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Abstract: We give, in a Markovian set-up, some examples of processes which are increasing in the convex order (we call them peacocks). We then establish some relation between the stochastic and convex orders.
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1 Introduction

1.1 Definitions

We start with a few definitions:
a) A real-valued process $(X_t, t \geq 0)$ is said to be *increasing in the convex order* if:

$$\forall t > 0, \quad \mathbb{E}\left[|X_t|\right] < \infty$$

and, for every convex function $\psi : \mathbb{R} \longrightarrow \mathbb{R}$:

$$t \in \mathbb{R}_+ \longmapsto \mathbb{E}\left[\psi(X_t)\right] \in] - \infty, +\infty] \quad \text{is increasing.} \tag{1}$$

This notion plays an important role in many applied domains of probability; see, e.g. Shaked-Shanthikumar [SS94, SS07]. We call such a process a peacock, an acronym derived from the French term: Processus Croissant pour l'Ordre Convexe. To prove (1), it suffices (see [HPRY, Chapter 1]) to consider only the class $\boldsymbol{C}$ of convex functions ψ such that:

$$\boldsymbol{C} := \{\psi \text{ is a convex function of } \mathcal{C}^2 \text{ class such that } \psi'' \text{ has compact support}\}.$$

Note that if $\psi \in \boldsymbol{C}$, then ψ' is a bounded function.

b) A real-valued process $(X_t, t \geq 0)$ is called a *1-martingale* if there exists a martingale $(M_t, t \geq 0)$ (defined on a suitable filtered probability space) which has the same one-dimensional marginals as $(X_t, t \geq 0)$, that is to say, for each fixed $t \geq 0$:

$$X_t \overset{(\text{law})}{=} M_t.$$

We say that such a martingale is associated to the process $(X_t, t \geq 0)$. From Jensen's inequality, it is clear that a 1-martingale is a peacock. Conversely, a remarkable result

due to Kellerer [Kel72] states that any peacock is a 1-martingale. However, the proofs presented in Kellerer's paper are not constructive, and in general, it is a difficult task to exhibit such a martingale.
In this paper, we shall only tackle the question of exhibiting peacocks, and mainly focus on examples derived from diffusions.

1.2 Some examples

Let $(B_s, s \geq 0)$ be a standard Brownian motion. Carr, Ewald and Xiao [CEX08] proved that the process:

$$\left(A_t := \frac{1}{t} \int_0^t \exp\left(B_s - \frac{s}{2} \right) ds = \int_0^1 \exp\left(B_{ts} - \frac{st}{2} \right) ds \ , \ t \geq 0 \right) \tag{2}$$

is a peacock. Baker-Yor [BY09] then exhibited a martingale which is associated to this peacock, and is constructed from the Wiener sheet. This example was the starting point of many recent developments which we try to synthesize; consider, for every $\lambda \geq 0$, a real-valued measurable process

$$Z_{\lambda,\bullet} := (Z_{\lambda,t}, \ t \geq 0)$$

such that

$$\forall \lambda \in \mathbb{R}_+, \ \forall t \in \mathbb{R}_+, \quad \mathbb{E}\left[e^{Z_{\lambda,t}} \right] < \infty,$$

and define, for any finite and positive measure μ on $\mathbb{R}_+$ the process:

$$\left(A_\lambda^{(\mu)} := \int_0^{+\infty} \frac{e^{Z_{\lambda,t}}}{\mathbb{E}\left[e^{Z_{\lambda,t}} \right]} \mu(dt), \ \lambda \geq 0 \right). \tag{3}$$

(Taking $Z_{\lambda,t} = B_{\lambda t}$ and $\mu(ds) = 1_{[0,1]}(ds)$, we recover (2).)
Now, this raises the following natural question:

Under which conditions is the process $\left(A_\lambda^{(\mu)}, \lambda \geq 0 \right)$ a peacock ?

It is known that $(A_\lambda^{(\mu)}, \lambda \geq 0)$ is a peacock in the following cases:

- $Z_{\lambda,t} = \lambda t X$ with X a r.v., see ([HPRY]),
- $Z_{\lambda,t} = \lambda L_t$ with $(L_t, t \geq 0)$ a Lévy process such that $\mathbb{E}\left[e^{L_1} \right] < \infty$, (see [HRY09b]).
- $Z_{\lambda,t} = G_{\lambda,t}$ with, for every $\lambda \geq 0$, $(G_{\lambda,t}, t \geq 0)$ a Gaussian process such that the function $\lambda \longrightarrow \mathbb{E}\left[G_{\lambda,t} G_{\lambda,s} \right]$ is increasing for every $s, t \geq 0$, (see [HRY09a]).

In this paper, we shall exhibit several other families of peacocks.

In Section 2, we introduce the notion of conditional monotonicity which will lead to a new large class of peacocks.

In Section 3, we give many examples, among which the processes with independent log-concave increments and the "well-reversible" diffusions at fixed times.

In Section 4, we present another condition, this time relying upon Laplace transforms, which implies the peacock property.

Finally, in Section 5, we present a result which links the stochastic and convex orders, and makes it possible to recover some of the peacocks presented above.

2 A class of peacocks under the conditional monotonicity hypothesis

In this section, we introduce and study the notion of conditional monotonicity, which already appear in [SS94, Chapter 4.B, p.114-126].

Definition 2.1 (Conditional monotonicity)**.** *A process* $(X_\lambda, \lambda \geq 0)$ *is said to be conditionally monotone if, for every* $n \in \mathbb{N}^*$*, every* $i \in \{1, \dots, n\}$*, every* $0 < \lambda_1 < \cdots < \lambda_n$ *and every bounded Borel function* $\phi : \mathbb{R}^n \longrightarrow \mathbb{R}$ *which increases (resp. decreases) with respect to each of its arguments, we have:*

$$\mathbb{E}[\phi(X_{\lambda_1}, X_{\lambda_2}, \dots, X_{\lambda_n})|X_{\lambda_i}] = \phi_i(X_{\lambda_i}), \tag{CM}$$

where $\phi_i : \mathbb{R} \longrightarrow \mathbb{R}$ *is a bounded increasing (resp. decreasing) function.*

Remark 2.2.
1) If there is an interval I of $\mathbb{R}$ such that, for every $\lambda \geq 0$, $X_\lambda \in I$, we may assume in Definition 2.1 that ϕ is merely defined on I^n, and ϕ_i is defined on I.
2) Note that $(X_\lambda, \lambda \geq 0)$ is conditionally monotone if and only if $(-X_\lambda, \lambda \geq 0)$ is conditionally monotone.
3) Let $\theta : \mathbb{R} \longrightarrow \mathbb{R}$ be a strictly monotone and continuous function. It is not difficult to see that if the process $(X_\lambda, \lambda \geq 0)$ is conditionally monotone, then so is $(\theta(X_\lambda), \lambda \geq 0)$.
4) We were careful to exclude the point $\lambda_1 = 0$ in this definition. This is explained by the fact that "well-reversible" diffusions (our main class of examples), can be only reversed a priori on $]0, \lambda_0]$: $(\overline{X}^{\lambda_0}_\lambda, 0 \leq \lambda < \lambda_0) := (X_{\lambda_0 - \lambda}, 0 \leq \lambda < \lambda_0)$, see Subsection 3.2.

To prove that a process is conditionally monotone, we can restrict ourselves to bounded Borel functions ϕ increasing with respect to each of their arguments. Indeed, replacing ϕ by $-\phi$, the result then holds also for bounded Borel functions decreasing with respect to each of their arguments.

Definition 2.3. *We denote by* $\mathcal{E}_n$ *the set of bounded Borel functions* $\phi : \mathbb{R}^n \longrightarrow \mathbb{R}$ *which are increasing with respect to each of their arguments.*

Theorem 2.4. *Let* $(X_\lambda, \lambda \geq 0)$ *a real-valued process which is right-continuous, conditionally monotone and which satisfies the following integrability conditions:*
For every compact $K \subset \mathbb{R}_+$ *and every* $t \geq 0$*:*

$$\Theta_{K,t} := \sup_{\lambda \in K} \exp(tX_\lambda) = \exp\left(t \sup_{\lambda \in K} X_\lambda\right) \text{ is integrable,} \tag{INT1}$$

and

$$k_{K,t} := \inf_{\lambda \in K} \mathbb{E}\left[\exp(tX_\lambda)\right] > 0. \tag{INT2}$$

We set $h_\lambda(t) = \log \mathbb{E}[\exp(tX_\lambda)]$*. Then, for every finite positive measure* μ *on* $\mathbb{R}_+$*:*

$$\left(A_t^{(\mu)} := \int_0^\infty e^{tX_\lambda - h_\lambda(t)} \mu(d\lambda)\ ,\ t \geq 0\right)$$

is a peacock.

Proof of Theorem 2.4

1. By (INT1), for every $\lambda \geq 0$ and every $t \geq 0$, $\mathbb{E}\left[\exp(tX_\lambda)\right] < \infty$. This easily implies, thanks to the dominated convergence theorem, that h_λ is continuous on $\mathbb{R}_+$, differentiable on $]0, +\infty[$, and

   $$h'_\lambda(t) e^{h_\lambda(t)} = \mathbb{E}\left[X_\lambda e^{tX_\lambda}\right]. \tag{4}$$

   Since $\mathbb{E}\left[e^{tX_\lambda - h_\lambda(t)}\right] = 1$, we obtain from (4):

   $$\mathbb{E}\left[(X_\lambda - h'_\lambda(t)) e^{tX_\lambda - h_\lambda(t)}\right] = 0. \tag{5}$$

   Moreover, we also deduce from (INT1) that, for every $t \geq 0$, the function $\lambda \geq 0 \longmapsto h_\lambda(t)$ is right-continuous.

2. Let $\varepsilon > 0$ and define $\mu^{(\varepsilon)}$ to be the restriction of μ to the interval $[\varepsilon, +\infty[$: $\mu^{(\varepsilon)} = \mu_{|[\varepsilon,+\infty[}$. We first consider the case

$$\mu^{(\varepsilon)} = \sum_{i=1}^{n} a_i \delta_{\lambda_i} \tag{6}$$

where $n \in \mathbb{N}^*$, $a_1 \geq 0, \ldots, a_n \geq 0$, and $\varepsilon \leq \lambda_1 < \ldots < \lambda_n$. Let $\psi \in \boldsymbol{C}$. For $t > 0$, we have:

$$\frac{\partial}{\partial t}\mathbb{E}\left[\psi\left(A_t^{(\mu^{(\varepsilon)})}\right)\right] = \mathbb{E}\left[\psi'\left(A_t^{(\mu^{(\varepsilon)})}\right)\sum_{i=1}^{n} a_i \left(X_{\lambda_i} - h'_{\lambda_i}(t)\right)\exp\left(tX_{\lambda_i} - h_{\lambda_i}(t)\right)\right]$$

Setting for $i \in \{1, \ldots, n\}$,

$$\Delta_i = \mathbb{E}\left[\psi'\left(A_t^{(\mu^{(\varepsilon)})}\right)\left(X_{\lambda_i} - h'_{\lambda_i}(t)\right)\exp\left(tX_{\lambda_i} - h_{\lambda_i}(t)\right)\right]$$

we shall show that $\Delta_i \geq 0$ for every $i \in \{1, \ldots, n\}$. Note that the function

$$(x_1, \ldots, x_n) \longmapsto \psi'\left(\sum_{j=1}^{n} a_j \exp\left(tx_j - h_{\lambda_j}(t)\right)\right)$$

is bounded and increases with respect to each of its arguments, i.e. belongs to $\mathcal{E}_n$. Hence, from the conditional monotonicity property of $(X_\lambda, \lambda \geq 0)$:

$$\begin{aligned}\Delta_i &= \mathbb{E}\left[\mathbb{E}\left[\psi'\left(A_t^{(\mu^{(\varepsilon)})}\right)(X_{\lambda_i} - h'_{\lambda_i}(t))e^{tX_{\lambda_i} - h_{\lambda_i}(t)}|X_{\lambda_i}\right]\right]\\ &= \mathbb{E}\left[(X_{\lambda_i} - h'_{\lambda_i}(t))e^{tX_{\lambda_i} - h_{\lambda_i}(t)}\phi_i(X_{\lambda_i})\right]\end{aligned}$$

where ϕ_i is a bounded increasing function. Besides, we have,

$$(X_{\lambda_i} - h'_{\lambda_i}(t))\left(\phi_i(X_{\lambda_i}) - \phi_i\left(h'_{\lambda_i}(t)\right)\right) \geq 0.$$

Therefore,

$$\begin{aligned}\Delta_i &\geq \phi_i\left(h'_{\lambda_i}(t)\right)\mathbb{E}\left[(X_{\lambda_i} - h'_{\lambda_i}(t))e^{tX_{\lambda_i} - h_{\lambda_i}(t)}\right]\\ &= 0 \quad \text{from (5).}\end{aligned}$$

3. We now assume that $\mu^{(\varepsilon)}$ has compact support contained in a compact interval K. Since the function $\lambda \longmapsto \exp(tX_\lambda - h_\lambda(t))$ is right-continuous and bounded from above by $k_{K,t}^{-1}\Theta_{K,t}$ which is finite a.s., there exists a sequence $(\mu_n^{(\varepsilon)}, n \geq 0)$ of measures of the form (6), with supp $(\mu_n^{(\varepsilon)}) \subset K$, $\int \mu_n^{(\varepsilon)}(d\lambda) = \int \mu^{(\varepsilon)}(d\lambda)$ and for every $t \geq 0$, $\lim\limits_{n\to+\infty} A_t^{(\mu_n^{(\varepsilon)})} = A_t^{(\mu^{(\varepsilon)})}$ a.s. Moreover, from (INT1) and (INT2):

$$|A_t^{(\mu_n^{(\varepsilon)})}| \leq \frac{\theta_{K,t}}{k_{K,t}}\int \mu(d\lambda),$$

and from Point 2, for $0 \leq s \leq t$:

$$\mathbb{E}[\psi(A_s^{(\mu_n^{(\varepsilon)})})] \leq \mathbb{E}[\psi(A_t^{(\mu_n^{(\varepsilon)})})].$$

Therefore, since ψ is sublinear, we can apply the dominated convergence theorem and pass to the limit when $n \to +\infty$ in this last inequality to obtain that $(A_t^{(\mu^{(\varepsilon)})}, t \geq 0)$ is a peacock.

4. In the general case, we set $\mu_n^{(\varepsilon)}(d\lambda) = 1_{[\varepsilon,n]}(\lambda)\mu(d\lambda)$ and observe that for n increasing and ε decreasing, $A^{(\mu_n^{(\varepsilon)})}$ is an increasing sequence of processes. Let ρ be defined by $\rho(x) = \int_0^x (x-z)\psi''(z)dz$. An integration by parts yields, for $0 \le s \le t$:

$$\begin{aligned}
&\mathbb{E}\left[\psi\left(A_t^{(\mu_n^{(\varepsilon)})}\right)\right] - \mathbb{E}\left[\psi\left(A_s^{(\mu_n^{(\varepsilon)})}\right)\right] \\
&\qquad = \mathbb{E}\left[\psi'(0)\left(A_t^{(\mu_n^{(\varepsilon)})} - A_s^{(\mu_n^{(\varepsilon)})}\right)\right] + \mathbb{E}\left[\rho\left(A_t^{(\mu_n^{(\varepsilon)})}\right)\right] - \mathbb{E}\left[\rho\left(A_s^{(\mu_n^{(\varepsilon)})}\right)\right] \\
&\qquad = \mathbb{E}\left[\rho\left(A_t^{(\mu_n^{(\varepsilon)})}\right)\right] - \mathbb{E}\left[\rho\left(A_s^{(\mu_n^{(\varepsilon)})}\right)\right] \ge 0
\end{aligned}$$

from Point 3), and since $\mathbb{E}\left[A_t^{(\mu_n^{(\varepsilon)})}\right] = \mathbb{E}\left[A_s^{(\mu_n^{(\varepsilon)})}\right]$. Now, since ρ is an increasing function on $\mathbb{R}_+$, the result follows from the monotone convergence theorem, applied first while letting n tend towards $+\infty$, and then while letting ε tend to 0.

□

Remark 2.5. Let $\theta : \mathbb{R} \longrightarrow \mathbb{R}$ be a strictly monotone and continuous function, and μ denote a finite positive measure. From Remark 2.2, under the assumption that $(\theta(X_\lambda), \lambda \ge 0)$ still satisfies conditions (INT1) and (INT2), we obtain, denoting $h_{\lambda,\theta}(t) = \log \mathbb{E}[\exp(t\theta(X_\lambda))]$, that the process

$$\left(A_t^{(\theta,\mu)} := \int_0^\infty e^{t\theta(X_\lambda) - h_{\lambda,\theta}(t)}\mu(d\lambda)\ ,\ t \ge 0\right)$$

is a peacock. Note that θ only needs to be continuous and strictly monotone on an interval containing the image of X_λ for every $\lambda \ge 0$.

Of course, Theorem 2.4 may have some practical interest only if we are able to exhibit enough examples of processes which enjoy the conditional monotonicity (CM) property. Below, we shall see that there exists a large class of diffusions which enjoy this property. But to start with, let us first give a few examples which consist of processes with independent increments and Lévy processes in particular.

3 Examples of processes satisfying the conditional monotonicity property

3.1 Processes with independent increments satisfying the conditional monotonicity property

We start by giving an assertion equivalent to (CM) when dealing with processes with independent (not necessarily time-homogeneous) increments.

Proposition 3.1. *Let $(X_\lambda, \lambda \ge 0)$ be a process with independent increments. Then, the conditional monotonicity hypothesis (CM) is equivalent to the following:*
For every $n \in \mathbb{N}^*$, every $0 < \lambda_1 < \cdots < \lambda_n$ and every function $\phi : \mathbb{R}^n \longrightarrow \mathbb{R}$ in $\mathcal{E}_n$, we have:

$$\mathbb{E}[\phi(X_{\lambda_1}, \ldots, X_{\lambda_n}) | X_{\lambda_n}] = \phi_n(X_{\lambda_n}) \qquad (\widetilde{\text{CM}})$$

where ϕ_n is an increasing bounded function.

Proof of Proposition 3.1
The proof is straightforward. Indeed, let $\phi \in \mathcal{E}_n$. For $i \in \{1, \ldots, n\}$, the hypothesis of

independent increments implies:

$$\begin{aligned}
&\mathbb{E}\left[\phi\left(X_{\lambda_1},\dots,X_{\lambda_n}\right)|X_{\lambda_i}\right]\\
&\quad=\mathbb{E}\left[\mathbb{E}\left[\phi\left(X_{\lambda_1},\dots,X_{\lambda_n}\right)|\mathcal{F}_{\lambda_i}\right]|X_{\lambda_i}\right]\\
&\quad=\mathbb{E}\left[\mathbb{E}\left[\phi\left(X_{\lambda_1},\dots,,X_{\lambda_i},X_{\lambda_{i+1}}-X_{\lambda_i}+X_{\lambda_i},\dots,X_{\lambda_n}-X_{\lambda_i}+X_{\lambda_i}\right)|\mathcal{F}_{\lambda_i}\right]|X_{\lambda_i}\right]\\
&\quad=\mathbb{E}\left[\widetilde{\phi}\left(X_{\lambda_1},\dots,X_{\lambda_i}\right)|X_{\lambda_i}\right]
\end{aligned}$$

where

$$\widetilde{\phi}(x_1,\dots,x_i)=\mathbb{E}\left[\phi\left(x_1,\dots,x_i,X_{\lambda_{i+1}}-X_{\lambda_i}+x_i,,\dots,X_{\lambda_n}-X_{\lambda_i}+x_i\right)\right]$$

belongs to $\mathcal{E}_i$.

□

3.1.1 The Gamma subordinator is conditionally monotone

The Gamma subordinator $(\gamma_\lambda,\lambda\geq 0)$ is characterized by:

$$\mathbb{E}\left[e^{-t\gamma_\lambda}\right]=\frac{1}{(1+t)^\lambda}=\exp\left(-\lambda\int_0^\infty(1-e^{-tx})\frac{e^{-x}}{x}dx\right).$$

In particular, γ_λ is a gamma random variable with parameter λ. From $(\widetilde{\text{CM}})$, we wish to show that for every $n\in\mathbb{N}^*$, every $0<\lambda_1<\dots<\lambda_n$ and every function $\phi:\mathbb{R}^n\longrightarrow\mathbb{R}$ in $\mathcal{E}_n$:

$$\mathbb{E}[\phi(\gamma_{\lambda_1},\dots,\gamma_{\lambda_n})|\gamma_{\lambda_n}]=\phi_n(\gamma_{\lambda_n}),\tag{7}$$

where ϕ_n is an increasing function. The explicit knowledge of the law of γ_λ and the fact that $(\gamma_\lambda,\lambda\geq 0)$ has time-homogeneous independent increments imply the well-known result that, given $\{\gamma_{\lambda_n}=x\}$, the vector $(\gamma_{\lambda_1},\gamma_{\lambda_2}-\gamma_{\lambda_1},\dots,\gamma_{\lambda_n}-\gamma_{\lambda_{n-1}})$ follows the Dirichlet law with parameters $(\lambda_1,\lambda_2-\lambda_1,\dots,\lambda_n-\lambda_{n-1})$ on $[0,x]$. In other words, the density f_n of $(\gamma_{\lambda_1},\gamma_{\lambda_2},\dots,\gamma_{\lambda_{n-1}})$ conditionally on $\{\gamma_{\lambda_n}=x\}$ equals:

$$\begin{aligned}
f_n(x_1,\dots,x_{n-1})=&\frac{C}{x^{\lambda_n-1}}x_1^{\lambda_1-1}(x_2-x_1)^{\lambda_2-\lambda_1-1}\dots\\
&(x_{n-1}-x_{n-2})^{\lambda_{n-1}-\lambda_{n-2}-1}(x-x_{n-1})^{\lambda_n-\lambda_{n-1}-1}1_{\mathbb{S}^{n,x}},
\end{aligned}$$

where $C:=C(\lambda_1,\dots,\lambda_n)$ is a positive constant and

$$\mathbb{S}^{n,x}=\{(x_1,\dots,x_{n-1})\in\mathbb{R}^{n-1}:\,0\leq x_1\leq\dots\leq x_{n-1}\leq x\}.$$

Hence,

$$\begin{aligned}
&\mathbb{E}[\phi(\gamma_{\lambda_1},\dots,\gamma_{\lambda_n})|\gamma_{\lambda_n}=x]\\
&\qquad=\int_{\mathbb{S}^{n,x}}\phi(x_1,\dots,x_{n-1},x)f(x_1,\dots,x_{n-1})dx_1\dots dx_{n-1}\\
&\qquad=C\int_{\mathbb{S}^{n,1}}\phi(xy_1,\dots,xy_{n-1},x)y_1^{\lambda_1-1}(y_2-y_1)^{\lambda_2-\lambda_1-1}\dots\\
&\qquad\quad(y_{n-1}-y_{n-2})^{\lambda_{n-1}-\lambda_{n-2}-1}(1-y_{n-1})^{\lambda_n-\lambda_{n-1}-1}dy_1\dots dy_{n-1}
\end{aligned}$$

after the change of variables: $x_i=xy_i$, $i=1,\dots,n-1$. It is then clear that since ϕ increases with respect to each of its arguments, this last expression is an increasing function with respect to x.

Corollary 3.2. *Let $(\gamma_\lambda,\lambda\geq 0)$ be the gamma subordinator. Then, for every finite positive measure μ on $\mathbb{R}_+$, and for every $p>0$, the process:*

$$\left(A_t^{(\mu,p)}:=\int_0^\infty e^{-t(\gamma_\lambda)^p-h_{\lambda,p}(t)}\mu(d\lambda)\ ,\ t\geq 0\right)\tag{8}$$

is a peacock. Here, the function $h_{\lambda,p}$ is defined as:

$$h_{\lambda,p}(t)=\log\mathbb{E}\left[\exp\left(-t(\gamma_\lambda)^p\right)\right].$$

Proof of Corollary 3.2
By Remark 2.5 with $\theta(x) = -x^p$ for $x \geq 0$, the process $(X_\lambda := -\gamma_\lambda^p, \lambda \geq 0)$ is conditionally monotone. Since it is a negative process, (INT1) is obviously satisfied. Moreover, since $(\gamma_\lambda, \lambda \geq 0)$ is an increasing process, (INT2) is easily verified. Finally, Theorem 2.4 holds. $\square$

Remark 3.3. Actually, for $p = 1$, Corollary 3.2 holds more generally with μ a signed measure, see [HRY09b].

3.1.2 The simple random walk is conditionally monotone

Let $(\varepsilon_i, i \in \mathbb{N}^*)$ be a sequence of independent and identically distributed r. v.'s such that, for every $i \in \mathbb{N}^*$:

$$\mathbb{P}(\varepsilon_i = 1) = p, \quad \mathbb{P}(\varepsilon_i = -1) = q \qquad \text{with } p, q > 0 \text{ and } p + q = 1.$$

Let $(S_n, n \in \mathbb{N})$ be the random walk defined by: $S_0 = 0$ and

$$S_n = \sum_{i=1}^{n} \varepsilon_i, \qquad \text{for every } n \in \mathbb{N}^*$$

We shall prove that $(S_n, n \in \mathbb{N})$ is conditionally monotone; i.e: for every $r \in [\![2, +\infty[\![$, every $0 < n_1 < n_2 < \cdots < n_r < +\infty$ and every function $\phi : \mathbb{R}^{r-1} \longrightarrow \mathbb{R}$ in $\mathcal{E}_{r-1}$,

$$k \in I_{n_r} \longmapsto \mathbb{E}[\phi(S_{n_1}, S_{n_2}, \dots, S_{n_{r-1}})|S_{n_r} = k] \text{ is an increasing function on } I_{n_r} \tag{9}$$

where $I_x \subset [\![-x, x]\!]$ denotes the set of all the values the r.v. S_x can take. It is not difficult to see that (9) holds if and only if: for every $N \in [\![2, +\infty[\![$ and every function $\phi : \mathbb{R}^{N-1} \longrightarrow \mathbb{R}$ in $\mathcal{E}_{N-1}$:

$$k \in I_N \longmapsto \mathbb{E}[\phi(S_1, \dots, S_{N-1})|S_N = k] \quad \text{is an increasing function on } I_N. \tag{10}$$

We shall distinguish two cases:
1) If N and k are even, we set $N = 2n\,(n \in [\![1, +\infty[\![)$ and $k = 2x\,(x \in [\![-n, n]\!])$. For every $n \in [\![1, +\infty[\![$ and every $x \in [\![-n, n]\!]$, let us denote by $\mathcal{J}_{2n}^{2x}$, the set of polygonal lines $\omega := (\omega_i, i \in [\![0, 2n]\!])$ such that $\omega_0 = 0$, $\omega_{p+1} = \omega_p \pm 1$, $(p \in [\![0, 2n-1]\!])$ and $\omega_{2n} = 2x$. Observe that any $\omega \in \mathcal{J}_{2n}^{2x}$ has $n + x$ positive slopes and $n - x$ negative ones. This implies that:

$$|\mathcal{J}_{2n}^{2x}| = C_{2n}^{n+x},$$

where $|\cdot|$ denotes cardinality. It is well known that, conditionally on $\{S_{2n} = 2x\}$, the law of the random vector $(S_1, S_2, \dots, S_{2n})$ is the uniform law on $\mathcal{J}_{2n}^{2x}$.

Let $n \in [\![1, +\infty[\![$ and $x \in [\![-n, n]\!]$ be fixed and consider, for every $i \in [\![1, n + x + 1]\!]$ the map:

$$\Pi_i : \mathcal{J}_{2n}^{2x+2} \longrightarrow \mathcal{J}_{2n}^{2x}$$

defined by: for every $\omega \in \mathcal{J}_{2n}^{2x+2}$, $\Pi_i(\omega)$ has the same negative slopes and the same positive slopes as ω except the i^{th} positive slope which is replaced by a negative one. For every $\omega \in \mathcal{J}_{2n}^{2x+2}$ and every function $\phi : \mathbb{R}^{2n} \longrightarrow \mathbb{R}$ in $\mathcal{E}_{2n}$,

$$\phi(\omega) \geq \phi(\Pi_i(\omega)).$$

Summing this relation, we obtain:

$$\begin{aligned}
(n + x + 1) \sum_{\omega \in \mathcal{J}_{2n}^{2x+2}} \phi(\omega) &\geq \sum_{\omega \in \mathcal{J}_{2n}^{2x+2}} \sum_{i=1}^{n+x+1} \phi(\Pi_i(\omega)) \\
&= \sum_{\omega \in \mathcal{J}_{2n}^{2x}} \sum_{i=1}^{n+x+1} \Pi_i^{-1}(\omega)\phi(\omega) \\
&= (n - x) \sum_{\omega \in \mathcal{J}_{2n}^{2x}} \phi(\omega).
\end{aligned}$$

Thus, we have proved the following:

Lemma 3.4. *For every $n \in N^*$ and every $\phi : \mathbb{R}^{2n} \longrightarrow \mathbb{R}$ in $\mathcal{E}_{2n}$,*

$$\frac{1}{|\mathcal{J}_{2n}^{2x+2}|} \sum_{\omega \in \mathcal{J}_{2n}^{2x+2}} \phi(\omega) \geq \frac{1}{|\mathcal{J}_{2n}^{2x}|} \sum_{\omega \in \mathcal{J}_{2n}^{2x}} \phi(\omega), \tag{11}$$

which means that $(S_n, n \in \mathbb{N})$ is conditionally monotone.

2) It is not difficult to establish a similar result when k and N are odd.

Corollary 3.5. *For every odd and positive integer p, and for every positive finite measure $\sum_{n\in\mathbb{N}} a_n \delta_n$ on $\mathbb{N}$:*

$$\left(\sum_{n=0}^{+\infty} a_n e^{-t(S_n)^p - h_{n,p}(t)}, t \geq 0\right) \quad \text{is a peacock.}$$

Here, the function $h_{n,p}$ is defined by: $h_{n,p}(t) = \log \mathbb{E}\left[\exp\left(-t(S_n)^p\right)\right]$.

3.1.3 The processes with independent log-concave increments are conditionally monotone

We first introduce the notions of PF_2 and log-concave random variables (see [DS96]).

Definition 3.6 ($\mathbb{R}$-valued PF_2 r.v.'s)**.**
An $\mathbb{R}$-valued random variable X is said to be PF_2 if:

1) X admits a probability density f,

2) for every $x_1 \geq x_2$, $y_2 \geq y_1$,

$$\det \begin{pmatrix} f(x_1 - y_1) & f(x_1 - y_2) \\ f(x_2 - y_1) & f(x_2 - y_2) \end{pmatrix} \geq 0.$$

Definition 3.7 ($\mathbb{Z}$-valued PF_2 r.v.'s)**.**
A $\mathbb{Z}$-valued random variable X is said to be PF_2 if, setting $f(x) = \mathbb{P}(X = x)$ $(x \in \mathbb{Z})$, one has: for every $x_1 \geq x_2$, $y_2 \geq y_1$,

$$\det \begin{pmatrix} f(x_1 - y_1) & f(x_1 - y_2) \\ f(x_2 - y_1) & f(x_2 - y_2) \end{pmatrix} \geq 0.$$

Definition 3.8 ($\mathbb{R}$-valued log-concave r.v.'s)**.**
An $\mathbb{R}$-valued random variable X is said to be log-*concave if:*

1 X admits a probability density f,

2 the function $\log f$ is concave; i.e., the second derivative of $\log f$ (in the distribution sense) is a negative measure.

Definition 3.9 ($\mathbb{Z}$-valued log-concave r.v.'s)**.**
A $\mathbb{Z}$-valued random variable X is said to be log-*concave if, with $f(x) = \mathbb{P}(X = x)$ $(x \in \mathbb{Z})$, one has: for every $n, n-1, n+1 \in \mathbb{Z}$,*

$$f^2(n) \geq f(n-1)f(n+1);$$

in other words, the discrete second derivative of $\log f$ is negative.

The following characterization of PF_2 random variables is well-known (see [DS96]).

Lemma 3.10. *An $\mathbb{R}$-valued (resp. $\mathbb{Z}$-valued) random variable X is PF_2 if and only if its probability density f satifies:*

1) The support of f is an (finite or infinite) interval $I \subset \mathbb{R}$ (resp. $I \subset \mathbb{Z}$),

2) $\log f$ is concave on I (resp. for every $n, n-1, n+1 \in I$, $f^2(n) \geq f(n-1)f(n+1)$).

We thus easily deduce the equivalence:

Theorem 3.11 (see [An97] or [DS96])**.** *An $\mathbb{R}$-valued (or $\mathbb{Z}$-valued) random variable is PF_2 if and only if it is $\log$-concave.*

Example and Counterexample 3.12. Many common density functions on $\mathbb{R}$ (or $\mathbb{Z}$) are PF_2. Indeed, the normal density, the uniform density, the exponential density, the negative binomial density, the Poisson density and the geometric density are PF_2. We refer to [An97] for more examples. Note that:

a) A Gamma random variable of parameter a (with density $f_a(x) = \dfrac{1}{\Gamma(a)} e^{-x} x^{a-1} 1_{[0,+\infty[}(x)$, $a > 0$) is not PF_2 if $a < 1$,

b) A Bernoulli random variable X such that $\mathbb{P}(X = 1) = p = 1 - \mathbb{P}(X = -1)$ is not PF_2.

The following result is due to Efron [Efr65] (see also [Sha87]).

Theorem 3.13. *Let $n \in [\![1, +\infty[\![$, $X_1, X_2, \dots, X_n$ be independent $\mathbb{R}$-valued (or $\mathbb{Z}$-valued) PF_2 random variables, $S_n = \sum_{i=1}^n X_i$, and $\phi : \mathbb{R}^n \to \mathbb{R}$ belonging to $\mathcal{E}_n$. Then,*

$$\mathbb{E}[\phi(X_1, X_2, \dots, X_n) | S_n = x] \quad \textit{is increasing in } x.$$

Thanks to Theorem 3.13, we obtain the following result:

Theorem 3.14. *Let $(Z_\lambda, \lambda \in \mathbb{R}_+$ or $\lambda \in \mathbb{N})$ be a $\mathbb{R}$-valued (or $\mathbb{Z}$-valued) process satisfying (INT1) and (INT2), with independent (not necessarily time-homogeneous) PF_2 increments. Then, $(Z_\lambda, \lambda \geq 0)$ is conditionally monotone, and for every positive measure μ on $\mathbb{R}_+$ (or $\mathbb{N}$) with finite total mass,*

$$\left(\int_0^{+\infty} e^{tZ_\lambda - h_\lambda(t)} \mu(d\lambda), t \geq 0 \right) \quad \textit{is a peacock,}$$

where the function h_λ is defined by: $h_\lambda(t) = \log \mathbb{E}\left[e^{tZ_\lambda}\right]$.

Proof of Theorem 3.14

It suffices to show that $(Z_\lambda, \lambda \geq 0)$ satisfies $(\widetilde{CM})$. Let $n \in [\![1, +\infty[\![$ and $\phi : \mathbb{R}^n \to \mathbb{R}$ belonging to $\mathcal{E}_n$. For every $0 < \lambda_1 < \lambda_2 < \dots < \lambda_n$ and $k \in \mathbb{R}$ (or $\mathbb{Z}$),

$$\mathbb{E}[\phi(Z_{\lambda_1}, Z_{\lambda_2}, \dots, Z_{\lambda_n}) | Z_{\lambda_n} = k] = \mathbb{E}\left[\widehat{\phi}(Z_{\lambda_1}, Z_{\lambda_2} - Z_{\lambda_1}, \dots, Z_{\lambda_n} - Z_{\lambda_{n-1}}) | Z_{\lambda_n} = k\right],$$

where the function $\widehat{\phi}$ is given by:

$$\widehat{\phi}(x_1, x_2, \dots, x_n) = \phi(x_1, x_2 + x_1, \dots, x_1 + x_2 + \dots + x_n).$$

It is obvious that $\widehat{\phi}$ belongs to $\mathcal{E}_n$. Thus, applying Theorem 3.13 with: $X_1 = Z_{\lambda_1}$ and $X_{i+1} = Z_{\lambda_{i+1}} - Z_{\lambda_i}$ $i = 1, \dots n-1$, one obtains the desired result.

□

Remark 3.15.

1) Theorem 3.14 does not apply neither in the case of the Gamma subordinator, nor in the case of the random walk whose increments are Bernoulli with values in $\{-1, 1\}$. Nevertheless, its conclusion remains true in these cases, see Subsections 3.1.1 and 3.1.2.

2) We deduce from Corollary 3.14 that the Poisson process and the random walk with geometric increments are conditionally monotone. We shall give below a direct proof, i.e. without using Theorem 3.13.

3.1.4 The Poisson process is conditionally monotone

Let $(N_\lambda, \lambda \geq 0)$ be a Poisson process with parameter 1 and let $(T_n, n \geq 1)$ be its successive jumps times. Then

$$N_\lambda = \#\{i \geq 1 : T_i \leq \lambda\}.$$

In order to prove that $(N_\lambda, \lambda \geq 0)$ is conditionally monotone, we shall show that for every $\lambda_1 < \cdots < \lambda_n$ and every function $\phi : \mathbb{R}^n \longrightarrow \mathbb{R}$ in $\mathcal{E}_n$, we have:

$$\mathbb{E}[\phi(N_{\lambda_1}, \dots, N_{\lambda_n})|N_{\lambda_n}] = \phi_n(N_{\lambda_n}), \tag{12}$$

where $\phi_n : \mathbb{R} \longrightarrow \mathbb{R}$ increases. But, conditionally on $\{N_{\lambda_n} = k\}$, the random vector $(T_1, \dots, T_k)$ is distributed as $(U_1, \dots, U_k)$, $U_1, \dots, U_k$ being the increasing rearrangement of k independent random variables, uniformly distributed on $[0, \lambda_n]$. We go from k to $k+1$ by adding one more point. Thus, with obvious notation, it is clear that: for all $\lambda \in [0, \lambda_n]$, $N_\lambda^{(k+1)} \geq N_\lambda^{(k)}$. Then, the conditional monotonicity property follows immediately.

Corollary 3.16. *Let $(N_\lambda, \lambda \geq 0)$ be a Poisson process and let μ be a finite positive measure on $\mathbb{R}_+$. Then, for every $p > 0$, the process:*

$$\left(A_t^{(\mu,p)} := \int_0^\infty e^{-t(N_\lambda)^p - h_{\lambda,p}(t)} \mu(d\lambda), t \geq 0\right) \tag{13}$$

is a peacock with:

$$h_{\lambda,p}(t) = \log \mathbb{E}\left[\exp\left(-t(N_\lambda)^p\right)\right].$$

3.1.5 The random walk with geometric increments is conditionally monotone

Let $(\varepsilon_i, i \in [\![1, +\infty[\![)$ be a sequence of independent geometric variables with the same parameter p; i.e, such that:

$$\mathbb{P}(\varepsilon_i = k) = p^k(1-p) \qquad (k \geq 0,\ 0 < p < 1).$$

We consider the random walk $(S_n, n \in \mathbb{N})$ defined by: $S_0 = 0$ and

$$S_n = \sum_{i=1}^n \varepsilon_i, \text{ for every } n \in \mathbb{N}^*.$$

For $n \in \mathbb{N}^*$, S_n is distributed as a negative binomial random variable with parameters n and p; more precisely:

$$\mathbb{P}(S_n = k) = C_{n+k-1}^k p^n (1-p)^k, \text{ for every } k \in \mathbb{N}.$$

As in Subsection 3.1.2, we only need to prove that: for every $N \in \mathbb{N}^*$ and every function $\phi : \mathbb{R}^N \longrightarrow \mathbb{R}$ in $\mathcal{E}_N$:

$$k \longmapsto \mathbb{E}[\phi(S_1, \dots, S_N)|S_{N+1} = k] \quad \text{ is an increasing function on } \mathbb{N}. \tag{14}$$

Let $\mathbf{J}_N^k$ denote the set:

$$\mathbf{J}_N^k := \{(x_1, \dots, x_N) \in \mathbb{N}^N : 0 \leq x_1 \leq \cdots \leq x_N \leq k\}. \tag{15}$$

For every $k \geq 0$ and $N \geq 1$, it is well known that $|\mathbf{J}_N^k| = C_{N+k}^k$. Now, we have:

$$\begin{aligned}
&\mathbb{E}\left[\phi\left(S_1,\ldots,S_N\right)|S_{N+1}=k\right] \\
=&\sum_{(l_1,\ldots,l_N)\in\mathbf{J}_N^k}\phi\left(l_1,\ldots,l_N\right)\frac{\mathbb{P}(S_1=l_1,\ldots,S_N=l_N,S_{N+1}=k)}{\mathbb{P}(S_{N+1}=k)} \\
=&\sum_{(l_1,\ldots,l_N)\in\mathbf{J}_N^k}\phi\left(l_1,\ldots,l_N\right)\frac{\mathbb{P}(S_1=l_1,S_2-S_1=l_2-l_1,\ldots,S_{N+1}-S_N=k-l_N)}{\mathbb{P}(S_{N+1}=k)} \\
=&\sum_{(l_1,\ldots,l_N)\in\mathbf{J}_N^k}\phi\left(l_1,\ldots,l_N\right)\frac{\mathbb{P}(S_1=l_1)\mathbb{P}(S_2-S_1=l_2-l_1)\ldots\mathbb{P}(S_{N+1}-S_N=k-l_N)}{\mathbb{P}(S_{N+1}=k)} \\
=&\sum_{(l_1,\ldots,l_N)\in\mathbf{J}_N^k}\phi\left(l_1,\ldots,l_N\right)\frac{p(1-p)^{l_1}p(1-p)^{l_2-l_1}\ldots p(1-p)^{k-l_N}}{C_{N+k}^k p^{N+1}(1-p)^k} \\
=&\frac{1}{C_{N+k}^k}\sum_{(l_1,\ldots,l_N)\in\mathbf{J}_N^k}\phi\left(l_1,\ldots,l_N\right) \\
=&\frac{1}{|\mathbf{J}_N^k|}\sum_{(l_1,\ldots,l_N)\in\mathbf{J}_N^k}\phi\left(l_1,\ldots,l_N\right).
\end{aligned}$$

Therefore, the law of the random vector $(S_1,\ldots,S_N)$ conditionally on $\{S_{N+1}=k\}$ is the uniform law on the set $\mathbf{J}_N^k$. Hence, we will obtain (14) if we prove that: for every $k \in \mathbb{N}$, every $N \in \mathbb{N}^*$ and every function $\phi : \mathbb{R}^N \longrightarrow \mathbb{R}_+$ in $\mathcal{E}_N$:

$$\frac{1}{|\mathbf{J}_N^k|}\sum_{x\in\mathbf{J}_N^k}\phi(x) \leq \frac{1}{|\mathbf{J}_N^{k+1}|}\sum_{x\in\mathbf{J}_N^{k+1}}\phi(x). \tag{16}$$

Let us notice that:

$$\mathbf{J}_N^0 = \{(\underbrace{0,\ldots,0}_{N \text{ times}})\}, \qquad \text{for every } N \in [\![1,+\infty[\![$$

and

$$\mathbf{J}_1^k = \{(0),(1),\ldots,(k)\}, \qquad \text{for every } k \in [\![0,+\infty[\![.$$

For $k \in [\![0,+\infty[\![$ and $N \in [\![1,+\infty[\![$, we define:

$$\mathbf{\Delta}_N^{k+1} := \mathbf{J}_N^{k+1} \setminus \mathbf{J}_N^k = \{(x_1,\ldots,x_N) \in \mathbf{J}_N^{k+1} : x_N = k+1\}. \tag{17}$$

and set $\mathbf{\Delta}_N^0 = \emptyset$. By Pascal's formula,

$$|\mathbf{\Delta}_N^{k+1}| = C_{k+1+N}^{k+1} - C_{k+N}^k = C_{N+k}^{k+1} = |\mathbf{J}_{N-1}^{k+1}|, \quad (\text{with } N \in [\![2,+\infty[\![).$$

On one hand, we consider, for $N \in [\![2,+\infty[\![$, the map $\Gamma : \mathbf{J}_{N-1}^{k+1} \longrightarrow \mathbf{\Delta}_N^{k+1}$ defined by:

$$\Gamma[(x_1,\ldots,x_{N-1})] = (x_1,\ldots,x_{N-1},k+1). \tag{18}$$

The map Γ is bijective, and for every non empty pair of subsets G and H of $\mathbf{J}_{N-1}^{k+1}$, there is the equivalence:

$$\left\{\begin{array}{l}\forall f : \mathbb{R}^{N-1} \longrightarrow \mathbb{R} \in \mathcal{E}_{N-1}, \\[2ex] \dfrac{1}{|G|}\sum\limits_{x\in G} f(x) \leq \dfrac{1}{|H|}\sum\limits_{x\in H} f(x)\end{array}\right. \iff \left\{\begin{array}{l}\forall \phi : \mathbb{R}^N \longrightarrow \mathbb{R} \in \mathcal{E}_N, \\[2ex] \dfrac{1}{|\Gamma(G)|}\sum\limits_{z\in\Gamma(G)}\phi(z) \leq \dfrac{1}{|\Gamma(H)|}\sum\limits_{z\in\Gamma(H)}\phi(z)\end{array}\right.$$

On the other hand, for $N \in [\![2,+\infty[\![$, let $\Lambda : \mathbf{\Delta}_N^k \longrightarrow \mathbf{\Delta}_N^{k+1}$ be the injection given by:

$$\Lambda[(x_1,\ldots,x_{N-1},k)] = (x_1,\ldots,x_{N-1},k+1). \tag{19}$$

For every $z \in \mathbf{\Delta}_N^k$ and function $\phi : \mathbb{R}^N \longrightarrow \mathbb{R}$ in $\mathcal{E}_N$,

$$\phi(z) \leq \phi(\Lambda(z)).$$

Therefore, for every non empty subset K of $\mathbf{\Delta}_N^k$,

$$\frac{1}{|K|} \sum_{z \in K} \phi(z) \leq \frac{1}{|\Lambda(K)|} \sum_{u \in \Lambda(K)} \phi(u). \tag{20}$$

since $|K| = |\Lambda(K)|$. Furthermore, one notices that:

$$\Gamma^{-1}[\Lambda(\mathbf{\Delta}_N^k)] = \mathbf{J}_{N-1}^k \quad \text{and} \quad \Gamma^{-1}(\mathbf{\Delta}_N^{k+1}) = \mathbf{J}_{N-1}^{k+1}$$

where Γ^{-1} denotes the inverse map of Γ. Hence, the following is easily obtained:

Lemma 3.17. *Let $k \in [\![1, +\infty[\![$ and $N \in [\![2, +\infty[\![$. Assume that for every function $f : \mathbb{R}^{N-1} \longrightarrow \mathbb{R}$ in $\mathcal{E}_{N-1}$:*

$$\frac{1}{|\mathbf{J}_{N-1}^k|} \sum_{x \in \mathbf{J}_{N-1}^k} f(x) \leq \frac{1}{|\mathbf{J}_{N-1}^{k+1}|} \sum_{x \in \mathbf{J}_{N-1}^{k+1}} f(x). \tag{21}$$

Then, for every function $\phi : \mathbb{R}^N \to \mathbb{R}$ in $\mathcal{E}_N$,

$$\frac{1}{|\mathbf{\Delta}_N^k|} \sum_{y \in \mathbf{\Delta}_N^k} \phi(y) \leq \frac{1}{|\mathbf{\Delta}_N^{k+1}|} \sum_{y \in \mathbf{\Delta}_N^{k+1}} \phi(y). \tag{22}$$

Now, we are able to prove (16) by induction on $N \in [\![1, +\infty[\![$ and $k \in [\![0, +\infty[\![$.

Proposition 3.18. *Let $k \in [\![0, +\infty[\![$, $N \in [\![1, +\infty[\![$ and let $\phi : \mathbb{R}^N \to \mathbb{R}$ be any function in $\mathcal{E}_N$. Then,*

$$\frac{1}{|\mathbf{J}_N^k|} \sum_{z \in \mathbf{J}_N^k} \phi(z) \leq \frac{1}{|\mathbf{J}_N^{k+1}|} \sum_{z \in \mathbf{J}_N^{k+1}} \phi(z); \tag{23}$$

in other words, $(S_n, n \in \mathbb{N})$ is conditionally monotone.

Proof of Proposition 3.18

1) It is obvious that (23) holds for $(k, N) \in [\![0, +\infty[\![\times \{1\}$, and for $(k, N) \in \{0\} \times [\![1, +\infty[\![$.
2) Let $(k, N) \in [\![1, +\infty[\![\times [\![2, +\infty[\![$. We assume that:

$$\forall\ (l, m) \in \mathcal{D} := [\![0, k-1]\!] \times [\![1, +\infty[\![\cup \{k\} \times [\![1, N-1]\!]$$

and any function $f : \mathbb{R}^m \to \mathbb{R}$ in $\mathcal{E}_m$:

$$\frac{1}{|\mathbf{J}_m^l|} \sum_{x \in \mathbf{J}_m^l} f(x) \leq \frac{1}{|\mathbf{J}_m^{l+1}|} \sum_{x \in \mathbf{J}_m^{l+1}} f(x). \tag{IH}$$

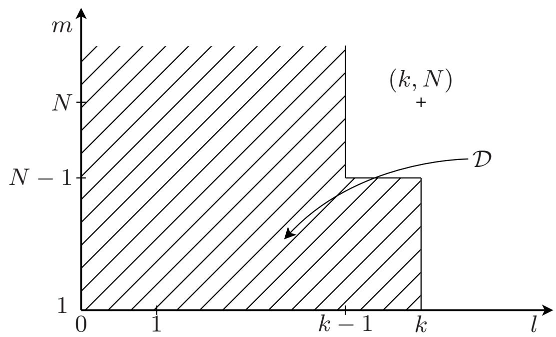

Fig.1 $\quad \mathcal{D} := [\![0, k-1]\!] \times [\![1, +\infty[\![\cup \{k\} \times [\![1, N-1]\!]$

By taking $(l,m)=(k,N-1)$ in (IH), lemma (3.17) yields:

$$\frac{1}{|\mathbf{\Delta}_N^k|}\sum_{y\in\mathbf{\Delta}_N^k}\phi(y)\leq\frac{1}{|\mathbf{\Delta}_N^{k+1}|}\sum_{y\in\mathbf{\Delta}_N^{k+1}}\phi(y). \tag{24}$$

On the other hand, from the definition of $\mathbf{\Delta}_N^{k+1}$, (23) is equivalent to:

$$\frac{1}{|\mathbf{J}_N^k|}\sum_{y\in\mathbf{J}_N^k}\phi(y)\leq\frac{1}{|\mathbf{\Delta}_N^{k+1}|}\sum_{y\in\mathbf{\Delta}_N^{k+1}}\phi(y). \tag{25}$$

Using (IH) with $(l,m)=(k-1,N)$, we have:

$$\frac{1}{|\mathbf{J}_N^k|}\sum_{y\in\mathbf{J}_N^k}\phi(y)\leq\frac{1}{|\mathbf{\Delta}_N^{k}|}\sum_{y\in\mathbf{\Delta}_N^{k}}\phi(y). \tag{26}$$

The comparison of (24) with (26) yields (25) which is equivalent to (23).

□

Corollary 3.19. *For every positive finite measure* $\sum_{n\in\mathbb{N}} a_n\delta_n$ *on* $\mathbb{N}$ *and every* $p>0$ *:*

$$\left(\sum_{n=0}^{+\infty} a_n e^{-t(S_n)^p-h_{n,p}(t)}, t\geq 0\right) \quad \textit{is a peacock,}$$

where the function $h_{n,p}$ *is defined by:* $h_{n,p}(t)=\log\mathbb{E}\left[\exp\left(-t(S_n)^p\right)\right]$.

Remark 3.20. The result in this example has to be compared with that of Subsection 3.1.1: we replace the gamma r.v's by geometric ones.

3.2 Diffusions which are "well-reversible" at fixed times are conditionally monotone.

Let us now present an important class of conditionally monotone processes: that of the "well-reversible" diffusions at a fixed time.

3.2.1 The diffusion $(X_\lambda,\lambda\geq 0;\mathbb{P}_x,x\in\mathbb{R})$

Let $\sigma:\mathbb{R}_+\times\mathbb{R}\to\mathbb{R}$ and $b:\mathbb{R}_+\times\mathbb{R}\to\mathbb{R}$ two Borel measurable functions and let $(B_u,u\geq 0)$ a standard Brownian motion starting from 0. We consider the SDE:

$$X_\lambda=x+\int_0^\lambda\sigma(s,X_s)\,dB_s+\int_0^\lambda b(s,X_s)\,ds,\ \lambda\geq 0. \tag{27}$$

We assume that:

(A1) For every $x\in\mathbb{R}$, this SDE admits a unique pathwise solution $(X_\lambda^{(x)},\lambda\geq 0)$, and furthermore the mapping $x\longmapsto(X_\lambda^{(x)},\lambda\geq 0)$ may be chosen measurable.

As a consequence of (A1), from Yamada-Watanabe's theorem, $(X_\lambda^{(x)},\lambda\geq 0)$ is a strong solution of equation (27), and it enjoys the strong Markov property; finally the transition kernel $P_\lambda(x,dy)=\mathbb{P}(X_\lambda^{(x)}\in dy)$ is measurable.
We now remark that, for $x\leq y$, the process $(X_\lambda^{(y)},\lambda\geq 0)$ is stochastically greater than $(X_\lambda^{(x)},\lambda\geq 0)$ in the following sense: for every $a\in\mathbb{R}$ and $\lambda\geq 0$,

$$\mathbb{P}\left(X_\lambda^{(y)}\geq a\right)\geq\mathbb{P}\left(X_\lambda^{(x)}\geq a\right). \tag{28}$$

Indeed, assuming that both $(X_\lambda^{(x)}, \lambda \geq 0)$ and $(X_\lambda^{(y)}, \lambda \geq 0)$ are defined on the same probability space, and setting

$$T = \inf\{\lambda \geq 0;\, X_\lambda^{(x)} = X_\lambda^{(y)}\}$$
$$(= +\infty \text{ if } \{\lambda \geq 0;\, X_\lambda^{(x)} = X_\lambda^{(y)}\} = \emptyset),$$

it is clear that, on $\{T = +\infty\}$,

$$X_\lambda^{(y)} \geq X_\lambda^{(x)} \qquad \text{(since } y \geq x)$$

while on $\{T < +\infty\}$, we have:

$$X_\lambda^{(y)} > X_\lambda^{(x)} \qquad \text{for every } \lambda \in [0, T[$$

and

$$X_\lambda^{(y)} = X_\lambda^{(x)} \qquad \text{for every } \lambda \in [T, +\infty[$$

since, as a consequence of our hypothesis (A1), (27) admits a unique strong Markovian solution.
On the other hand, (28) is equivalent to: for every bounded and increasing (resp. decreasing) function, and for every $\lambda \geq 0$:

$$x \to \mathbb{E}_x[\phi(X_\lambda)] = \int_{\mathbb{R}} P_\lambda(x, dy)\phi(y) \text{ is increasing (resp. decreasing).} \tag{29}$$

Lemma 3.21. *Let $((X_\lambda)_{\lambda\geq 0}, (\mathcal{F}_\lambda)_{\lambda\geq 0}, (\mathbb{P}_x)_{x\in\mathbb{R}})$ be a Markov process in $\mathbb{R}$ which satisfies (28). Then, for every $n \geq 1$, every $0 < \lambda_1 < \dots < \lambda_n$, every $i \in \{1, \dots, n\}$, every function $\phi : \mathbb{R}^n \to \mathbb{R}$ in $\mathcal{E}_n$, and for every $x \geq 0$,*

$$\mathbb{E}_x[\phi(X_{\lambda_1}, \dots, X_{\lambda_n})|\mathcal{F}_{\lambda_i}] = \widetilde{\phi}_i(X_{\lambda_1}, \dots, X_{\lambda_i}), \tag{30}$$

where $\widetilde{\phi}_i : \mathbb{R}^i \to \mathbb{R}$ belongs to $\mathcal{E}_i$. In particular,

$$x \to \mathbb{E}_x[\phi(X_{\lambda_1}, \dots, X_{\lambda_n})] \textit{ is increasing.} \tag{31}$$

Proof of Lemma 3.21
If $i = n$, (30) is obvious. If $i = n - 1$, then (30) is satisfied since:

$$\mathbb{E}_x[\phi(X_{\lambda_1}, \dots, X_{\lambda_{n-1}}, X_{\lambda_n})|\mathcal{F}_{\lambda_{n-1}}] = \int_{\mathbb{R}} \phi(X_{\lambda_1}, \dots, X_{\lambda_{n-1}}, y) P_{\lambda_n - \lambda_{n-1}}(X_{\lambda_{n-1}}, dy)$$

and then, for $i = n-1$, (30) follows immediately from (29). Thus, Lemma 3.21 follows by iteration of this argument.

□

Observe that as a consequence of Lemma 3.21, the conditional monotonicity property (CM) for these diffusions is equivalent to $(\widetilde{\text{CM}})$.

3.2.2 Time-reversal at a fixed time

Let $x \in \mathbb{R}$ fixed. We assume that:

(A2) For every $\lambda > 0$, $\sigma(\lambda, \cdot)$ is a differentiable function and X_λ admits a $\mathcal{C}^{1,2}$ density function p on $]0, +\infty[\times\mathbb{R}$.

By setting

$$a(\lambda, y) := \sigma^2(\lambda, y) \qquad \text{for every } \lambda \geq 0 \text{ and } y \in \mathbb{R},$$

we define successively, for any fixed $\lambda_0 > 0$ and for $y \in \mathbb{R}$:

$$\begin{cases} \overline{a}^{\lambda_0}(\lambda, y) = a(\lambda_0 - \lambda, y), \quad (0 \leq \lambda \leq \lambda_0) \\ \overline{b}^{\lambda_0}(\lambda, y) = -b(\lambda_0 - \lambda, y) + \dfrac{1}{p(\lambda_0 - \lambda, y)} \dfrac{\partial}{\partial y}\big(a(\lambda_0 - \lambda, y)\, p(\lambda_0 - \lambda, y)\big), \quad (0 \leq \lambda < \lambda_0) \end{cases} \tag{32}$$

and the differential operator $L_\lambda^{\lambda_0}$, $(0 \le \lambda < \lambda_0)$:

$$L_\lambda^{\lambda_0} f(x) = \frac{1}{2}\overline{a}^{\lambda_0}(\lambda, y) f''(y) + \overline{b}^{\lambda_0}(\lambda, y) f'(y) \quad \text{for } f \in \mathcal{C}_b^2.$$

Under some suitable conditions on a and b, U.G. Haussmann and E. Pardoux [HP86] (see also P.A. Meyer [Mey94]) proved that:

(A3) The process $(\overline{X}_\lambda^{\lambda_0}, 0 \le \lambda < \lambda_0)$ obtained by time-reversing $(X_\lambda, 0 < \lambda \le \lambda_0)$ at time λ_0:

$$(\overline{X}_\lambda^{\lambda_0}, 0 \le \lambda < \lambda_0) := (X_{\lambda_0 - \lambda}, 0 \le \lambda < \lambda_0)$$

is a diffusion and there exists a Brownian motion $(\overline{B}_u, 0 \le u \le \lambda_0)$, independent of X_{λ_0}, such that $(\overline{X}_\lambda^{\lambda_0}, 0 \le \lambda < \lambda_0)$ solves the SDE:

$$\begin{cases} dY_\lambda &= \overline{\sigma}^{\lambda_0}(\lambda, Y_\lambda) d\overline{B}_\lambda + \overline{b}^{\lambda_0}(\lambda, Y_\lambda) d\lambda \qquad (0 \le \lambda < \lambda_0) \\ Y_0 &= X_{\lambda_0} \qquad (\text{with } \overline{\sigma}^{\lambda_0}(\lambda, y) = \sigma(\lambda_0 - \lambda, y)). \end{cases} \tag{33}$$

Note that the coefficients $\overline{b}^{\lambda_0}$ and $\overline{\sigma}^{\lambda_0}$ depend on x.

(A4) We assume furthermore that the SDE (33) admits a unique strong solution on $[0, \lambda_0[$; thus, this strong solution is strongly Markovian.

Note that, a priori, the solution of (33) is only defined on $[0, \lambda_0[$, but it can be extended on $[0, \lambda_0]$ by setting $\overline{X}_{\lambda_0}^{\lambda_0} = x$.

3.2.3 Our hypotheses and the main result

Our goal here is not to give optimal hypotheses under which the assertions (A1)-(A4) are satisfied. We refer the reader to [HP86] or [MNS89] for more details. Instead, we shall present two hypotheses (H1) and (H2), either of them implying the preceding assertions:

(H1) We assume that:

i) the functions $(\lambda, y) \longmapsto \sigma(\lambda, y)$ and $(\lambda, y) \longmapsto b(\lambda, y)$ are of $\mathcal{C}^{1,2}$ class on $]0, +\infty[\times\mathbb{R}$, locally Lipschitz continuous in y uniformly in λ, and the solution of (27) does not explode on $[0, \lambda_0]$,

ii) there exists $\alpha > 0$ such that:

$$a(\lambda, y) \equiv \sigma^2(\lambda, y) \ge \alpha \qquad \text{for every } y \in \mathbb{R} \text{ and } 0 \le \lambda \le \lambda_0,$$

and

$$\frac{\partial^2 a}{\partial y^2} \in L^\infty(]0, \lambda_0] \times \mathbb{R}_+).$$

(H2) We assume that:

i) the functions σ and b are of $\mathcal{C}^{1,2}$ class, locally Lipschitz continuous in y uniformly in λ, and the solution of (27) does not explode on $[0, \lambda_0]$,

ii) the functions a and b are of $\mathcal{C}^\infty$ class on $]0, +\infty[\times\mathbb{R}$ in (λ, y) and the differential operator

$$\overline{L} = \frac{\partial}{\partial \lambda} + L_\lambda$$

is hypoelliptic (see Ikeda-Watanabe [IW89, p.411] for the definition and properties of hypoelliptic operators), where $(L_\lambda, \lambda \ge 0)$ is the generator of the diffusion (27):

$$L_\lambda = \frac{1}{2} a(\lambda, \cdot) \frac{d}{dy^2} + b(\lambda, \cdot) \frac{d}{dy}. \tag{34}$$

Then, under either (H1) or (H2), the assertions $(A_i)_{i=1\dots4}$ of both paragraphs 3.2.1 and 3.2.2 are satisfied, see [HP86]. In particular, $(\overline{X}_\lambda^{\lambda_0}, 0 \leq \lambda < \lambda_0)$ is a strong solution of equation (33), see P.A. Meyer [Mey94]. Let us now give the main result of this subsection.

Theorem 3.22. *Under either (H1) or (H2), and for every $x \in \mathbb{R}$, the process $(X_\lambda, \lambda \geq 0)$ is conditionally monotone under $\mathbb{P}_x$.*

Proof of Theorem 3.22
Let $n \in \mathbb{N}^*$ and let $\phi : \mathbb{R}^n \to \mathbb{R}$ in $\mathcal{E}_n$. For every $0 < \lambda_1 < \cdots < \lambda_n$ and every $i \in \{1, \dots, n\}$:
$\mathbb{E}_x[\phi(X_{\lambda_1}, \dots, X_{\lambda_n})|X_{\lambda_i} = z]$

$$
\begin{aligned}
&= \mathbb{E}_x\left[\mathbb{E}_x\left[\phi(X_{\lambda_1}, \dots, X_{\lambda_n})|\mathcal{F}_{\lambda_i}\right]|X_{\lambda_i} = z\right] \\
&= \mathbb{E}_x\left[\widetilde{\phi}_i(X_{\lambda_1}, \dots, X_{\lambda_i})|X_{\lambda_i} = z\right] \\
&\quad \text{(by Lemma 3.21, where } \widetilde{\phi}_i : \mathbb{R}^i \to \mathbb{R} \text{ belongs to } \mathcal{E}_i \\
&= \overline{\mathbb{E}}_x\left[\widetilde{\phi}_i(\overline{X}^{\lambda_i}_{\lambda_i-\lambda_1}, \dots, \overline{X}^{\lambda_i}_0)|\overline{X}^{\lambda_i}_0 = z\right] \quad \text{(by time-reversal at } \lambda_i) \\
&= \overline{\mathbb{E}}_z\left[\widetilde{\phi}_i(\overline{X}^{\lambda_i}_{\lambda_i-\lambda_1}, \dots, \overline{X}^{\lambda_i}_{\lambda_i-\lambda_{i-1}}, z)\right]
\end{aligned}
$$

and, by applying (31) to the reversed process $(\overline{X}_\lambda^{\lambda_i}, 0 \leq \lambda < \lambda_i)$, this last expression is a bounded function which increases with respect to z.

□

Corollary 3.23. *Let $(X_\lambda, \lambda \geq 0)$ the unique strong solution of (27), taking values in $\mathbb{R}_+$, where b and σ satisfy either (H1) or (H2). Then, for every finite positive measure μ and for every $p > 0$, the process:*

$$
\left(A_t^{(\mu,p)} := \int_0^\infty e^{-t(X_\lambda)^p - h_{\lambda,p}(t)} \mu(d\lambda), t \geq 0\right) \tag{35}
$$

is a peacock, with:

$$
h_{\lambda,p}(t) = \log \mathbb{E}_x\left[\exp\left(-t(X_\lambda)^p\right)\right].
$$

Proof of Corollary 3.23
As $(X_\lambda, \lambda \geq 0)$ is a continuous positive process, conditions (INT1) and (INT2) are satisfied, and we may apply Theorems 3.22 and 2.4.

□

3.2.4 A few examples of diffusions which are "well-reversible" at fixed times

Example 3.24 (Brownian motion with drift ν)**.**
We take $\sigma \equiv 1, b(s, y) = \nu$ and $X_\lambda = x + B_\lambda + \nu\lambda$. Then,

$$
p(t, x, y) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{(y - (x + \nu t))^2}{2t}\right),
$$

and $(\overline{X}_\lambda^{\lambda_0}, 0 \leq \lambda < \lambda_0)$ is the solution of:

$$
Y_\lambda = \overline{X}_0^{\lambda_0} + \overline{B}_\lambda + \int_0^\lambda \frac{x - Y_u}{\lambda_0 - u} du
$$

with $(\overline{B}_\lambda, 0 \leq \lambda < \lambda_0)$ independent from $\overline{X}_0^{\lambda_0} = X_{\lambda_0}^{(x)}$.
See Jeulin-Yor [JY79] for similar computations.

Example 3.25 (Bessel processes of dimension $\delta \geq 2$)**.**
We take $\sigma \equiv 1$ and $b(s,y) = \dfrac{\delta - 1}{2y}$, with $\delta = 2(\nu+1)$, $\delta \geq 2$. Then,
i) for $x > 0$:

$$p(t,x,y) = \frac{1}{t}\frac{y^{\nu+1}}{x^{\nu}} \exp\left(-\frac{x^2+y^2}{2t}\right) I_\nu\left(\frac{xy}{t}\right),$$

where I_ν denote the modified Bessel function of index ν (see Lebedev [Leb65, p.110] for the definition of I_ν), and $(\overline{X}_\lambda^{\lambda_0}, 0 \leq \lambda < \lambda_0)$ is the solution of:

$$Y_\lambda = \overline{X}_0^{\lambda_0} + \overline{B}_\lambda + \int_0^\lambda \left(\frac{1}{2Y_u} - \frac{Y_u}{\lambda_0 - u} + \frac{x}{\lambda_0 - u}\frac{I_\nu'}{I_\nu}\left(\frac{xY_u}{\lambda_0 - u}\right)\right) du.$$

ii) for $x = 0$:

$$p(t,0,y) = \frac{1}{2^\nu t^{\nu+1}\Gamma(\nu+1)} y^{2\nu+1} \exp\left(\frac{-y^2}{2t}\right),$$

and $(\overline{X}_\lambda^{\lambda_0}, 0 \leq \lambda < \lambda_0)$ is the solution of:

$$Y_\lambda = \overline{X}_0^{\lambda_0} + \overline{B}_\lambda + \int_0^\lambda \left(\frac{2\nu+1}{2Y_u} - \frac{Y_u}{\lambda_0 - u}\right) du$$

This examples has a strong likelihood with Bessel processes with drift, see Watanabe [Wat75].

Example 3.26 (Squared Bessel processes of dimension $\delta > 0$)**.**
We take $\sigma(s,y) = 2\sqrt{y}$ and $b \equiv \delta$. Then:
i) for $x > 0$:

$$p(t,x,y) = \frac{1}{2t}\left(\frac{y}{x}\right)^{\nu/2} \exp\left(-\frac{x+y}{2t}\right) I_\nu\left(\frac{\sqrt{xy}}{t}\right),$$

and $(\overline{X}_\lambda^{\lambda_0}, 0 \leq \lambda < \lambda_0)$ is the solution of:

$$Y_\lambda = \overline{X}_0^{\lambda_0} + 2\int_0^\lambda \sqrt{Y_u} dB_u + 2\lambda - 2\int_0^\lambda \left(\frac{Y_u}{\lambda - u} - \frac{\sqrt{xY_u}}{\lambda_0 - u}\frac{I_\nu'}{I_\nu}\left(\frac{\sqrt{xY_u}}{\lambda_0 - u}\right)\right) du.$$

ii) for $x = 0$:

$$p(t,0,y) = \left(\frac{1}{2t}\right)^{\delta/2} \frac{1}{\Gamma(\delta/2)} y^{\frac{\delta}{2}-1} \exp\left(-\frac{y}{2t}\right),$$

and $(\overline{X}_\lambda^{\lambda_0}, 0 \leq \lambda < \lambda_0)$ is the solution of:

$$Y_\lambda = \overline{X}_0^{\lambda_0} + 2\int_0^\lambda \sqrt{Y_u} dB_u + \delta\lambda - \int_0^\lambda \frac{2Y_u}{\lambda - u} du.$$

Note that we could also have obtained this example by squaring the results on Bessel processes.

Remark 3.27. All the above examples have a strong link with initial enlargement of a filtration (by the terminal value). We refer the reader to Mansuy-Yor [MY06] for further examples.

4 Another class of Markovian peacocks

We shall introduce another set of hypotheses on the Markov process $(X_\lambda, \lambda \geq 0)$ which ensures that:

$$\left(A_t^{(\mu)} := \int_0^\infty e^{-tX_\lambda - h_\lambda(t)} \mu(d\lambda), t \geq 0\right)$$

is a peacock.

Definition 4.1. *A right-continuous Markov process $(X_\lambda, \lambda \geq 0; \mathbb{P}_x, x \in \mathbb{R}_+)$, with values in $\mathbb{R}_+$, is said to satisfy conditon (L) if both (i) and (ii) below are satisfied:*

i) This process increases in the stochastic order with respect to the starting point x; in other words, for every $a \geq 0$ and $\lambda \geq 0$, and for every $0 \leq x \leq y$:

$$\mathbb{P}_y(X_\lambda \geq a) \geq \mathbb{P}_x(X_\lambda \geq a). \tag{36}$$

ii) The Laplace transform $\mathbb{E}_x[e^{-tX_\lambda}]$ is of the form:

$$\mathbb{E}_x[e^{-tX_\lambda}] = C_1(t,\lambda)\exp(-x\, C_2(t,\lambda)), \tag{37}$$

where C_1 and C_2 are two positive functions such that:

- *For every $t > 0$ and $\lambda \geq 0$,*

$$\frac{\partial}{\partial t} C_2(t,\lambda) > 0. \tag{38}$$

- *For every $t \geq 0$ and every compact K, there exist two constants $k_K(t) > 0$ and $\widetilde{k}_K(t) < +\infty$ such that:*

$$k_K(t) \leq \inf_{\lambda\in K} C_1(t,\lambda); \quad \sup_{\lambda \in K} C_2(t,\lambda) \leq \widetilde{k}_K(t). \tag{39}$$

Taking $x = 0$ in (37), we see that $C_1(\centerdot, \lambda)$ is completely monotone (and hence infinitely differentiable) on $]0, +\infty[$ and continuous at 0. Consequently, $C_2(\centerdot, \lambda)$ is also infinitely differentiable on $]0, +\infty[$ and continuous at 0. Moreover, we have for $t > 0$ and $\lambda \geq 0$:

$$\mathbb{E}_x\left[X_\lambda e^{-tX_\lambda}\right] = \left(-\frac{\partial}{\partial t}C_1(t,\lambda) + xC_1(t,\lambda)\frac{\partial}{\partial t}C_2(t,\lambda)\right)\exp\left(-xC_2(t,\lambda)\right)$$

and we introduce:

$$\begin{cases} \alpha(t,\lambda) & := -\dfrac{\partial}{\partial t}C_1(t,\lambda) \geq 0 \\ \beta(t,\lambda) & := C_1(t,\lambda)\dfrac{\partial}{\partial t}C_2(t,\lambda) > 0 \end{cases} \tag{40}$$

We can now state the main result of this subsection.

Theorem 4.2. *Let $(X_\lambda, \lambda \geq 0; \mathbb{P}_x, x \in \mathbb{R}_+)$ be a Markov process which satisfies condition (L). Then, for every $x \geq 0$ and every finite positive measure μ on $\mathbb{R}_+$,*

$$\left(A_t^{(\mu)} := \int_0^\infty e^{-tX_\lambda - h_\lambda(t)}\mu(d\lambda), t \geq 0\right)$$

is a peacock under $\mathbb{P}_x$. Here, the function h_λ is defined by:

$$h_\lambda(t) = \log\left(\mathbb{E}_x\left[e^{-tX_\lambda}\right]\right).$$

Before proving Theorem 4.2, let us give two examples of processes $(X_\lambda, \lambda \geq 0; \mathbb{P}_x, x \in \mathbb{R}_+)$ which satisfy condition (L).

Example 4.3. Let $(X_\lambda, \lambda \geq 0; \mathbb{Q}_x, x \in \mathbb{R}_+)$ be the square of a δ-dimensional Bessel process (denoted BESQ$^\delta$, $\delta \geq 0$, see [RY99, Chapter XI]). This process satisfies condition (L) since:

- It is stochastically increasing with respect to x; indeed, it solves a SDE which enjoys both existence and uniqueness properties, hence the strong Markov property (see paragraph 3.2.1).

- For every $t > 0$, We have:

$$\mathbb{Q}_x \left[e^{-tX_\lambda} \right] = \frac{1}{(1+2t\lambda)^{\frac{\delta}{2}}} \exp\left(-\frac{tx}{1+2t\lambda} \right),$$

which yields Point *ii)* of Definition 4.1.

In particular, for $(X_t, t \geq 0)$ a squared Bessel process of dimension 0, $(A_t^{(\mu)}, t \geq 0)$ is a peacock. This case was outside the scope of Example 3.26.

Example 4.4 (A generalization of the preceding example for $\delta = 0$)**.**
Let $(X_\lambda, \lambda \geq 0; \mathbb{P}_x, x \in \mathbb{R}_+)$ be a continuous state branching process (denoted CSBP) (see [LG99]).We denote by $P_\lambda(x, dy)$ the law of X_λ under $\mathbb{P}_x$, (with $x \neq 0$), and by $*$ the convolution product. Then (P_λ) satisfies:

$$P_\lambda(x, \centerdot) * P_\lambda(x', \centerdot) = P_\lambda(x + x', \centerdot) \quad \text{for every } \lambda \geq 0,\ x \geq 0 \text{ and } x' \geq 0$$

which easily implies (36) (see [LG99, p.21-23]). On the other hand, one has:

$$\mathbb{E}_x \left[e^{-tX_\lambda} \right] = \exp(-x\, C(t, \lambda)), \tag{41}$$

where the function $C : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfies:

- for every $\lambda \geq 0$, the function $C(\centerdot, \lambda)$ is continuous on $\mathbb{R}_+$ and differentiable on $]0, +\infty[$, and
$$\frac{\partial C}{\partial t}(t, \lambda) > 0 \quad \text{for every } t > 0,$$
- For every $t \geq 0$ and every compact K, there exists a constant $k_K(t) < \infty$ such that:
$$\sup_{\lambda \in K} C(t, \lambda) \leq k_K(t). \tag{42}$$

Thus, $(X_\lambda, \lambda \geq 0)$ satisfies (37).

Corollary 4.5. *Let $(X_\lambda, \lambda \geq 0; \mathbb{P}_x, x \in \mathbb{R}_+)$ be either a BESQ$^\delta$ or a CSBP. Then, for any finite positive measure μ on $\mathbb{R}_+$, and for every $x \geq 0$:*

$$\left(A_t^{(\mu)} := \int_0^\infty e^{-tX_\lambda - h_\lambda(t)} \mu(d\lambda), t \geq 0 \right)$$

is a peacock under $\mathbb{P}_x$ with:

$$h_\lambda(t) = \log\left(\mathbb{E}_x \left[e^{-tX_\lambda} \right] \right).$$

Remark 4.6. This example generalizes the previous one in the following sense. Let $(Y_t, t \geq 0)$ be a Lévy process of characteristic exponent $\psi(\lambda) = c\lambda^\alpha$, $(c > 0, \alpha \in]1, 2])$:

$$\mathbb{E}\left[e^{-\lambda Y_t} \right] = \exp\left(-ct\lambda^\alpha \right).$$

We denote by $(H_t, t \geq 0)$ the height process associated to $(Y_t, t \geq 0)$. This process admits a family of local times $(L_t^a(H), t \geq 0, a \geq 0)$ and, denoting by $\tau_r(H) := \inf\{s \geq 0; L_s^0(H) > r\}$ its right-continuous inverse, it is known (see [LG99]) that the process $(L^a_{\tau_r(H)}, a \geq 0)$ is a stable CSBP of index α. Then, observe that for $\alpha = 2$ and $c = \frac{1}{2}$, $(Y_t := B_t, t \geq 0)$ is a standard Brownian motion started from 0, $(H_t, t \geq 0) \overset{\text{(law)}}{=} (|B_t|, t \geq 0)$ has the same law as a reflected Brownian motion, and that, from the Ray-Knight theorem, $(L^a_{\tau_r(H)}, a \geq 0)$ is a squared Bessel process of dimension 0 started fom r.
We refer the interested reader to [HPRY, Chapter 4] for a description of other peacocks constructed from CSBP, and their associated martingales.

Proof of Theorem 4.2

Let $(X_\lambda, \lambda \geq 0)$ be a process which enjoys condition (L).

1. $(-X_\lambda, \lambda \geq 0)$ being a negative process, condition (INT1) clearly holds. Moreover, by (39), (INT2) also holds. Thus, following the proof of Theorem 2.4, it suffices to show that $(A_t^{(\mu)}, t \geq 0)$ is a peacock when μ is a finite linear combination of Dirac measures with positive coefficients.
2. For $t \geq 0$, $a_1 \geq 0, \dots, a_n \geq 0$ and $0 < \lambda_1 < \dots < \lambda_n$, we set:
$$A_t := \sum_{i=1}^{n} a_i e^{-tX_{\lambda_i} - h_{\lambda_i}(t)}.$$
Let $\psi \in \boldsymbol{C}$. One has:
$$\frac{\partial}{\partial t}\mathbb{E}_x[\psi(A_t)] = -\mathbb{E}_x\left[\psi'(A_t)\sum_{i=1}^{n} a_i e^{-tX_{\lambda_i} - h_{\lambda_i}(t)}(h'_{\lambda_i}(t) + X_{\lambda_i})\right]$$
and, we shall prove as in the proof of Theorem 2.4 that, for every $i \in \{1, \dots, n\}$, $\Delta_i \leq 0$, with:
$$\begin{aligned}\Delta_i &= \mathbb{E}_x\left[\psi'(A_t)e^{-tX_{\lambda_i} - h_{\lambda_i}(t)}(h'_{\lambda_i}(t) + X_{\lambda_i})\right] \\ &= \mathbb{E}_x\left[\psi'(A_t)e_{\lambda_i}(X_{\lambda_i})\right],\end{aligned}$$
and where we have set
$$e_{\lambda_i}(z) := e^{-tz - h_{\lambda_i}(t)}(h'_{\lambda_i}(t) + z).$$
We note, since $\mathbb{E}\left[e^{-tX_{\lambda_i} - h_{\lambda_i}(t)}\right] = 1$, that:
$$\mathbb{E}_x[e_{\lambda_i}(X_{\lambda_i})] = 0. \tag{43}$$
Since the function
$$(x_1, \dots, x_n) \to \psi'\left(\sum_{j=0}^{n} a_j e^{-tx_j - h_{\lambda_j}(t)}\right)$$
is bounded and decreases with respect to each of its arguments, it suffices to show that: for every bounded Borel function $\phi : \mathbb{R}^n \to \mathbb{R}_+$ which decreases with respect to each of its arguments, and for every $i \in \{1, \dots, n\}$,
$$\mathbb{E}_x[\phi(X_{\lambda_1}, \dots, X_{\lambda_n})e_{\lambda_i}(X_{\lambda_i})] \leq 0. \tag{44}$$
3. We now show (44).
a) We may suppose $i = n$. Indeed, thanks to (36) and to Lemma 3.21, we have, for $i < n$:
$$\begin{aligned}\mathbb{E}_x[\phi(X_{\lambda_1}, \dots, X_{\lambda_n})e_{\lambda_i}(X_{\lambda_i})] &= \mathbb{E}_x[\mathbb{E}_x[\phi(X_{\lambda_1}, \dots, X_{\lambda_n})|\mathcal{F}_{\lambda_i}]e_{\lambda_i}(X_{\lambda_i})] \\ &= \mathbb{E}_x[\widetilde{\phi}_i(X_{\lambda_1}, \dots, X_{\lambda_i})e_{\lambda_i}(X_{\lambda_i})],\end{aligned}$$
where $\widetilde{\phi}_i : \mathbb{R}^i \to \mathbb{R}$ is a bounded Borel function which decreases with respect to each of its arguments.
b) On the other hand, one has:
$$\begin{aligned}&\mathbb{E}_x[\widetilde{\phi}_i(X_{\lambda_1}, \dots, X_{\lambda_i})e_{\lambda_i}(X_{\lambda_i})] \\ &\quad= \mathbb{E}_x[\widetilde{\phi}_i(X_{\lambda_1}, \dots, X_{\lambda_i})e^{-tX_{\lambda_i} - h_{\lambda_i}(t)}(h'_{\lambda_i}(t) + X_{\lambda_i})] \\ &\quad\leq \mathbb{E}_x[\widetilde{\phi}_i(X_{\lambda_1}, \dots, X_{\lambda_{i-1}}, -h'_{\lambda_i}(t))e_{\lambda_i}(X_{\lambda_i})] \\ &\qquad (\text{since } \widetilde{\phi}_i(X_{\lambda_1}, \dots, X_{\lambda_i})(h'_{\lambda_i}(t) + X_{\lambda_i}) \leq \widetilde{\phi}_i(X_{\lambda_1}, \dots, -h'_{\lambda_i}(t))(h'_{\lambda_i}(t) + X_{\lambda_i})) \\ &\quad= \mathbb{E}_x\left[\widetilde{\widetilde{\phi}}_i(X_{\lambda_1}, \dots, X_{\lambda_{i-1}})e_{\lambda_i}(X_{\lambda_i})\right],\end{aligned}$$

where $\widetilde{\widetilde{\phi}}_i : \mathbb{R}^{i-1} \to \mathbb{R}$ is a bounded Borel function which decreases with respect to each of its arguments, and is defined by:

$$\widetilde{\widetilde{\phi}}_i(z_1, \dots, z_{i-1}) = \widetilde{\phi}_i(z_1, \dots, z_{i-1}, -h'_{\lambda_i}(t)). \tag{45}$$

$c)$ We now end the proof of Theorem 4.2 by showing the following lemma.

Lemma 4.7. *For every $i \in \{1, \dots, n\}$ and $j \in \{0, 1, \dots, i-1\}$, let $\phi : \mathbb{R}^j \to \mathbb{R}$ be a bounded Borel function which decreases with respect to each of its arguments. Then,*

$$\mathbb{E}_x[\phi(X_{\lambda_1}, \dots, X_{\lambda_j}) e_{\lambda_i}(X_{\lambda_i})] \le 0. \tag{46}$$

In particular,

$$\mathbb{E}_x[\phi(X_{\lambda_1}, \dots, X_{\lambda_{i-1}}) e_{\lambda_i}(X_{\lambda_i})] \le 0. \tag{47}$$

Proof of Lemma 4.7
We prove this lemma by induction on j.
• For $j = 0$, ϕ is constant and one has:

$$\mathbb{E}_x[\phi\, e_{\lambda_i}(X_{\lambda_i})] = \phi\, \mathbb{E}_x[e_{\lambda_i}(X_{\lambda_i})] = 0 \quad \text{(from (43))}$$

• On the other hand, if one assumes that (46) holds for $0 \le j < i-1$, then

$$\begin{aligned}
&\mathbb{E}_x[\phi(X_{\lambda_1}, \dots, X_{\lambda_j}, X_{\lambda_{j+1}}) e_{\lambda_i}(X_{\lambda_i})] \\
&= \mathbb{E}_x[\phi(X_{\lambda_1}, \dots, X_{\lambda_j}, X_{\lambda_{j+1}}) P_{\lambda_i - \lambda_{j+1}} e_{\lambda_i}(X_{\lambda_{j+1}})] \\
&\quad \text{(by the Markov property)} \\
&= \mathbb{E}_x[\phi(X_{\lambda_1}, \dots, X_{\lambda_j}, X_{\lambda_{j+1}}) e^{-X_{\lambda_{j+1}} C_2(t, \lambda_i - \lambda_{j+1}) - h_{\lambda_i}(t)} \\
&\qquad\qquad . \left(\alpha(t, \lambda_i - \lambda_{j+1}) + X_{\lambda_{j+1}} \beta\,(t, \lambda_i - \lambda_{j+1})\right)] \qquad (48) \\
&\quad \left(\text{where, from (37) and (40), } \beta > 0, \text{ and } \alpha \text{ depends on } a \text{ and } h'_{\lambda_i}\right) \\
&\le \mathbb{E}_x\left[\phi\left(X_{\lambda_1}, \dots, X_{\lambda_j}, -\frac{\alpha(t, \lambda_i - \lambda_{j+1})}{\beta\,(t, \lambda_i - \lambda_{j+1})}\right) P_{\lambda_i - \lambda_{j+1}} e_{\lambda_i}(X_{\lambda_{j+1}})\right] \\
&= \mathbb{E}_x\left[\widetilde{\phi}(X_{\lambda_1}, \dots, X_{\lambda_j}) e_{\lambda_i}(X_{\lambda_i})\right] \le 0 \quad \text{(by the induction hypothesis),}
\end{aligned}$$

where $\widetilde{\phi} : \mathbb{R}^j \to \mathbb{R}$ is defined by:

$$\widetilde{\phi}(z_1, \dots, z_j) = \phi\left(z_1, \dots, z_j, -\frac{\alpha(t, \lambda_i - \lambda_{j+1})}{\beta\,(t, \lambda_i - \lambda_{j+1})}\right).$$

□

5 Stochastic and convex orders

The purpose of this Section is different from that of the previous Sections. Here, we do not look a priori for peacocks, but rather study a link between the stochastic and convex orders. As a byproduct, this will provide us with some new peacocks.

Definition 5.1. *Let μ and ν be two probability measures on $\mathbb{R}_+$. We shall say that μ is stochastically greater than ν, and we write:*

$$\mu \overset{(st)}{\ge} \nu$$

if for every $t \ge 0$,

$$F_\mu(t) := \mu([0, t]) \le F_\nu(t) := \nu([0, t]).$$

In [HPRY], the authors prove that if $(M_t, t \geq 0)$ is a martingale in H^1_{loc} (thus, it is a peacock), and $\alpha : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is a continuous and strictly increasing function such that $\alpha(0) = 0$, then the process

$$\left(\frac{1}{\alpha(t)} \int_0^t M_u d\alpha(u) \;,\; u \geq 0 \right)$$

is a peacock. In other words, for every $0 \leq s \leq t$:

$$\int_0^{+\infty} M_u \left(\frac{1}{\alpha(t)} 1_{[0,t]}(u) \right) d\alpha(u) \overset{\text{(c)}}{\geq} \int_0^{+\infty} M_u \left(\frac{1}{\alpha(s)} 1_{[0,s]}(u) \right) d\alpha(u).$$

Now, it is clear that:

$$\left(\frac{1}{\alpha(t)} 1_{[0,t]}(u) \right) d\alpha(u) \overset{\text{(st)}}{\geq} \left(\frac{1}{\alpha(s)} 1_{[0,s]}(u) \right) d\alpha(u),$$

and this leads to the following question: which processes $(X_t, \geq 0)$ satisfy, for every couple of probabilities (μ, ν) such that $\mu \overset{\text{(st)}}{\geq} \nu$, the property:

$$A^{(\mu)} := \int_0^{+\infty} X_u \mu(du) \overset{\text{(c)}}{\geq} A^{(\nu)} := \int_0^{+\infty} X_u \nu(du) \quad ? \tag{49}$$

Note that such a process $(X_t, t \geq 0)$ must be a peacock. Indeed, taking for $0 \leq s \leq t$, $\mu = \delta_t$ and $\nu = \delta_s$, we deduce from (49) that $X_t \overset{\text{(c)}}{\geq} X_s$, i.e. $(X_t, t \geq 0)$ is a peacock.
Here is a partial answer to this question:

Theorem 5.2. *Let $(X_t, t \geq 0)$ be an integrable right-continuous process satisfying both following conditions:*

i) For every bounded and increasing function $\phi : \mathbb{R} \longrightarrow \mathbb{R}_+$ and every $0 \leq s \leq t$, $\mathbb{E}[\phi(X_t)|\mathcal{F}_s]$ is an increasing function of X_s.

ii) For every $n \in \mathbb{N}^*$, every $0 < t_1 < \cdots < t_n$ and every $\phi : \mathbb{R}^n \to \mathbb{R}$ in $\mathcal{E}_n$, we have:

$$\mathbb{E}\left[\phi\left(X_{t_1}, \ldots, X_{t_n}\right)\left(X_{t_{n+1}} - X_{t_n}\right)\right] \geq 0.$$

Let μ and ν two probability measures on $\mathbb{R}_+$ such that $\mu \overset{(st)}{\geq} \nu$. Moreover, we assume that either:

μ and ν have compact supports, and for every compact $K \subset \mathbb{R}_+$, $\sup_{t \in K} X_t$ is integrable,

or:

$\sup_{t \geq 0} X_t$ is integrable.

Then:

$$A^{(\mu)} := \int_0^{+\infty} X_u \mu(du) \overset{(c)}{\geq} A^{(\nu)} := \int_0^{+\infty} X_u \nu(du).$$

Remark 5.3.
a) Observe that condition *ii*) implies that the process $(X_t, t \geq 0)$ is a peacock. Indeed, if ψ is a convex function of $\mathcal{C}^1$ class, then, for $0 \leq s \leq t$:

$$\mathbb{E}[\psi(X_t)] - \mathbb{E}[\psi(X_s)] \geq \mathbb{E}[\psi'(X_s)(X_t - X_s)] \geq 0.$$

In particular, $\mathbb{E}[X_t]$ does not depend on t.
b) Note also that condition *i*) implies that $(X_t, t \geq 0)$ is Markovian.

Before proving Theorem 5.2, we shall give some examples of processes which satisfy both conditions i) and ii).

Example 5.4. Let X be a r.v. such that for every $t \geq 0$, $\mathbb{E}[e^{tX}] < \infty$. We define $(\xi_t^X = \exp(tX - h_X(t)), t \geq 0)$ where $h_X(t) = \log \mathbb{E}[e^{tX}]$. Then, $(\xi_t^X, t \geq 0)$ satisfies the conditions of Theorem 5.2. Indeed, condition i) is obvious, and condition ii) follows from:

$$\mathbb{E}\left[\phi(\xi_{t_1}^X, \dots, \xi_{t_n}^X)(\xi_{t_{n+1}}^X - \xi_{t_n}^X)\right] \geq \phi(e^{t_1\beta_n - h_X(t_1)}, \dots, e^{t_n\beta_n - h_X(t_n)})\mathbb{E}\left[\xi_{t_{n+1}}^X - \xi_{t_n}^X\right] = 0,$$

where $\beta_n = \dfrac{h_X(t_{n+1}) - h_X(t_n)}{t_{n+1} - t_n}$. In particular, we recover that, if $\alpha : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is a continuous and strictly increasing function such that $\alpha(0) = 0$, then the process

$$\left(\frac{1}{\alpha(t)} \int_0^t e^{uX - h_X(u)} d\alpha(u)\ ,\ t \geq 0\right)$$

is a peacock.

Example 5.5 (Martingales)**.**
Clearly, martingales satisfy condition ii). Here are some examples of martingales satisfying also condition i):

a) Let $(X_t, t \geq 0)$ be an integrable process with independent and centered increments. Then

$$\mathbb{E}[\phi(X_t)|\mathcal{F}_s] = \mathbb{E}[\phi(X_s + X_t - X_s)|\mathcal{F}_s] = \mathbb{E}[\phi(x + Z)],$$

where $X_s = x$ and $Z \overset{\text{(law)}}{=} X_t - X_s$, is an increasing function of x.

b) Let $(L_t, t \geq 0)$ be an integrable right-continuous process with independent increments, and such that, for every $\lambda, t \geq 0$, $\mathbb{E}[e^{\lambda L_t}] < \infty$. Then, the process

$$\left(X_t := e^{\lambda L_t - h_{L_t}(\lambda)}, t \geq 0\right) \quad \text{where } h_{L_t}(\lambda) = \log \mathbb{E}\left[e^{\lambda L_t}\right]$$

is a martingale which, as in item a), satisfies condition i).

c) Let $(X_t, t \geq 0)$ be a diffusion process which satisfies an equation of type

$$X_t^{(x)} = x + \int_0^t \sigma(X_s^{(x)})\, dB_s.$$

Then condition i) follows from the stochastic comparison theorem (see Point (A1)).

Example 5.6 ("Well-reversible" diffusions)**.**
Let $(Z_t, t \geq 0)$ be a "well-reversible" diffusion satisfying (27) and such that b is an increasing function. Then $(X_t := Z_t - \mathbb{E}[Z_t], t \geq 0)$ satisfies both conditions i) and ii). Indeed, condition i) is clearly satisfied from Lemma 3.21. As for condition ii), setting $h(t) = \mathbb{E}[Z_t]$, we have by time reversal at t_{n+1}:

$$\begin{aligned}
&\mathbb{E}\left[\phi\left(X_{t_1}, \dots, X_{t_n}\right)\left(X_{t_{n+1}} - X_{t_n}\right)\right] \\
&=\overline{\mathbb{E}}\left[\phi\left(\overline{X}_{t_{n+1}-t_1}^{(t_{n+1})}, \dots, \overline{X}_{t_{n+1}-t_n}^{(t_{n+1})}\right)(\overline{X}_0^{(t_{n+1})} - \overline{X}_{t_{n+1}-t_n}^{(t_{n+1})})\right] \\
&=\overline{\mathbb{E}}\left[\mathbb{E}\left[\phi\left(\overline{X}_{t_{n+1}-t_1}^{(t_{n+1})}, \dots, \overline{X}_{t_{n+1}-t_n}^{(t_{n+1})}\right)|\overline{\mathcal{F}}_{t_{n+1}-t_n}\right](\overline{X}_0^{(t_{n+1})} - \overline{X}_{t_{n+1}-t_n}^{(t_{n+1})})\right] \\
&=\overline{\mathbb{E}}\left[\widetilde{\phi}\left(\overline{X}_{t_{n+1}-t_n}^{(t_{n+1})}\right)(\overline{X}_0^{(t_{n+1})} - \overline{X}_{t_{n+1}-t_n}^{(t_{n+1})})\right] \\
&\quad \text{where } \widetilde{\phi} \text{ is an increasing function,} \\
&=\mathbb{E}\left[\widetilde{\phi}\left(X_{t_n}\right)\left(X_{t_{n+1}} - X_{t_n}\right)\right].
\end{aligned}$$

Now from (27):

$$\mathbb{E}\left[\widetilde{\phi}\left(Z_{t_n} - h(t_n)\right)\left(\int_{t_n}^{t_{n+1}} \mu(s, Z_s)dB_s + \int_{t_n}^{t_{n+1}} b(s, Z_s)ds - h(t_{n+1}) + h(t_n)\right)\right]$$

$$= \int_{t_n}^{t_{n+1}} \mathbb{E}\left[\widetilde{\phi}\left(Z_{t_n} - h(t_n)\right)\left(b(s, Z_s) - h'(s)\right)\right] ds$$
$$= \int_{t_n}^{t_{n+1}} \mathbb{E}\left[\widetilde{\phi}\left(Z_{t_n} - h(t_n)\right)\left(\widetilde{b}(s, Z_{t_n}) - h'(s)\right)\right] ds$$

where $x \longmapsto \widetilde{b}(s,x) := \mathbb{E}[b(s, Z_s)|Z_{t_n} = x]$ is an increasing function such that $\mathbb{E}\left[\widetilde{b}(s, Z_{t_n})\right] = \mathbb{E}[b(s, Z_s)] = h'(s)$. Denoting by $\widetilde{b}_s^{-1}$ its right-continuous inverse, we finally obtain:

$$\mathbb{E}\left[\widetilde{\phi}\left(X_{t_n}\right)\left(X_{t_{n+1}} - X_{t_n}\right)\right] \geq \int_{t_n}^{t_{n+1}} \widetilde{\phi}\left(\widetilde{b}_s^{-1}(h'(s)) - h(t_n)\right) \mathbb{E}\left[\widetilde{b}(s, Z_{t_n}) - h'(s)\right] ds = 0.$$

Example 5.7. Let $(B_t, t \geq 0)$ be a Brownian motion started from 0 and φ be a strictly increasing odd function of $\mathcal{C}^2$ class such that φ' is convex. It is known, see [HPRY, Chapter 1, Section 5] that the process $(\varphi(B_t), t \geq 0)$ is a peacock. As a consequence of Example 5.6 and of Theorem 5.2 applied with $\mu(du) = \dfrac{1}{\alpha(t)} 1_{[0,t]}(u) d\alpha(u)$ and $\nu(du) = \dfrac{1}{\alpha(s)} 1_{[0,s]}(u) d\alpha(u)$, where $\alpha : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is a continuous and strictly increasing function such that $\alpha(0) = 0$, we deduce that the process

$$\left(\frac{1}{\alpha(t)} \int_0^t \varphi(B_u) d\alpha(u)\ ,\ t \geq 0\right)$$

is also a peacock. Indeed, from Itô's formula, $(\varphi(B_u), u \geq 0)$ satisfies (27) with $b = \frac{1}{2}\varphi'' \circ \varphi^{-1}$ increasing.

Proof of Theorem 5.2

1. Since
$$A^{(\mu)} := \int_0^\infty X_s\, d\mu(s) = \int_0^\infty X_s\, dF_\mu(s) = \int_0^1 X_{F_\mu^{-1}(u)}\, du,$$
it suffices, by approximation of dF_μ with a linear combination of Dirac measures (as in the proof of Theorem 2.4), to show that for every $n \in \mathbb{N}^*$, for every $a_1, a_2, \dots, a_n$ and for every $t_1 \geq s_1, \dots, t_n \geq s_n$,
$$\sum_{i=1}^n a_i X_{t_i} \overset{\text{(c)}}{\geq} \sum_{i=1}^n a_i X_{s_i}. \tag{50}$$

2. Let $\psi : \mathbb{R} \to \mathbb{R}$ in $\boldsymbol{C}$. By convexity, we have:
$$\psi\left(\sum_{i=1}^n a_i X_{t_i}\right) = \psi\left(\sum_{i=1}^n a_i X_{s_i} + \sum_{i=1}^n a_i (X_{t_i} - X_{s_i})\right)$$
$$\geq \psi\left(\sum_{i=1}^n a_i X_{s_i}\right) + \psi'\left(\sum_{i=1}^n a_i X_{s_i}\right) \sum_{j=1}^n a_j (X_{t_j} - X_{s_j}).$$
Then, taking the expectation leads to:
$$\mathbb{E}\left[\psi\left(\sum_{i=1}^n a_i X_{t_i}\right)\right] \geq \mathbb{E}\left[\psi\left(\sum_{i=1}^n a_i X_{s_i}\right)\right] + \mathbb{E}\left[\psi'\left(\sum_{i=1}^n a_i X_{s_i}\right) \sum_{j=1}^n a_j (X_{t_j} - X_{s_j})\right]. \tag{51}$$
We set $\phi(x_1, \dots, x_n) := \psi'\left(\sum_{i=1}^n a_i x_i\right)$. Thus, $\phi \in \mathcal{E}_n$. Let j be fixed and assume that:
$$0 < s_1 < \ldots < s_j < \ldots < s_{j+r} < t_j < s_{j+r+1} < \ldots < s_n.$$

We write:

$$\begin{aligned}\phi(X_{s_1},\dots,X_{s_n})(X_{t_j}-X_{s_j}) \\ &=\phi(X_{s_1},\dots,X_{s_n})(X_{t_j}-X_{s_{j+r}}+X_{s_{j+r}}-\dots+X_{s_{j+1}}-X_{s_j}) \\ &=\phi(X_{s_1},\dots,X_{s_n})(X_{t_j}-X_{s_{j+r}})+\sum_{k=0}^{r-1}\phi(X_{s_1},\dots,X_{s_n})(X_{s_{j+k+1}}-X_{s_{j+k}})\end{aligned}$$

and we study the expectation of each term separately. From condition i), we obtain by iteration:

$$\begin{aligned}\mathbb{E}\left[\phi(X_{s_1},\dots,X_{s_n})(X_{t_j}-X_{s_{j+r}})\right] &= \mathbb{E}\left[\widetilde{\phi}(X_{s_1},\dots,X_{s_{j+r}},X_{t_j})(X_{t_j}-X_{s_{j+r}})\right] \\ &\geq \mathbb{E}\left[\widetilde{\phi}(X_{s_1},\dots,X_{s_{j+r}},X_{s_{j+r}})(X_{t_j}-X_{s_{j+r}})\right] \\ &\geq 0\end{aligned}$$

from condition ii). The other terms can be dealt with in the same way.

□

Remark 5.8.
1) Note that, in general, the process $\left(\frac{1}{t}\int_0^t X_u du, t\geq 0\right)$ may be a peacock even if $(X_t, t\geq 0)$ is not a peacock. For example, this is the case for the process $(X_t = e^{-t}G, t\geq 0)$ where G is a centered Gaussian r.v. Similarly, $(X_u, u\geq 0)$ may be a peacock while $\left(\frac{1}{t}\int_0^t X_u du, t\geq 0\right)$ is not; for example, take the process $(X_t = (1_{[0,1]}(t) - 1_{]1,+\infty[}(t))G, t\geq 0)$ where G is a centered Gaussian r.v.
2) Theorem 5.2 answers partially a question raised in [HRY10], namely, for which martingales does (49) hold ?

Concluding remark 5.9. In this paper, our aim has been to give several examples of peacocks. On the other hand, we did not exhibit associated martingales (see Point b) of the introduction). We refer the interested reader to [HPRY] where numerous martingales associated to given peacocks are presented. However, for most of the peacocks presented in this paper, we do not know how to exhibit an associated martingale.

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