $R \geq 1$ and $\frac{1}{2} |f_0|_{W^{1,1}(\mathbb{R}^2)} (\exp(RT) - 1)/R < 1$. We take for example:

$$R \geq \max(1, |f_0|_{W^{1,1}(\mathbb{R}^2)}) \quad \text{and} \quad T < \frac{1}{R} \ln\left(1 + \frac{2\,R}{|f_0|_{W^{1,1}(\mathbb{R}^2)}}\right).$$

Considering the two conditions, we obtain that given

$$R \geq \max(1, |f_0|_{W^{1,1}(\mathbb{R}^2)}, M_{T_*}, M'_{T_*}), \quad T \leq \min\left(T_*,\ \frac{1}{R} \ln\left(\frac{R}{|f_0|_{W^{1,1}(\mathbb{R}^2)}}\right)\right),$$

the mapping $\phi_2 \circ \phi_1$ goes from B'_R into B'_R and is a contraction for the norm $\|\cdot\|_{L^\infty(U_T)}$. By Lemma 9, B'_R is a complete space for this norm. Utilizing the contraction mapping principle, the mapping $\phi_2 \circ \phi_1$ admits a unique fixed point $E \in B'_R$. If we denote $f = \phi_1(E)$, the pair $(E, f) \in X(T) \times L(T)$ is a mild solution to (1–3).

3.2.3. Estimation of the existence time Assume that T_* is large enough; we define $R_0 = \max(1, M_{T_*}, M'_{T_*})$. The function $x \mapsto \ln(ax)/x$ admits a unique maximum at the point $x = e/a$, and its value is a/e. Thus, the greatest value of expression $\frac{1}{R} \ln(\frac{R}{|f_0|_{W^{1,1}}})$ is attained at $R = e\,|f_0|_{W^{1,1}}$ and equal to $(e\,|f_0|_{W^{1,1}})^{-1}$.

There are two possibilities. If $|f_0|_{W^{1,1}} \geq R_0/e$, we can take $R = R_1 := e\,|f_0|_{W^{1,1}}$ and $T = T_1 := (e\,|f_0|_{W^{1,1}})^{-1}$ in §3.2.2. The estimate (16) then shows $|f(T_1)|_{W^{1,1}} = e\,|f_0|_{W^{1,1}}$. So, §3.2.2 proves the existence and uniqueness of the solution to the Vlasov–Poisson problem with initial data $f(T_1)$ during the time $T_2 := (e\,|f(T_1)|_{W^{1,1}})^{-1} = (e^2\,|f_0|_{W^{1,1}})^{-1}$. The local existence and uniqueness result allows one to glue together solutions to (1–2) that coincide at some time T_1. Thus, the solution generated by the initial data f_0 exists during $T_1 + T_2$. By induction, we obtain an existence time at least equal to:

$$\frac{1}{|f_0|_{W^{1,1}}} \left(\frac{1}{e} + \frac{1}{e^2} + \cdots + \frac{1}{e^n} + \cdots \right) = \frac{1}{(e-1)\,|f_0|_{W^{1,1}}},$$

provided this quantity is still less or equal to T_*. Now, if $|f_0|_{W^{1,1}} \leq R_0/e$, the existence time given by §3.2.2 is maximal for $R = R_0$ and equal to is equal to $T_0 := \frac{1}{R_0} \ln(\frac{R_0}{|f_0|_{W^{1,1}}})$. Applying (16), we obtain $|f(T_0)|_{W^{1,1}} = e^{R_0 T_0} |f_0|_{W^{1,1}} = R_0 > R_0/e$. Thus we can use the previous argument to show that the solution to the Vlasov–Poisson problem with initial data $f(T_0)$ exists for a time at least equal to $((e-1)R_0)^{-1}$. Finally, the total existence time is no less than

$$\frac{1}{R_0} \left(\ln\left(\frac{R_0}{|f_0|_{W^{1,1}}}\right) + \frac{1}{(e-1)}\right),$$

if this is no more than T_*.

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3.2.4. Blow-up versus globality Consider the maximal solution (f, E) to (1–3) with data f_0 and n_b, i.e. that with the largest existence interval $(0, T_{ex})$. Assume that T_{ex} is finite. Then, we show that $\lim_{t \to T_{ex}} |f(t)|_{W^{1,1}} = +\infty$. Otherwise, there would exist a sequence $(t_k)_k$ such that $t_k \to T_{ex}$ and $|f(t_k)|_{W^{1,1}} \leq C$. The argument of §3.2.3 would imply that the solution can be extended to an interval $(t_k, t_k + \tau)$, with τ bounded away from 0. Thus, it could be extended past T_{ex}, a contradiction.

Furthermore, at any instant t, the existence time given by §3.2.3 cannot be greater than $T_{ex} - t$. For t close enough to T_{ex}, $|f(t)|_{W^{1,1}}$ is large enough, and the existence time is equal to $(e - 1)^{-1} |f(t)|_{W^{1,1}}^{-1}$; so we get the estimate:

$$|f(t)|_{W^{1,1}} \geq \frac{1}{(e - 1)(T_{ex} - t)}.$$

Now, the second bound in (27) shows that $\|\partial_x E(t)\|_\infty = \|\rho[f(t)] - n_b(t)\|_\infty$ blows up when $t \to T_{ex}$. Quantitatively, this norm is greater than $t^{-1} \ln(|f(t)|_{W^{1,1}}/|f_0|_{W^{1,1}})$. As the norm of n_b remains bounded on the finite interval $(0, T_*)$, there holds:

$$\|\rho[f(t)]\|_\infty \geq C \, |\ln(T_{ex} - t)|,$$

for some constant C. Conversely, any maximal solution such that $|f(t)|_{W^{1,1}}$ or $\|\rho[f(t)]\|_\infty$ remain bounded as $t \to T_{ex}$ is global, i.e., $T_{ex} = +\infty$.

4. Proof of Theorem 2

4.1. Preliminary results

Here we collect some well-known results on the functions of $W^{1,1}(\mathbb{R}^2)$ and $BV(\mathbb{R}^2)$. The following proposition can be found, for example, in [10, pp. 3–4]:

Proposition 11 $W^{1,1}(\mathbb{R}^2) \subset BV(\mathbb{R}^2)$ *and* $\forall f \in W^{1,1}(\mathbb{R}^2)$, $|f|_{W^{1,1}} = TV[f]$.

The following two theorems are taken from [10], p. 7 and p. 14:

Theorem 12 *Let* $f \in L^1(\mathbb{R}^2)$ *and* $(f_n)_n$ *be a sequence in* $BV(\mathbb{R}^2)$ *which converges to* f *in* $L^1(\mathbb{R}^2)$. *Then:*

$$TV[f] \leq \liminf_{n \to +\infty} TV[f_n].$$

Theorem 13 *Let* $f \in BV(\mathbb{R}^2)$. *There exists a sequence* $(f_n)_n$ *in* $C^\infty(\mathbb{R}^2) \cap BV(\mathbb{R}^2)$ *such that:*

$$\lim_{n \to +\infty} \|f_n - f\|_{L^1(\mathbb{R}^2)} = 0 \quad \text{and} \quad \lim_{n \to +\infty} TV[f_n] = TV[f].$$

4.2. A priori estimates

Theorem 14 *Let* $f \in L^1(V_T)$ *be the unique mild solution to the linear Vlasov equation associated to* $E \in X(T)$ *with initial condition* $f_0 \in BV(\mathbb{R}^2)$. *Then,* $\forall s \in [0, T]$, $f(s) \in BV(\mathbb{R}^2)$ *and*

$$TV[f(s)] \leq TV[f_0] \exp(C(E)s).$$

Thus, integrating from 0 *to* T:

$$\int_0^T TV[f(\tau)] \, d\tau \leq TV[f_0] \frac{\exp(C(E)T) - 1}{C(E)}.$$

Remark that the estimate (15) *is still valid, as it only uses the* L^1 *character of* f_0 *and* f.

Proof. Let E, f_0, and f be as in the statement of the theorem. Theorem 13 yields the existence of a sequence $(f_0^n)_n$ in $C^\infty(\mathbb{R}^2) \cap BV(\mathbb{R}^2)$ such that:

$$\lim_{n \to +\infty} \|f_0^n - f_0\|_{L^1(\mathbb{R}^2)} = 0 \quad \text{and} \quad \lim_{n \to +\infty} TV[f_0^n] = TV[f_0].$$

In particular, we have: $\forall n \in \mathbb{N}, f_0^n \in W^{1,1}(\mathbb{R}^2)$.

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Let f_n be the unique mild solution to the linear Vlasov equation associated to E with initial condition f_0^n. Using Theorem 7 and Proposition 11, we get:

$$\forall s \in [0,T], \quad TV[f_n(s)] \leq TV[f_0^n] \exp(C(E)s).$$

We have $Y_E(f_n - f) = Y_E(f_n) - Y_E(f) = 0$, so we can use Theorem 6 and obtain:

$$\int_{\mathbb{R}^2} |f_n(s) - f(s)| = \int_{\mathbb{R}^2} |f_0^n - f_0|.$$

Therefore, $\lim f_n(s) = f(s)$ in $L^1(\mathbb{R}^2)$, for almost every $s \in [0,T]$. Applying Theorem 12 then yields:

$$TV[f(s)] \leq \liminf_{n \longrightarrow +\infty} TV[f_n(s)] \leq \liminf_{n \longrightarrow +\infty} TV[f_0^n] \exp(C(E)s)$$
$$= TV[f_0] \exp(C(E)s),$$

which implies $f(s) \in BV(\mathbb{R}^2)$. ∎

4.3. Construction of a contraction mapping

We now get down to the non-linear Vlasov–Poisson problem. We define the mappings ρ, ϕ_1 and ϕ_2 as in §3.2, and we find sufficient conditions for $\phi_2 \circ \phi_1$ to be a contraction mapping from B_R' to itself.

Let $f_0 \in BV(\mathbb{R}^2)$, $E \in X(T)$ and $f = \phi_1(E)$. By Theorem 14, $f \in L^{bv}(T)$ and

$$\|\phi_1(E)\|_{L^\infty(0,T;L^1(\mathbb{R}^2))} = \|f_0\|_1 \, ; \quad \|\nabla\phi_1(E)\|_{L^\infty(0,T;L^1(\mathbb{R}^2))} \leq TV[f_0] \, e^{C(E)T}.$$

Let us examine the mapping ϕ_2. The bound (28) is still valid. Moreover, if f is non-negative, the inequality $|\partial_x\phi_2(f)(t,x)| \leq \max(\rho[f(t)](x), n_b(t,x))$ still holds. Combining Lemma 10, Proposition 11 and Theorem 13, one easily proves:

Lemma 15 Let $g \in BV(\mathbb{R}^2)$. Then, $\rho[g] \in L^\infty(\mathbb{R})$ and $\|\rho[g]\|_\infty \leq TV[g]$.

Thus, one proves the counterpart of the estimate (29):

$$\|\phi_2(f)\|_{X(T)} \leq \max\left\{ \tfrac{1}{2}\left[\|f\|_{L^\infty(0,T;L^1(\mathbb{R}^2))} + \|n_b\|_{L^\infty(0,T;L^1(\mathbb{R}))}\right], \|n_b\|_{L^\infty([0,T]\times\mathbb{R})}, \sup_{t\in[0,T]} TV[f(t)] \right\}.$$

Together with the estimate on ϕ_1, we obtain:

$$\forall E \in X(T), \quad \|\phi_2 \circ \phi_1(E)\|_{X(T)} \leq \max(M_T, M_T', TV[f_0] \, e^{C(E)T}).$$

Now we establish that the mapping $\phi_2 \circ \phi_1$ is Lipschitz continuous in the norm of $L^\infty([0,T] \times \mathbb{R})$. Let E_1, $E_2 \in X(T)$; we denote $f_1 = \phi_1(E_1)$ and $f_2 = \phi_1(E_2)$. Moreover, as we did in the proof of Theorem 14, we approximate f_0 by a sequence $(f_0^n)_n$ whose terms lie in $W^{1,1}(\mathbb{R}^2)$, and such that

$$\lim_{n \longrightarrow +\infty} \|f_0^n - f_0\|_{L^1(\mathbb{R}^2)} = 0 \quad \text{and} \quad \lim_{n \longrightarrow +\infty} TV[f_0^n] = TV[f_0].$$

The solutions to the linear Vlasov equation with field E_1 (resp. E_2) and initial condition f_0^n will be denoted f_1^n (resp. f_2^n). Applying the $W^{1,1}$ estimate (31) to these functions yields:

$$\|f_1^n - f_2^n\|_{L^\infty(0,T;L^1(\mathbb{R}^2))} \leq \|E_1 - E_2\|_{L^\infty(U_T)} \, TV[f_0^n] \, \frac{\exp(C(E_2)T) - 1}{C(E_2)}. \tag{34}$$

As seen in the proof of Theorem 14, we have

$$\|f_i^n(s) - f_i(s)\|_{L^1(\mathbb{R}^2)} = \|f_0^n - f_0\|_{L^1(\mathbb{R}^2)}, \quad \text{for a.e. } s \in [0,T], \text{ and } i = 1, \, 2.$$

Thus, f_i^n converges toward f_i in $L^\infty([0,T]; L^1(\mathbb{R}^2))$. Passing to the limit in (34), we obtain:

$$\|f_1 - f_2\|_{L^\infty(0,T;L^1(\mathbb{R}^2))} \leq \|E_1 - E_2\|_{L^\infty(U_T)} \, TV[f_0] \, \frac{\exp(C(E_2)T) - 1}{C(E_2)}.$$

The L^1 estimate (32) still holds, which finally implies:

$$\|\phi_2 \circ \phi_1(E_1) - \phi_2 \circ \phi_1(E_2)\|_{L^\infty(U_T)} \leq$$
$$\tfrac{1}{2} \|E_1 - E_2\|_{L^\infty(U_T)} \, TV[f_0] \, \frac{\exp(C(E_2)T) - 1}{C(E_2)}. \tag{35}$$

Reasoning like in §3.2.2, we infer that $\phi_2 \circ \phi_1$ admits a unique fixed point in B_R' for suitable values of R and T (using the contraction mapping principle of Banach), then we deduce the local existence and uniqueness of a mild solution to (1–3). The existence time is estimated as in §3.2.3, and the alternative between blow-up and globality is established as in §3.2.4, with obvious adaptations.

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5. Concluding remarks

We have established a result of local existence and uniqueness of a mild solution to the one-dimensional Vlasov–Poisson system. The hypotheses on the data of this problem were improved: the initial data is not assumed to have a compact support, as in [12], or an integrable majorizing function, as in [5, 7, 13], but only to be of bounded variation. Nothing is assumed, either, about its moments in v, unlike most results in higher dimensions [8, 14, 15]. That said, the BV character of the solution implies that the spatial density $\rho[f]$ remains bounded as long as such a solution exists: this property is specific to the dimension one. Therefore, the hypothesis $f_0 \in BV(\mathbb{R}^2)$ is close to the minimal assumption guaranteeing that E and $\partial_x E$ are uniformly bounded, and thus the possibility of the existence of a mild solution in dimension one. Unfortunately, our proof does not extend immediately to higher dimensions.

The drawback is that we were not able to establish global existence. We have seen that this is equivalent to prove that $\rho[f]$ remains bounded on U_T for an arbitrary T. This is where the more restrictive assumptions made in the literature come in. Interestingly, certain results of uniqueness of weak solutions [15, 16] also rest upon the boundedness of the spatial density. The result of this article is, so to say, in the same vein.

Finally, we notice that the arguments presented in this paper can be extended with slight modifications to many-species Vlasov–Poisson systems.

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