3-facial colouring of plane graphs

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Frédéric Havet[*] Jean-Sébastien Sereni[‡] Riste Škrekovski[§]

Abstract

A plane graph is ℓ-facially k-colourable if its vertices can be coloured with k colours such that any two distinct vertices on a facial segment of length at most ℓ are coloured differently. We prove that every plane graph is 3-facially 11-colourable. As a consequence, we derive that every 2-connected plane graph with maximum face-size at most 7 is cyclically 11-colourable. These two bounds are just one higher than those that are proposed by the $(3\ell+1)$-Conjecture and the Cyclic Conjecture.

1 Introduction

The concept of facial colorings, introduced by Král', Madaras, and Škrekovski [11, 12], extends the well-known concept of cyclic colorings. A *facial segment* of a plane graph G is a sequence of vertices in the order obtained when traversing a part of the boundary of a face. The *length* of a facial segment is the number of its edges. Two vertices u and v of G are *ℓ-facially adjacent* if there exists a facial segment of length at most ℓ between them. An *ℓ-facial coloring* of G is a function which assigns a color to each vertex of G such that any two distinct ℓ-facially adjacent vertices are assigned with distinct colors. Notice that a vertex of G that is ℓ-facially adjacent to itself does not prevent G from being colored. A graph admitting an ℓ-facial coloring with k colors is called *ℓ-facially k-colorable*.

The following conjecture is called the $(3\ell+1)$-conjecture [11].

Conjecture 1 (Král', Madaras, and Škrekovski). *Every plane graph is ℓ-facially colorable with $3\ell+1$ colors.*

Observe that the bound offered by Conjecture 1 is tight: as shown by Figure 1, for every $\ell \geq 1$, there exists a plane graph that is not ℓ-facially 3ℓ-colorable.

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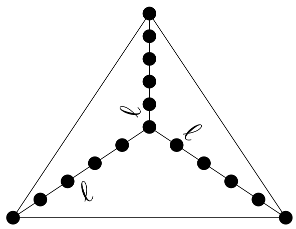

Figure 1: The plane graph $G_\ell = (V,E)$: each thread represents a path of length ℓ. The graph G_ℓ is not ℓ-facially 3ℓ-colorable: any two vertices are ℓ-facially adjacent, and therefore any ℓ-facial coloring must use $|V| = 3\ell+1$ colors.

Conjecture 1 can be considered as a counterpart for ℓ-facial coloring of the following famous conjecture by Ore and Plummer [13] concerning the cyclic coloring. A plane graph G is *cyclically k-colorable* if it admits a vertex coloring with k colors such that any two vertices incident to the same face are assigned distinct colors.

Conjecture 2 (Ore and Plummer). *Every plane graph is cyclically $\lfloor \frac{3\Delta^*}{2} \rfloor$-colorable, where Δ^* is the size of the largest face of G.*

Note that Conjecture 1 implies Conjecture 2 for odd values of Δ^*. The best known result towards Conjecture 2 has been obtained by Sanders and Zhao [16], who proved the bound $\lceil \frac{5\Delta^*}{3} \rceil$.

Define $f_c(x)$ to be the minimum number of colors needed to cyclically color every plane graph of maximum face size x. The value of $f_c(x)$ is known for $x \in \{3,4\}$: $f_c(3) = 4$ (the problem of finding $f_c(3)$ being equivalent to the four color theorem proved by Appel and Haken [1]) and $f_c(4) = 6$ (see [3, 5]). It is also known that $f_c(5) \in \{7,8\}$ and $f_c(6) \leq 10$ [6], and that $f_c(7) \leq 12$ [4].

Conjecture 1 is trivially true for $\ell = 0$, and is equivalent to the four color theorem for $\ell = 1$. It is open for all other values of ℓ. As noted by Král', Madaras, and Škrekovski [11], if Conjecture 1 were true for $\ell = 2$, it would have several interesting corollaries. Besides giving the exact value of $f_c(5)$ (which would then be 7), it would allow the upper bound on the number of colors needed to 1-diagonally color every plane quadrangulation to decrease from 16 to 14 (by applying a method from [11]). (For more details on this problem, consult [9, 14, 15, 11].) It would also imply Wegner's conjecture on 2-distance colorings (i.e., colorings of squares of graphs) restricted to plane cubic graphs since colorings of the square of a plane cubic graph are precisely its 2-facial colorings (refer to the book by Jensen and Toft [10, Problem 2.18] for more details on Wegner's conjecture).

Let $f_f(\ell)$ be the minimum number of colors needed to ℓ-facially color every plane graph. Note that $f_c(2\ell+1) \leq f_f(\ell)$. So far, no value of ℓ is known for which this inequality is strict. The following problem is offered by [11].

Problem 1. *Is it true that, for every integer $\ell \geq 1$, $f_c(2\ell+1) = f_l(\ell)$?*

Another conjecture that should maybe be mentioned is the so-called 3ℓ-conjecture proposed by Dvořák, Škrekovski, and Tancer [7], stating that every plane triangle-free graph is ℓ-facially

3ℓ-colorable. As for the $(3\ell+1)$-conjecture, if this conjecture were true, then its bound would be tight and it would have several interesting corollaries (see [7] for more details).

Král', Madaras, and Škrekovski [11] proved that every plane graph has an l-facial coloring using at most $\left\lfloor \frac{18}{5}\ell \right\rfloor + 2\ \left\lfloor \frac{18}{5}\ell \right\rfloor + 2$ colors (and this bound is decreased by 1 for $\ell \in \{2,4\}$). So, in particular, every plane graph has a 3-facial 12-coloring. In this paper, we improve this last result by proving the following theorem.

Theorem 1. *Every plane graph is* 3*-facially* 11*-colorable.*

To prove this result, we suppose that it is false. In section 2, we exhibit some properties of a minimal graph (regarding the number of vertices) that contradict Theorem 1. Relying on these properties, we use the discharging method in section 3 to obtain a contradiction.

2 Properties of (3,11)-minimal graphs

Let us start this section by introducing some definitions. A vertex of degree d (at least d, at most d) is said to be a *d-vertex* (a *$(\geq d)$-vertex*, a *$(\leq d)$-vertex*, respectively). The notion of a *d-face* (a *$(\leq d)$-face*, a *$(\geq d)$-face*, respectively) is defined analogously regarding the size of a face. An *ℓ-path* is a path of length ℓ.

Two faces are *adjacent*, or *neighboring*, if they share a common edge. A 5-face is *bad* if it is incident to at least four 3-vertices. It is said to be *very bad* if it is incident to five 3-vertices.

If u and v are 3-facially adjacent, then u is a 3*-facial neighbor of* v. The set of all 3-facial neighbors of v is $\mathcal{N}_3(v)$. The 3*-facial degree* of v is $\deg_3(v) = |\mathcal{N}_3(v)|$. A vertex is *dangerous* if it has degree three and is incident to a face of size three or four. A 3-vertex is *safe* if it is not dangerous, i.e., is not incident to a (≤ 4)-face.

Let $G = (V,E)$ be a plane graph, and $\mathcal{U} \subseteq V$. Let $G_3[\mathcal{U}]$ be the graph with vertex set $\mathcal{U}$ such that xy is an edge in $G_3[\mathcal{U}]$ if and only if x and y are 3-facially adjacent vertices in G. If c is a partial coloring of G and u an uncolored vertex of G, we let $L_c(u)$ (or just $L(u)$) be the set $\{x \in \{1,2,\ldots,11\} :$ for all $v \in \mathcal{N}_3(u), c(v) \neq x\}$. The graph $G_3[\mathcal{U}]$ is *L-colorable* if there exists a proper vertex coloring of the vertices of $G_3[\mathcal{U}]$ such that for every $u \in \mathcal{U}$ we have $c(u) \in L(u)$.

The next two results are used by Král', Madaras, and Škrekovski [11].

Lemma 1. *Let v be a vertex whose incident faces in a* 2*-connected plane graph G are $f_1, f_2, \ldots, f_d$. Then*

$$\deg_3(v) \leq \left(\sum_{i=1}^{d} \min(|f_i|, 7) \right) - 2d,$$

where $|f_i|$ is the size of the face f_i.

Suppose that Theorem 1 is false: a $(3,11)$*-minimal* graph G is a plane graph that is not 3-facially 11-colorable, with $|V(G)| + |E(G)|$ as small as possible.

Lemma 2. *Let G be a* $(3,11)$*-minimal graph. Then the following hold:*

1. *G is* 2*-connected.*

2. *G has no separating cycle of length at most* 7.

3. *G contains no adjacent* f_1*-face and* f_2*-face with* $f_1 + f_2 \leq 9$.

4. *G has no vertex whose* 3*-facial degree is less than* 11*. In particular, the minimum degree of G is at least three.*

5. *G contains no edge uv separating two* (≥ 4)*-faces with* $\deg_3(u) \leq 11$ *and* $\deg_3(v) \leq 12$*. In particular, if two adjacent dangerous vertices do not lie on a same* (≤ 4)*-face, then none of them is incident to a* 3*-face.*

In the remainder of this section, we give additional local structural properties of $(3,11)$-minimal graphs.

Lemma 3. *Let G be a* $(3,11)$*-minimal graph. Suppose that v and w are two adjacent* 3*-vertices of G, both incident to a same* 5*-face and a same* 6*-face. Then the size of the third face incident to w is at least* 7.

Proof. By contradiction, suppose that the size of the last face incident to w is at most 6. Then, according to Lemma 1, we infer that $\deg_3(v) \leq 12$ and $\deg_3(w) \leq 11$, but this contradicts Lemma 2(v). □

A *reducible configuration* is a (plane) graph that cannot be an induced subgraph of a $(3,11)$-minimal graph. The usual method to prove that a configuration is reducible is the following: first, we suppose that a $(3,11)$-minimal graph G contains a prescribed induced subgraph H. Then we contract some subgraphs $H_1, H_2, \ldots, H_k$ of H. In most of the cases, $k \leq 2$. This yields a proper minor G' of G, which by the minimality of G admits a 3-facial 11-coloring c'. The goal is to derive from c' a 3-facial 11-coloring c of G, which would give a contradiction. To do so, each noncontracted vertex v of G keeps its color $c'(v)$. Let h_i be the vertex of G' created by the contraction of the vertices of H_i: some vertices of H_i are assigned the color $c'(h_i)$ (in doing so, we must take care that these vertices are not 3-facially adjacent in G). Last, we show that the remaining uncolored vertices can also be colored.

In other words, we show that the graph $G_3[\mathcal{U}]$ is L-colorable, where for each $u \in \mathcal{U}$, $L(u)$ is the list of the colors that are assigned to no vertex in $\mathcal{N}_3(u) \setminus \mathcal{U}$ (defined in section 1) and $\mathcal{U}$ is the set of uncolored vertices. In most of the cases, the vertices of $\mathcal{U}$ will be greedily colored.

In all figures of the paper, the following conventions are used: a triangle represents a 3-vertex, a square represents a 4-vertex, and a circle may be any kind of vertex whose degree is at least the maximum between three and the one it has in the figure. The edges of each subgraph H_i are drawn in bold, and the circled vertices are the vertices of $\mathcal{U} = \{u_1, u_2, \ldots\}$. A dashed edge between two vertices indicates a path of length at least one between those two vertices. An (in)equality written in a bounded region indicates a face whose size achieves the (in)equality. Last, vertices that are assigned the color $c'(h_i)$ are v, w, and t if a unique subgraph is contracted or x_1, x_2 for $i = 1$ and y_1, y_2 for $i = 2$ if two subgraphs are contracted.

Note that the graph G' may contain loops and parallel edges. One way to consider them is as follows. If there is a face of size at most two, then we can just remove the loop, or one of the parallel edges. Otherwise, there is a separating cycle C of length at most two, and we can first color the subgraph of G' induced by C and the vertices inside C, and then the subgraph of G' induced by C and the vertices outside C.

Lemma 4. *Configurations in Figures* 2, 3, *and* 4 *are reducible.*

Proof. Let H be an induced subgraph of G. We suppose that H is isomorphic to one of the configurations stated and derive a way to construct a 3-facial 11-coloring of G, a contradiction.

$L1$ Suppose that H is isomorphic to the configuration (L1) of Figure 2. The edge u_2u_3 cannot be incident to a 3-face, since otherwise the edge u_2u_5 would contradict Lemma 2.5. More precisely, it would be incident to two (≥ 7)-faces by Lemma 2.3, and the 3-facial degree of u_2 and u_5 would be at most 11. Let H_1 be the subgraph induced by the bold edges. Contract the vertices of H_1, thereby creating a new vertex h_1. By minimality of G, let c' be a 3-facial 11-coloring of the obtained graph. Assign to each vertex x not in H_1 the color $c'(x)$, and to each of v,w,t the color $c'(h_1)$. Observe that, since the edge u_2u_3 does not lie on a 3-face, no two vertices among v,w,t are 3-facially adjacent in G; otherwise there would be a (≤ 7)-separating cycle in G, thereby contradicting Lemma 2.2. According to Lemma 1, $\deg_3(u_1) \leq 15$, $\deg_3(u_i) \leq 14$ if $i \in \{2,3\}$, and $\deg_3(u_i) \leq 11$ if $i \in \{4,5\}$. Note that any two vertices of $\mathcal{U} = \{u_1,u_2,\ldots,u_5\}$ are 3-facially adjacent; that is, $G_3[\mathcal{U}] \simeq K_5$. Hence, the number of colored 3-facial neighbors of u_1 is at most 11; i.e., $|\mathcal{N}_3(u_1) \setminus \{u_2,u_3,u_4,u_5\}| \leq 11$. Moreover, at least two of them are assigned the same color, namely v and w. Therefore, $|L(u_1)| \geq 1$. For $i \in \{2,3\}$, the vertex u_i has at most 10 colored 3-facial neighbors. Furthermore, at least two 3-facial neighbors of u_2 are identically colored, namely w and t. Thus, $|L(u_2)| \geq 2$. Now, observe that at least three 3-facial neighbors of u_3 are colored the same, namely $v,w,$ and t. Hence, $|L(u_3)| \geq 3$. For $i \in \{4,5\}$, the vertex u_i has at most 7 colored 3-facial neighbors. Thus, $|L(u_4)| \geq 4$, and because at least two 3-facial neighbors of u_5 are identically colored (w and t), $|L(u_5)| \geq 5$. So, the graph $G_3[\mathcal{U}]$ is greedily L-colorable, according to the ordering u_1,u_2,u_3,u_4,u_5. This allows us to extend c to a 3-facial 11-coloring of G.

$L2$ Suppose that H is isomorphic to the configuration (L2) of Figure 2. Assume first that the edge u_2u_3 is not incident to a 3-face. Let c' be a 3-facial 11-coloring of the minor of G obtained by contracting the bold edges into a single vertex h_1. Let $c(x) = c'(x)$ for every vertex $x \neq h_1$. Define $c(v) = c(w) = c(t) = c'(h_1)$. The obtained coloring is still 3-facial since no two vertices among v,w,t are 3-facially adjacent in G by Lemma 2.2, and because of our assumption. Note that $G_3[\mathcal{U}] \simeq K_5$. In particular, each vertex u_i has four uncolored 3-facial neighbors. By Lemma 1, $\deg_3(u_1) \leq 15, \deg_3(u_i) \leq 14$ if $i \in \{2,3\}$, and $\deg_3(u_i) \leq 11$ if $i \in \{4,5\}$. Moreover, each of u_1 and u_2 has at least two 3-facial neighbors colored the same; for u_1, these vertices are w,t, and for u_2 they are w,v. So, there exists at least one color which is assigned to no vertex of $\mathcal{N}_3(u_1)$ and at least two colors assigned to no vertex of $\mathcal{N}_3(u_2)$. Also, u_3 has at least three 3-facial neighbors colored the same, namely w, v, and t; hence at least three colors are assigned to no vertex of $\mathcal{N}_3(u_3)$. Therefore, $|L(u_1)| \geq 1, |L(u_2)| \geq 2$, and $|L(u_3)| \geq 3$. Furthermore, $|L(u_4)| \geq 4$ and $|L(u_5)| \geq 5$ because w and t are both 3-facial neighbors of u_5. So $G_3[\mathcal{U}]$ is L-colorable, and hence G is 3-facially 11-colorable.

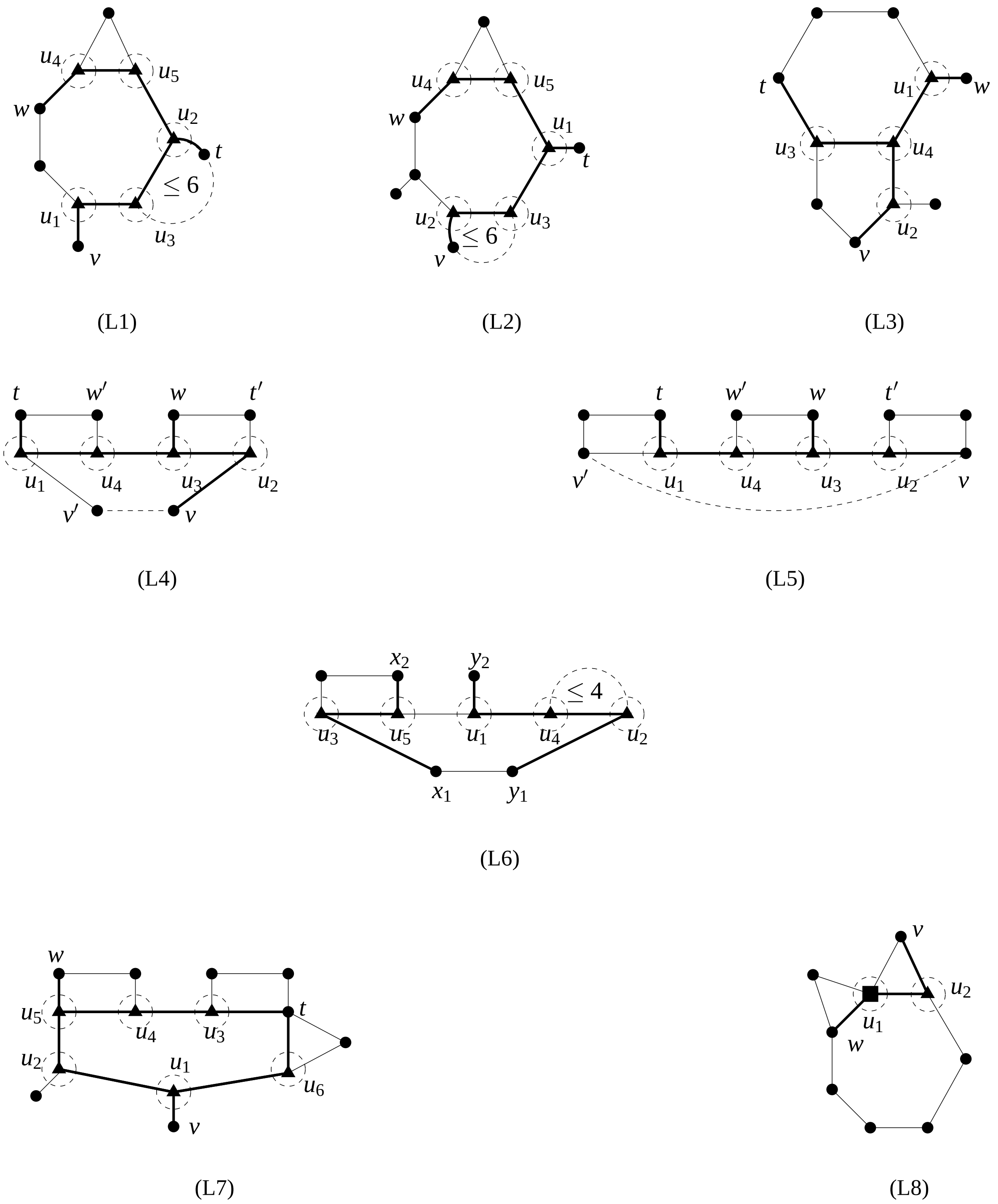

Figure 2: Reducible configurations (L1)–(L8).

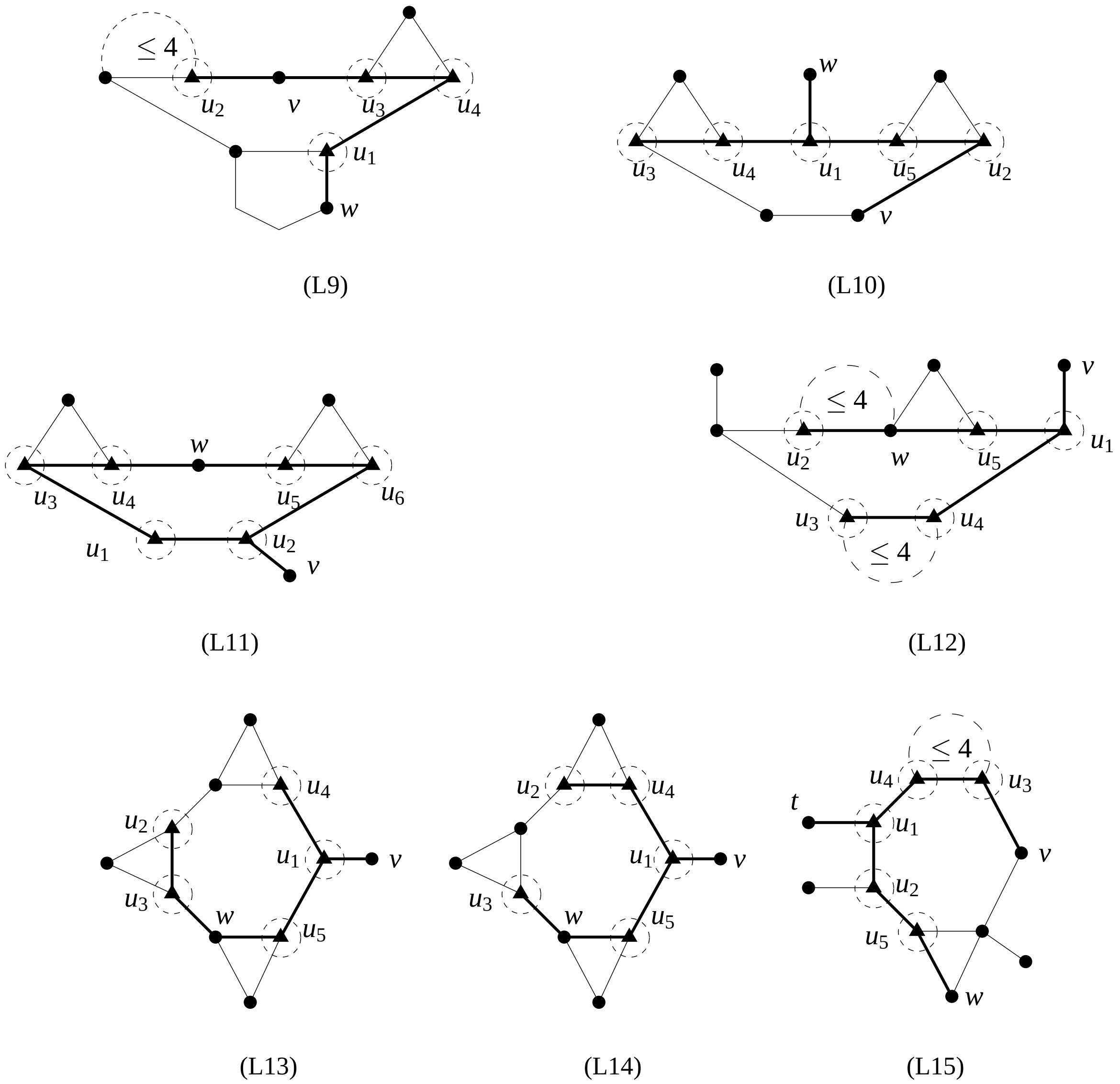

Figure 3: Reducible configurations (L9)–(L15).

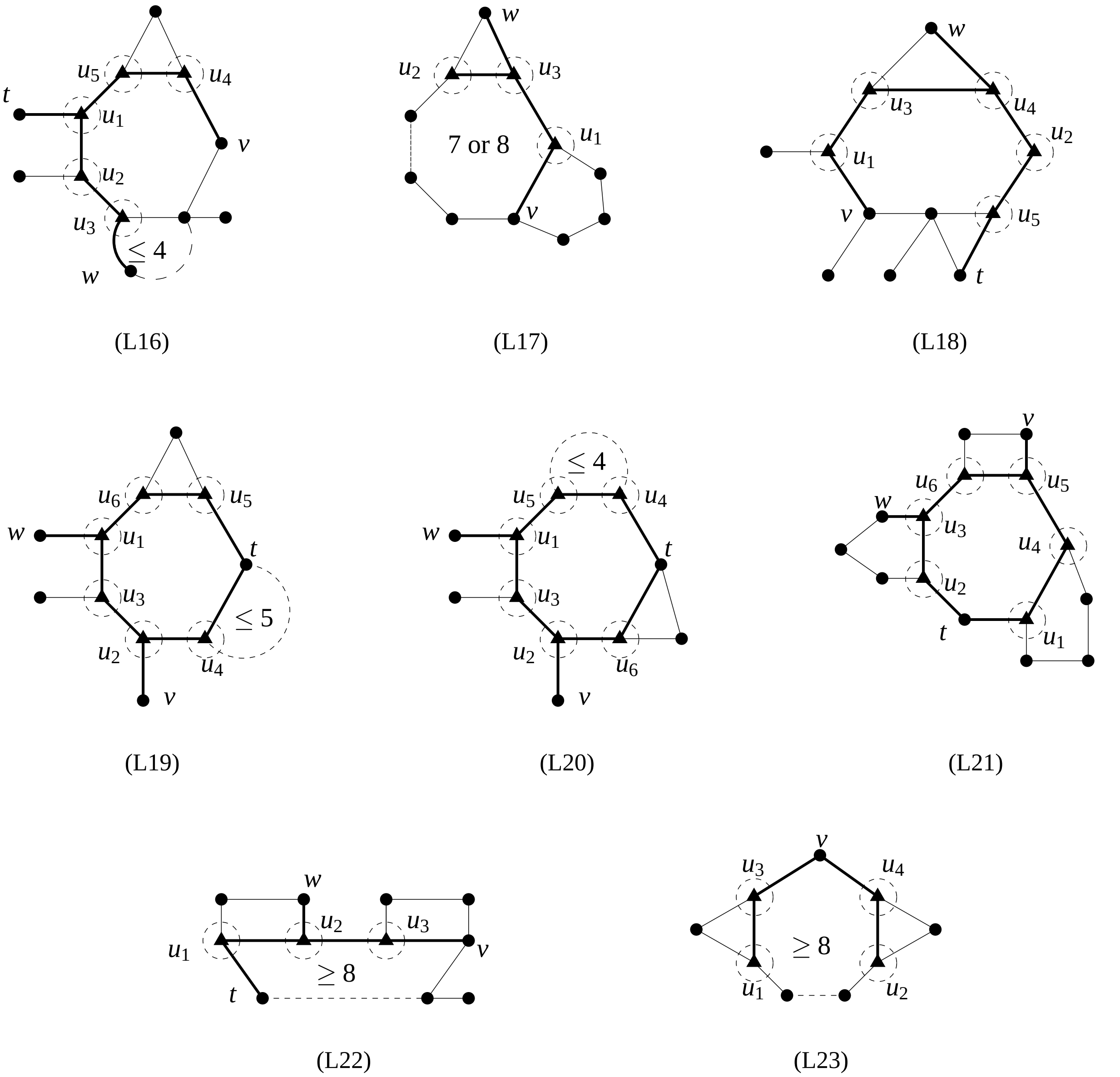

Figure 4: Reducible configurations (L16)–(L23).

If the edge u_2u_3 is incident to a 3-face, then the same proof works, except that at the beginning the edge u_2v is not contracted. Thus, only the vertices w and t have the same color, but the partial coloring extends as previously to G since u_2 and u_3 both have now 3-facial degree 11.

*L*3 Suppose that H is isomorphic to the configuration (L3) of Figure 2. Contract the bold edges into a new vertex h_1, and let c' be a 3-facial 11-coloring of the obtained graph. This coloring can be extended to a 3-facial 11-coloring c of G as follows: first, let $c(v) = c(w) = c(t) = c'(h_1)$. Note that no two of these vertices can be 3-facially adjacent in G without contradicting Lemma 2.2. By Lemma 1, $\deg_3(u_1) \leq 14$, $\deg_3(u_2) \leq 13$ and for $i \in \{3,4\}$, and $\deg_3(u_i) \leq 12$. Observe that $G_3[\mathcal{U}] \simeq K_4$. Moreover, each of u_1, u_2, u_3 has a set of two 3-facial neighbors colored by $c'(h_1)$. These sets are $\{w,t\}$, $\{w,v\}$, and $\{v,t\}$ for u_1, u_2, and u_3, respectively. Thus, $|L(u_1)| \geq 1$, $|L(u_2)| \geq 2$, and $|L(u_3)| \geq 3$. Also $|L(u_4)| \geq 4$ because u_4 has at least three identically colored 3-facial neighbors, namely v, w, and t. Hence, $G_3[\mathcal{U}]$ is L-colorable, and so G is 3-facially 11-colorable.

*L*4 First, observe that if $v \in \mathcal{N}_3(t)$, then $v' \notin \mathcal{N}_3(t')$, since G is a plane graph. So, by symmetry, we may assume that v and t are not 3-facially adjacent in G. Now, contract the bold edges into a new vertex h_1. Again, let c' be a 3-facial 11-coloring of the obtained graph, and define c to be equal to c' on all vertices of $V(G) \setminus \{v,w,t,u_1,u_2,u_3,u_4\}$. Let $c(v) = c(w) = c(t) = c'(h_1)$. Note that the partial coloring c is still 3-facial due to the above assumption. The graph $G_3[\mathcal{U}]$ is isomorphic to K_4, and according to Lemma 1, $\deg_3(u_i) \leq 12$ for all $i \in \{1,2,3,4\}$. Moreover, for $i \in \{2,3\}$, the vertex u_i has at least two 3-facial neighbors that are colored the same, namely, v and w. Last, the vertex u_4 has at least three such 3-facial neighbors, namely v, w, t. Therefore, $|L(u_1)| \geq 2$, $|L(u_i)| \geq 3$ for $i \in \{2,3\}$, and $|L(u_4)| \geq 4$. So, $G_3[\mathcal{U}]$ is L-colorable, and hence G is 3-facially 11-colorable.

*L*5 The same remark as in the previous configuration allows us to assume that $t \notin \mathcal{N}_3(v)$. Again, the graph obtained by contracting the bold edges into a new vertex h_1 admits a 3-facial 11-coloring c'. As before, define a 3-facial 11-coloring c of the graph induced by $V(G) \setminus \mathcal{U}$. Then, for every $i \in \{1,2,3,4\}, \deg_3(u_i) \leq 12$ and $G_3[\mathcal{U}] \simeq K_4$. Thus, $|L(u_1)| \geq 2$ and $|L(u_2)| \geq 2$. Note that u_3 has at least two identically colored 3-facial neighbors, namely v and w, so $|L(u_3)| \geq 3$. Last, the vertex u_4 has at least three such neighbors, hence $|L(u_4)| \geq 4$. Therefore, the graph $G_3[\mathcal{U}]$ is L-colorable, and so the graph G admits a 3-facial 11-coloring.

*L*6 Let H_1 be the path $x_1u_3u_5x_2$, H_2 the path $y_1u_2u_4u_1y_2$, and c' a 3-facial coloring of the graph obtained from G by contracting each path H_i into a vertex h_i. Notice that $c'(h_1) \neq c'(h_2)$. For every $v \notin V(H_1) \cup V(H_2)$, let $c(v) = c'(v)$. Observe that x_1 and x_2 cannot be 3-facially adjacent in G, otherwise G would have a separating (≤ 7)-cycle, contradicting Lemma 2.2. Note that the same holds for y_1 and y_2; therefore defining $c(x_1) = c(x_2) = c'(h_1)$ and $c(y_1) = c(y_2) = c'(h_2)$ yields a partial 3-facial 11-coloring of G, since $c'(h_1) \neq c'(h_2)$. It remains to color the vertices of $\mathcal{U} = \{u_1, u_2, \ldots, u_5\}$. Note that $G_3[\mathcal{U}] \simeq K_5$. According to Lemma 2.2, $\deg_3(u_1) \leq 15$ and $\deg_3(u_i) \leq 12$ if $i \geq 2$. The number of colored 3-facial neighbors of u_1, i.e., its number of 3-facial neighbors in $V(G)\setminus\{u_2,u_3,u_4,u_5\}$, is at most 11 because each u_i with $i\geq 2$ is a 3-facial neighbor of u_1. Furthermore, u_1 has two 3-facial neighbors colored with the same color, namely x_1 and x_2. Hence, $|L(u_1)|\geq 1$. The vertex u_2 has four uncolored 3-facial neighbors, so $|L(u_2)|\geq 3$. For $i\in\{3,4\}$, the vertex u_i has at least two 3-facial neighbors colored the same, namely x_1,x_2 for u_3, and y_1,y_2 for u_4, so $|L(u_i)|\geq 4$. Finally, observe that u_5 has two pairs of identically colored 3-facial neighbors; the first pair being x_1,x_2 and the second y_1,y_2. Thus, $|L(u_5)|\geq 5$, and hence the graph $G_3[\mathcal{U}]$ is L-colorable, which yields a contradiction.

*L*7 We contract the bold edges into a new vertex h_1, take a 3-facial 11-coloring of the graph obtained, and define a 3-facial 11-coloring c of $V(G)\setminus\mathcal{U}$ as usual. By Lemma 1, $\deg_3(u_i)\leq 15$ if $i\in\{1,2\}$, $\deg_3(u_i)\leq 12$ if $i\in\{3,4,5\}$, and $\deg_3(u_6)\leq 11$. Moreover, $G_3[\mathcal{U}]\simeq K_6$. As v,w, and t are colored the same, and $\{v,w\}\subset\mathcal{N}_3(u_2)$, $\{w,t\}\subset\mathcal{N}_3(u_i)$ for $i\in\{4,5\}$, and $\{v,t\}\subset\mathcal{N}_3(u_6)$, we obtain $|L(u_i)|\geq i$ for every $i\in\{1,2,3,4,5,6\}$. Thus, the graph $G_3[\mathcal{U}]$ is L-colorable, and hence G admits a 3-facial 11-coloring.

*L*8 We contract the bold edges into a new vertex, take a 3-facial 11-coloring of the graph obtained, and define a 3-facial 11-coloring of $V(G)\setminus\mathcal{U}$ as usual. Then, $G_3[\mathcal{U}]\simeq K_2$. Moreover, $\deg_3(u_1)\leq 12$ and $\deg_3(u_2)\leq 11$. Furthermore, $\{v,w\}\subset\mathcal{N}_3(u_i)$ for $i\in\{1,2\}$. Thus, we infer $|L(u_i)|\geq i$ for $i\in\{1,2\}$. Therefore, $G_3[\mathcal{U}]$ is L-colorable.

*L*9 We contract the bold edges into a new vertex, take a 3-facial 11-coloring of the graph obtained, and define a 3-facial 11-coloring of $V(G)\setminus\mathcal{U}$ as usual. Then, $G_3[\mathcal{U}]\simeq K_4$. Moreover, $\deg_3(u_1)\leq 13$, $\deg_3(u_2)\leq 12$, and $\deg_3(u_i)\leq 11$ for $i\in\{3,4\}$. Furthermore, $\{v,w\}\subset\mathcal{N}_3(u_i)$ for $i\in\{1,4\}$. Thus, we infer $|L(u_i)|\geq 2$ for $i\in\{1,2\}$, and $|L(u_i)|\geq i$ for $i\in\{3,4\}$. Therefore, $G_3[\mathcal{U}]$ is L-colorable.

*L*10 We contract the bold edges into a new vertex h_1, take a 3-facial 11-coloring of the graph obtained, and define a 3-facial 11-coloring c of $V(G)\setminus\mathcal{U}$ as usual. By Lemma 1, $\deg_3(u_1)\leq 15$ and $\deg_3(u_i)\leq 11$ if $i\in\{2,3,4,5\}$. Moreover, $G_3[\mathcal{U}]\simeq K_5$. As v and w are colored the same, and $\{v,w\}\subset\mathcal{N}_3(u_i)$ for $i\in\{1,4,5\}$, we obtain $|L(u_1)|\geq 1$, $|L(u_i)|\geq 4$ if $i\in\{2,3\}$, and $|L(u_i)|\geq 5$ if $i\in\{4,5\}$. Thus, the graph $G_3[\mathcal{U}]$ is L-colorable, and hence G admits a 3-facial 11-coloring.

*L*11 Let c' be a 3-facial 11-coloring of the graph G' obtained by contracting the bold edges into a new vertex h_1. Define $c(x)=c'(x)$ for every vertex $x\in V(G)\cap V(G')$, and let $c(v)=c(w)=c'(h_1)$. By Lemma 1, $\deg_3(u_i)\leq 15$ for $i\in\{1,2\}$ and $\deg_3(u_i)\leq 11$ for $i\in\{3,4,5\}$. Moreover, $G_3[\mathcal{U}]\simeq K_6$. Hence, $|L(u_1)|\geq 1$ and $|L(u_i)|\geq i$ for $i\in\{3,4,5\}$. As v and w are colored the same, and $\{v,w\}\subset\mathcal{N}_3(u_i)$ for $i\in\{2,6\}$, we infer that $|L(u_2)|\geq 2$ and $|L(u_6)|\geq 6$. Thus, the graph G is 3-facially 11-colorable.

$L12$ Let us define the partial 3-facial 11-coloring c as always, regarding the bold edges and the vertices v and w. From Lemma 1 we obtain that $\deg_3(u_1) \le 15$, $\deg_3(u_i) \le 12$ for $i \in \{2,3,4\}$, and $\deg_3(u_5) \le 11$. Moreover, since $G_3[\mathcal{U}] \simeq K_5$ and $\{v,w\} \subset \mathcal{N}_3(u_i)$ for $i \in \{1,4,5\}$, we obtain $|L(u_1)| \ge 1$, $|L(u_i)| \ge 3$ for $i \in \{2,3\}$, $|L(u_4)| \ge 4$, and $|L(u_5)| \ge 5$. Therefore, $G_3[\mathcal{U}]$ is L-colorable.

$L13$ Define the partial 3-facial 11-coloring c as usual, regarding the bold edges and the vertices v and w. By Lemma 1, $\deg_3(u_1) \le 15$ and $\deg_3(u_i) \le 11$ for $i \in \{2,3,4,5\}$. Moreover, since $G_3[\mathcal{U}] \simeq K_5$ and $\{v,w\} \subset \mathcal{N}_3(u_i)$ for $i \in \{1,5\}$, we obtain $|L(u_1)| \ge 1$, $|L(u_i)| \ge 4$ for $i \in \{2,3,4\}$, and $|L(u_5)| \ge 5$. Therefore, $G_3[\mathcal{U}]$ is L-colorable.

$L14$ Let us define the partial 3-facial 11-coloring c as always, regarding the bold edges and the vertices v and w. Again, $G_3[\mathcal{U}] \simeq K_5$. From Lemma 1 we obtain that $\deg_3(u_1) \le 15$ and $\deg_3(u_i) \le 11$ if $i \in \{2,3,4,5\}$. Moreover, since $\{v,w\} \subset \mathcal{N}_3(u_i)$ for $i \in \{1,5\}$, we obtain $|L(u_1)| \ge 1$, $|L(u_i)| \ge 4$ for $i \in \{2,3,4\}$, and $|L(u_5)| \ge 5$. Therefore, $G_3[\mathcal{U}]$ is L-colorable.

$L15$ Define the partial 3-facial 11-coloring c as always, regarding the bold edges and the vertices v, w, and t. Then, $G_3[\mathcal{U}] \simeq K_5$ and $\deg_3(u_i) \le 15$ for $i \in \{1,2\}$, $\deg_3(u_i) \le 12$ for $i \in \{3,4\}$, and $\deg_3(u_5) \le 11$. Moreover, notice that $\{v,t\} \subset \mathcal{N}_3(u_i)$ for $i \in \{1,4\}$, $\{v,w,t\} \subset \mathcal{N}_3(u_2)$, and $\{v,w\} \subset \mathcal{N}_3(u_5)$. Thus, we obtain $|L(u_1)| \ge 1$, $|L(u_2)| \ge 2$, $|L(u_3)| \ge 3$, $|L(u_4)| \ge 4$, and $|L(u_5)| \ge 5$. Therefore, $G_3[\mathcal{U}]$ is L-colorable.

$L16$ Define the partial 3-facial 11-coloring c as always, regarding the bold edges and the vertices v, w, and t. Then, $G_3[\mathcal{U}] \simeq K_5$ and $\deg_3(u_i) \le 15$ for $i \in \{1,2\}$, $\deg_3(u_3) \le 12$, and $\deg_3(u_i) \le 11$ for $i \in \{4,5\}$. Moreover, notice that $\{v,t\} \subset \mathcal{N}_3(u_i)$ for $i \in \{1,5\}$, $\{v,w,t\} \subset \mathcal{N}_3(u_2)$, and $\{v,w\} \subset \mathcal{N}_3(u_3)$. Thus, we obtain $|L(u_1)| \ge 1$, $|L(u_2)| \ge 2$, $|L(u_i)| \ge 4$ for $i \in \{3,4\}$, and $|L(u_5)| \ge 5$. Therefore, $G_3[\mathcal{U}]$ is L-colorable.

$L17$ Let us define the partial 3-facial 11-coloring c as always, regarding the bold edges and the vertices v and w. Then, $G_3[\mathcal{U}] \simeq K_3$, $\deg_3(u_1) \le 13$, and $\deg_3(u_i) \le 11$ for $i \in \{2,3\}$. Moreover, $\{v,w\} \subset \mathcal{N}_3(u_i)$ for $i \in \{1,2,3\}$. Thus, we obtain $|L(u_1)| \ge 1$ and $|L(u_i)| \ge 3$ for $i \in \{2,3\}$. Therefore, $G_3[\mathcal{U}]$ is L-colorable.

$L18$ Again, $G_3[\mathcal{U}] \simeq K_5$ and $\deg_3(u_i) \le 15$ for $i \in \{1,2\}$, while $\deg_3(u_i) \le 11$ for $i \in \{3,4,5\}$. Furthermore, $\{v,w\} \subset \mathcal{N}_3(u_i)$ for $i \in \{1,3,4\}$, $\{v,t\} \subset \mathcal{N}_3(u_5)$, and $\{v,w,t\} \subset \mathcal{N}_3(u_2)$. Thus, we deduce $|L(u_1)| \ge 1$, $|L(u_2)| \ge 2$, and $|L(u_i)| \ge 5$ for $i \in \{3,4,5\}$. Therefore, $G_3[\mathcal{U}]$ is L-colorable.

$L19$ Here, $G_3[\mathcal{U}] \simeq K_6$. Also, $\deg_3(u_i) \le 15$ for $i \in \{1,2,3\}$, $\deg_3(u_4) \le 13$, and $\deg_3(u_i) \le 11$ for $i \in \{5,6\}$. Furthermore, $\{w,t\} \subset \mathcal{N}_3(u_i)$ for $i \in \{1,6\}$, $\{v,w,t\} \subset \mathcal{N}_3(u_3)$, and $\{v,t\} \subset \mathcal{N}_3(u_i)$ for $i \in \{2,4\}$. Thus, we infer $|L(u_i)| \ge 2$ for $i \in \{1,2\}$, $|L(u_3)| \ge 3$, $|L(u_4)| \ge 4$, $|L(u_5)| \ge 5$, and $|L(u_6)| \ge 6$. Therefore, $G_3[\mathcal{U}]$ is L-colorable.

*L*20 Again $G_3[\mathcal{U}] \simeq K_6$. Also, $\deg_3(u_i) \leq 15$ for $i \in \{1,2,3\}$, $\deg_3(u_i) \leq 12$ for $i \in \{4,5\}$, and $\deg_3(u_6) \leq 11$. Furthermore, $\{w,t\} \subset \mathcal{N}_3(u_i)$ for $i \in \{1,5\}$, $\{v,w,t\} \subset \mathcal{N}_3(u_3)$, and $\{v,t\} \subset \mathcal{N}_3(u_i)$ for $i \in \{2,6\}$. Thus, we infer $|L(u_i)| \geq 2$ for $i \in \{1,2\}$ and $|L(u_i)| \geq i$ for $i \in \{3,4,5,6\}$. Therefore, $G_3[\mathcal{U}]$ is L-colorable.

*L*21 In this case, $G_3[\mathcal{U}] \simeq K_6$. Also, $\deg_3(u_i) \leq 13$ for $i \in \{1,2,3,4\}$, and $\deg_3(u_i) \leq 12$ for $i \in \{5,6\}$. Furthermore, $\{v,t\} \subset \mathcal{N}_3(u_i)$ for $i \in \{4,5\}$, $\{v,w,t\} \subset \mathcal{N}_3(u_6)$, and $\{w,t\} \subset \mathcal{N}_3(u_i)$ for $i \in \{2,3\}$. Thus, we infer $|L(u_1)| \geq 3$, $|L(u_i)| \geq 4$ for $i \in \{2,3,4\}$, $|L(u_5)| \geq 5$, and $|L(u_6)| \geq 6$. Therefore, $G_3[\mathcal{U}]$ is L-colorable.

*L*22 In this case, $G_3[\mathcal{U}] \simeq K_3$. Also, $\deg_3(u_i) \leq 12$ for $i \in \{1,2,3\}$. Moreover, $\{v,w,t\} \subset \mathcal{N}_3(u_i)$ for $i \in \{1,2,3\}$. Thus, we infer $|L(u_i)| \geq 3$ for $i \in \{1,2,3\}$. Therefore, $G_3[\mathcal{U}]$ is L-colorable.

*L*23 Define the partial coloring c as always, regarding the bold edges and the vertex v. Note that $G_3[\mathcal{U}]$ is isomorphic to the complete graph on four vertices minus one edge K_4^-, since $u_1 \notin \mathcal{N}_3(u_2)$ (because the face has size at least 8). By Lemma 1, $\deg_3(u_i) \leq 11$ for every $i \in \{1,2,3,4\}$. Thus, $|L(u_i)| \geq 2$ for $i \in \{1,2\}$, and $|L(u_i)| \geq 3$ for $i \in \{3,4\}$. Hence, the graph $G_3[\mathcal{U}]$ is L-colorable. This assertion can be directly checked, or seen as a consequence of a theorem independently proved by Borodin [2] and Erdős, Rubin, and Taylor [8] (see also [17]), stating that a connected graph is degree-choosable unless it is a *Gallai tree*, that is, each of its blocks is either complete or an odd cycle. □

Corollary 1. *Every* $(3,11)$*-minimal graph G has the following properties:*

1. *Let* f_1, f_2 *be two* 5*-faces of G with a common edge xy. Then, x and y are not both* 3*-vertices.*
2. *Let* f *be a* 7*-face whose every incident vertex is a* 3*-vertex. If* f *is adjacent to a* 3*-face, then every other face adjacent to* f *is a* (≥ 7)*-face.*
3. *Two dangerous vertices incident to a same* 6*-face are not adjacent.*
4. *There cannot be four consecutive dangerous vertices incident to a same* (≥ 6)*-face.*
5. *A very bad face is adjacent to at least three* (≥ 7)*-faces.*
6. *A bad face is adjacent to at least two* (≥ 7)*-faces.*

Proof. 1. By Lemma 2.5, $\deg_3(x) + \deg_3(y) \geq 23$. By Lemma 1, the 3-facial degree of a 3-vertex incident to two 5-faces is at most 11. Hence at least one of x and y is a (≥ 4)-vertex.

2. First note that, according to Lemma 2.3, the faces adjacent to both f and the 3-face have sizes at least 7. Hence, f is adjacent to at most four (≤ 6)-faces. Now, the assertion directly follows from the reducibility of the configurations (L1) and (L2) of Figure 2.

3. Suppose on the contrary that x and y are two such vertices. By Lemma 2.3, a 6-face is not adjacent to a 3-face; hence both x and y are incident to a 4-face. Then, $\deg_3(x) \leq 11$ and $\deg_3(y) \leq 11$, which contradicts Lemma 2.5.

4. Suppose that the assertion is false. Then, according to the third item of this corollary, the graph G must contain the configuration (L4) or (L5) of Figure 2, which are both reducible.

5. Let f be a very bad face. By the first item of this corollary and Lemma 3, two adjacent (≤ 6)-faces cannot be both adjacent to f. Hence, f is adjacent to at most two such faces.

6. Let f be a bad face, and $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\}$ its incident vertices in clockwise order. Without loss of generality, assume that, for every $i \in \{1,2,3,4\}$, α_i is a 3-vertex. For $i \in \{1,2,3,4\}$, let f_i be the face adjacent to f and incident to both α_i and α_{i+1}. According to the first item of this corollary and Lemma 3, at most two faces among f_1, f_2, f_3, f_4 can be (≤ 6)-faces. This concludes the proof.

3 Proof of Theorem 1

Suppose that Theorem 1 is false, and let G be a $(3,11)$-minimal graph. We obtain a contradiction by using the discharging method. Here is an overview of the proof: each vertex and face is assigned an initial charge. The total sum of the initial charges is known to be negative by Euler's formula. Then, some redistribution rules are defined, and each vertex and face gives or receives some charge according to these rules. The total sum of the charges is not changed during this step, but at the end we show, by case analysis, that the charge of each vertex and each face is nonnegative, a contradiction.

Initial charge First, we assign a charge to each vertex and face. For every $v \in V(G)$, we define the initial charge

$$\mathrm{ch}(v) = d(v) - 4,$$

where $d(v)$ is the degree of the vertex v in G. Similarly, for every $f \in F(G)$, where $F(G)$ is the set of faces of G, we define the initial charge

$$\mathrm{ch}(f) = r(f) - 4,$$

with $r(f)$ the size of the face f. By Euler's formula the total sum is

$$\sum_{v \in V(G)} \mathrm{ch}(v) + \sum_{f \in F(G)} \mathrm{ch}(f) = -8.$$

Rules We use the following discharging rules to redistribute the initial charge.

RULE R1. *A (≥ 5)-face sends $1/3$ to each of its incident safe vertices and $1/2$ to each of its incident dangerous vertices.*

RULE R2. *A (≥ 7)-face sends $1/3$ to each adjacent 3-face.*

RULE R3. *A (≥ 7)-face sends $1/6$ to each adjacent bad face.*

RULE R4. *A 6-face sends $1/12$ to each adjacent very bad face.*

RULE R5. *A (≥ 5)-vertex v gives $2/3$ to an incident face f if and only if there exist two 3-faces both incident to v and both adjacent to f. (Note that the size of such a face f is at least 7.)*

We now prove that the final charge $\mathrm{ch}^*(x)$ of every $x \in V(G) \cup F(G)$ is non-negative. Therefore, we obtain

$$-8 = \sum_{v \in V(G)} \mathrm{ch}(v) + \sum_{f \in F(G)} \mathrm{ch}(f) = \sum_{v \in V(G)} \mathrm{ch}^*(v) + \sum_{f \in F(G)} \mathrm{ch}^*(f) \geq 0,$$

a contradiction.

Final charge of vertices First, as noticed in Lemma 2.4, G has minimum degree at least three. Let v be an arbitrary vertex of G. We prove that its final charge $\mathrm{ch}^*(v)$ is nonnegative. To this end, we consider a few cases regarding its degree. So, suppose first that v is a 3-vertex. If v is a safe vertex, then by Rule R1 its final charge is $\mathrm{ch}^*(v) = -1 + 3 \cdot \frac{1}{3} = 0$. Similarly, if v is dangerous, then $\mathrm{ch}^*(v) = -1 + 2 \cdot \frac{1}{2} = 0$. If v is a 4-vertex, then it neither receives nor sends any charge. Thus, $\mathrm{ch}^*(v) = \mathrm{ch}(v) = 0$.

Finally, suppose that v is of degree $d \geq 5$. Notice that v may send charge only by Rule R5. This may occur at most $d/2$ times if d is even, and at most $\lfloor d/2 \rfloor - 1$ times if d is odd (since two 3-faces are not adjacent). Thus, $\mathrm{ch}^*(v) \geq d - 4 - \lfloor \frac{d}{2} \rfloor \cdot \frac{2}{3}$, which is nonnegative if $d \geq 6$. For $d = 5$, $\mathrm{ch}^*(v) \geq 5 - 4 - \frac{2}{3} > 0$.

Final charge of faces Let f be an arbitrary face of G. We define `fce` and `bad` to be the number of 3-faces and the number of bad faces, respectively, adjacent to f. We define `sfe` and `dgs` to be the number of safe vertices and the number of dangerous vertices, respectively, incident to f. We prove that the final charge $\mathrm{ch}^*(f)$ of f is nonnegative. To this end, we consider a few cases regarding the size of f.

***f* is a 3-face** It is adjacent only to (≥ 7)-faces by Lemma 2.3. Thus, by Rule R2, f receives $1/3$ from each of its three adjacent faces, so we obtain $\mathrm{ch}^*(f) = 0$.

***f* is a 4-face** It neither receives nor sends any charge. Thus, $\mathrm{ch}^*(f) = \mathrm{ch}(f) = 0$.

***f* is a 5-face** Then, f is adjacent only to (≥ 5)-faces due to Lemma 2.3. So a 5-face may send charge only to its incident 3-vertices, which are all safe. Consider the following cases regarding the number `sfe` of such vertices:

`sfe` ≤ 3: Then, $\mathrm{ch}^*(v) \geq 1 - 3 \cdot \frac{1}{3} = 0$.

`sfe` $= 4$: In this case, f is a bad face. According to Corollary 1.6, at least two of the faces that are adjacent to f have size at least 7. Thus, according to Rule R3, f receives $1/6$ from at least two of its adjacent faces. Hence, we conclude that $\mathrm{ch}^*(v) \geq 1 - 4 \cdot \frac{1}{3} + 2 \cdot \frac{1}{6} = 0$.

`sfe` $= 5$: Then f is a very bad face, and so, according to Corollary 1.5, at least three faces adjacent to f have size at least 7. Moreover, all faces adjacent to f have size at least 6, by Lemma 2.3 and Corollary 1.1. By Rules R3 and R4, it follows that the neighboring faces of f send at least $4 \cdot 1/6$ to f, which implies that $\mathrm{ch}^*(v) \geq 1 - 5 \cdot \frac{1}{3} + 4 \cdot \frac{1}{6} = 0$.

f is a 6-face By Lemma 2.3, $\texttt{fce} = 0$. Let $\texttt{vbd}$ be the number of very bad faces adjacent to f. The final charge of f is $2 - \texttt{dgs} \cdot \frac{1}{2} - \texttt{sfe} \cdot \frac{1}{3} - \texttt{vbd} \cdot \frac{1}{12}$ due to Rules R1 and R4.

According to Corollary 1.3, two dangerous vertices on f cannot be adjacent, so there are at most three dangerous vertices on f. Observe also that $\texttt{vbd} \leq \texttt{sfe}/2$ by Corollary 1.1 and because a very bad face adjacent to f is incident to two safe vertices of f. Let us consider the final charge of f regarding its number of dangerous vertices.

$\texttt{dgs} = 3$: Since a safe vertex is not incident to a (≤ 4)-face, there is at most one safe vertex incident to f, i.e., $\texttt{sfe} \leq 1$. Thus, $\texttt{vbd} = 0$, and hence $\text{ch}^*(f) \geq 2 - 3 \cdot \frac{1}{2} - \frac{1}{3} > 0$.

$\texttt{dgs} = 2$: Then, $\texttt{sfe} \leq 3$. Let us distinguish two cases according to the value of $\texttt{sfe}$.

- $\texttt{sfe} = 3$: Notice that $\texttt{vbd} = 0$; otherwise it would contradict the reducibility of (L3). Hence, $\text{ch}^*(f) \geq 2 - 2 \cdot \frac{1}{2} - 3 \cdot \frac{1}{3} = 0$.
- $\texttt{sfe} \leq 2$: In this case, there is at most one very bad face adjacent to f, so $\text{ch}^*(f) \geq 2 - 2 \cdot \frac{1}{2} - 2 \cdot \frac{1}{3} - \frac{1}{12} > 0$.

$\texttt{dgs} = 1$: Then, $\texttt{sfe} \leq 4$ and $\texttt{vbd} \leq 1$ because (L3) is reducible. So, $\text{ch}^*(f) \geq 2 - \frac{1}{2} - \frac{4}{3} - \frac{1}{12} > 0$.

$\texttt{dgs} = 0$: If $\texttt{sfe} \geq 5$ then, because (L3) is reducible, $\texttt{vbd} = 0$; therefore $\text{ch}^*(f) \geq 2 - \frac{6}{3} = 0$. And, if $\texttt{sfe} \leq 4$, then $\texttt{vbd} \leq 2$, so $\text{ch}^*(f) \geq 2 - 4 \cdot \frac{1}{3} - 2 \cdot \frac{1}{12} > 0$.

f is a 7-face The final charge of f is at least $3 - \texttt{dgs} \cdot \frac{1}{2} - (\texttt{fce} + \texttt{sfe}) \cdot \frac{1}{3} - \texttt{bad} \cdot \frac{1}{6}$.

According to Corollary 1.4, four dangerous vertices cannot be consecutive on f; hence there cannot be more than five dangerous vertices on f. Let $\alpha_1, \alpha_2, \ldots, \alpha_7$ be the vertices of f in clockwise order. Let $\mathcal{D}$ be the set of dangerous vertices of f, so $\texttt{dgs} = |\mathcal{D}|$. We look at the final charge of f, regarding its number $\texttt{dgs}$ of dangerous vertices.

$\texttt{dgs} = 5$: Up to symmetry, $\mathcal{D} = \{\alpha_1, \alpha_2, \alpha_3, \alpha_5, \alpha_6\}$. Suppose first that α_5 and α_6 are not incident to a same (≤ 4)-face. Then, there can be neither a safe vertex incident to f nor a bad face adjacent to f, because a safe vertex is not incident to a (≤ 4)-face, and also a bad face is not adjacent to a (≤ 4)-face. Moreover, by Corollary 1.3, there is no 3-face adjacent to f. Therefore, $\text{ch}^*(f) \geq 3 - \frac{5}{2} > 0$. Now, if α_5 and α_6 are incident to a same (≤ 4)-face, then α_4 and α_7 must be (≥ 4)-vertices, by the reducibility of (L6) and because none of them is a dangerous vertex. Hence, there is no safe vertex and no bad face adjacent to f, so its charge is $\text{ch}^*(f) \geq 3 - \frac{5}{2} - \frac{1}{3} > 0$.

$\texttt{dgs} = 4$: We consider several subcases, according to the relative position of the dangerous vertices on f. Recall that, by Corollary 1.4, there are at most three consecutive dangerous vertices. Without loss of generality, we need only to consider the following three possibilities:

- $\mathcal{D} = \{\alpha_1, \alpha_2, \alpha_3, \alpha_5\}$: The charge of f is $\text{ch}^*(f) = 1 - (\texttt{fce} + \texttt{sfe}) \cdot \frac{1}{3} - \texttt{bad} \cdot \frac{1}{6}$. Moreover, $\texttt{sfe} \leq 2$, $\texttt{bad} \leq 1$, and $\texttt{fce} + \texttt{sfe} \leq 3$ by Corollary 1.3 and because a safe vertex is not incident to a (≤ 4)-face. So, $\text{ch}^*(f)$ is negative if and only if $\texttt{sfe} = 2, \texttt{bad} = 1$, and $\texttt{fce} = 1$. But in this case, the obtained configuration is (L7), which is reducible.

$\mathcal{D} = \{\alpha_1, \alpha_2, \alpha_4, \alpha_5\}$: As a bad face is neither adjacent to a (≤ 4)-face nor incident to a dangerous vertex, we obtain that $\texttt{bad} \leq 1$. Observe also that, as α_3 is not dangerous, it has degree at least four by the reducibility of (L6) and (L10). Thus, $\texttt{sfe} \leq 2$. Suppose first that $\texttt{bad} = 1$, then $\texttt{sfe}$ is one or two. According to the reducibility of (L9), we infer $\texttt{sfe} + \texttt{fce} \leq 2$. Hence, $\mathrm{ch}^*(f) \geq 3 - 4 \cdot \frac{1}{2} - 2 \cdot \frac{1}{3} - \frac{1}{6} > 0$. Suppose now that $\texttt{bad} = 0$. We have $\texttt{fce} \leq 3$ and $\texttt{sfe} \leq 2$. If $\texttt{fce} = 3$, then $\texttt{sfe} = 0$, and if $\texttt{fce} = 2$, then $\texttt{sfe} \leq 1$ according to the reducibility of (L11). So, $\texttt{fce} + \texttt{sfe} \leq 3$. Therefore, $\mathrm{ch}^*(f) \geq 3 - 4 \cdot \frac{1}{2} - (\texttt{fce} + \texttt{sfe}) \cdot \frac{1}{3} \geq 0$.

$\mathcal{D} = \{\alpha_1, \alpha_2, \alpha_4, \alpha_6\}$: In this case, there is no bad face adjacent to f. Furthermore, by Corollary 1.3, $\texttt{fce} \leq 3$ and $\texttt{sfe} \leq 2$, as the dangerous vertices α_4 and α_6 prevent at least one nondangerous vertex from being safe. Observe that $\texttt{fce} + \texttt{sfe} \neq 5$ since otherwise it would contradict the reducibility of (L12). According to the reducibility of (L12), if $\texttt{fce} + \texttt{sfe} = 4$, then $\texttt{fce} = 3$ and no two 3-faces have a common vertex. Hence, the obtained configuration is isomorphic to (L13) or (L14), which are both reducible. So, $\texttt{fce} + \texttt{sfe} \leq 3$, and thus $\mathrm{ch}^*(f) \geq 3 - 2 - (\texttt{fce} + \texttt{sfe}) \cdot \frac{1}{3} \geq 0$.

$\texttt{dgs} = 3$: Again, we consider several subcases according to the relative position of the dangerous vertices on f:

$\mathcal{D} = \{\alpha_1, \alpha_2, \alpha_3\}$: Then $\texttt{fce} + \texttt{sfe} \leq 3$ by Corollary 1.3, and $\texttt{bad} \leq 2$. Thus, $\mathrm{ch}^*(f) \geq 3 - 3 \cdot \frac{1}{2} - 3 \cdot \frac{1}{3} - 2 \cdot \frac{1}{6} > 0$.

$\mathcal{D} = \{\alpha_1, \alpha_2, \alpha_4\}$: Then, $\texttt{fce} \leq 4$. We now examine the situation according to each possible value of $\texttt{fce}$.

$\texttt{fce} = 4$: Necessarily, $\texttt{sfe} \leq 1$ and $\texttt{bad} = 0$. Now, if $\texttt{sfe} = 0$, then $\mathrm{ch}^*(f) \geq 3 - 3 \cdot \frac{1}{2} - 4 \cdot \frac{1}{3} > 0$. And, if $\texttt{sfe} = 1$, then the safe vertex must be α_3. Moreover, α_5 must be a (≥ 5)-vertex because (L8) is reducible. Hence, f is incident to α_5 between two 3-faces, so by Rule R5 the vertex α_5 gives $\frac{2}{3}$ to f. Thus, $\mathrm{ch}^*(f) \geq 3 - 3 \cdot \frac{1}{2} - 5 \cdot \frac{1}{3} + \frac{2}{3} > 0$.

$\texttt{fce} = 3$: Suppose first that one of the dangerous vertices is incident to a 4-face. Necessarily, $\texttt{sfe} \leq 1$ and $\texttt{bad} \leq 1$. Thus, $\mathrm{ch}^*(f) \geq 3 - 3 \cdot \frac{1}{2} - 4 \cdot \frac{1}{3} - \frac{1}{6} = 0$. Suppose now that no dangerous vertex is incident to a 4-face. In particular, $\texttt{sfe} \leq 2$. If $\texttt{sfe} = 2$, then the obtained configuration contradicts the reducibility of (L18). Hence, $\texttt{sfe} \leq 1$ and $\texttt{bad} \leq 1$. Therefore, $\mathrm{ch}^*(f) \geq 3 - 3 \cdot \frac{1}{2} - 4 \cdot \frac{1}{3} - \frac{1}{6} = 0$.

$\texttt{fce} = 2$: We prove that $\texttt{sfe} \leq 2$. This is true if α_1 and α_2 are not incident to a same 3-face. So, we may assume that the edge $\alpha_1\alpha_2$ lies on a 3-face. But then we obtain the inequality due to the reducibility of (L18) and (L19). Using Corollary 1.1, the reducibility of (L17), and $\texttt{sfe} \leq 2$, we infer that $\texttt{bad} \leq 1$. Hence, $\mathrm{ch}^*(f) \geq 3 - 3 \cdot \frac{1}{2} - 4 \cdot \frac{1}{3} - \frac{1}{6} = 0$.

$\texttt{fce} = 1$: Then $\texttt{sfe} \leq 3$ and $\texttt{bad} \leq 2$. If $\texttt{sfe} = 3$ and $\texttt{bad} = 2$, the obtained configuration contradicts the reducibility of (L19) or of (L20). So, $\mathrm{ch}^*(f) \geq 3 - 3 \cdot \frac{1}{2} - 4 \cdot \frac{1}{3} - \frac{1}{6} = 0$.

$\texttt{fce} = 0$: Again, $\texttt{sfe} \leq 3$ and $\texttt{bad} \leq 2$, so $\mathrm{ch}^*(f) \geq 3 - 3 \cdot \frac{1}{2} - 3 \cdot \frac{1}{3} - 2 \cdot \frac{1}{6} > 0$.

$\mathcal{D} = \{\alpha_1, \alpha_2, \alpha_5\}$: As in the previous case, $\texttt{fce} \leq 4$, and we look at all the possible cases according to the value of $\texttt{fce}$. Since a bad face is not incident to a dangerous vertex, notice that only edges $\alpha_3\alpha_4$ and $\alpha_6\alpha_7$ can be incident to a bad face. In particular, $\texttt{bad} \leq 2$.

$\texttt{fce} = 4$: In this case, $\texttt{sfe} = 0$ and $\texttt{bad} = 0$. Therefore, $\mathrm{ch}^*(f) = 3 - 3 \cdot \frac{1}{2} - 4 \cdot \frac{1}{3} > 0$.

$\texttt{fce} = 3$: If one of the dangerous vertices is incident to a 4-face, then $\texttt{sfe} = 0$; hence $\texttt{bad} = 0$. Thus, $\mathrm{ch}^*(f) \geq 3 - 3 \cdot \frac{1}{2} - 3 \cdot \frac{1}{3} \geq 0$. So now we infer that $\texttt{sfe}$ cannot be 2; otherwise it would contradict the reducibility of (L15). Therefore, $\texttt{sfe}$ is at most one, and so $\texttt{bad} \leq 1$ by Corollary 1.1. Thus, $\mathrm{ch}^*(f) \geq 3 - 3 \cdot \frac{1}{2} - 4 \cdot \frac{1}{3} - \frac{1}{6} = 0$.

$\texttt{fce} = 2$: According to the reducibility of (L15) and (L16), $\texttt{sfe} \leq 2$. As $\mathrm{ch}^*(f) = 3 - 3 \cdot \frac{1}{2} - (\texttt{fce} + \texttt{sfe}) \cdot \frac{1}{3} - \texttt{bad} \cdot \frac{1}{6}$, we deduce $\mathrm{ch}^*(f) < 0$ if and only if $\texttt{sfe} = 2$ and $\texttt{bad} = 2$. In this case, the obtained configuration is (L17), which is reducible.

$\texttt{fce} = 1$: Because (L15) and (L16) are reducible, $\texttt{sfe} \leq 2$. So, $\mathrm{ch}^*(f) \geq 3 - 3 \cdot \frac{1}{2} - 3 \cdot \frac{1}{3} - 2 \cdot \frac{1}{6} > 0$.

$\texttt{fce} = 0$: Then $\texttt{sfe} \leq 3$, and so $\mathrm{ch}^*(f) \geq 3 - 3 \cdot \frac{3}{2} - 3 \cdot \frac{1}{3} - 2 \cdot \frac{1}{6} > 0$.

$\mathcal{D} = \{\alpha_1, \alpha_3, \alpha_5\}$: In this case, $\texttt{sfe} \leq 2$ since a safe vertex is not incident to a (≤ 4)-face, and $\texttt{bad} \leq 1$, since a bad face cannot be incident to a dangerous vertex. Moreover, $\texttt{fce} \leq 4$. Let us examine the possible cases regarding the value of $\texttt{fce}$.

$\texttt{fce} = 4$: Observe that $\texttt{sfe} \leq 1$ and $\texttt{bad} = 0$. Note also that one of $\alpha_2, \alpha_4, \alpha_6, \alpha_7$ is adjacent to a dangerous vertex, and incident to f between two triangles. Hence, by the reducibility of (L8), it has degree at least five, and by Rule R5, it sends $\frac{2}{3}$ to f. Thus, $\mathrm{ch}^*(f) \geq 3 - 3 \cdot \frac{1}{2} - 5 \cdot \frac{1}{3} + \frac{2}{3} > 0$.

$\texttt{fce} = 3$: If $\texttt{sfe} \leq 1$, then $\mathrm{ch}^*(f) \geq 3 - 3 \cdot \frac{1}{2} - 4 \cdot \frac{1}{3} - \frac{1}{6} = 0$. And, if $\texttt{sfe} = 2$, then, up to symmetry, the two safe vertices are either α_6 and α_7 or α_2 and α_6. In the former case, one of α_2, α_4 is incident to f at the intersection of two 3-faces. Furthermore, it must be a (≥ 5)-vertex due to the reducibility of (L8). In the latter case, the same holds for α_4 due to the reducibility of (L8). Hence, in both cases the face f receives $2/3$ from one of its incident vertices by Rule R5. Recall that $\texttt{bad} \leq 1$, and therefore $\mathrm{ch}^*(f) \geq 3 - 3 \cdot \frac{1}{2} - 5 \cdot \frac{1}{3} - \frac{1}{6} + \frac{2}{3} > 0$.

$\texttt{fce} \leq 2$: As $\texttt{sfe} \leq 2$ and $\texttt{bad} \leq 1$, we infer that $\mathrm{ch}^*(f) \geq 3 - 3 \cdot \frac{1}{2} - 4 \cdot \frac{1}{3} - \frac{1}{6} = 0$.

$\texttt{dgs} = 2$: Again, we consider several subcases, regarding the position of the dangerous vertices on f.

$\mathcal{D} = \{\alpha_1, \alpha_2\}$: Observe that $\texttt{bad} \leq 3$, and according to Corollary 1.3, $\texttt{fce} + \texttt{sfe} \leq 6$. We consider three cases, according to the value of $\texttt{fce} + \texttt{sfe}$:

$\texttt{fce} + \texttt{sfe} = 6$: All the vertices incident to f have degree three, and f is adjacent to a 3-face. Thus, by Corollary 1.2, f is not adjacent to any (≤ 6)-face. In particular, no bad face is adjacent to f; i.e., $\texttt{bad} = 0$. Hence, $\mathrm{ch}^*(f) \geq 3 - 1 - 6 \cdot \frac{1}{3} = 0$.

$\mathtt{fce}+\mathtt{sfe}=5$: If $\mathtt{bad}\le 2$, then $\mathrm{ch}^*(f)\ge 3-1-5\cdot\frac{1}{3}-2\cdot\frac{1}{6}=0$. Otherwise, $\mathtt{bad}=3$. Note that the edge $\alpha_1\alpha_2$ must be incident to a (≤ 4)-face. If this face is of size four, then we obtain configuration (L21). Suppose now that this face is of size three. Since there are no three consecutive bad faces around f, we can assume that each of the edges $\alpha_3\alpha_4$ and $\alpha_6\alpha_7$ lies on a bad face. By the reducibility of (L17), we conclude that α_3 and α_7 have degree at least four. But then, $\mathtt{fce}+\mathtt{sfe}<5$.

$\mathtt{fce}+\mathtt{sfe}\le 4$: In this case, $\mathrm{ch}^*(f)\ge 3-1-4\cdot\frac{1}{3}-3\cdot\frac{1}{6}>0$.

$\mathcal{D}=\{\alpha_1,\alpha_3\}$ *or* $\mathcal{D}=\{\alpha_1,\alpha_4\}$: Again $\mathtt{fce}+\mathtt{sfe}\le 6$, and we consider two cases regarding the value of $\mathtt{fce}+\mathtt{sfe}$. Since a bad face is not incident to a dangerous vertex, we infer that $\mathtt{bad}\le 3$.

$\mathtt{fce}+\mathtt{sfe}=6$: Suppose first that $\mathcal{D}=\{\alpha_1,\alpha_3\}$. Let $P_1=\alpha_1\alpha_2\alpha_3$ and $P_2=\alpha_3\alpha_4\alpha_5\alpha_6\alpha_7\alpha_1$. In order to assure $\mathtt{fce}+\mathtt{sfe}=6$, observe that all edges of P_1 are incident to 3-faces, and all inner vertices of P_2 are safe, or vice versa. Thus, α_2 or α_4 is a (≥ 5)-vertex by the reducibility of (L8). Hence, it gives $\frac{2}{3}$ to f by Rule R5. Therefore, $\mathrm{ch}^*(f)\ge 3-2\cdot\frac{1}{2}-6\cdot\frac{1}{3}-3\cdot\frac{1}{6}+\frac{2}{3}>0$.
Suppose now that $\mathcal{D}=\{\alpha_1,\alpha_4\}$. Similarly as above, one can show that α_2 or α_5 is a (≥ 5)-vertex that donates $\frac{2}{3}$ to f. Hence, $\mathrm{ch}^*(f)\ge 3-2\cdot\frac{1}{2}-6\cdot\frac{1}{3}-\frac{3}{6}+\frac{2}{3}>0$.

$\mathtt{fce}+\mathtt{sfe}\le 5$: Notice that $\mathtt{bad}\le 2$. Therefore, $\mathrm{ch}^*(f)\ge 3-2\cdot\frac{1}{2}-5\cdot\frac{1}{3}-2\cdot\frac{1}{6}=0$.

$\mathtt{dgs}=1$: Then $\mathtt{fce}+\mathtt{sfe}\le 6$ and, by Corollary 1.1, we infer that $\mathtt{bad}\le 3$. So, $\mathrm{ch}^*(f)\ge 3-\frac{1}{2}-6\cdot\frac{1}{3}-3\cdot\frac{1}{6}=0$.

$\mathtt{dgs}=0$: By Corollary 1.1, $\mathtt{fce}+\mathtt{sfe}\le 7$ and $\mathtt{bad}\le 4$. So, $\mathrm{ch}^*(f)\ge 3-7\cdot\frac{1}{3}-4\cdot\frac{1}{6}=0$.

***f* is an 8-face** By Lemma 2.5 and because (L22) is reducible, there cannot be three consecutive dangerous vertices on f. Hence, $\mathtt{dgs}\le 5$. Let α_i, $i\in\{1,2,\dots,8\}$, be the vertices incident to f in clockwise order, and let $\mathcal{D}$ be the set of dangerous vertices incident to f.

$\mathtt{dgs}=5$: Up to symmetry, $\mathcal{D}=\{\alpha_1,\alpha_2,\alpha_4,\alpha_5,\alpha_7\}$. Since a bad face is not incident to a dangerous vertex, necessarily $\mathtt{bad}=0$. For $i\in\{1,4\}$, let f_i be the face adjacent to f and incident to both α_i and α_{i+1}. Since (L23) is reducible, at most one of f_1 and f_4 is a 3-face. Furthermore, at most two of $\alpha_3,\alpha_6,\alpha_8$ can be safe vertices, since at least one of α_6,α_8 is a (≥ 4)-vertex. Therefore, $\mathtt{fce}\le 2$, $\mathtt{sfe}\le 2$, and so $\mathrm{ch}^*(f)\ge 4-5\cdot\frac{1}{2}-4\cdot\frac{1}{3}>0$.

$\mathtt{dgs}=4$: Up to symmetry the set of dangerous vertices comprises $\{\alpha_1,\alpha_2,\alpha_4,\alpha_5\}$, $\{\alpha_1,\alpha_2,\alpha_5,\alpha_6\}$, $\{\alpha_1,\alpha_2,\alpha_4,\alpha_6\}$, $\{\alpha_1,\alpha_2,\alpha_4,\alpha_7\}$, or $\{\alpha_1,\alpha_3,\alpha_5,\alpha_7\}$. In any case, $\mathtt{bad}\le 2$ and $\mathtt{fce}+\mathtt{sfe}\le 6$. Moreover, $\mathtt{fce}+\mathtt{sfe}+\mathtt{bad}\le 6$. Indeed, if $\mathtt{bad}=0$, then the inequality holds by the prior remark. This solves the fifth case for $\mathcal{D}$ since a bad face is not incident to a dangerous vertex. If $\mathtt{bad}=2$, then we are in the first or second case for $\mathcal{D}$. We infer that $\mathtt{fce}+\mathtt{sfe}\le 4$ by the reducibility of (L17) and (L23), and by Corollary 1.1. Finally, if $\mathtt{bad}=1$, then we deduce that $\mathtt{fce}+\mathtt{sfe}\le 5$ by the reducibility of (L17). Hence, $\mathrm{ch}^*(f)\ge 4-\frac{4}{2}-\frac{6}{3}=0$.

$\mathtt{dgs} = 3$: Then, $\mathtt{fce} + \mathtt{sfe} \leq 6$ and $\mathtt{bad} \leq 3$. So, $\mathrm{ch}^*(f) \geq 4 - \frac{3}{2} - \frac{6}{3} - \frac{3}{6} = 0$.

$\mathtt{dgs} = 2$: Then, $\mathtt{fce} + \mathtt{sfe} \leq 7$, and by Corollary 1.1, $\mathtt{bad} \leq 4$. Thus, $\mathrm{ch}^*(f) \geq 4 - \frac{2}{2} - \frac{7}{3} - \frac{4}{6} = 0$.

$\mathtt{dgs} = 1$: Again, $\mathtt{fce} + \mathtt{sfe} \leq 7$ and $\mathtt{bad} \leq 4$, so $\mathrm{ch}^*(f) \geq 4 - \frac{1}{2} - \frac{7}{3} - \frac{4}{6} > 0$.

$\mathtt{dgs} = 0$: By Corollary 1.1, $\mathtt{bad} \leq 5$. So, $\mathrm{ch}^*(f) \leq 4 - \frac{8}{3} - \frac{5}{6} > 0$.

f is a (≥ 9)-face Let f be a k-face with $k \geq 9$. We use the following averaging scheme for the rules followed by f: the face f sends the charge to incident vertices or adjacent faces *through* its incident edges. More precisely, for Rules R2 and R3, if f' is a bad face or a triangle incident to f, then we say that f sends the corresponding charge *through* the edge that is incident to both f and f'. As for Rule R1, let u be a vertex incident to f, and let v and w be its two neighbors on f. If u is a safe vertex, then f sends $\frac{1}{6}$ to u through each of the two edges uv and uw. If u is dangerous, let uv be the edge incident with a (≤ 4)-face: if v is dangerous, then f sends $\frac{1}{9}$ to u through uv and $\frac{7}{18}$ to u through uw. Otherwise (i.e., when v is a (≥ 4)-vertex), f sends $\frac{2}{9}$ to u through uv and $\frac{5}{18}$ to u through uw.

By Lemma 2.5 and the reducibility of (L22), there cannot be three consecutive dangerous vertices on f. So, we deduce that all the vertices incident to f receive the same charge as if f applied the original Rule R1. We prove now that f sends at most $\frac{5}{9}$ to each of its edges, and hence $\mathrm{ch}^*(f) \geq k(1 - \frac{5}{9}) - 4 \geq 0$ since $k \geq 9$.

Let uv be an edge incident to f. We consider three cases.

uv is incident to a bad face f': Then, u and v are not dangerous. So f sends through uv $\frac{1}{6}$ to u plus $\frac{1}{6}$ to v and $\frac{1}{6}$ to f'. Thus, the charge sent by f through the edge uv is at most $3 \cdot \frac{1}{6} = \frac{1}{2} < \frac{5}{9}$.

uv is incident to a triangle f': In this case, f sends $\frac{1}{3}$ to f'. If neither of u or v is dangerous, then f sends nothing more through uv. If exactly one of u and v is dangerous, say u, then f sends $\frac{2}{9}$ to u through uv. Thus, the charge sent by f through uv is $\frac{1}{3} + \frac{2}{9} = \frac{5}{9}$. Finally, assume that both u and v are dangerous. Then, f sends $\frac{1}{9}$ to each of u and v through uv. Hence, f sends $\frac{1}{3} + 2 \cdot \frac{1}{9} = \frac{5}{9}$ through uv.

uv is incident to neither a bad face nor a triangle: Again, if neither of u or v is dangerous, then f sends at most $2 \cdot \frac{1}{6} = \frac{1}{3}$ through uv. Suppose that both u and v are dangerous. If uv is incident to a 4-face, then f sends $2 \cdot \frac{1}{9} = \frac{2}{9}$ through uv. Otherwise, let t be the neighbor of u on f different from v, and let w be the neighbor of v on f different from u. By Lemma 2.5, each of tu and vw is incident to a 4-face, and t and w are not dangerous since (L22) is reducible. Therefore, f sends $\frac{5}{18}$ to each of u and v through uv, and thus f sends $\frac{5}{9}$ through uv. Finally, if exactly one of u and v is dangerous, say u, then f sends at most $\frac{7}{18}$ to u through uv, and at most $\frac{1}{6}$ to v through uv. In total, f sends at most $\frac{7}{18} + \frac{1}{6} = \frac{5}{9}$ through uv.

The proof of Theorem 1 is now complete.

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