Connectivity in Sub-Poisson Networks

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***Abstract*: We consider a class of point processes, which we call *sub-Poisson*; these are point processes that can be directionally-convexly (dcx) dominated by some Poisson point process. The dcx order has already been shown in [4] useful in comparing various point process characteristics, including Ripley's and correlation functions as well as shot-noise fields generated by point processes, indicating in particular that smaller in the dcx order processes exhibit more regularity (less clustering, less voids) in the repartition of their points. Using these results, in this paper we study the impact of the dcx ordering of point processes on the properties of two continuum percolation models, which have been proposed in the literature to address macroscopic connectivity properties of large wireless networks. As the first main result of this paper, we extend the classical result on the existence of phase transition in the percolation of the Gilbert's graph (called also the Boolean model), generated by a homogeneous Poisson point process, to the class of homogeneous sub-Poisson processes. We also extend a recent result of the same nature for the SINR graph, to sub-Poisson point processes. Finally, we show examples the so-called perturbed lattices, which are sub-Poisson. More generally, perturbed lattices provide some spectrum of models that ranges from periodic grids, usually considered in cellular network context, to Poisson ad-hoc networks, and to various more clustered point processes including some doubly stochastic Poisson ones.**

***Index Terms*—percolation, dcx order, Gilbert's graph, Boolean model, SINR graph, wireless network, Poisson point process, perturbed lattice, determinantal point process, connectivity, capacity**

I. INTRODUCTION

A *network*, in the simplest terms, is a collection of points in some space (e.g. on the Euclidean plane), called *nodes* or *vertexes*, and a collection of node pairs, called *edges*. The presence of an edge between two nodes indicates that they can directly communicate with each other. The mathematical name for this network model is *graph*.

A class of networks that has recently attracted particular interest in the wireless communication context, is called *ad-hoc networks*. It is distinguished by the fact that the network nodes are not subject to any regular (say periodic) geometric emplacement in the space but can be rather seen as a snapshot of some *random point pattern*, called also *point process* (pp) in the mathematical formalism typically used in this context.

Connectivity, i.e. possibility of indirect, multi-hop, communication between distant nodes, is probably the first issue that has to be addressed when considering ad-hoc networks. An ubiquitous assumption when studying this problem is that the node randomness is modeled by a spatial *Poisson* pp. This latter situation can be characterized by independence and Poisson distribution of the number of nodes observed in disjoint subsets of the space. Poisson assumption in the above context is often too simplistic, however analysis or even modeling of networks without this assumption is in most cases very difficult.

In this paper we introduce some class of point processes that might be roughly described as exhibiting less variable point patterns than Poisson pp. We call point processes (pp's) of this class *sub-Poisson point processes*. Our objective is two-fold. On one hand we want to argue that this class of pp's allows to extend some classical results regarding network connectivity. On the other hand we want to bring attention to sub-Poisson pp's, as they might be useful for modeling of ad-hoc networks, which have nodes more regularly distributed than in Poisson pp.

A. Sub-Poisson Random Variables

A random variable is called sub-Poisson, if its variance is not larger than its mean (with the equality holding true for Poisson variable). Intuitively, if strictly sub-Poisson variables were to describe the number of nodes in different subsets of the space then the resulting point patterns would exhibit less clustering (or bunching) than the Poisson point pattern having the same number of points per unit of space volume. This in turn, still intuitively, should have positive impact on the connectivity and perhaps capacity and other network performance metrics, the reason being that the perfectly regular, periodic patterns are commonly considered (and sometimes can be proved) as being optimal. The aim of the present article is to provide rigorous results on the comparison of the connectivity and capacity properties of certain sub-Poisson networks to these of the respective Poisson networks.

B. dcx Sub-Poisson Point Processes

The statistical variability of random variables (say with the same mean) can be compared only to some limited extent by looking at their variances, but more fully by convex ordering. Under this order one can compare the expected values of *all* convex functions of these variables. In multi-dimensions there is no one single notion of convexity. Besides different statistical variability of marginal distributions, two random vectors (think of number of nodes in different subsets of the space) can exhibit different dependence properties on their coordinates. The most evident example here is comparison of the vector composed of several copies of one random variable to a vector composed of independent copies sampled from the same distribution. Among several notions of "convex-like" ordering of random vectors the so called *directionally convex* (dcx) order, allowing one to compare expectations of all dcx functions (see Section II-A below) of these vectors, takes into account both the dependence structure of random vectors and the variability of their marginals. It can be naturally extended to random fields by comparison of all finite dimensional distributions, as well as to random pp's and even locally finite random measures by viewing them as non-negative fields of measure-values on all bounded Borel subsets of the space; cf. [4].

Using this latter formalism, we say that a pp is *dcx-sub-Poisson* (or simply *sub-Poisson* when there is no ambiguity) if it is dcx-smaller than Poisson pp having the same mean measure, i.e., the mean number of nodes in any given set. We will also say *sub-Poisson network* in the case when the nodes of the graph modeling the network are distributed according to some sub-Poisson pp.

We shall also see that there are classes of pp whose so-called *joint intensities* (densities of the higher order moment measures, when they exist; see Section II-B) are smaller than these of Poisson pp. Using this latter (and weaker) property we define the class of *weakly sub-Poisson point processes*.

Another weakening of the comparability assumption, to the expectations of increasing (or decreasing) dcx functions allows to define *increasing dcx* sub-Poisson pp's, which can be seen as being less variable than some Poisson pp having possibly smaller (or larger) mean measure. We shall abbreviate increasing dcx by $idcx$ and similarly *decreasing dcx* by $ddcx$.

Our choice of the dcx order to define sub-Poisson pp's has its roots in [4], where one shows various results as well as examples indicating that the dcx order implies ordering of several clustering characteristics known in spatial statistics such as Ripley's K-function or second moment densities. Namely, a pp that is larger in the dcx order exhibits more clustering (while having the same mean number of points in any given set).

C. Connectivity of Sub-Poisson Networks

Full connectivity (multi-hop communication between any two nodes) of an ad-hoc network with many nodes is typically hard to maintain, and so, a more modest question of existence of a large enough, connected subset of nodes (called component) is studied. A possible approach to this problem, proposed in [9], and based on the mathematical theory of *percolation*, consists in studying existence of an *infinite* (called *giant*) *component* of the infinite graph modeling a network. Existence of such a component is interpreted as an indication that the connectivity of the modeled ad-hoc network scales well with its number of nodes.

1) Percolation of Gilbert's Network: Percolation models have been extensively studied both in mathematical and communication literature. The model proposed in [9], called now *Gilbert's model* [1], is now considered as the classical continuum model in percolation theory. It assumes that each node has a given fixed range of communication ρ and the direct connection between any two nodes is feasible if they are within this distance ρ from each other, regardless of the positions of other nodes in the network. The known answer to the percolation problem in this model is given under the assumption that the nodes are distributed according to a homogeneous Poisson pp having a *density* of λ nodes on average per unit of volume (or surface). The result says that there exists a *non-degenerated critical communication range* $0 < \rho_c = \rho_c(\lambda) < \infty$, such that for $\rho \leq \rho_c$ there is no giant connected component of the network (the model does not percolate), while for $\rho > \rho_c$ there is exactly one such component (the model percolates), both statements holding true almost surely; i.e., for almost all Poisson realizations of the network. An equivalent statement of the above result says that for a given communication range ρ there exists a non-degenerated critical density of nodes $0 < \lambda_c = \lambda_c(\rho) < \infty$, below which the model does not percolate and above which it does so almost surely.

As one of the main results of this paper we will prove an extension of the above result, which says that the *critical communication range of any homogeneous sub-Poisson Gilbert's network model is not degenerated*. Moreover, this critical communication range is bounded away from zero and infinity by the constants which depend only of the mean density of nodes and not on the finer structure of the sub-Poisson pp of network nodes. Partial results, regarding only non-degeneracy at zero or at the infinity can be proved for, respectively, $ddcx-$ and $idcx$-sub Poisson networks.

2) Percolation of SINR Networks: A more adequate percolation model of wireless communication network, called *SINR* graph, was studied more recently in [5, 6]. It allows one to take into account the interference intrinsically related to wireless communications. The interference power is modeled by the *shot-noise* field generated by the pp of transmitting nodes. Each pair of nodes in the new model is joined by an edge when the signal power to this shot-noise plus some other (external) noise power ratio is large enough. The resulting random graph does not have the independence structure of the Gilbert's model and increasing the communication range (equivalent in this model to increasing the signal emitted power and hence the value of the shot-noise) is not necessarily beneficial for connectivity. Similar observation regards the increasing of the node density. In fact, the above SINR network model has two essential parameters: the density λ of Poisson pp of nodes and the shot-noise reduction factor γ. The percolation domain is characterized in the Cartesian product of these two parameters. The key result of [6] says that whenever λ is larger than the critical value corresponding to the percolation of the model with interference perfectly canceled out (for $\gamma = 0$, which simplifies the model to the Gilbert's one) then there exists a critical value $\gamma_c = \gamma_c(\lambda) > 0$ such that the model percolates for $\gamma < \gamma_c$ and does not percolate for $\gamma > \gamma_c$. As the second main result of this paper, we extend the above result, also to sub-Poisson SINR networks.

[1]also *Boolean model* or *random geometric graph*

D. Comparison of Shot-Noise Fields

Our proofs rely on the comparison of extremal and additive shot-noise fields generated by dcx-ordered pp's studied in [4]. While the connection to the SINR model is evident (additive shot-noise is an element of this model) the connection to the Gilbert's model is perhaps less evident and relays on the fact that this latter model can be represented as a upper level-set of some extremal shot-noise field.

More precisely, from [4, Propsition 4.1] one can conclude that the probability of n given locations in the space *not* being within the communication range of any of the nodes in the Gilbert's model is higher for the network whose nodes are modeled by a pp larger in dcx order. Using this property, suitable discretization of the model and the Peierls argument (cf. [10, pp. 17–18] or [1, Proposition 14.1.4]) one can prove finiteness of the critical transmission range in the sub-Poisson Gilbert's network. The strict positivity of this range can be proved comparing the expected number of paths from the origin to the boundary of an increasing box, again in some suitable discretization of the model. This latter comparison can be done relying only on weak sub-Poisson assumption.

The remaining part of this paper is organized as follows: In Section II we provide necessary notions and notation. Sub-Poisson Gilbert's model is studied in Section III and sub-Poisson SINR model in Section IV. In Section V we make some remarks on the impact of dcx ordering on various model characteristics usually called capacities, including the so-called capacity functional, being one of the fundamental characteristics studied in stochastic geometry, as well as on some other capacity quantifiers in the information-theoretic sense. In Section VI we show some examples of sub-Poisson and weakly sub-Poisson pp, in particular the so-called *perturbed lattices*. Conclusions as well as open questions are presented in Section VII.

II. Preliminaries

A. dcx Order

We say that a function $f : \mathbb{R}^d \to \mathbb{R}$ is *directionally convex* (dcx) if for every $x, y, p, q \in \mathbb{R}^d$ such that $p \le x, y \le q$ (i.e., $p \le x \le q$ and $p \le y \le q$ with inequalities understood component-wise) and $x + y = p + q$ one has $f(x) + f(y) \le f(p) + f(q)$. Also, we shall abbreviate *increasing* and dcx functions by $idcx$ and *decreasing* and dcx by $ddcx$.

For two real-valued random vectors X and Y of the same dimension, X is said to be dcx smaller than Y, (denoted by $X \le_{dcx} Y$) if $\mathbf{E}(f(X)) \le \mathbf{E}(f(Y))$ for every dcx function f for which both expectations are finite. In full analogy one defines $idcx$ and $ddcx$ orders of random vectors considering $idcx$ and $ddcx$ functions, respectively. These orders clearly depend only on the distributions of the vectors. Two real valued stochastic processes (or random fields) are said $dcx, idcx$ or $ddcx$ ordered if any finite-dimensional distributions of these processes are ordered. This definition extends also to locally finite random measures (in particular pp's), by ordering random values of these measures (in particular numbers of points) on any finite collection of subsets of the state space. More precisely, we say for two pp's Φ_1, Φ_2 that $\Phi_1 \le_{dcx(idcx,ddcx)} \Phi_2$ if for every finitely many bounded Borel subsets $B_1, \ldots, B_n$ we have that,

$$(\Phi_1(B_1), \ldots, \Phi_1(B_n)) \le_{dcx(idcx,ddcx)} (\Phi_2(B_1), \ldots, \Phi_2(B_n)).$$

It was shown in [4] that verifying the above property for all *mutually disjoint bounded Borel subsets* B_i is a sufficient condition for the respective ordering of the pp's.

Using the above definition, we say that Φ *is sub-Poisson if* $\Phi \le_{dcx} \Phi_{Poi}$*, where* Φ_{Poi} *is some Poisson pp*. Noting that for the linear function $f(x) = x$, both f and $-f$ is convex (and thus dcx in one dimension) one can observe that Φ_{Poi} needs to have the same mean measure as Φ; i.e., if $\Phi \le_{dcx} \Phi_{Poi}$ then $\mathbf{E}[\Phi(B)] = \mathbf{E}[\Phi_{Poi}(B)]$ for every bounded Borel subset B. In particular, *a stationary pp* Φ *is sub-Poisson if it is dcx smaller than Poisson pp* Φ_λ *of the same intensity* $\lambda = \mathbf{E}[\Phi(B)]/|B|$, where $|B|$ is the d dimensional Lebesgue measure of the set B. From now on Φ_λ or Φ_μ will always denote homogeneous Poisson pp of intensity λ or μ. We will also say that pp Φ is homogeneous if its mean measure is equal, up to a constant, to the Lebesque measure; i.e. $\mathbf{E}[\Phi(B)] = \lambda|B|$, for some constant λ all bounded Borel sets B. This is a weaker assumption than the stationarity of Φ.

We say that Φ is $idcx$($ddcx$)-sub-Poisson if $\Phi \le_{idcx(ddcx)} \Phi_{Poi}$, where Φ_{Poi} is some Poisson pp. In this case the mean measure of Φ is smaller or equal (larger or equal) to that of Φ_{Poi}.

B. Joint Intensities

The joint intensities $\rho^{(k)}(x_1, \ldots, x_k)$ of a pp are defined by the following relation for every finitely many disjoint bounded Borel sets $B_1, \ldots, B_n$:

$$\mathbf{E}\Big[\prod_{i=1}^{k} \Phi(B_i)\Big] = \int \ldots \int_{B_1 \times \cdots \times B_k} \rho^{(k)}(x_1, \ldots, x_k) \mathrm{d}x_1 \ldots \mathrm{d}x_k,$$

provided they exist (i.e., the respective moment measures admit densities). A pp Φ with joint intensities of all orders k is said to be *weakly sub-Poisson* if there exists a constant λ such that for all $k \ge 1$,

$$\rho^{(k)}(x_1, \ldots, x_k) \le \lambda^k \quad \text{a.e..} \tag{2.1}$$

Due to the $idcx$ property of the function $f(x_1, \ldots, x_k) = \prod_i x_i^+$, ordering $\Phi_1 \le_{idcx} \Phi_2$ implies that for all $k \ge 1$, $\mathbf{E}[\prod_{i=1}^k \Phi_1(B_i)] \le \mathbf{E}[\prod_{i=1}^k \Phi_2(B_i)]$ for any disjoint bounded Borel sets $B_1, \ldots, B_k$. Hence, the respective joint intensities $\rho_j^{(k)}$ of pp's Φ_j, $j = 1, 2$, provided they exist, obey for all $k \ge 1$, $\rho_1^{(k)}(x_1, \ldots, x_k) \le \rho_2^{(k)}(x_1, \ldots, x_k)$ a.e.. Since $\rho^{(k)}(x_1, \ldots, x_k) = \lambda^k$ for Poisson pp Φ_λ, we are justified in using the term weakly sub-Poisson for pp Φ satisfying (2.1).

III. Percolation of Sub-Poisson Gilbert's Networks

Given a pp Φ on $\mathbb{R}^d$ and a non-negative constant $r > 0$ one defines the (*spherical*) Boolean model generated by Φ of ball of radius r as the union $C(\Phi, r) = \bigcup_{X\in\Phi} B_X(r)$, where $B_x(r)$ is the ball centered at $x \in \mathbb{R}^d$ of radius r. We say that $C(\Phi, r)$ *percolates* if there exists an unbounded, connected subset of $C(\Phi, r)$. This definition extends to any random subset of $\mathbb{R}^d$. One defines the *critical radius* $r_c(\Phi)$ of the Boolean model as $r_c(\Phi) := \inf\{r > 0 : \mathbf{P}\{\, C(\Phi, r) \text{ percolates}\} > 0\}$. Here is the first result of this paper.

Proposition 3.1: *There exist universal constants $0 < c(\lambda)$ and $C(\lambda) < \infty$ depending only on λ (and the dimension d) such that if $\Phi \le_{idcx} \Phi_\lambda$ then $c(\lambda) \le r_c(\Phi)$ and if $\Phi \le_{ddcx} \Phi_\lambda$ then $r_c(\Phi) \le C(\lambda)$. Thus, for a homogeneous dcx-sub-Poisson Φ of intensity λ we have $0 < c(\lambda) \le r_c(\Phi) \le C(\lambda) < \infty$, where the constants depend only on λ. The lower bound holds also when Φ is weakly sub-Poisson.*

The proof is given in the Appendix. The above result can be extended to the so-called k-percolation models.

Note that the connectivity structure of the Boolean model $C(\Phi, r)$ corresponds to that of the Gilbert's network with nodes in Φ and communication range $\rho = 2r$; cf. Section I-C1. Thus the critical communication range $\rho_c = \rho_c(\Phi)$ in this latter network is twice the critical radius of the corresponding Boolean model.

While the finiteness of the critical radius of the Boolean model (and thus communication range of the sub-Poisson Gilbert's network) is intuitively a desired property, its positivity at first glance might be seen from the networking point of view as irrelevant, if not a disadvantage. A deeper inspection of wireless communication mechanisms shows however that sometimes a non-percolation might be also a desired property. The following modification of the Gilbert's model, that can be seen as a toy version of the SINR model studied in Section IV, sheds some light on this latter statement.

Example 3.2 (Gilbert's carrier-sense network): Consider a planar ad-hoc network consisting of nodes modeled by a point process Φ_B on the plane (in $\mathbb{R}^2$) and having communication range ρ. This process corresponds to a *back-bone* of the network, whose percolation we are looking for. Consider also an auxiliary pp Φ_I of *interferers* also on $\mathbb{R}^2$. Consider the following modification of the Gilbert's connection rule: any two given nodes of Φ_B can directly communicate (are joined be an edge) when they are within the communication range ρ from each other, however only when there is no interfering node (point of Φ_I) within the *sensing range* $R > \rho$ of any of these two nodes. Note that any connected component of this network is included in some connected subset of the *complement* $\mathcal{O} = \mathbb{R}^d \setminus C(\Phi_I, R-\rho)$ of the Boolean model $C(\Phi_I, R-\rho)$. Percolation of this *vacant region* $\mathcal{O}$ of the spherical Boolean model is thus a necessary condition for the percolation of our modification of the Gilbert's network. Now, the vacant region of the planar Boolean model cannot percolate if the Boolean model itself does (cf [13, Theorem 4.4]). Thus a non-percolation of some Boolean model related to the interferes Φ_I is a necessary condition for the percolation of the communication network on Φ_B.

IV. Percolation of Sub-Poisson SINR Networks

In this section, we shall work only on the plane $\mathbb{R}^2$. We slightly modify the definition of SINR network introduced in [5] allowing for *external interferers*. The parameters of the model are non-negative numbers P (signal power), N (environmental noise), γ (interference reduction factor), T (SINR threshold) and an attenuation function $l(r)$ of the distance $r \ge 0$ satisfying $0 \le l(r) \le 1$, continuous, strictly decreasing [2] on its support, with $l(0) \ge \frac{TN}{P}$ and $\int_0^\infty x l(x) \mathrm{d}x < \infty$. These assumptions are exactly as in [6].

Given a pp Φ, the (unit-power) interference generated by Φ at location x is defined as $I_\Phi(x) := \sum_{X\in\Phi\setminus\{x\}} l(|X - x|)$. More generally this object is also called (additive) shot-noise field generated on by Φ withe *response function* l; cf [1, Ch. 2.2 and 2.3]. The SINR from x to y with interference from Φ is defined as

$$\mathrm{SINR}(x, y, \Phi, \gamma) := \frac{Pl(|x-y|)}{N + \gamma P I_{\Phi\setminus\{x\}}(y)}. \tag{4.2}$$

Let Φ_B and Φ_I be two pp's on the plane. We do not assume any particular dependence between Φ_B and Φ_I. In particular one may think of $\Phi_B \subset \Phi_I$. Let $P, N, T > 0$ and $\gamma \ge 0$. The SINR network with *back-bone* Φ_B and *interferers* Φ_I is defined as a graph $G(\Phi_B, \Phi_I, \gamma)$ with nodes in Φ_B and edges joining any two nodes $X, Y \in \Phi_B$ when $\mathrm{SINR}(Y, X, \Phi_I, \gamma) > T$ and $\mathrm{SINR}(X, Y, \Phi_I, \gamma) > T$. The SNR graph (i.e, the graph without interference) is defined as $G(\Phi_B) = G(\Phi_B, \Phi_I, 0)$, which is equivalent also to taking $\Phi_I = \emptyset$. Observe that the SNR graph $G(\Phi_B)$ corresponds to the Gilbert's network with nodes in Φ_B of communication range $\rho_l = l^{-1}(\frac{TN}{P})$. Percolation in the above graphs is existence of an infinite connected component in the graph-theoretic sense.

1) Poisson Back-Bone: Firstly, we consider the case when the backbone nodes are distributed according to Poisson pp $\Phi_B = \Phi_\lambda$, for some $\lambda > 0$. We shall use $G(\lambda, \Phi_I, \gamma)$ and $G(\lambda)$ to denote the corresponding SINR and SNR graphs respectively. Recall from I-C1 that $\lambda_c(\rho)$ is the critical intensity for percolation of the Poisson Gilbert's network of communication range ρ. The following result guarantees the existence of $\gamma > 0$ such that for any homogeneous sub-Poisson pp of interferers Φ_I the SINR network $G(\lambda, \Phi, \gamma)$ percolates provided $G(\lambda)$ percolates.

Proposition 4.1: *Let $\lambda > \lambda_c(\rho_l)$ and $\Phi \le_{idcx} \Phi_\mu$ for some $\mu > 0$. Then there exists $\gamma_c = \gamma_c(\lambda, \mu, P, T, N) > 0$ such that $G(\lambda, \Phi, \gamma)$ percolates for $\gamma < \gamma_c$.*

The proof is given in the Appendix.

Recall that we have not assumed the independence of Φ_I and $\Phi_B = \Phi_\lambda$. In particular, one can take $\Phi_I = \Phi_\lambda \cup \Phi'$ where Φ' are some *external* interferers. If Φ' is $idcx$-sub-Poisson and independent of Φ_λ then Φ_I is also $idcx$-sub-Poisson (cf [4, Proposition 3.2]). The result of Proposition 4.1 in the special case of $\Phi' = \emptyset$ was proved in [6]. The present extension allows

[2] So it is rather path *gain* function.

for any $idcx$-sub-Poisson pattern of independent external interferers. The proof of our result can be also modified (which will not be presented in this version of the paper) to allow for external interferer's Φ' possibly dependent of the backbone. In this full generality the result says that *any external pattern of homogeneous $idcx$-sub-Poisson interferers added to the SINR network of [6] cannot make the giant component of this network to disappear, provided the interference cancellation factor γ is appropriately adjusted*. Since an $idcx$-sub-Poisson point process can be of arbitrarily large intensity, the above observation can be loosely rephrased in the following form: *It is not the density of interferers that matters for the network connectivity, but their structure; $idcx$-sub-Poisson interferers do not hurt essentially the network connectivity*.

2) Sub-Poisson Back-Bone: We shall now consider the case when the backbone nodes are formed by a sub-Poisson pp. In this case, we can give a weaker result, namely that with appropriately increased signal power P, the SINR graph will percolate for small interference parameter $\gamma > 0$. This corresponds to an early version of the result for the Poisson SINR network, proved in [5], where the percolation of the SINR network is guaranteed for the intensity of nodes possibly larger than the critical one in the corresponding SNR network.

Proposition 4.2: *Let $\Phi_B \leq_{ddcx} \Phi_\lambda$ for some $\lambda > 0$ and $\Phi_I \leq_{idcx} \Phi_\mu$ for some $\mu > 0$ and also assume that $l(x) > 0$ for all $x \geq 0$. Then there exist $P, \gamma > 0$ such that $G(\Phi_B, \Phi_I, \gamma)$ percolates.*

The proof is given in the Appendix.

As in Proposition 4.1, we have not assumed the independence of Φ_I and Φ_B. In particular one can take $\Phi_I = \Phi_B \cup \Phi'$, where Φ' and Φ_B are independent and dcx-sub-Poisson.

V. dcx Ordering and Capacity

Now we want to make some remarks on the relation between dcx ordering and some model characteristics usually called capacities, including the so-called stochastic-geometric capacity functional, as well as some other capacity quantifiers in the information-theoretic sense. Our main tool is the following result proved in [4, Theorem 2.1]. It says that $\Phi_1 \leq_{dcx} \Phi_2$ implies the same ordering of the respective shot-noise fields (with an arbitrary non-negative response function)

$$(I_{\Phi_1}(x_1), \ldots, I_{\Phi_1}(x_n)) \leq_{dcx} (I_{\Phi_2}(x_1), \ldots, I_{\Phi_2}(x_n)) \quad (5.3)$$

for any $x_i, i = 1, \ldots, n$. This implies in particular that

$$\mathbf{E}\Big[\exp\Big\{s\sum_{i=1}^{n} I_{\Phi_1}(x_i)\Big\}\Big] \leq \mathbf{E}\Big[\exp\Big\{s\sum_{i=1}^{n} I_{\Phi_2}(x_i)\Big\}\Big] \quad (5.4)$$

for *both positive and negative s*.

A. dcx Ordering and the Capacity Functional

Capacity functional $T_\Xi(B)$ of a random set Ξ is defined as $T_\Xi(B) = \mathbf{P}\{\Xi \cap B \neq \emptyset\}$ for all bounded Borel sets B. A fundamental result of stochastic geometry, called the Choquet's theorem (cf [12]) says that the capacity functional defines entirely the distribution of a random closed set. The complement of it, $V_\Xi(B) = 1 - T_\Xi(B)$, is called *void probability functional*.

Inequality (5.4) allows to compare capacity functionals and void probabilities of Boolean models generated by dcx ordered pp's. Indeed, note that

$$\begin{aligned} V_{C(\Phi,r)}(B) &= \mathbf{E}\Big[\prod_{X\in\Phi} \mathbb{1}(|X - B| > r)\Big] \\ &= \mathbf{E}\Big[\exp\Big\{\sum_{X\in\Phi} h(X)\Big\}\Big], \end{aligned}$$

where $h(x) = \log(\mathbb{1}(|X - B| > r))$. Note that the latter expression has a form of the shot-noise variable and thus using the inequality analogous to (5.4) for shot-noise with the response function $h(\cdot)$ (cf [4, Theorem 2.1] for such a generalization) we observe that if $\Phi_1 \leq_{dcx} \Phi_2$ then

$$\begin{aligned} V_{C(\Phi_1,r)}(B) &\leq V_{C(\Phi_2,r)}(B) \\ T_{C(\Phi_1,r)}(B) &\geq T_{C(\Phi_2,r)}(B). \end{aligned}$$

In other words, one can say that dcx smaller pp's exhibit smaller voids. In Section VI-B we will show some simulation examples which illustrate this statement.

B. dcx Ordering and Network Capacity

We focus now on capacity quantifiers in the sense of communication theory. Inequality (5.4) with $s > 0$ and $\Phi_1 = \Phi_I$ and Poisson $\Phi_2 = \Phi_\mu$ will be used in the proof of Proposition 4.1 (see Appendix) to show that the *lower level-sets $\{x : I_{\Phi_I}(x) \leq M\}$ of the interference field generated by Φ_I percolate through Peierls argument* (cf. [1, Proposition 14.1.4]) for sufficiently large M. Similarly (5.4) with $s < 0$ can be used to prove that the *upper level-sets of the interference field $I_{\Phi_I}(x)$ generated by Φ_I percolate through Peierls argument for sufficiently large level values M.*

Having observed this *double* impact of sub-Poisson assumption on Φ_I it is not evident whether the threshold value γ_c of the interference reduction factor is larger in sub-Poisson network than in the corresponding Poisson one. Note that γ_c can be related to the information-theoretic capacity (throughput) that can be sustained on the links of the SINR graph. In what follows we will try to explain how sub-Poisson assumption can impact some quantifiers of the network capacity.

1) Ordering of Independent Interference Field: It is quite natural to consider a network capacity characteristic $C = f(I(x_1), \ldots, I(x_n))$ that depends on some interference field I through some $ddcx$ function f. We will give a few simple examples of such characteristics in what follows. Then, a *larger in $ddcx$ order interference field (more variable!) $I(x)$ leads to larger average capacity* $\mathbf{E}[C]$. In particular in the case of shot-noise interference field $I_{\Phi_I}(x)$, by (5.3) we conclude that *larger in dcx order pp Φ_I (clusters more!) leads to larger average capacity* $\mathbf{E}[C]$.

Let us illustrate this somewhat surprising observation by two simple examples.

Example 5.1 (Shannon Capacity): Let $C(I) = \log(1 + F_0/(N + I))$ for some constants $F_0, N > 0$ and random I.

Clearly this is a decreasing convex function of I and larger in convex order I gives larger mean capacity $\mathbf{E}[C(I)]$. The smallest value of $\mathbf{E}[C(I)]$ given $\mathbf{E}[I]$ is attained for constant $I \equiv \mathbf{E}[I]$.

Example 5.2 (Outage capacity of a channel with fading): Assume a random channel fading F with convex tail distribution function $G(t) = \mathbf{P}\{F > t\}$. Consider $C(I) = \mathbf{P}\{F/(N+I) > T\}$ for some constant $N, T > 0$. Assuming independence of F and I we have $\mathbf{E}[C] = \mathbf{E}[G(T(N+I))]$, which is by our assumption expectation of a convex function of I and the same conclusion can be made as in Example 5.1. The above general form of the expression for the outage capacity can be found in many more detailed models based on pp, in particular in the Bipolar model of spatial Aloha in [2, Chapter 16]. See also [4] for a multidimensional version of this observation. Similar conclusion can be made for the ergodic Shannon capacity $\mathbf{E}[\log(1 + F/(N+I))]$ of this channel.

2) Ordering of the Back-Bone: Let us take one step further and consider the interference created by the original pattern of network points (the back-bone process Φ_B) rather than an external (independent) interference field. In this case the interference $I_{\Phi_B}(x)$ at receiver x usually has to be considered under the so-called Palm distribution of Φ_B *given* the location of the emitter (in Φ_B) of this receiver x (again see e.g. the Bipolar model of spatial Aloha in [2, Chapter 16] for a detailed example). The problem is that dcx ordering of Φ_B implies only $idcx$ ordering of the respective Palm versions of Φ_B; cf. [4]. The fundamental reason for the required "extra" *increasing* property of the comparable functions is that a smaller in dcx pp Φ_B (having less clustering or even some point "repulsion") will have under Palm probability potentially *fewer points in vicinity of the conditioned point*. This potentially decreases interference created locally near this point, thus potentially increases our capacity characteristic. More formally: having $idcx$ ordered of Φ_B under Palm probability $\mathbf{P}^0$ and the capacity expressed as a $ddcx$ function of I_{Φ_B} we cannot conclude any inequality for $\mathbf{E}^0[C(I_{\Phi_B})]$.

The situation is naturally inverted when we are dealing with pp which are dcx larger than Poisson pp. A detailed analysis in [8] of the outage capacity in the Bipolar model generated by some Poisson-Poisson cluster pp known as Neyman-Scott pp (which is dcx larger than Poisson pp) confirms the above observations. Namely, for smaller transmission distance the negative impact of clustering (locally more interferers) decreases the outage capacity, while for larger transmission distance the positive impact of interferers being more clustered increases this capacity.

3) Multi-hop Capacity Models: Percolation models have been also shown useful to study the transport (multi-hop) capacity of ad-hoc networks. For example, in [7], by using a specific multi-hop transmission strategy that involves percolation theory models, it was shown that a Poisson SINR network can achieve capacity rate of the order of $1/\sqrt{n}$ bits per unit of time and per node, thus closing the gap with respect to the rate $1/\sqrt{n \log n}$ shown achievable in Poisson networks in [11]. An interesting and open question (particularly in view of what was shown above) is whether these capacity results can be extended to sub-Poisson networks.

VI. Examples of Sub-Poisson Point Processes

From [4, Section 5.2, 5.3], we have a rich source of examples of Cox (doubly stochastic Poisson) pp's comparable in dcx and $ddcx$ order. In particular, we know that the so called Lévy-based Cox pp (with Poisson-Poisson cluster pp as a special case) is dcx larger than the Poisson pp of the same mean intensity. They can be called thus (dcx*-super-Poisson*). However, note that any Cox pp, whose (random) realizations of the intensity are almost surely bounded by some constant, can by coupled with (constructed as a subset of) a Poisson pp with intensity equal to this constant. Consequently such Cox processes are trivially $idcx$-sub-Poisson.

A. Sub-Poisson Point processes

In the remaining part of this section we concentrate on the construction of examples of dcx-sub-Poisson pp's.

Example 6.1 (Perturbed lattice): Consider some lattice (e.g. the planar hexagonal one, i.e., the usual "honeycomb" model often considered in cellular network context). Let us "perturb" this "ideal" pattern of points as follows. For each point of this lattice, say z_i, let us generate independently, from some given distribution, a random number, say N_i, of nodes. Moreover, instead of putting these nodes at X_i, let us translate each of these nodes independently from z_i, by vectors, having some given spatial distribution (say for simplicity, of bounded support). The resulting network, called *perturbed lattice*, can be seen as *replicating and dispersing points from the original lattice*. Interesting observations are as follows.

- If the number of replicas N_i are Poisson random variables, then the perturbed lattice is a Poisson p.p.
- If moreover the node displacement is uniform in the Voronoi cell of the original lattice, then the resulting perturbed lattice is homogeneous Poisson process Φ_λ with λ related to $\mathbf{E}[N]$ and the original lattice density.
- Now, if N_i are convexly smaller than some Poisson variable [3] then the perturbed lattice is dcx-sub-Poisson. This is so because the perturbed lattice pp is an independent sum of countably many pp formed by the perturbations of every vertex of the lattice. This means that $\Phi = \bigcup_{z_i} \Phi_{z_i}$ where Φ_{z_i} are independent pp. From [4, Proposition. 3.2], we have that if Φ_{z_i} are dcx ordered for every z_i, then so is Φ. Since each of these pp is formed by N_i i.i.d. perturbations, it can be shown that these pp's are dcx ordered if their respective N_i's are convexly ordered. This follows from proving that the following function g is convex in its argument n: $g(n) = \mathbf{E}(f(\Phi_{z_i}(A_1), \ldots, \Phi_{z_i}(A_n))|N = n)$ for any dcx function f, disjoint bounded Borel subsets $A_1 \ldots, A_n$ and any fixed z_i of the original lattice.

[3] i.e., expectations of all convex functions of $N = N_i$ are smaller than the respective expectations for Poisson variable of the same mean; a special case is when $N_i = const$ is constant

- Consequently, if N_i's are convexly smaller than some Poisson random variable and moreover the node displacement is uniform in the Voronoi cell of the original lattice, then the resulting perturbed lattice is dcx-smaller than the respective homogeneous Poisson process Φ_λ.

The interest in the above perturbed lattice models in the networking context stems from the fact that they provide some spectrum of models that ranges from periodic grids, usually considered in cellular network context, to ad-hoc networks almost exclusively considered under Poisson assumptions. In Section VI-B we will show some samples of perturbed lattices.

On the theoretical side, the interest in perturbed lattices stems from their relations to zeros of Gaussian analytic functions (GAFs) (see [14, 15]). More precisely [16] shows that zeros of some GAFs have the same distribution as points of some "non-independently" perturbed lattice.

Another class of pp's, which can be shown as *weakly sub-Poisson*, are stationary determinantal pp's. For a quick introduction refer [3].

Example 6.2 (Determinantal pp's): These pp's are defined by their joint intensities satisfying the following relation for all $k \geq 1$:

$$\rho^{(k)}(x_1, \ldots, x_k) = \det\Big(K(x_i - x_j)\Big)_{i,j},$$

for some Hermitian, non-negative definite, locally square integrable kernel $K : \mathbb{R}^d \to \mathbb{C}$. Then by Hadamard's inequality [4], we have that *stationary determinantal pp are weakly sub-Poisson*. These pp are considered as examples of pp whose points "exhibit repulsion". Though zeros of GAFs are related to determinantal pp's, curiously enough only zeros of i.i.d. Gaussian power series (i.e, $f(z) = \sum_{n=0}^{\infty} a_n z^n$, with a_n i.i.d. standard complex normal) are proved to be determinantal pp's (see [14]).

B. Simulations

Now, we will show some examples of simulated patterns of perturbed lattices. We consider hexagonal pattern of original (unperturbed) points (cf. Figure 1, upper-left plot). The replicas are always displaced uniformly in the Voronoi cell of the given point of the original point. Consequently, when the numbers of replicas N have Poisson distribution then the corresponding perturbed lattice is Poisson pp. (cf. Figure 1, middle-right plot). We always take $\mathbf{E}[N] = 1$. In order to generate sub-Poisson pp's we take N having Binomial distribution $Bin(n, 1/n)$. This family of distributions can be shown *convexly increasing in* n and dcx *upper-bounded by Poisson* $Poi(1)$ distribution. Note on Figure 1 that in the case $Bin(1, 1)$, where we have only point displacement (no replications), the resulting perturbed lattice looks much more "regular" (less visible voids, less clusters) than Poisson pp. This regularity becomes less evident when n increases and already for $n \geq 3$ it is difficult to distinguish the perturbed lattice patterns from Poisson pp. However, as we will see latter studying Gilbert graphs on them (Figures 2, and 3), up to approximately $n = 5$ they still have significantly different clustering structure than this of Poisson pp.

The last raw of Figure 1 shows three examples of perturbed lattices dcx *larger* than Poisson pp. On the first two plots in the last row we show examples of a doubly stochastic Poisson pp (Cox pp), where the mean value L of the number of replicas N is first sampled independently for each original lattice point, with $\Pr\{ L = n \} = 1 - \Pr\{ L = 0 \} = 1/n$ and then, given L, N has Poisson distribution $Poi(L)$. This construction is an example of the Ising-Poisson cluster pp considered in [4, Section 5.1]. Another possibility to generate the number of replicas N convexly larger than Poisson variable is to "scale-up" the Poisson variable of a smaller intensity. In the lower-right plot of Figure 1 we take $N = nN'$ where $N' \sim Poi(1/n)$.

In order to discover more fine properties of the clustering structure of the perturbed lattices with binomial number of replicas, on Figures 2 and 3 we consider a larger simulation window comprising about 40^2 points of the original hexagonal lattice. We generate points of the perturbed lattices with $N \sim Bin(n, 1/n)$ for various $n = 1, \ldots 8$ and for each n we plot the Gibert graphs of different communication radius. The largest component in the simulation window is highlighted. Moreover, we show on the bar-plots the empirical fraction of the number of points in 10 largest components. Observing values of ρ for which the largest component starts significantly out-number all other components, we conjecture that *the critical radius* $\rho_c(n)$ *for the percolation of the Gilbert graph on the considered perturbed lattices is increasing in* n. Note that for the unperturbed lattice $\rho_c = 1$ (the distance between adjacent nodes in our hexagonal lattice) and for Poisson pp of the same intensity ρ_c is known to be close to 1.112.

VII. Concluding Remarks

We have extended two results on nontrivial phase transition in percolation of Gilbert's and SINR graphs from Poisson to dcx-sub-Poisson pp's. This means that, regarding existence of this phase transition, Poisson pp represents a worst-case scenario within this class of pp's.

A natural question in this context is as follows. Consider $\Phi_1 \leq_{dcx} \Phi_2$. Does this ordering imply that the corresponding critical communication ranges in the Gilbert's model are ordered as well $\rho_c(\Phi_1) \leq \rho_c(\Phi_2)$? The answer in the full generality is negative, as we expect some counterexamples of dcx-super-Poisson pp's with the critical communication ranges degenerated to 0 or ∞. However, we conjecture that the critical communication range of a stationary dcx-sub-Poisson pp's of intensity λ is smaller than that of a Poisson pp of intensity λ.

In this paper we have also discussed the impact on dcx ordering on some local quantifiers of the capacity in networks. An interesting open question in this context is the relation between dcx ordering and the transport capacity in these networks.

Finally, we brought attention to the so called perturbed lattices, which can provide a spectrum of pp's monotone in dcx order that ranges from periodic grids, usually considered

[4] $\det(a_{ij})_{ij} \leq \prod_i a_{ii}$ for Hermitian, non-negative definite matrices (a_{ij})

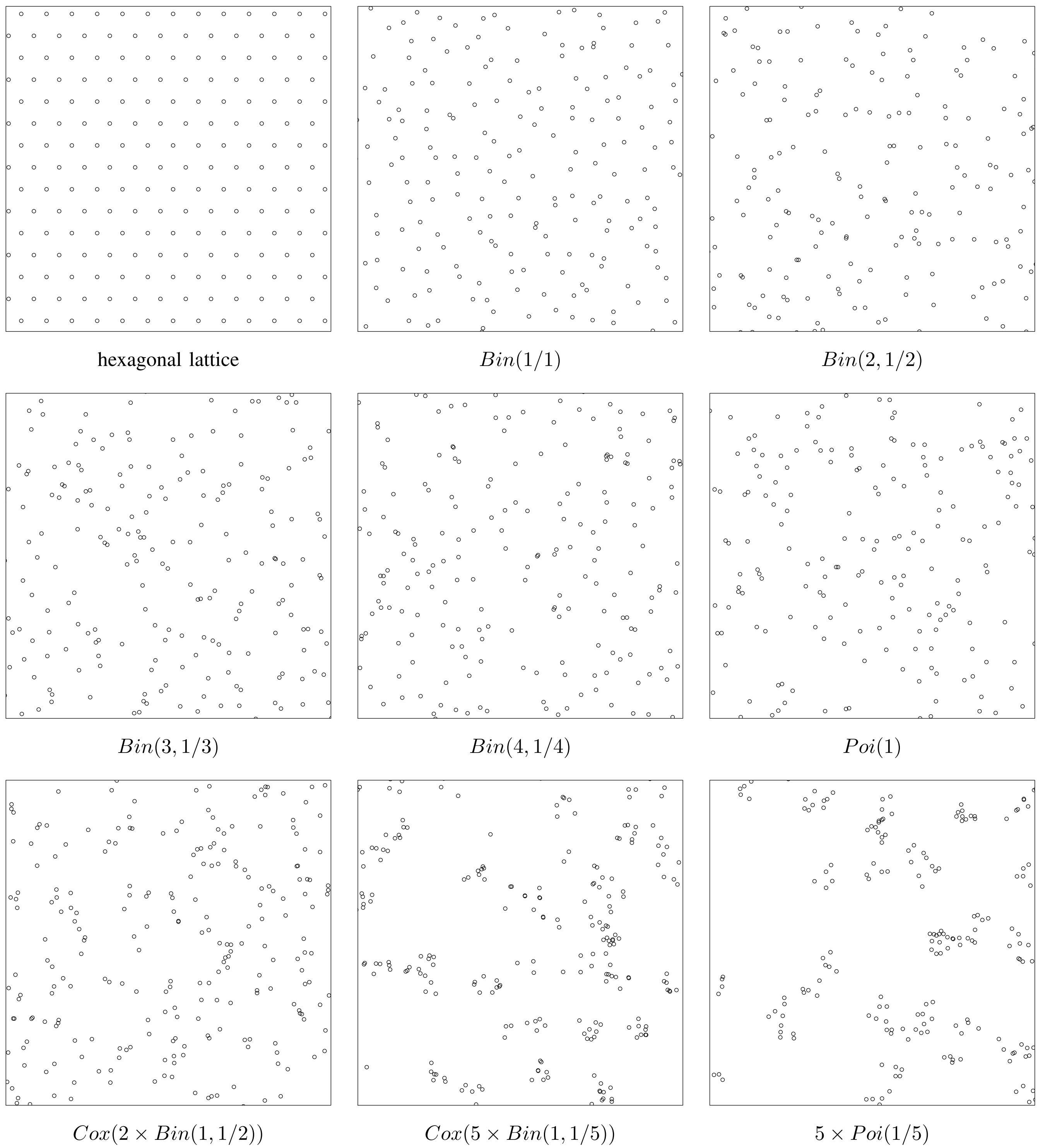

Fig. 1. ”Unperturbed” hexagonal lattice and sub-Poisson perturbed lattices with the number of replicas N having binomial distribution $B(n,1/n)$. In the last raw three examples of super-Poisson perturbed lattices: with N having double stochastic Poisson (Cox) distribution of random mean L having Bernoulli distribution $n \times Bin(1,1/n)$ (i.e., $\Pr\{ L = n \} = 1 - \Pr\{ L = 0 \} = 1/n$) and “rescaled” Poisson distribution $N \sim n \times Poi(1/n)$.

in cellular network context, to Poisson pp’s and to various more clustered pp’s including doubly stochastic Poisson pp’s.

Appendix

In this section we prove our main results of this paper.

Proof: (*of Proposition* 3.1) Denote by $\mathbb{Z}^d(r)$, the discrete graph formed by the vertexes of $r\mathbb{Z}^d = \{rz : z \in \mathbb{Z}^d\}$, where

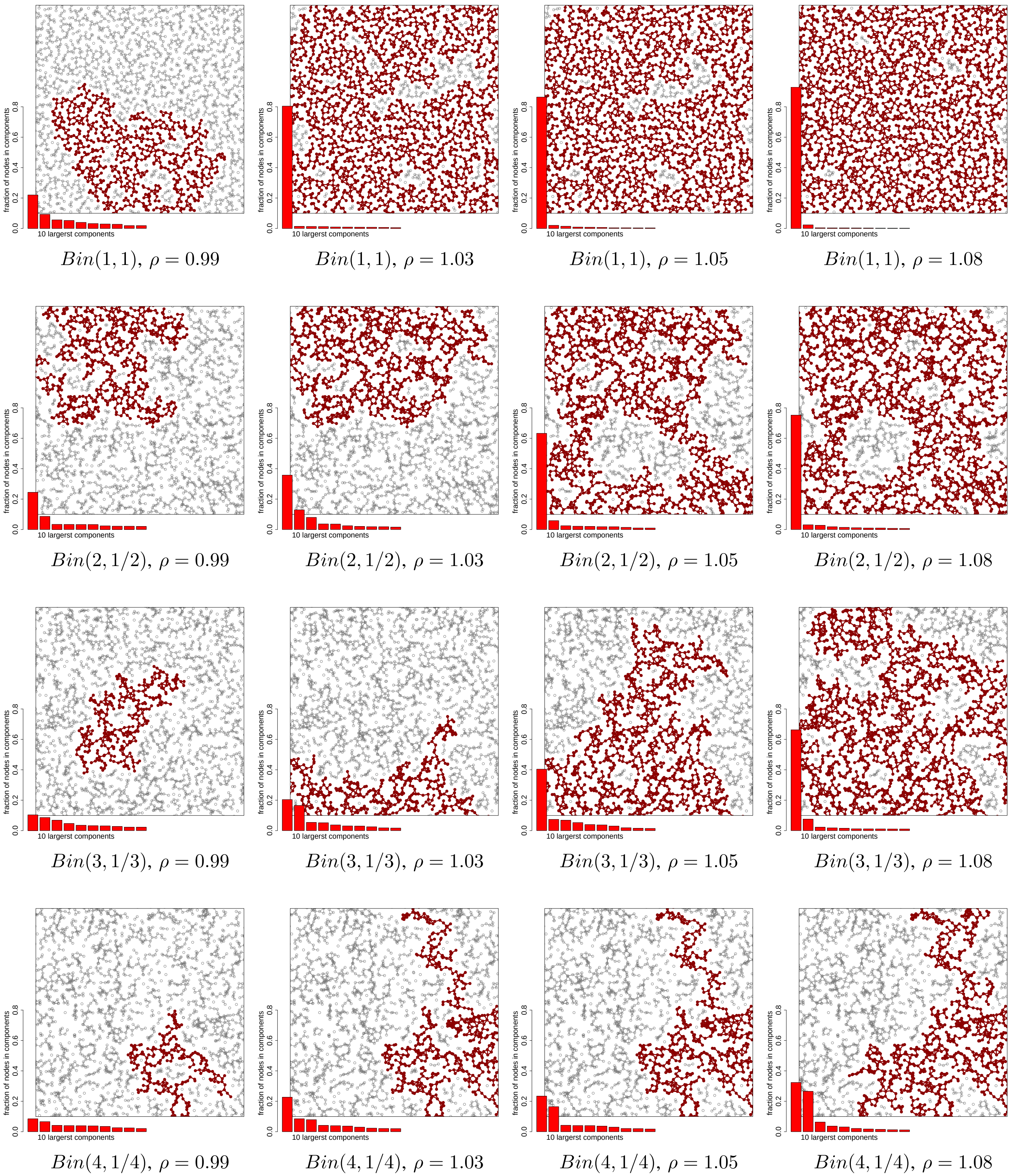

Fig. 2. Gilbert graph with communication range ρ and nodes in perturbed lattice pp with Binomial $Bin(n, 1/n)$ number of replicas uniformly distributed in hexagonal cells. The largest component in the simulation window is highlighted. Bar-plots show the fraction of nodes in ten largest components.

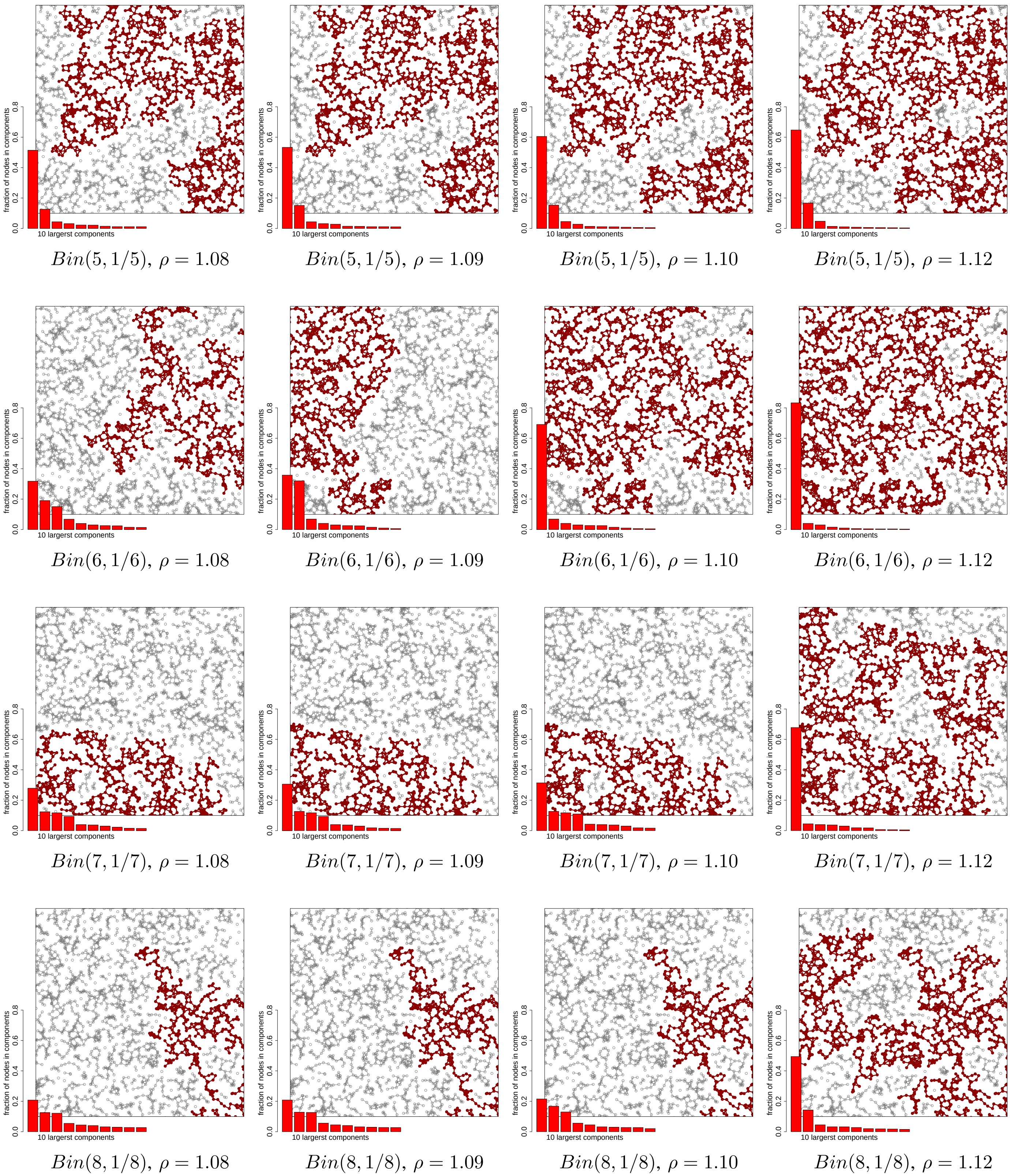

Fig. 3. Gilbert graph with communication range ρ and nodes in perturbed lattice pp with Binomial $Bin(n, 1/n)$ number of replicas uniformly distributed in hexagonal cells. The largest component in the simulation window is highlighted. Bar-plots show the fraction of nodes in ten largest components.

$\mathbb{Z}^d$ is the d-dimensional integer lattice, and edges between $z_i, z_j \in \mathbb{Z}^d(r)$ such that $\{z_i+[-\frac{r}{2},\frac{r}{2}]^d\}\cap\{z_j+[-\frac{r}{2},\frac{r}{2}]^d\} \neq \emptyset$. Now we define a site percolation on $\mathbb{Z}^d(r)$ induced by the pp $\Phi : X_r(z) = \mathbf{1}[\Phi(Q_r(z)) \geq 1]$ where $Q_r(z) = (-\frac{r}{2},\frac{r}{2}]^d$. Note that when $X_{2r}(z)$ does not percolate, $C(\Phi, r)$ does not percolate and if $X_{\frac{r}{\sqrt{d}}}(z)$ percolates, so does $C(\Phi, r)$.

The standard technique to show non-percolation is to show that the expected number of (self-avoiding) paths of length n starting at the origin in the random sub-graph induced by opened sites of the percolation model tends to zero as $n \to \infty$. The probability of a path $(z_1,\ldots,z_n)$ of length n in $\mathbb{Z}^d(2r)$ being supported by open sites is $\mathbf{P}\{\prod_i \Phi(Q_{2r}(z_i)) \geq 1\} \leq \mathbf{E}(\prod_i \Phi(Q_{2r}(z_i))) \leq (\lambda(2r)^d)^n$ where the first inequality is due to Markov's inequality and the last inequality is by weak sub-Poisson property of Φ, which in particular is implied by $idcx$-sub-Poisson property (cf. (2.1)). Since the number of paths starting at the origin, of length n in $\mathbb{Z}^d(2r)$, is bounded by $(3^d-2)^n$, we have that the expected number of paths of length n is at most $((3^d-2)\lambda(2r)^d)^n$ and this tends to 0 for r small enough. This shows that there exists $c(\lambda) > 0$ (depending also on the dimension) such that $r_c(\Phi) \geq c(\lambda)$.

For the upper bound, we use the Peierls argument (cf. [10, pp. 17–18]) on the site percolation model induced by $X_{\frac{r}{\sqrt{d}}}(z)$ on $\mathbb{Z}^d(\frac{r}{\sqrt{d}})$. To use this argument, one needs to estimate the probability of the site-percolation model not intersecting a path $(z_1,\ldots,z_n)$. This probability can be expressed as $\mathbf{P}\{\Phi(\bigcup_{i=1}^n Q_{\frac{r}{\sqrt{d}}}(z_i)) = 0\}$.

and is smaller than $\exp\{-\lambda n(\frac{r}{\sqrt{d}})^d\}$ by the $ddcx$-sub-Poisson property of Φ and [4, Proposition 4.1]. Now by choosing r large enough this probability can be made as small as we wish and so the Peierls argument can be used. ■

Proof: (*of Proposition* 4.1) We follow the proof given in [6]. Assuming $\lambda > \lambda_c(\rho_l)$, one observes first that the graph $G(\lambda)$ also percolates with any slightly larger constant noise $N' = N+\delta'$, for some $\delta' > 0$. Essential for the original proof of the result is to show that the level-set $\{x : I_{\Phi_I}(x) \leq M\}$ of the interference field percolates (contains an infinite connected component) for sufficiently large M. Suppose that it is true. Then taking $\gamma = \delta'/M$ one has percolation of the level-set $\{y : \gamma I_{\Phi_I}(y) \leq \delta'\}$. The main difficulty consists in showing that $G(\lambda)$ with noise $N' = N+\delta'$ percolates *within* an infinite connected component of $\{y : I_{\Phi_I}(y) \leq \delta'\}$. This was done in [6], by mapping both models $G(\lambda)$ and the level-set of the interference to a discrete lattice and showing that both discrete approximations not only percolate but actually satisfies a stronger, sufficient condition for percolation, related to the Peierls argument [1, Proposition 14.1.4]. We follow exactly the same steps and the only fact that we have to prove, regarding the interference, is that there exists a constant $\epsilon < 1$ such that for arbitrary $n \geq 1$ and arbitrary choice of locations $x_1,\ldots,x_n$ one has $\mathbf{P}\{I_{\Phi_I}(x_i) > M, i = 1,\ldots,n\} \leq \epsilon^n$. To this regard, as in [6], using the Chernoff bound we dominate this probability by $e^{-snM}\mathbf{E}[\exp\{\sum_{i=1}^n sI_{\Phi_I}(x_i)\}]$ with arbitrary $s > 0$. The crucial observation for our extension of the original proof is that $f(u_1,\ldots,u_n) = \exp[s(u_1+\ldots+u_n)]$ is an $idcx$ function. By the assumption $\Phi_I \leq_{idcx} \Phi_\mu$ and [4, Theorem 2.1] we have

$$\mathbf{E}\Big[\exp\Big\{s\sum_{i=1}^n I_{\Phi_I}(x_i)\Big\}\Big] \leq \mathbf{E}\Big[\exp\Big\{s\sum_{i=1}^n I_{\Phi_\mu}(x_i)\Big\}\Big]$$

and we can use the explicit form of the Laplace transform of the Poisson shot-noise (the right-hand-side in the above inequality) to prove, exactly as in [6], that for sufficiently small s it is not larger than K^n for some constant K which depends on μ but not on M. This completes the proof. ■

Proof: (*of Proposition* 4.2) In this scenario, increased power is equivalent to increased radius in the Gilbert's model associated with the SINR model. From this observation, it follows that by using the discrete mapping and arguments as in the proof of Proposition 3.1, we obtain that with increased power the associated Gilbert's model percolates. Then, we use the approach from the proof of Proposition 4.1 to obtain a $\gamma > 0$ such that the SINR network percolates as well. ■

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