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Some new less conservative criteria for impulsive synchronization of a hyperchaotic Lorenz system based on small impulsive signals

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Abstract: In this Letter the issue of impulsive Synchronization of a hyperchaotic Lorenz system is developed. We propose an impulsive synchronization scheme of the hyperchaotic Lorenz system including chaotic systems. Some new and sufficient conditions on varying impulsive distances are established in order to guarantee the synchronizabillity of the systems using the synchronization method. In particular, some simple conditions are derived for synchronizing the systems by equal impulsive distances. The boundaries of the stable regions are also estimated. Simulation results show the proposed synchronization method to be effective.

Key words**:** chaos; impulsive synchronization; hyperchaotic Lorenz system

1. Introduction

Hyperchaos, which has more than one positive Lyapunov exponent, has increasingly aroused the interest of many researches due to its great potential in technological applications in many fields, including secure communications and lasers. Hence, the generation, control, synchronization and application of hyperchaos have recently become a hot topic for research in this regard [1-6]. A large variety of hyperchaotic systems have been presented over the past few decades. For example, the hyperchaotic Chua's circuit [1] and Rossler system [2] are two representative hyperchaotic systems. Recently, several new hyperchaotic systems have been proposed. Li *et al* proposed a new hyperchaotic system by introducing an additional state in a third-order generalized Lorenz chaotic system [3]. Chen *et al* proposed a new hyperchaotic system by adopting a state feedback control to Lu's chaotic system [4]. These hyperchaotic systems are new and each has their own properties. Therefore, it is very important to explore further the control and synchronization of these new hyperchaotic systems for engineering applications.

Due to their high complexity and properties, hyperchaotic systems have significant potential in several fields. For example the presence of more than one positive Lyapunov exponent or unstable direction in these systems generates more complexity in secure communications [7]. In [7, 8] the authors investigated the synchronization of hyperchaotic systems by transmitting just one scalar signal. The other point to study is that in the synchronization problem of the slave system, most of the methods rely on receiving the master system signal continuously which is not generally the case in communications. Impulsive synchronization is one of the methods proposed to overcome this problem [9-11]. In [9] the conditions under which chaotic and hyperchaotic systems can be synchronized by impulses determined from samples of their state variables were studied. In [10], a detailed mathematical analysis was provided to explain how the asymptotic stability of the sporadically driven system depends on the driving period in linear systems. The sensitivity of the synchronization with respect to noise was also investigated for coupled chaotic systems. It was shown that the synchronization might be enhanced through the use of sporadic driving in special cases. In [11], the lag synchronization of hyperchaotic systems using the sporadic method was studied. Impulsive controllers seem to have a simple structure, and the controller is discontinuous which can be useful for digital communication systems. The research on impulsive synchronization in [12, 14] is based on the theory of comparison systems, but it is difficult to estimate the interval of the impulsive control for some systems using this theory. The impulsive synchronization of Chua's oscillator and a hyperchaotic circuit has been studied in [15]. The experimental results in [15] show that the accuracy of impulsively controlled synchronization depends on both the period and the width of the impulse. Furthermore, the robustness of impulsive synchronization to additive noise was also experimentally studied in [13, 16]. Itoh *et al*. gave a sufficient condition for impulsive synchronization of continuous systems under the assumption that the synchronization errors are sufficiently small, but this result does not hold for chaotic systems with strong nonlinearities [17]. The impulsive synchronization method is also applicable to systems which cannot endure continuous disturbances. Using this method, the slave system receives the information from the master system only at discrete times and the amount of conveyed information is, therefore, decreased, which is suitable in practice because of reduced control cost.

However, few analysis results (if any) have been reported for synchronization of hyperchaotic systems governed by ordinary differential equations (ODE). In this paper, we pay particular attention to the investigation of synchronization in identical hyperchaotic systems using only small impulses. A fourth-order coupled hyperchaotic Lorenz system is taken as an example to demonstrate the results.

New and less conservative criteria are also proposed to synchronize systems with varying impulse distances, and a simple and sufficient condition is derived to achieve synchronization based on equal impulse distances. The boundaries of the stable regions are also determined, and numerical simulation results are given to show the feasibility and effectiveness of the used method.

The organization of the paper is as follows. In Section 2, the theory of impulsive synchronization is explained. The synchronization of the hyperchaotic Lorenz system using a small impulse is discussed. The numerical simulation results are given to show the feasibility and effectiveness of the method. Boundaries of the stable regions are estimated in Section 3 and Section 4, respectively. Finally, some conclusions are drawn in Section 5.

2. The theory of impulsive synchronization

In impulsive systems, the master system is described by the following relation

$$\dot{x} = f\left(t, x\right) \tag{1}$$

$f : R_+ \times R^n \to R^n$ is a continuous function with respect to its arguments and $x \in R^n$ represents the state variables. The slave system is characterized by

$$\begin{aligned} &\dot{y} = f\left(t, y\right), && t \neq t_i \\ &\Delta y = y\left(t_i^+\right) - y\left(t_i^-\right) = y\left(t_i^+\right) - y\left(t_i\right) = B_i e, && t = t_i \\ &y\left(t_0^+\right) = y_0, && i = 1,2,3,\cdots \end{aligned} \tag{2}$$

f is the same function as above, $y \in R^n$ is left continuous at $t = t_i$, B_i are $n \times n$ matrices, and $e = \left[e_1, e_2, \cdots, e_n\right]^T = \left[y_1 - x_1, y_2 - x_2, \cdots, y_n - x_n\right]^T$. Define a discrete instant set $\{t_i\}$ that satisfies $t_1 < t_2 < \cdots < t_i < t_{i+1} < \cdots, t_i \to \infty$ as $i \to \infty$. t_i is the discrete time instants at which the master signal is transmitted to the slave system. The states of the slave system are changed at these instants in accordance with the synchronization errors. Subtracting (2) from (1), provides results for synchronization error dynamics. Since the states of the master system are continuous in time, Δx will be zero at the time instants t_i

$$\begin{aligned} \dot{e} &= f(t,y) - f(t,x), \quad t \neq t_i, \\ \Delta e &= B_i e, \qquad\qquad\quad t = t_i. \end{aligned} \tag{3}$$

The goal is to find some conditions on the control gains, B_i and the impulsive distances $\delta_{i+1} = t_{i+1} - t_i < \infty$ $(i = 1, 2, 3, \cdots)$ such that the slave system (2) is synchronized asymptotically with the master system (1) for any initial condition.

Remark 1. Several hyperchaotic systems satisfy (3). For example, the fourth Rossler's system [4], the Chen's hyperchaotic system [5] and the hyperchaotic Lorenz system [6] all belong to the class defined by (3).

3. The impulsive synchronization of the hyperchaotic Lorenz system

Here we investigate the impulsive synchronization of the hyperchaotic Lorenz system [6]. The system is descried as follows:

$$\begin{aligned} \dot{x}_1 &= a(x_2 - x_1), \\ \dot{x}_2 &= bx_1 + cx_2 - x_1x_3 + x_4, \\ \dot{x}_3 &= -dx_3 + x_1x_2, \\ \dot{x}_4 &= -kx_1. \end{aligned} \tag{4}$$

Where x_1, x_2, x_3 and x_4 are state variable, and a, b, c, d and k are system parameters. The hyperchaotic Lorenz system shows hyperchaotic behavior when $a = 35, b = 7$, $c = 12$, $d = 3$, and $k = 5$.

First we decompose the system dynamics to its linear and nonlinear parts. Thus (4) is rewritten as

$$\dot{x} = Ax + \phi(x), \tag{5}$$

where $\phi(x)$ represents the nonlinear part of the dynamics.

$$A=\begin{bmatrix} -a & a & 0 & 0 \\ b & c & 0 & 1 \\ 0 & 0 & -d & 0 \\ -k & 0 & 0 & 0 \end{bmatrix}, \quad \phi(x)=\begin{bmatrix} 0 \\ -x_1x_3 \\ x_1x_2 \\ 0 \end{bmatrix}. \tag{6}$$

Therefore, the error dynamics in (3) can be written as

$$\begin{aligned} \dot{e} &= Ae+\psi(x,y), & t &\neq t_i, \\ \Delta e &= B_i e, & t &= t_i, \end{aligned} \tag{7}$$

in which

$$\psi(x,y)=\phi(y)-\phi(x)=\begin{bmatrix} 0 \\ -y_1y_3+x_1x_3 \\ y_1y_2-x_1x_2 \\ 0 \end{bmatrix} \tag{8}$$

and t_i are the instants that the impulsive controls are implemented.

Remark 2. From the analysis above, it follows that it is sufficient for synchronizing chaos that the origin of (7) is asymptotically stable. It is worth noting that the origin is one of the equilibrium points of system (7). Also, the origin is the unique equilibrium of system (7) $\Delta e = B_i e$ implies $e(t_i^+) \neq e(t_i)$ unless $e(t_i^+)=e(t_i)=0.$

Regardless of their initial conditions, chaotic systems have bounded states so that one can find a positive number M such that $|x_i(t)| \leq M$ and $|y_i(t)| \leq M$ for any initial conditions. This fact is used in the proof of the following theorem.

Theorem. *Let* β_i *and* λ *be the largest eigenvalues of* $(I+B_i)^T(I+B_i)$, $i=1,2,3,\cdots$, *and* $(A+A^T)$, *respectively. If there exists a constant* $\alpha>1$ *such that*

$$\ln(\alpha\beta_i)+(\lambda+2M)\delta_i \leq 0, \quad i=1,2,3,\cdots. \tag{9}$$

then the slave system (2) will be globally asymptotically synchronous with the master system (1).

Proof. Let the candidate Lyapuov function be in the form of

$$V(e)=e^Te. \tag{10}$$

The time derivative along the trajectory (7) is

$$
\begin{aligned}
\dot{V}(e) &= \dot{e}^T e + e^T \dot{e} \\
&= \left(Ae + \psi(x,y)\right)^T e + e^T \left(Ae + \psi(x,y)\right) \\
&= e^T\left(A^T + A\right)e - 2e_1e_2x_3 + 2e_1e_3x_2 \\
&\le e^T\left(A^T + A\right)e + 2M\left|e_1e_2\right| + 2M\left|e_1e_3\right| \\
&\le e^T\left(A^T + A\right)e + M\left(2\left|e_1e_2\right| + 2\left|e_1e_3\right| + 2\left|e_2e_3\right|\right) \\
&\le \lambda V\left(e(t)\right) + 2M\left(e_1^2 + e_2^2 + e_3^2\right) \\
&= (\lambda + 2M)V\left(e(t)\right), \qquad t \in \left(t_{i-1}, t_i\right] \text{ for } i = 1, 2, 3, \cdots . \qquad (11)
\end{aligned}
$$

This implies that

$$
V\left(e(t)\right) \le V\left(e\left(t_{i-1}^+\right)\right)\mathrm{e}^{(\lambda+2M)(t-t_{i-1})}, \; t \in \left(t_{i-1}, t_i\right] \text{ for } i = 1, 2, \cdots . \qquad (12)
$$

Now from (7)

$$
\begin{aligned}
V\left(e\left(t_i^+\right)\right) &= \left[(I + B_i)e(t_i)\right]^T (I + B_i)e(t_i) \\
&= e^T(t_i)\left[(I + B_i)^T (I + B_i)\right]e(t_i) \\
&\le \beta_i e^T(t_i)e(t_i) = \beta_i V\left(e(t_i)\right) \qquad (13)
\end{aligned}
$$

When $i = 1$ in inequality (12), then for any $t \in \left(t_0, t_1\right]$,

$$
V\left(e(t)\right) \le V\left(e\left(t_0^+\right)\right)e^{(\lambda+2M)(t-t_0)}. \qquad (14)
$$

This leads to

$$
V\left(e(t_1)\right) \le V\left(e\left(t_0^+\right)\right)e^{(\lambda+2M)(t_1-t_0)}. \qquad (15)
$$

Also from (13) we have

$$
V\left(e\left(t_1^+\right)\right) \le \beta_1 V\left(e(t_1)\right) \le \beta_1 V\left(e\left(t_0^+\right)\right)e^{(\lambda+2M)(t_1-t_0)}. \qquad (16)
$$

In the same way for $t \in \left(t_1, t_2\right]$ we have

$$
\begin{aligned}
V\left(e(t)\right) &\le V\left(e\left(t_1^+\right)\right)e^{(\lambda+2M)(t-t_1)} \\
&\le \beta_1 V\left(e\left(t_0^+\right)\right)e^{(\lambda+2M)(t-t_0)}. \qquad (17)
\end{aligned}
$$

In general for any $t \in \left(t_i, t_{i+1}\right]$ one finds that

$$
V\left(e(t)\right) \le \beta_1\beta_2\cdots\beta_i V\left(e\left(t_0^+\right)\right)e^{(\lambda+2M)(t-t_0)}. \qquad (18)
$$

From the assumption given in the theorem we have

$$\alpha\beta_i e^{(\lambda+2M)\delta_i} \le 1, \quad i = 1, 2, \cdots . \tag{19}$$

Thus for $t \in (t_i, t_{i+1}]$, $i = 1, 2, \cdots$, we have

$$V(e(t)) \le \beta_1\beta_2 \cdots \beta_i V(e(t_0^+))e^{(\lambda+2M)(t-t_0)}$$

$$= V(e(t_0^+))\left[\beta_1 e^{(\lambda+2M)\delta_1}\right]\cdots \times \left[\beta_i e^{2(\lambda(c)+M)\delta_i}\right]e^{(\lambda+2M)(t-t_i)}$$

$$\le V(e(t_0^+))\frac{1}{\alpha^i}e^{(\lambda+2M)(t-t_i)}. \tag{20}$$

This implies that the origin in system (7) is globally asymptotically stable or the slave system is synchronized with the master system asymptotically for any initial conditions. By this we conclude the proof of the theorem.

To be convenient the gain matrices B_i and the impulsive distances δ_i can be chosen to be constant. Thus we have the following corollary.

Corollary. *Suppose* $\delta_i = \delta > 0$ *and gain matrices* $B_i = B(i = 1, 2, \cdots)$. *If there exists a constant* $\alpha > 1$ *such that*

$$\ln(\alpha\beta) + (\lambda + 2M)\delta \le 0. \tag{21}$$

Then the slave system (2) is globally asymptotically synchronous with the master system (1).

4. Numerical simulations

In order to demonstrate and verify the performance of the proposed method, some numerical simulations are presented in this section. The hyperchaotic Lorenz system is given in (4) where a , b , c , d and k are the real constants. Typical phase portraits of this system are shown as Fig. 1, Fig. 2, and Fig. 3.. This system indicates hyperchaotic behavior when $a = 35$, $b = 7$, $c = 12$, $d = 3$, and $k = 5$, it is a forced dissipative system with bounded states $(M \le 22.4927)$ as $t \to \infty$. then

$$\left(A^T + A\right) = \begin{bmatrix} -2a & a+b & 0 & -k \\ a+b & 2c & 0 & 1 \\ 0 & 0 & -2d & 0 \\ -k & 1 & 0 & 0 \end{bmatrix}$$

$$=\begin{bmatrix} -70 & 42 & 0 & -5 \\ 42 & 24 & 0 & 1 \\ 0 & 0 & -6 & 0 \\ -5 & 1 & 0 & 0 \end{bmatrix}, \tag{22}$$

This eigenvalues of this matrix are $-86.3246,\ -6.000,\ 0.2748,$ and $40.0498.$

Thus $\lambda = 40.0498.$

If B_i is a constant matrix $\begin{bmatrix} k & 0 & 0 & 0 \\ 0 & k & 0 & 0 \\ 0 & 0 & k & 0 \\ 0 & 0 & 0 & k \end{bmatrix}$,

It is evident that $\beta = (1+k)^2$. From (9), the estimation of bounds of stable regions are given by

$$0 \le \delta \le -\frac{\ln(\alpha\beta)}{\lambda + 2M} = -\frac{\ln\alpha + \ln(k+1)^2}{85.0352}. \tag{23}$$

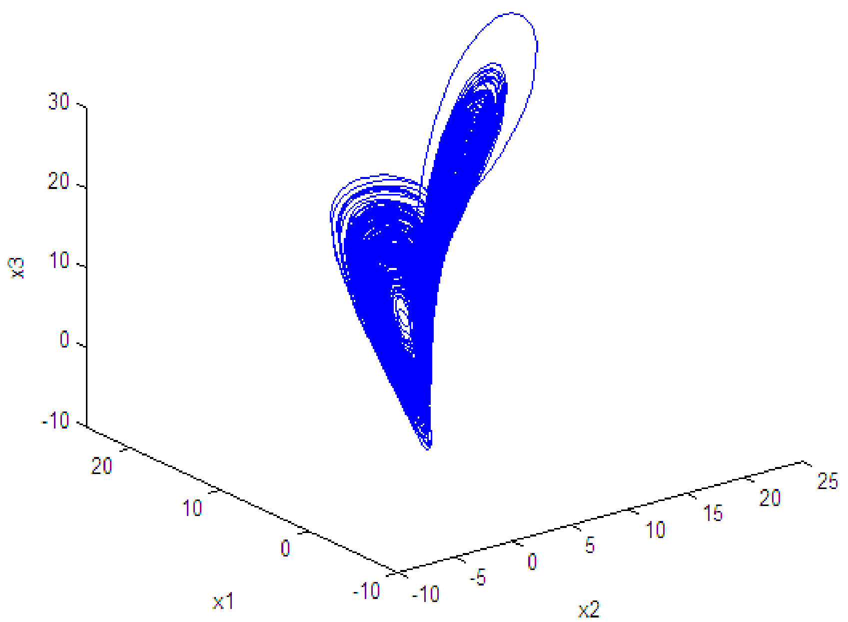

Fig. 1. Phase graph of hyperchaotic Lorenz system with parameters $a = 35, b = 7,$ $c = 12, d = 3,$ Initial condition $[0.05, 0.02, 0.001, 0.05]$.

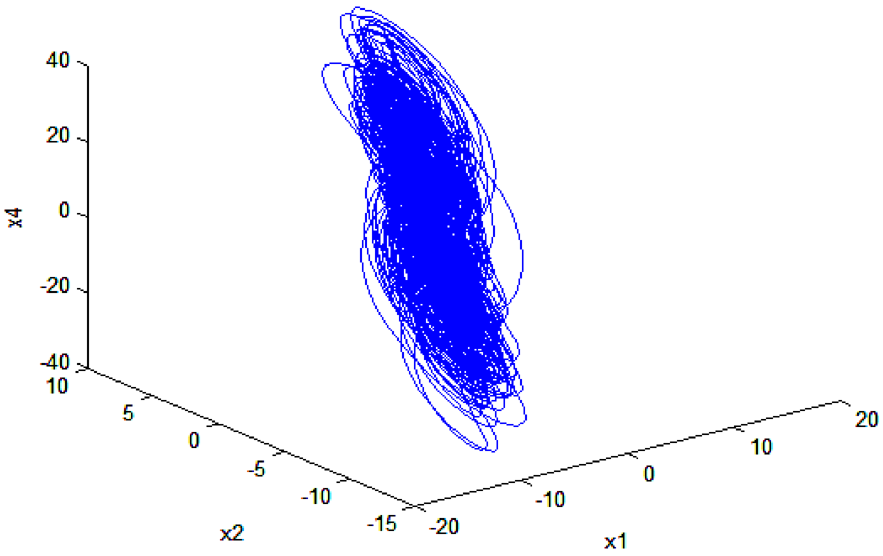

Fig. 2. Phase graph of hyperchaotic Lorenz system with parameters $a=35, b=7, \ c=12,$

$k=5$. initial condition $[0.05, 0.02, 0.001, 0.05]$.

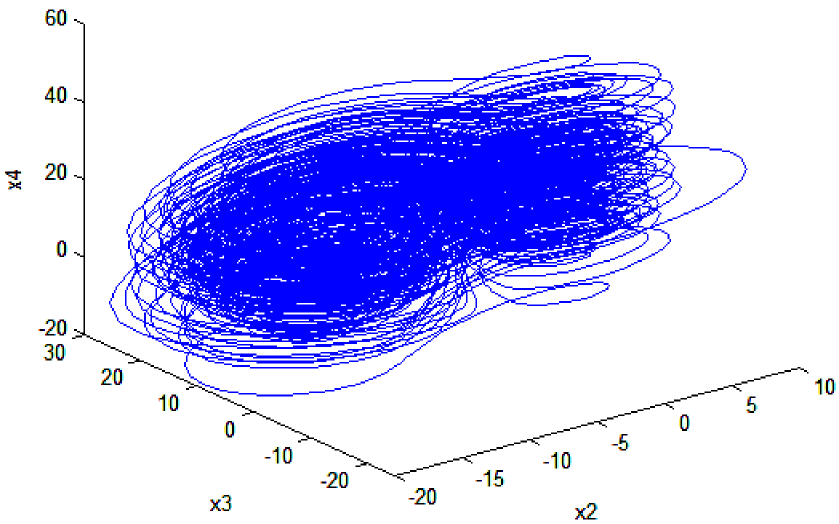

Fig. 3. Phase graph of hyperchaotic Lorenz system with parameters $b=7, \ c=12,$ d=3,

$k = 5$. initial condition $[0.05, 0.02, 0.001, 0.05]$.

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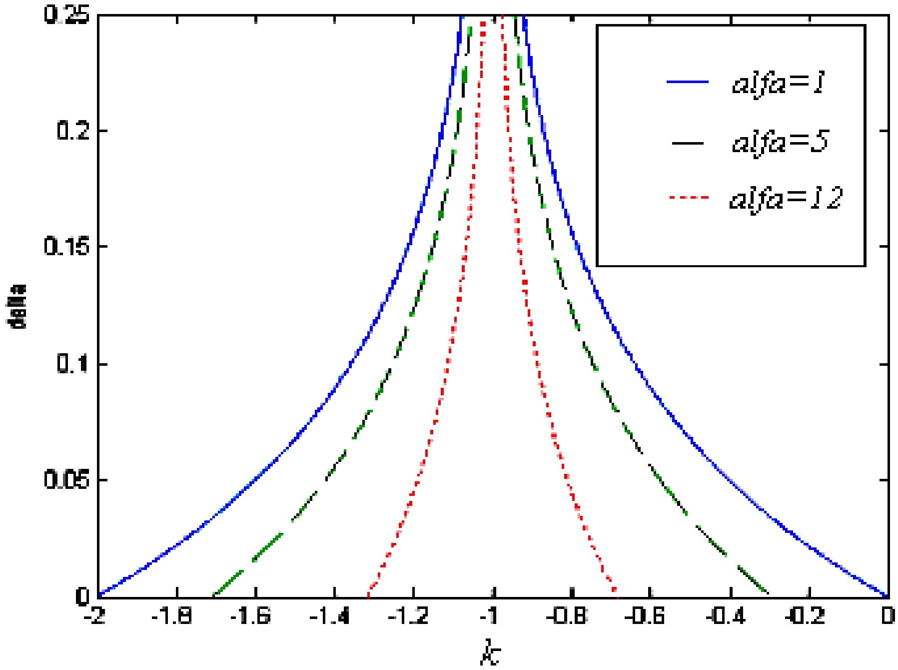

Fig. 4. The boundaries of the stable region for different values of α .

Fig. 4 shows the stable regions for different values of α and k, The entire region below the curve corresponding to given parameter α is the predicted stable region. It can be seen that for $\alpha \to \infty$ the stable region approaches a vertical line $k=-1$. For example if $\alpha=5$ and $k=-1.5$ then $0 \le \delta \le 0.0026$. Fig. 5 shows the curve of impulse time interval $\delta=0.002$ with respect to control parameter $\alpha=5$ and $k=-1.5$.

In the next simulation we choose the gain matrix $B_i = B$ a diagonal matrix as follows:

$B = diag(-1.8,-1.6,-0.8,-0.4)$. Thus we can obtain $\beta=0.64$. For $\alpha=1.2$ and $\delta=0.001$, the condition given in (21) $\ln(\alpha\beta)+(\lambda+2M)\delta \le 0$ is satisfied. The numerical simulation result for this case with $\alpha=1.2$, and $\delta=0.001$ is shown in Fig. 6. As expected in both cases asymptotic synchronization of the hyperchaotic Lorenz system is achieved.

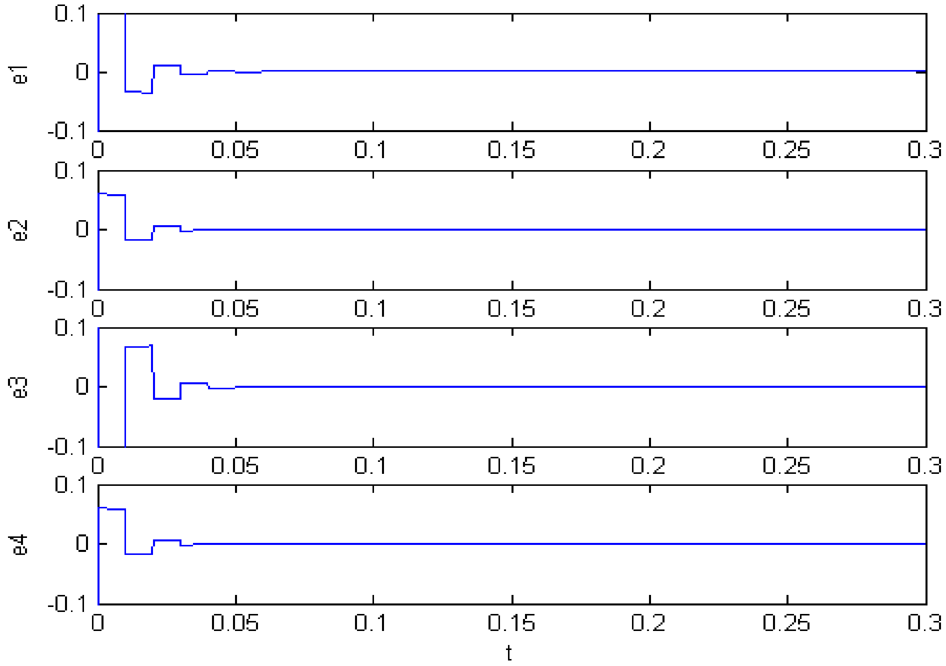

Fig. 5. Time response of the synchronization error system with $\alpha = 5, k = -1.5, \delta = 0.002.$

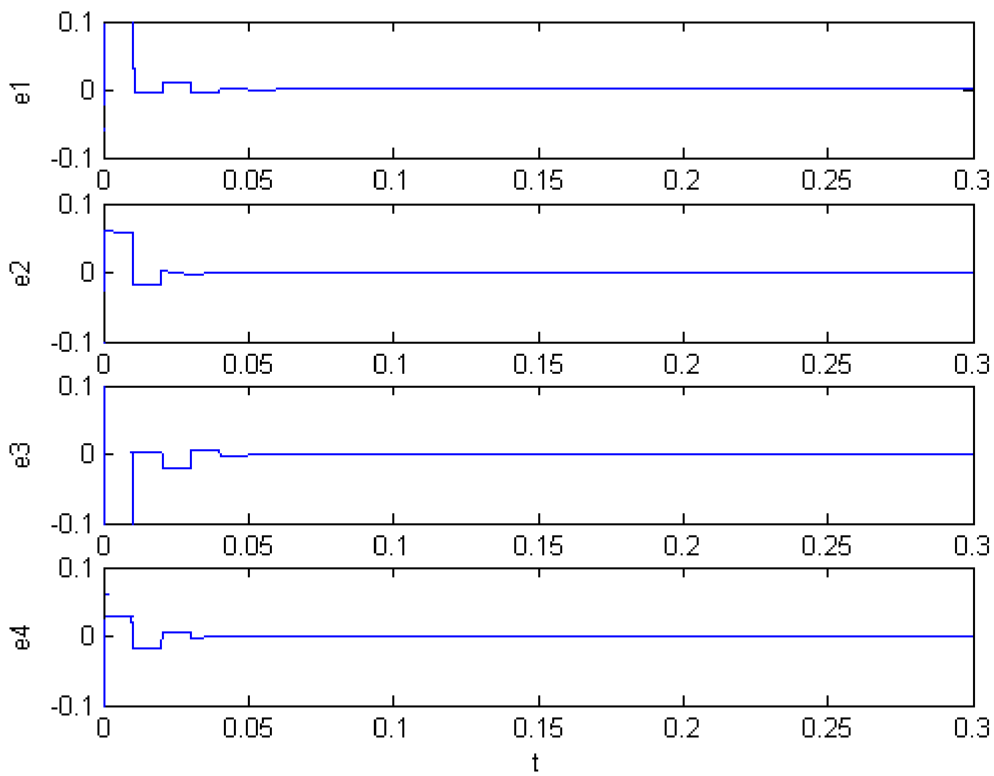

Fig. 6. Time response of the synchronization error system with $\alpha = 1.2, \quad \beta = 0.64, \quad \delta = 0.001.$

Remark 3. We have investigated the issue on the synchronization of hyperchaotic Lorenz system via an impulsive method. In comparison with the schemes reported in the literature, e.g., [14, 15], our method does not require complex mathematical analysis. Moreover, we can see that the stable conditions in this paper are simpler and less conservative. Simulation results in this paper show that the proposed synchronization method is effective and less conservative.

5. Conclusions

In this paper, some simple conditions are obtained in synchronizing systems by equal impulsive distances to guarantee that the impulsive synchronization is globally asymptotically synchronous. The effectiveness of the suggested method has been shown by computer simulation. Since the upper bound of the impulsive interval is related to the system parameters and the impulsive control coefficients, the estimate of the bound is simpler than the method derived from comparison systems. The theory of impulsive synchronization is implementation many systems, especially for the synchronization of chaos in secure communication systems.

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