An introduction to constructive algebraic analysis and its applications

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Résumé : Ce texte est une extension des notes de cours que j'ai préparés pour les *Journées Nationales de Calcul Formel* qui ont eu lieu au CIRM, Luminy (France) du 3 au 7 Mai 2010. Le but principal de ce cours était d'introduire la communauté française du calcul formel à l'analyse algébrique constructive, et particulièrement à la théorie des D-modules algébriques, à ses applications à la théorie mathématique des systèmes et à ses implantations dans des logiciels de calcul formel tels que Maple ou GAP4. Parce que l'analyse algébrique est une théorie mathématique qui utilise différentes techniques venant de la théorie des modules, de l'algèbre homologique, de la théorie des faisceaux, de la géométrie algébrique et de l'analyse microlocale, il peut être difficile d'entrer dans ce domaine, nouveau et fascinant, des mathématiques. En effet, il existe peu de textes introductifs. Nous sommes rapidement conduits aux livres de Björk qui, à première vue, peuvent sembler difficiles aux membres de la communauté de calcul formel et aux mathématiciens appliqués. Je pense que le problème vient moins de la difficulté technique de la littérature existante que du manque d'introductions pédagogiques qui donnent une idée globale du domaine, montrent quels types de résultats et d'applications on peut attendre et qui développent les différents calculs à mener sur des exemples explicites. A leur humble niveau, ces notes de cours ont pour but de combler ce manque, tout du moins en ce qui concerne les idées de base de l'analyse algébrique. Puisque l'on ne peut enseigner bien que les choses que l'on a bien comprises, j'ai choisi de restreindre cette introduction à mes travaux sur les aspects constructifs de l'analyse algébrique et sur ses applications.

Mots-clés : Analyse algébrique, systèmes linéaires d'équations aux dérivées partielles, systèmes linéaires fonctionnels, théorie des modules, algèbre homologique, algèbre constructive, calcul formel, théorie des systèmes, théorie du contrôle.

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An introduction to constructive algebraic analysis and its applications

Abstract: This text is an extension of lectures notes I prepared for *les Journées Nationales de Calcul Formel* held at the CIRM, Luminy (France) on May 3-7, 2010. The main purpose of these lectures was to introduce the French community of symbolic computation to the constructive approach to algebraic analysis and particularly to algebraic D-modules, its applications to mathematical systems theory and its implementations in computer algebra systems such as Maple or GAP4. Since algebraic analysis is a mathematical theory which uses different techniques coming from module theory, homological algebra, sheaf theory, algebraic geometry, and microlocal analysis, it can be difficult to enter this fascinating new field of mathematics. Indeed, there are very few introducing texts. We are quickly led to Björk's books which, at first glance, may look difficult for the members of the symbolic computation community and for applied mathematicians. I believe that the main issue is less the technical difficulty of the existing references than the lack of friendly introduction to the topic, which could have offered a general idea of it, shown which kind of results and applications we can expect and how to handle the different computations on explicit examples. To a very small extent, these lectures notes were planned to fill this gap, at least for the basic ideas of algebraic analysis. Since, we can only teach well what we have clearly understood, I have chosen to focus on my work on the constructive aspects of algebraic analysis and its applications.

Key-words: Algebraic analysis, linear systems of partial differential equations, linear functional systems, module theory, homological algebra, constructive algebra, symbolic computation, systems theory, control theory.

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Introduction

This text is an extension of lectures notes I prepared for *les Journées Nationales de Calcul Formel* held at the CIRM, Luminy (France) on May 3-7, 2010. The main purpose of these lectures was to introduce the French community of symbolic computation to the constructive approach to algebraic analysis and particularly to algebraic D-modules, its applications to mathematical systems theory and its implementations in computer algebra systems such as Maple or GAP4. Since algebraic analysis is a mathematical theory which uses different techniques coming from module theory, homological algebra, sheaf theory, algebraic geometry, and microlocal analysis, it can be difficult to enter this fascinating new field of mathematics. Indeed, there are very few introducing texts (to our knowledge, the best one is [66] with a few chapters of [13]). We are quickly led to Bjork's first book [10] which, at first glance, may look difficult for the members of the symbolic computation community and for applied mathematicians. I believe that the main issue is less the technical difficulty than the lack of friendly introduction to the topic, which could have offered a general idea of it, shown which kind of results and applications we can expect and how to handle the different computations on explicit examples. Indeed, even if algebraic analysis aims at studying linear systems of algebraic or analytic partial differential equations ("the Courant-Hilbert ([23]) for the new generation" according to [46]), no examples illustrate the main results of the books [10, 11, 13, 45, 46, 66]. And when the term "applications" appears in the title of a book on algebraic analysis such as Bjork's second book "Analytic D-modules and Applications" ([11]), the term "applications" has to be taken in the sense of applications to other pure fields of mathematics such as algebraic geometry, analytic geometry, symplectic geometry... To a very small extent, these lectures notes were planned to fill this gap, at least for the basic ideas of algebraic analysis such as those appearing in [45]. Since, we can only teach well what we have clearly understood, I have chosen to focus on my work on the constructive aspects of algebraic analysis and its applications.

Let us shortly outline the main ideas developed in these lectures notes. In [16], Chyzak (INRIA Rocquencourt), Robertz (RWTH Aachen University) and I developed an approach to linear systems theory based on the concept of an *Ore algebra* introduced in [18], which is a particular case of the so-called *Ore extensions* in noncommutative algebra (see, e.g., [71]). An Ore algebra is a polynomial ring which is not too badly noncommutative (in particular, the commutation rules do not involve monomials of higher degree). This class contains the ring of partial differential operators, the ring of differential difference operators, the ring of differential time-delay operators... (see Section 1.1). Based on the concept of Ore algebras, we developed in [16, 17] an algebraic analysis approach to linear systems over Ore algebras. In particular, this approach allowed us to develop a unified mathematical framework for different classes of mathematical systems encountered in control theory, to study certain of their built-in properties in an intrinsic way by means of module theory (see Section 1.6), to develop generic algorithms for the study of these module properties and to implement them in the Maple package OreModules ([17]) based on the noncommutative Gröbner bases computation available in Maple (thanks to the work of Chyzak ([18])). In particular, we were able to extend the results of Kashiwara ([45]) (see also [89]) concerning the characterization of module properties (e.g., existence of torsion elements, torsion-free, reflexive, projective, stably free) in terms of the vanishing of certain extension modules from the rings of partial differential operators to certain classes of Ore algebras (see Section 1.3). Recently, I came to realize that these results were already known by Auslander ([2]), one of "the kings" of modern algebra. These classical concepts of module theory have important interpretations in systems theory in terms of the existence of parametrizations of the linear system associated with the studied module (when the functional space of the linear system is rich enough ([78])) (see Section 1.4). Surprisingly, we cannot find these interesting interpretations in any textbooks on module theory although they could motivate one to introduce them in module theory. It is certainly one of finest consequences of connecting module theory to linear systems theory. For instance, the differential module associated with the classical curl operator (used in mathematical physics) is torsion-free since it is parametrized by the gradient operator, and the divergence operator defines a reflexive differential module since it is parametrized by the curl operator and the curl operator is parametrized by the gradient operator. The implementation of the results developed in [16, 89] (see Section 1.3) can be used to obtain explicit parametrizations of underdetermined linear systems of partial differential equations appearing in mathematical physics (e.g., electromagnetism, hydrodynamics, linear elasticity, field theory). In particular, they can be used to solve questions or remarks raised in these literatures (see, e.g., Example 1.4.9). Moreover, these techniques received natural applications in the study of variational problems and optimal control theory ([93]) (see Section 1.6). In algebra, a well-known but difficult issue is to recognize whether or not a finitely generated projective module is free. This problem has been studied lengthily in number theory, algebraic geometry, algebraic and topological K-theory, noncommutative geometry... For instance, in 1955, Serre asked whether or not a finitely generated projective module over a commutative polynomial ring D with coefficients in a field was free (Serre's conjecture ([55])). Equivalently, Serre's question asks whether or not every matrix with entries in D and which admits a right-inverse over D could be completed to a square unimodular polynomial matrix over D, namely, to a matrix whose determinant is a nonzero constant. Surprisingly, this rather elementary question took more than twenty years to be solved by Quillen ([107]) and Suslin ([114]). Explicit computation of bases of free modules is an even more complicated issue. Motivated by many applications of basis computation in mathematical systems theory, Fabiańska (RWTH Aachen University) and I studied constructive proofs of the Quillen-Suslin theorem (e.g., [30, 61, 62]) and one of which was implemented by Fabiańska in the QuillenSuslin package (see Section 1.5). A straightforward consequence of the exciting proofs of the Quillen-Suslin theorem is that a flat multidimensional system is equivalent to the 1-dimensional system obtained by setting all but one of the functional operators to particular values (e.g., 0) in the matrix of functional operators defining the system ([29]). Hence, a flat differential time-delay system is equivalent to the corresponding differential system without delays (i.e., the lengths of the time-delay operators can be set to 0). Moreover, using Quillen-Suslin theorem, we were able to constructively solve Lin-Bose's generalization of Serre's conjecture ([60]) which asks whether or not a matrix with entries in D which is such that the ideal formed by its maximal minors is generated by one element $d \in D \setminus \{0\}$ can be completed to a square matrix whose determinant is d. Equivalently, we can ask whether or not this matrix R can be factorized as $R = R'' R'$, where $\det(R'') = d$ and R' admits a right-inverse over D. A theorem due to Stafford ([110]) states that projective modules over the Weyl algebras of partial differential operators with either polynomial or rational function coefficients over a field k of characteristic 0 (e.g., $k = \mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$) are free when their ranks are at least 2. In collaboration with Robertz, we developed in [103] a constructive algorithm of this result based on the famous Stafford's result asserting that every left or right ideal over one of the two Weyl algebras can be generated by two elements ([110]) (Section 1.5). All these results were implemented in the Stafford package ([103]). Finally, the extension of Stafford's theorems to the case of the rings of partial differential operators with either formal power series or locally convergent series (i.e., germs of real analytic or holomorphic functions) seems to be open (e.g., following personal discussions with Stafford). Recently, Robertz and I were able to prove the simplest case in ([106]), namely, every projective module over the ring of ordinary differential operators with either formal power series or locally convergent series, whose rank is at least 2, is free (Section 1.5). This result has interesting applications in control theory and answers a question raised in [70] about the flatness of analytic linear control systems.

As explained in [16, 89], the obstruction for the existence of "potential-like" parametrizations of an underdetermined linear functional system is defined by the existence of autonomous elements, i.e., by torsion elements in the finitely presented module associated with the linear system (at least when the system functional space is rich enough). However, we can wonder if the concept of "potential-like" parametrization can be extended to include more general parametrizations such as parametrizations which depend on arbitrary constants, arbitrary functions of one independent variable, arbitrary functions of two independent variables, ..., arbitrary "potentials", namely, arbitrary functions of all the independent variables. For underdetermined nonlinear systems of ordinary differential equations, the general parametrization was first studied by Monge ([72]) and further developed by Hadamard ([39]), Hilbert, Cartan, Zervos... For more details, see [119]. In a long series of papers, the Monge problem was extended to the case of nonlinear systems of partial differential equations by Goursat. See [36, 37, 38] and the references therein. In [101, 102, 104, 105], Robertz and I, we studied the Monge problem for linear functional systems such as partial differential equations, differential time-delay systems... and its applications to optimal control problems and variational problems. In particular, we show how the concept of Baer's extensions, also used in homological algebra to define the first extension functor (Section 2.1), can be used to parametrize all the finitely presented modules which contain a given torsion module and such that the cokernels of the corresponding injections are a given torsion-free module (Section 2.2). In systems theory, this result can be used to parametrize all the linear systems which contain a given parametrizable linear system and such that the cokernels of the corresponding injections are a given autonomous system. In particular, this result allows us to obtain a block-triangular representation of a general linear system which is useful for computing a Monge parametrization of this system. Indeed, we first have to integrate a determined/overdetermined linear system and then solve an inhomogeneous underdetermined linear system whose homogeneous part is parametrizable. Using these techniques, within a systematic way, we can found again different explicit Monge parametrizations obtained by Rouchon and his co-authors for different differential time-delay systems ([26, 74, 79]). The main problem for computing a Monge parametrization is then twofold. First, we have to compute the general solution of the determined/overdetermined linear system (e.g., closed-form solutions as studied in the symbolic computation community), which is generally impossible. Secondly, we have to find a particular solution of the inhomogeneous underdetermined linear system (the parametrization of the homogeneous part can be computed as explained in Section 1.4). In a particular situation, related to the splitting of the canonical short exact sequence existing between the torsion submodule $t(M)$ of the module M and $M/t(M)$, a particular solution can easily be computed. Now, to study the integration of an overdetermined linear system, we can use the interesting concept of purity filtration introduced in the literature of algebraic geometry and algebraic analysis (see, e.g., [11]). A purity filtration of module over a ring of partial differential operators is a filtration of the module which is based on the dimension of the annihilator of the elements of the module (Section 2.3). This concept has interesting applications in systems theory as explained in [85, 89, 95, 97, 98]. But, following, for instance, [11], the computation of the purity filtration can be obtained by means of a spectral sequence computation. This approach has recently been followed by Barakat in [5] who successfully implemented the corresponding spectral sequences within a powerful package `homalg` ([4]) of GAP4 dedicated to constructive homological algebra. In [97, 98], we proved that a direct way can be used to compute the purity filtration of the differential module by simply extending the characterization of the torsion submodule $t(M)$ in terms of the first extension module of the Auslander transpose of the module with value in the base ring (see Section 2.4). Using the results on Baer's extensions developed in [104, 105], we can obtain a block-triangular representation of the differential module which generalizes the one explained above based on $t(M)$ and $M/t(M)$. In particular, each diagonal block of this presentation has a fixed dimension (i.e., the dimension of the annihilator of the corresponding module has a precise dimension). To our knowledge, this equivalent presentation of the module is the best form for integrating in closed-form solutions linear systems of partial differential equations. The corresponding algorithm have recently been implemented in the PurityFiltration package ([98]) which was used to integrate linear systems of partial differential equations which could not be computed by means of the classical computer algebra systems such as Maple. For more details, see [98]. Hence, using the PurityFiltration package, we can compute Monge parametrizations for linear systems of partial differential equations. Finally, I think that the work developed in [97, 98] shows that a constructive approach to algebraic analysis can help simplifying the formulation of certain results stated in classical textbooks (e.g., the use of the spectral sequences for the purity filtration), which also advocates for pursuing this approach (see [64] for a common philosophy) and can help new comers to enter into this field of mathematics.

For matrices with entries in the noncommutative polynomial ring of ordinary differential operators with coefficients in a differential field (e.g., field of rational functions) or in the ring of difference (resp., q-difference) operators with coefficients in a difference field (e.g., field of rational functions), the factorization, reduction and decomposition problems have lengthily been studied in the symbolic computation community. These problems respectively aim at studying when a matrix of functional operators (e.g., ordinary differential operators, difference operators, q-difference operators) can be either factorize as the product of two matrices or is equivalent to either a block-triangular or a block-diagonal matrix. For more details, see [7, 94, 113] and the references therein. The corresponding algorithms were implemented in different packages of computer algebra systems which can be used to obtain closed-form solutions of the corresponding linear functional systems. One of the approaches to the study of these problems, developed by Singer in [113], is based on the concept of the eigenring of a linear functional system (see also [7, 19, 94]). I soon realized that they could be studied within an algebraic analysis approach which allowed me to consider more general systems such as determined/overdetermined/underdetermined linear functional systems (Section 3.1). Cluzeau (ENSIL, University of Limoges) and I developed this approach and we explained in [19] that a natural generalization of the concept of eigenring is the endomorphism ring of the left module finitely presented by the matrix under study, namely, the ring of endomorphisms (Section 3.2). The abelian group of left homomorphisms from one finitely presented left module to another one can be computed when the polynomial ring of functional operators is commutative or when the differential module is holonomic ([77, 115]). If the underlying module is neither holonomic nor defined over a commutative polynomial ring (e.g., the conjugate Beltrami equations, linearization of the Navier-Stokes equations around the parabolic Poiseuille profile), then we can only compute a kind of "filtration" of the endomorphism ring (Section 3.2). Most of the examples of linear systems of partial differential equations studied in engineering sciences, mathematical physics and applied mathematics do not define holonomic differential modules (see, e.g., [23, 51, 52, 53]). Fortunately, they are mainly defined by matrices with entries in a commutative polynomial ring of partial differential operators (e.g., Maxwell equations, Dirac equations, Navier-Lamé equations, Stokes equations, Oseen equations). It can be easily shown that a left homomorphism between two finitely presented left modules induces an abelian group homomorphism between the linear systems defined by these modules. In particular, an element of the endomorphism ring defines an internal transformation of the linear system and an element of the group of the left automorphisms is a kind of Galois-like transformations (see [94, 113] for the connection between eigenrings and differential Galois theory). These facts advocate for the computation of homomorphisms, endomorphisms and automorphisms. As explained in [19], computing homomorphisms is also relevant to find quadratic conservation laws of linear systems of partial differential equations studied in mathematical physics (Section 3.3). Indeed, a left homomorphism from the adjoint module to the primal module naturally defines a quadratic conservation law. It is worth pointing out that the computation of general conservation laws requires the knowledge of solutions of the adjoint module, which is in general a difficult issue. But, if we are only interested in quadratic conservation laws, then only Gröbner basis computations are needed. Within the algebraic analysis approach, Cluzeau and I were able to characterize the existence of factorizations (e.g., in terms of there existence of a non-generic solution), the existence of reductions and decompositions (in terms of the existence of idempotents of the endomorphism ring). See Sections 3.5, 3.6 and 3.7. These results can be used to factorize, reduce and decompose the solution space of a linear functional system. The computational issues are generally difficult and are still mainly open in the general case. However, implementing the different algorithms in the OreMorphisms package ([20]), we were to able to factorize, reduce and decompose many explicit linear functional systems studied in the literature of control theory and mathematical physics. Finally, the explicit computation of the reductions and decompositions requires the basis computation of certain free modules, and thus of the packages Jacobson ([25]), QuillenSuslin ([29]) and Stafford ([103]).

Mathematical models of physical systems are generally obtained after a long chain of physical reasonings (e.g., obtained by means of a variational formulation, from an equilibrium of forces and momentum). One consequence is that the system we obtain after this chain is generally not "minimal", i.e., it is generally formed by a non-minimal set of equations and unknowns. Symbolic computation can play an important role in the rewriting and the preconditioning of the corresponding system of equations (e.g., using Gröbner and Janet basis techniques, purity filtration techniques). For instance, an important issue is to be able to compute an equivalent representation of a (determined/overdetermined/underdetermined) linear functional system which is simpler in the sense it contains fewer equations and fewer unknowns and the entries of the new system are "small". Motivated by the complete intersection problem studied in algebraic geometry and algebra, Serre investigated in [112] the possibility to find finite presentations of a given module (of projective dimension less or equal to 1) which are defined by smallest possible ranks. This problem is called *Serre's reduction problem*. Following Serre's ideas, the constructive approach to this important issue was initiated in [14, 21] (see Section 4.2). The techniques developed in [14, 21] are particularly interesting for a finitely presented module whose Auslander transpose is either a finite-dimensional vector space over the base field or a holonomic differential module. Observing that generically, this case holds for a torsion-free module finitely presented by a full row rank matrix with entries in a commutative polynomial in two variables over a field, we were able to compute Serre's reduction for many different examples of differential time-delay systems studied in the literature (see, e.g., [47, 73, 74, 75]). The computation of an explicit Serre's reduction (if it exists) uses the basis computation of certain free modules (see Section 4.3). Therefore, the constructive algorithms developed in [29, 30, 61, 62, 103] as well as the packages JACOBSON ([25]), QUILLENSUSLIN ([29]) and STAFFORD ([103]) play important roles in the computation of Serre's reductions. Finally, using the fact that a torsion module over the ring D of ordinary differential operators with either polynomial, formal power series or locally convergent power series coefficients is holonomic and thus cyclic (Section 2.3), [21] proves that every left D-module finitely presented by a full row rank thin rectangular matrix can be defined by only one relation, i.e., the corresponding linear system of ordinary differential equations can be defined by one ordinary differential equation.

Finally, in Section 5, we shortly demonstrate the implementations of the different algorithms in the Maple packages OREMODULES ([17]), JACOBSON ([25]), QUILLENSUSLIN ([29]), STAFFORD ([103]), PURITYFILTRATION ([98]), OREMORPHISMS ([20]) and SERRE ([21]).

Chapter 1

Algebraic analysis approach to mathematical systems theory

"La science ne s'apprend pas : elle se comprend. Elle n'est pas lettre morte et les livres n'assurent pas sa pérennité : elle est une pensée vivante. Pour s'intéresser à elle, puis la maîtriser, notre esprit doit, habilement guidé, la redécouvrir, de même que notre corps a dû revivre, dans le sein maternel, l'évolution qui créa notre espèce ; non point tous ses détails, mais son schéma. Aussi n'y a-t-il qu'une façon efficace de faire acquérir par nos enfants les principes scientifiques qui sont stables, et les procédés techniques qui évoluent rapidement : c'est donner à nos enfants l'esprit de recherche."

Jean Leray, dans M. Schmidt, *Hommes de Sciences : 28 portraits*, Hermann, 1990.

The purpose of this chapter is to give a short introduction to basic ideas, concepts and results of constructive algebraic analysis. Algebraic analysis, pioneered by Malgrange and the Japanese school of Sato, is a mathematical theory which studies linear systems of partial differential equations (PDEs) based on module theory, homological algebra and sheaf theory (see [10, 11, 13, 45, 46, 66, 67] and the references therein). Basic algebraic analysis has recently been studied within a constructive viewpoint (see, e.g., [5, 16, 19, 66, 77, 78, 85, 89, 97, 98, 103, 104, 115]). The module-theoretic approach to linear ordinary differential (OD) or partial differential (PD) systems developed within the algebraic analysis approach gives a powerful mathematical framework for the study of the structural properties of general linear differential systems (determined, overdetermined, underdetermined). In particular, the module characterizations of the structural properties developed in this approach are intrinsic in the sense that they do not depend on particular representations of the linear PD system. Using powerful tools of homological algebra, we can obtain general characterizations for the module properties (e.g., existence of torsion elements, torsion-free, reflexive, projective, stably free, free). Using constructive algebra (e.g., noncommutative Gröbner or Janet bases), those homological characterizations can be made constructive and can be implemented in dedicated symbolic computation packages (e.g., OreModules, OreMorphisms, Jacobson, QuillenSuslin, Stafford, Serre, PurityFiltration). Finally, the module properties have important interpretations in mathematical systems theory and mathematical physics (e.g., existence of autonomous elements or (minimal/injective/chain of) parametrizations).

1.1 Linear systems and finitely presented left D-modules

We recall that the definition of a *left D-module* (resp., *right D-module*) M is the same as the one of a k-vector space but where the field k is replaced by a ring D and the elements of D act on the left (resp., right) of M, namely, for all $m_1, m_2 \in M$ and all $d_1, d_2 \in D$, we have $d_1\, m_1 + d_2\, m_2 \in M$ (resp., $m_1\, d_1 + m_2\, d_2 \in M$). In particular, a k-vector space is a k-module and an abelian group is a $\mathbb{Z}$-module. For more details, see, e.g., [15, 65, 109].

Within algebraic analysis (see, e.g., [10, 11, 13, 16, 45, 46, 66, 85] and the references therein), a linear functional system (e.g., linear systems of ODEs or PDEs, OD time-delay equations, difference equations) can be studied by means of module theory and homological algebra ([15, 65, 109]). More precisely, if D is a noncommutative polynomial ring of functional operators (e.g., OD or PD operators, time-delay operators, shift operators, difference operators), $R \in D^{q \times p}$ a $q \times p$ matrix with entries in D and $\mathcal{F}$ a left D-module, then the *linear functional system*

$$\ker_{\mathcal{F}}(R.) \triangleq \{\eta \in \mathcal{F}^p \mid R\, \eta = 0\}$$

i.e., the abelian group formed by the $\mathcal{F}$-solutions of $R\, \eta = 0$, can be studied by means of the left D-module $M \triangleq D^{1 \times p}/(D^{1 \times q}\, R)$ *finitely presented* by the matrix R. Indeed, Malgrange's remark ([67]) asserts the existence of the following abelian group isomorphism (i.e., $\mathbb{Z}$-isomorphism)

$$\ker_{\mathcal{F}}(R.) \cong \hom_D(M, \mathcal{F}),$$

where $\hom_D(M, \mathcal{F})$ is the abelian group of left D-*homomorphisms* from M to $\mathcal{F}$ (i.e., maps $f : M \longrightarrow \mathcal{F}$ satisfying $f(d_1\, m_1 + d_2\, m_2) = d_1\, f(m_1) + d_2\, f(m_2)$ for all $d_1, d_2 \in D$ and all $m_1, m_2 \in M$) and $\cong$ denotes an *isomorphism*, namely, a bijective homomorphism.

Let us describe this isomorphism. To do that, we first give an explicit description of M in terms of *generators and relations*. Let $\pi : D^{1 \times p} \longrightarrow M = D^{1 \times p}/(D^{1 \times q}\, R)$ be the canonical projection onto M, namely, the left D-homomorphism which sends a row vector of $D^{1 \times p}$ of length p to its residue class $\pi(\lambda)$ in M, $\{f_j\}_{j=1,\ldots,p}$ the *standard basis* of $D^{1 \times p}$, namely, f_j is the row vector of length p defined by 1 at the j^{th} entry and 0 elsewhere, and $y_j = \pi(f_j)$ the residue class of f_j in M for $j = 1, \ldots, p$. Since every element $m \in M$ is the residue class of an element $\lambda = (\lambda_1 \ \ldots \ \lambda_p) \in D^{1 \times p}$, then, using the left D-linearity of the left D-homomorphism π, we get

$$m = \pi(\lambda) = \pi\left(\sum_{j=1}^{p} \lambda_j\, f_j\right) = \sum_{j=1}^{p} \lambda_j\, \pi(f_j) = \sum_{j=1}^{p} \lambda_j\, y_j,$$

which shows that $\{y_j\}_{j=1,\ldots,p}$ is a family of generators of the left D-module M. Moreover, if we denote by $R_{i\bullet}$ the i^{th} row of the matrix R, then $R_{i\bullet} \in D^{1 \times q}\, R$, which yields $\pi(R_{i\bullet}) = 0$ and thus

$$\pi(R_{i\bullet}) = \pi\left(\sum_{j=1}^{p} R_{ij}\, f_j\right) = \sum_{j=1}^{p} R_{ij}\, \pi(f_j) = \sum_{j=1}^{p} R_{ij}\, y_j = 0, \quad i = 1, \ldots, q, \tag{1.1}$$

which shows that the set of generators $\{y_j\}_{j=1,\ldots,p}$ of M satisfies the left D-linear relations (1.1) and all their left D-linear combinations. If $y = (y_1 \ \ldots \ y_p)^T \in M^p$, then (1.1) becomes $R\, y = 0$.

Now, let $\chi : \ker_{\mathcal{F}}(R.) \longrightarrow \hom_D(M, \mathcal{F})$ be the $\mathbb{Z}$-homomorphism defined by $\chi(\eta) = \phi_\eta$ for all $\eta \in \ker_{\mathcal{F}}(R.)$, where $\phi_\eta(\pi(\lambda)) = \lambda\, \eta \in \mathcal{F}$ for all $\lambda \in D^{1 \times p}$. The $\mathbb{Z}$-homomorphism ϕ_η is well-defined since $\pi(\lambda) = \pi(\lambda')$ yields $\pi(\lambda - \lambda') = 0$, i.e., $\lambda - \lambda' = \mu\, R$ for a certain $\mu \in D^{1 \times q}$, and thus $\phi_\eta(\pi(\lambda)) = \lambda\, \eta = \lambda'\, \eta + \mu\, R\, \eta = \lambda'\, \eta = \phi_\eta(\pi(\lambda'))$. Moreover, χ is injective since $\phi_\eta = 0$

yields $\lambda\,\eta = 0$ for all $\lambda \in D^{1\times p}$, and thus $\eta_j = f_j\,\eta = 0$ for all $j = 1,\dots,p$, i.e., $\eta = 0$. It is also surjective since for all $\phi \in \hom_D(M,\mathcal{F})$, $\eta = (\phi(y_1)\ \dots\ \phi(y_p))^T \in \mathcal{F}^p$ satisfies $\chi(\eta) = \phi$ and:

$$\forall\, i = 1,\dots,q, \quad \sum_{j=1}^{p} R_{ij}\,\eta_j = \sum_{j=1}^{p} R_{ij}\,\phi(y_j) = \phi\left(\sum_{j=1}^{p} R_{ij}\,y_j\right) = \phi(0) = 0 \quad \Rightarrow \quad \eta \in \ker_{\mathcal{F}}(R.).$$

Thus, the $\mathbb{Z}$-homomorphism χ is an isomorphism and $\chi^{-1} : \hom_D(M,\mathcal{F}) \longrightarrow \ker_{\mathcal{F}}(R.)$ is defined by $\chi^{-1}(\phi) = (\phi(y_1)\ \dots\ \phi(y_p))^T$ for all $\phi \in \hom_D(M,\mathcal{F})$. Let us sum up Malgrange's remark.

Theorem 1.1.1 ([67])**.** *Let D be a ring, $R \in D^{q\times p}$ a matrix, $M = D^{1\times p}/(D^{1\times q}\,R)$ the left D-module finitely presented by R, $\pi : D^{1\times p} \longrightarrow M$ the canonical projection onto M, $\{f_j\}_{j=1,\dots,p}$ the standard basis of $D^{1\times p}$, $y_j = \pi(f_j)$ for $j = 1,\dots,p$, and $\mathcal{F}$ a left D-module. Then, we have the following abelian group isomorphism:*

$$\begin{array}{rcl} \hom_D(M,\mathcal{F}) & \longrightarrow & \ker_{\mathcal{F}}(R.) = \{\eta \in \mathcal{F}^p \mid R\,\eta = 0\} \\ \phi & \longmapsto & \eta = (\phi(y_1)\ \dots\ \phi(y_p))^T. \end{array} \tag{1.2}$$

Hence, there is a one-to-one correspondence between the elements of $\hom_D(M,\mathcal{F})$ *and of* $\ker_{\mathcal{F}}(R.)$.

Remark 1.1.1. Theorem 1.1.1 shows that the linear functional system $\ker_{\mathcal{F}}(R.)$ can be studied by means of the finitely presented left D-module $M = D^{1\times p}/(D^{1\times q}\,R)$ and the left D-module $\mathcal{F}$: M intrinsically defines the linear system of equations defined by the matrix $R \in D^{q\times p}$ and $\mathcal{F}$ is the functional space where we seek the solutions of the linear functional system.

A *differential ring* $(A, \{\delta_1,\dots,\delta_n\})$ is a commutative ring A equipped with *commuting derivations* $\delta_i : A \longrightarrow A$ for $i = 1,\dots,n$, namely, maps satisfying

$$\forall\, a_1,\, a_2 \in A, \quad \delta_i \circ \delta_j = \delta_j \circ \delta_i, \quad \delta_i(a_1 + a_2) = \delta_i(a_1) + \delta_i(a_2), \quad \delta_i(a_1\,a_2) = \delta_i(a_1)\,a_2 + a_1\,\delta_i(a_2),$$

for all $i, j = 1,\dots,n$. If we take $a_1 = a_2 = 1$, then the above equality yields $\delta_i(1) = 2\,\delta_i(1)$, i.e., $\delta_i(1) = 0$. If A is a field and $a \in A \setminus \{0\}$, then $\delta_i(a)\,a^{-1} + a\,\delta_i(a^{-1}) = \delta_i(a\,a^{-1}) = \delta_i(1) = 0$, which shows that the derivation δ_i satisfies $\delta_i(a^{-1}) = -a^{-2}\,\delta_i(a)$. A is called a *differential field*.

In what follows, we shall mainly focus on the differential ring $\left(A, \left\{\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_n}\right\}\right)$, where $A = k[x_1,\dots,x_n]$, $k[\![x_1,\dots,x_n]\!]$ (i.e., the ring of formal power series at 0 with coefficients in k), where k is a field of characteristic 0 (e.g., $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$), $k\{x_1,\dots,x_n\}$ where $k = \mathbb{R}$ or $\mathbb{C}$ (i.e., the ring of locally convergent power series at 0 or the ring of germs of real analytic or holomorphic functions at 0) or the differential field $A = k$ or $k(x_1,\dots,x_n)$, where k is a field.

The *ring of PD operators* in $\partial_1,\dots,\partial_n$ with coefficients in the differential ring $(A, \{\delta_1,\dots,\delta_n\})$, simply denoted by $D = A\langle\partial_1,\dots,\partial_n\rangle$, is the noncommutative polynomial ring in the ∂_i's with coefficients in the commutative differential ring A satisfying:

$$\forall\, a \in A, \quad \forall\, i, j = 1,\dots,n, \quad \partial_i\,\partial_j = \partial_j\,\partial_i, \quad \partial_i\,a = a\,\partial_i + \delta_i(a).$$

An element $d \in D$ can be written as $d = \sum_{|\nu|=0,\dots,r} a_\nu\,\partial^\nu$, where $a_\nu \in A$, $\nu = (\nu_1\ \dots\ \nu_n)^T \in \mathbb{N}^n$, $|\nu| = \nu_1 + \ldots + \nu_n$ and $\partial^\nu = \partial_1^{\nu_1} \ldots \partial_n^{\nu_n}$.

The *first* (resp., *second*) *Weyl algebra* is defined by $A_n(k) = k[x_1,\dots,x_n]\langle\partial_1,\dots,\partial_n\rangle$ (resp., $B_n(k) = k(x_1,\dots,x_n)\langle\partial_1,\dots,\partial_n\rangle$). If $n = 1$, then we shall simply use the notations $\delta = \frac{d}{dt}$ instead of δ_1, ∂ instead of ∂_1 and $k[t]$, $k(t)$, $k[\![t]\!]$ and $k\{t\}$ instead of $k[x_1]$, $k(x_1)$, $k[\![x_1]\!]$ and $k\{x_1\}$.

More generally, we can consider the noncommutative polynomial rings $D = A\langle\partial_1, \ldots, \partial_m\rangle$ of functional operators ∂_i for $i = 1, \ldots, m$, where $A = k[x_1, \ldots, x_n]$, k is a field,

$$\forall\, i,\, j = 1, \ldots, m, \quad \forall\, l = 1, \ldots, n, \quad \partial_i\, \partial_j = \partial_j\, \partial_i, \quad \partial_i\, x_l = (a_{il}\, x_l + b_{il})\, \partial_i + c_{il}, \tag{1.3}$$

and $a_{il} \in k \setminus \{0\}$, $b_{il} \in k$, $c_{il} \in A$ and $\deg(c_{il}) \leq 1$, such as *Ore algebras* ([18]). For instance, the ring of OD time-delay operators or the ring of OD and difference operators are Ore algebras.

Example 1.1.1. The linearization of the Navier-Stokes equations around the parabolic Poiseuille profile is defined by the following linear PD system with polynomial coefficients:

$$\left\{\begin{array}{l} \partial_t\, \delta u_1 + 4\, y\, (1 - y)\, \partial_x\, \delta u_1 - 4\, (2\, y - 1)\, \delta u_2 - \nu\, (\partial_x^2 + \partial_y^2)\, du_1 + \partial_x\, \delta p = 0, \\ \partial_t\, \delta u_2 + 4\, y\, (1 - y)\, \partial_x\, \delta u_2 - \nu\, (\partial_x^2 + \partial_y^2)\, \delta u_2 + \partial_y\, \delta p = 0, \\ \partial_x\, \delta u_1 + \partial_y\, \delta u_2 = 0. \end{array}\right. \tag{1.4}$$

Here, δu_i (resp., δp) denotes a perturbation of the i^{th} component of the speed $\vec{u} = (u_1 \quad u_2)^T$ (resp., of the pressure). If $D = A_3(\mathbb{Q}(\nu))$ is the first Weyl algebra of PD operators in ∂_t, ∂_x and ∂_y with coefficients in $\mathbb{Q}(\nu)[t, x, y]$, then (1.4) is defined by the following matrix of PD operators

$$R = \left(\begin{array}{ccc} \partial_t + 4\, y\, (1 - y)\, \partial_x - \nu\, (\partial_x^2 + \partial_y^2) & -4\, (2\, y - 1) & \partial_x \\ 0 & \partial_t + 4\, y\, (1 - y)\, \partial_x - \nu\, (\partial_x^2 + \partial_y^2) & \partial_y \\ \partial_x & \partial_y & 0 \end{array}\right) \in D^{3 \times 3},$$

and the generators $\{\delta u_1 = \pi(f_1), \delta u_2 = \pi(f_2), \delta p = \pi(f_3)\}$ of the finitely presented left D-module $M = D^{1\times 3}/(D^{1\times 3}\, R)$ satisfy the left D-linear relations generated by (1.4), where $\{f_j\}_{j=1,2,3}$ is the standard basis of $D^{1\times 3}$ and $\pi : D^{1\times 3} \longrightarrow M$ the canonical projection onto M. Finally, if $\mathcal{F}$ is a left D-module (e.g., $C^\infty(\mathbb{R}_+ \times \mathbb{R}^2)$), then the $\mathcal{F}$-solutions of the linear system (1.4), i.e., $\ker_{\mathcal{F}}(R.) = \{\eta = (\delta u_1 \quad \delta u_2 \quad \delta p)^T \in \mathcal{F}^3 \mid R\, \eta = 0\}$, is $\mathbb{Z}$-isomorphic to $\hom_D(M, \mathcal{F})$.

Remark 1.1.2. Sheaf theory (e.g., sheaves of finitely presented differential modules) can be used to study locally algebraic or analytic linear systems of PD equations and the ring D of PD operators can also be replaced by the sheaf $\mathcal{E}$ of germs of microdifferential operators ([45, 46]).

If M and $\mathcal{F}$ are two left D-modules, then $\hom_D(M, \mathcal{F})$ has an abelian group structure but is usually not a left or a right D-module. Indeed, if $\hom_D(M, \mathcal{F})$ has a left D-module structure defined by $(d\, f)(m) = f(d\, m)$, for all $d \in D$ and all $m \in M$, then, according to the definition of a left D-module, for all d, $d' \in D$ and for all $f \in \hom_D(M, \mathcal{F})$, we have $(d\, d')\, f = d\, (d'\, f)$ and:

$$\left\{\begin{array}{l} (d\, d'\, f)(m) = f(d\, d'\, m), \\ (d\, (d'\, f))(m) = (d'\, f)(d\, m) = f(d'\, d\, m), \end{array}\right. \quad \Rightarrow \quad f(d\, d'\, m) = f(d'\, d\, m).$$

But, $f(d\, d'\, m)$ and $f(d'\, d\, m)$ are not necessarily equal for all d, $d' \in D$ and all $m \in M$.

Example 1.1.2. Let us consider the first Weyl algebra $D = A_1(\mathbb{Q}(m, \sigma))$, $R = (\partial + (t - m)/\sigma^2)$, the finitely presented left D-module $M = D/(D\, R)$ and the left D-module $\mathcal{F} = C^\infty(\mathbb{R})$. Then, the Gaussian distribution $\eta = e^{-\frac{(t-m)^2}{2\,\sigma^2}}$ belongs to $\ker_{\mathcal{F}}(R.)$ since we can easily check that:

$$\partial\, \eta + \frac{(t - m)}{\sigma^2}\, \eta = 0.$$

But, neither $\partial\,\eta$ nor $t\,\eta$ belong to $\ker_{\mathcal{F}}(R.)$:

$$\begin{cases} \partial\,(\partial\,\eta) + \dfrac{(t-m)}{\sigma^2}\,\partial\,\eta = -\dfrac{(t-m)}{\sigma^2}\,\partial\,\eta - \dfrac{1}{\sigma^2}\,\eta + \dfrac{(t-m)}{\sigma^2}\,\partial\,\eta = -\dfrac{1}{\sigma^2}\,\eta \neq 0, \\ \partial\,(t\,\eta) + \dfrac{(t-m)}{\sigma^2}\,(t\,\eta) = t\,\left(\partial\,\eta + \dfrac{(t-m)}{\sigma^2}\,\eta(t)\right) + \eta = \eta \neq 0. \end{cases}$$

Therefore, $\ker_{\mathcal{F}}(R.) = \{\eta \in \mathcal{F} \mid R\,\eta = 0\}$ has no left D-module structure which, by Theorem 1.1.1, implies that $\hom_D(M, \mathcal{F})$ is only an abelian group and a $\mathbb{Q}(m, \sigma)$-vector space.

If D is a commutative ring, then $\hom_D(M, \mathcal{F})$ inherits a D-module structure defined by:

$$\forall\, d \in D, \quad \forall\, m \in M, \quad (d\,f)(m) = f(d\,m).$$

We recall that a ring D is called a *domain* if it does not contain non-trivial zero divisors, i.e., $d_1\,d_2 = 0$ implies $d_1 = 0$ or $d_2 = 0$. Moreover, D is a *left noetherian ring* if every left ideal of D (i.e., every left D-submodule of D) is finitely generated, i.e., can be generated by a finite family of generators as a left D-module. Similarly, we can define the concept of a *right noetherian ring*. A ring is simply called *noetherian* if it is both a left and a right noetherian ring ([54, 109]). A result due to Goldie ([71]) proves that a left (resp., right) noetherian domain is a *left* (resp., *right*) *Ore domain*, namely, a domain satisfying the left (resp., *right*) *Ore property*, i.e., for all $d_1,\, d_2 \in D \setminus \{0\}$, there exist $e_1,\, e_2 \in D \setminus \{0\}$ such that $e_1\,d_1 = e_2\,d_2$ (resp., $d_1\,e_1 = d_2\,e_2$).

Example 1.1.3. The rings $A\langle\partial_1, \ldots, \partial_n\rangle$ of PD operators with coefficient in the differential ring

- $A = k$, where k is a field,
- $A = k[x_1, \ldots, x_n]$, $k(x_1, \ldots, x_n)$ or $k[\![x_1, \ldots, x_n]\!]$, where k is a field,
- $A = k\{x_1, \ldots, x_n\}$, where $k = \mathbb{R}$ or $\mathbb{C}$,

are noetherian domains, and thus Ore domains ([71]). Moreover, if k is a computable field (e.g., $\mathbb{Q}$ or $\mathbb{F}_p$ for a prime p), $A = k$, $k[x_1, \ldots, x_n]$ or $k(x_1, \ldots, x_n)$, and $R \in D^{q \times p}$, then, for any admissible term order, Buchberger's algorithm terminates and it computes a Gröbner basis of the left D-submodule $D^{1 \times q}\,R$ of $D^{1 \times p}$ for the corresponding term order. For more details, see, e.g., [18, 35, 58] and the references therein. A similar result holds for the Ore algebras satisfying (1.3). For an introduction to Gröbner basis techniques, see [8, 18, 58] and the references therein. Finally, Janet basis techniques can also be used to constructively study module theory over the same classes of noncommutative polynomial rings (e.g., rings of PD operators) ([12, 41, 84, 108]).

We recall a few definitions of module theory we shall use in what follows (see, e.g., [54, 109]).

Definition 1.1.1. Let D be a left noetherian domain and M a *finitely generated* left D-module, namely, M can be generated by a finite family of elements of M as a left D-module.

1. M is *free* if there exists $r \in \mathbb{N} = \{0, 1, \ldots\}$ such that $M \cong D^{1 \times r}$. Then, r is called the *rank* of the free left D-module M and is denoted by $\operatorname{rank}_D(M)$.
2. M is *stably free* if there exist $r,\, s \in \mathbb{N}$ such that $M \oplus D^{1 \times s} \cong D^{1 \times r}$. Then, $r - s$ is called the *rank* of the stably free left D-module M.
3. M is *projective* if there exist $r \in \mathbb{N}$ and a left D-module N such that $M \oplus N \cong D^{1 \times r}$, where $\oplus$ denotes the direct sum of left D-modules.
4. M is *reflexive* if the following canonical left D-homomorphism

$$\begin{array}{rcl} \varepsilon : M & \longrightarrow & \hom_D(\hom_D(M, D), D), \\ m & \longmapsto & \varepsilon(m), \end{array}$$

where $\varepsilon(m)(f) = f(m)$ for all $f \in \hom_D(M, D)$ and all $m \in M$, is a left D-isomorphism.

5. M is *torsion-free* if the *torsion left D-submodule of M*

$$t(M) = \{m \in M \mid \exists\, d \in D \setminus \{0\} : \; d\, m = 0\}$$

is reduced to 0, i.e., if $t(M) = 0$. The elements of $t(M)$ are the *torsion elements* of M.

6. M is *torsion* if $t(M) = M$, i.e., if every element of M is a torsion element of M.
7. M is *cyclic* if M is generated by $m \in M$, i.e., $M = D\, m \triangleq \{d\, m \mid d \in D\}$.

Remark 1.1.3. The fact that $t(M)$ is a left D-submodule of M is a consequence of the left Ore property of D (which comes from the left noetherian domain property). Indeed, for all $m_1,\, m_2 \in t(M)$ and all $d_1,\, d_2 \in D$, we need to prove that $d_1\, m_1 + d_2\, m_2 \in t(M)$. Since $m_1,\, m_2 \in t(M)$, there exist $p_1,\, p_2 \in D \setminus \{0\}$ such that $p_1\, m_1 = 0$ and $p_2\, m_2 = 0$. Using the left Ore property of D, there exist non-trivial $r_1,\, r_2,\, s_1,\, s_2,\, t_1,\, t_2 \in D$ satisfying:

$$r_1\, p_1 = s_1\, d_1, \quad r_2\, p_2 = s_2\, d_2, \quad t_1\, s_1 = t_2\, s_2.$$

Therefore, we get

$$(t_1\, s_1)\,(d_1\, m_1 + d_2\, m_2) = t_1\,(s_1\, d_1)\, m_1 + t_2\,(s_2\, d_2)\, m_2 = t_1\, r_1\,(p_1\, m_1) + t_2\, r_2\,(p_2\, m_2) = 0,$$

which shows that $d_1\, m_1 + d_2\, m_2 \in t(M)$ since $t_1\, s_1 \in D \setminus \{0\}$.

In the forthcoming Theorem 1.3.1, we shall explain how the module properties introduced in Definition 1.1.1 can be constructively checked when Gröbner basis techniques are available for a noncommutative polynomial ring D. We shall then give explicit examples.

A free left D-module $M \cong D^{1\times r}$ is clearly stably free since we can take $s = 0$ in 2 of Definition 1.1.1 and a stably free left D-module is projective since we can take $N = D^{1\times s}$ in 3 of Definition 1.1.1. Moreover, if M is a projective left D-module, then M is a reflexive left D-module since M is a direct summand of a finite free left D-module $F \cong D^{1\times r}$ and F is a reflexive left D-module. If M is a reflexive left D-module and $m \in t(M)$, then there exists $d \in D \setminus \{0\}$ such that $d\, m = 0$, and thus $d\, f(m) = f(d\, m) = f(0) = 0$ for all $f \in \hom_D(M, D)$, i.e., $f(m) = 0$ since $d \neq 0$, $f(m) \in D$ and D is a domain, which shows that $\varepsilon(m)(f) = f(m) = 0$ for all $f \in \hom_D(M, D)$ and proves that $\varepsilon(m) = 0$, i.e., $m \in \ker \varepsilon = 0$, and thus $t(M) = 0$.

Proposition 1.1.1 ([109]). *A free left D-module is stably free, a stably free left D-module is projective, a projective left D-module is reflexive and a reflexive left D-module is torsion-free.*

The converses of the results of Proposition 1.1.1 are generally not true. However, it holds in particular interesting situations.

Theorem 1.1.2 ([54, 107, 110, 114]).

1. *If D is a* principal left ideal domain, *namely, every left ideal of the domain D is cyclic (e.g., the ring $A\langle\partial\rangle$ of OD operators with coefficients in a differential field A such as $A = k$, $k(t)$ and $k[\![t]\!][t^{-1}]$, where k is a field of characteristic 0, or $k\{t\}[t^{-1}]$, where $k = \mathbb{R}$ or $\mathbb{C}$), then every finitely generated torsion-free left D-module is free.*
2. *If $D = k[x_1, \ldots, x_n]$ is a commutative polynomial ring with coefficients in a field k, then every finitely generated projective D-module is free (Quillen-Suslin theorem).*
3. *If D is the Weyl algebra $A_n(k)$ or $B_n(k)$, where k is a field of characteristic 0, then every finitely generated projective left D-module is stably free and every finitely generated stably free left D-module of rank at least 2 is free (Stafford's theorem).*

In 1955, Serre wrote "On ignore s'il existe des A-modules projectifs de type fini qui ne soient pas libres", where $A = k[x_1, \ldots, x_n]$ and k a field (page 243 of [111]). In 1976, this remark, called "Serre's conjecture" ([55]), was independently solved by Quillen ([107]) and Suslin ([114]).

The purpose of the next sections is to explain how to check whether or not a finitely presented module M over a noetherian domain D is respectively torsion-free, projective, stably free or free, and give applications of these concepts to mathematical systems theory.

1.2 Finite free resolutions and extension functor

> "S'il est vrai que la mathématique est la reine des sciences, qui est la reine de la mathématique ? La suite exacte !", Henri Cartan, Oberwolfach, 1952.
>
> "…If I could only understand the beautiful consequence following from the concise proposition $d^2 = 0$", Henri Cartan, Laudatio on receiving the Doctor Honoris Causa degree at Oxford University, 1980.

To simplify the notations, the set $\mathcal{F}^{p\times 1}$ of column vectors of length p with coefficients in $\mathcal{F}$ will be denoted by $\mathcal{F}^p$. Let us recall basic concepts of homological algebra (see, e.g., [15, 65, 109]).

Definition 1.2.1. 1. A *complex* of left (resp., right) D-modules, denoted by

$$M_\bullet \quad \ldots \xrightarrow{d_{i+2}} M_{i+1} \xrightarrow{d_{i+1}} M_i \xrightarrow{d_i} M_{i-1} \xrightarrow{d_{i-1}} \ldots, \tag{1.5}$$

is a sequence of left (resp., right) D-homomorphisms $d_i : M_i \longrightarrow M_{i-1}$ between left (resp., right) D-modules which satisfy $\operatorname{im} d_{i+1} \subseteq \ker d_i$, i.e., $d_i \circ d_{i+1} = 0$ for all $i \in \mathbb{Z}$.

2. The *defect of exactness* of (1.5) at M_i is the left (resp., right) D-module defined by:

$$H_i(M_\bullet) \triangleq \ker d_i / \operatorname{im} d_{i+1}.$$

3. The complex (1.5) is said to be *exact at* M_i if $H_i(M_\bullet) = 0$, i.e., $\ker\, d_i = \operatorname{im} d_{i+1}$, and *exact* if $\ker\, d_i = \operatorname{im} d_{i+1}$ for all $i \in \mathbb{Z}$. An exact complex is also called an *exact sequence*.

4. The exact sequence of the form $0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$, i.e., f is injective, $\ker g = \operatorname{im} f$ and g is surjective, is called a *short exact sequence*.

5. A *finite free resolution* of the left D-module M is an exact sequence of the form

$$\ldots \xrightarrow{.R_4} D^{1\times r_3} \xrightarrow{.R_3} D^{1\times r_2} \xrightarrow{.R_2} D^{1\times r_1} \xrightarrow{.R_1} D^{1\times r_0} \xrightarrow{\pi} M \longrightarrow 0, \tag{1.6}$$

where $R_i \in D^{r_i \times r_{i-1}}$ and $.R_i : D^{1\times r_i} \longrightarrow D^{1\times r_{i-1}}$ is the left D-homomorphism defined by $(.R_i)(\lambda) = \lambda\, R_i$ for all $\lambda \in D^{1\times r_i}$.

6. A *finite free resolution* of a right D-module N is an exact sequence of the form

$$0 \longleftarrow N \xleftarrow{\kappa} D^{s_0} \xleftarrow{S_1.} D^{s_1} \xleftarrow{S_2.} D^{s_2} \xleftarrow{S_3.} D^{s_3} \xleftarrow{S_4.} \ldots, \tag{1.7}$$

where $S_i \in D^{s_{i-1}\times s_i}$ and $S_i. : D^{s_i} \longrightarrow D^{s_{i-1}}$ is defined by $(S_i.)(\eta) = S_i\, \eta$ for all $\eta \in D^{s_i}$.

7. A short exact sequence $0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$ of left D-modules is said to *split* if one of the following equivalent assertions holds:
   – There exists a left D-homomorphism $h : M'' \longrightarrow M$ such that $g \circ h = \mathrm{id}_{M''}$.
   – There exists a left D-homomorphism $k : M \longrightarrow M'$ such that $k \circ f = \mathrm{id}_{M'}$.

– There exists a left D-isomorphism from $M' \oplus M''$ to M, i.e., $M \cong M' \oplus M''$.
We denote the previous split short exact sequence by the following diagram:

$$0 \longrightarrow M' \underset{\overset{\longleftarrow}{k}}{\overset{f}{\longrightarrow}} M \underset{\overset{\longleftarrow}{h}}{\overset{g}{\longrightarrow}} M'' \longrightarrow 0. \tag{1.8}$$

Example 1.2.1. If D is a noetherian domain and M is a finitely generated left D-module, then we have the short exact sequence $0 \longrightarrow t(M) \xrightarrow{i} M \xrightarrow{\rho} M/t(M) \longrightarrow 0$ of left D-modules, where i (resp., ρ) denotes the canonical injection (resp., projection).

Example 1.2.2. If M is a left D-module, $m \in M$ and $\mathrm{ann}_D(m) = \{d \in D \mid d\,m = 0\}$ the *annihilator* of m, then $\mathrm{ann}_D(m)$ is a left ideal of D and the following short exact sequence holds

$$0 \longrightarrow \mathrm{ann}_D(m) \longrightarrow D \xrightarrow{f} D\,m \longrightarrow 0$$

where the left D-homomorphism f is defined by $f(d) = d\,m$ for all $d \in M$. Hence, we get $D\,m = \mathrm{im}\, f \cong \mathrm{coim}\, f \triangleq D/\mathrm{ann}_D(m)$. If $\mathrm{ann}_D(m) = 0$, then $D\,m \cong D$, which proves that $D\,m$ is a free left D-module of rank 1. If $\mathrm{ann}_D(m) \neq 0$, then $D\,m$ is a torsion left D-module since $D/\mathrm{ann}_D(m)$ is a torsion left D-module generated by the residue class of 1 in $D/\mathrm{ann}_D(m)$.

If D is a left noetherian ring and M a finitely generated left D-module, then M admits a finite free resolution. Indeed, if $\{y_j\}_{j=1,\ldots,r_0}$ is a finite family of generators of M, then we can define the left D-homomorphism $\pi : D^{1\times r_0} \longrightarrow M$ by $\pi(f_j) = y_j$ for all $j = 1, \ldots, r_0$, where $\{f_j\}_{j=1,\ldots,r_0}$ is the standard basis of the free left D-module $D^{1\times r_0}$ of rank r_0. Then, we have the following short exact sequence:

$$0 \longrightarrow \ker \pi \xrightarrow{i} D^{1\times r_0} \xrightarrow{\pi} M \longrightarrow 0.$$

Now, $\ker \pi$ is a left D-submodule of the noetherian left D-module $D^{1\times r_0}$, a fact implying that $\ker \pi$ is a finitely generated left D-module (see, e.g., [54, 109]). Hence, there exists a finite family of generators of $\ker \pi$. Stacking these row vectors of length r_0 into a matrix, we obtain a matrix $R_1 \in D^{r_1\times r_0}$ such that $\ker \pi = D^{1\times r_1} R_1$, which yields the following long exact sequence:

$$0 \longrightarrow \ker_D(.R_1) \longrightarrow D^{1\times r_1} \xrightarrow{.R_1} D^{1\times r_0} \xrightarrow{\pi} M \longrightarrow 0.$$

$\ker_D(.R_1)$ is called the *(first) syzygy left D-module* of $D^{1\times r_1} R_1$. We obtain that a finitely generated left module over a left noetherian ring is finitely presented. Repeating the same process, we obtain a finite free resolution (1.6) of the left D-module M (*syzygy module computation*).

Within mathematical systems theory, we note that the matrix $R_2 \in D^{r_2\times r_1}$ defined by $\ker_D(.R_1) = D^{1\times r_2} R_2$ is a generating set of the *compatibility conditions* of the inhomogeneous linear system $R_1\,\eta = \zeta$ since, for every $\lambda \in \ker_D(.R_1)$, we have $\lambda\,\zeta = \lambda\,(R_1\,\eta) = (\lambda\,R_1)\,\eta = 0$. Hence, the compatibility conditions of $R_1\,\eta = \zeta$ are generated by $R_2\,\zeta = 0$. If Gröbner bases exist for finitely generated left D-submodules of $D^{1\times r_i}$ and for elimination term orders, then a finite free resolution (1.6) of M can be inductively computed by eliminating η from the inhomogeneous linear system $R_i\,\eta = \zeta$ to get $R_{i+1}\,\zeta = 0$. For more details, see, e.g., [16, 17].

We give the sketch of an algorithm which computes syzygy modules ([16]).

Algorithm 1.2.1. – **Input:** A noncommutative polynomial ring D for which Buchberger's algorithm terminates for any admissible term order and a finitely generated left D-submodule L of $D^{1\times p}$ defined by a matrix $R \in D^{q\times p}$, i.e., $L = D^{1\times q} R$.

- **Output:** A matrix $S \in D^{r\times q}$ such that $\ker_D(.R) = D^{1\times r}\, S$.

1. Introduce the indeterminates $\eta_1, \ldots, \eta_p$, $\zeta_1, \ldots, \zeta_q$ over D and define the following set:

$$P = \left\{ \sum_{j=1}^{p} R_{ij}\, \eta_j - \zeta_i \mid i = 1, \ldots, q \right\}.$$

2. Compute the Gröbner basis G of P in the free left D-module generated by the η_j's and the ζ_i's for $j = 1, \ldots, p$ and $i = 1, \ldots, q$, namely, $\bigoplus_{j=1}^{p} D\,\eta_j \oplus \bigoplus_{i=1}^{q} D\,\zeta_i$, with respect to a term order which eliminates the η_j's.
3. Compute the intersection $G \cap (\bigoplus_{i=1}^{q} D\,\zeta_i) = \{\sum_{i=1}^{q} S_{ki}\,\zeta_i \mid k = 1, \ldots, r\}$ by selecting the elements of G containing only the ζ_i's and form the matrix $S = (S_{ij}) \in D^{r\times q}$.

Example 1.2.3. In mathematical physics ([51, 52]), it is well-known that the compatibility conditions of the gradient operator in $\mathbb{R}^3$ are defined by the curl operator, and the compatibility conditions of the curl operator are defined by the divergence operator. It means that the $D = \mathbb{Q}[\partial_1, \partial_2, \partial_3]$-module $M = D/(D\,\partial_1 + D\,\partial_2 + D\,\partial_3)$ admits the following finite free resolution

$$0 \longrightarrow D \xrightarrow{.R_3} D^{1\times 3} \xrightarrow{.R_2} D^{1\times 3} \xrightarrow{.R_1} D \xrightarrow{\pi} M \longrightarrow 0, \tag{1.9}$$

with the notations $R_1 = (\partial_1 \quad \partial_2 \quad \partial_3)^T$, $R_3 = R_1^T$ and:

$$R_2 = \begin{pmatrix} 0 & -\partial_3 & \partial_2 \\ \partial_3 & 0 & -\partial_1 \\ -\partial_2 & \partial_1 & 0 \end{pmatrix} \in D^{3\times 3}. \tag{1.10}$$

The long exact sequence (1.9) is the well-known differential sequence "gradient-curl-divergence" which corresponds to the *Poincaré sequence* for the exterior derivative ([82, 84]). In what follows, we shall also use the following classical notations $\vec{\nabla}\,\xi = R_1\,\xi$, $\vec{\nabla} \wedge \eta = R_2\,\eta$ and $\vec{\nabla}\,.\,\zeta = R_3\,\zeta$.

Example 1.2.4. Let us consider the following linear PD system (Janet's system) ([84]):

$$\left\{ \begin{array}{l} \partial_3^2\, y - x_2\,\partial_1^2\, y = 0, \\ \partial_2^2\, y = 0. \end{array} \right. \tag{1.11}$$

If $D = A_3(\mathbb{Q})$ is the first Weyl algebra, then the presentation matrix R of (1.11) is defined by:

$$R_1 = \begin{pmatrix} \partial_3^2 - x_2\,\partial_1^2 \\ \partial_2^2 \end{pmatrix}.$$

Using Algorithm 1.2.1, the left D-module $M = D/(D^{1\times 2}\, R_1)$ admits the free resolution

$$0 \longrightarrow D \xrightarrow{.R_3} D^{1\times 2} \xrightarrow{.R_2} D^{1\times 2} \xrightarrow{.R_1} D \xrightarrow{\pi} M \longrightarrow 0,$$

with the following notations:

$$R_2 = \begin{pmatrix} \partial_2^3 & 3\,\partial_1^2 + x_2\,\partial_1^2\,\partial_2 - \partial_2\,\partial_3^2 \\ -2\,x_2\,\partial_1^2\,\partial_2^2\,\partial_3^2 - 2\,x_2\,\partial_2\,\partial_1^4 + x_2^2\,\partial_1^4\,\partial_2^2 + \partial_2^2\,\partial_3^4 + 2\,\partial_1^2\,\partial_2\,\partial_3^2 + 2\,\partial_1^4 & x_2^3\,\partial_1^6 + 3\,x_2\,\partial_1^2\,\partial_3^4 - \partial_3^6 - 3\,x_2^2\,\partial_1^4\,\partial_3^2 \end{pmatrix},$$

$$R_3 = \begin{pmatrix} x_2^2\,\partial_1^4 - 2\,x_2\,\partial_1^2\,\partial_3^2 + \partial_3^4 & -\partial_2 \end{pmatrix}.$$

We refer the reader to [82, 83, 84, 85] for an introduction to Spencer's formal theory of PDEs which studies the existence of canonical resolutions of linear systems based on intrinsic properties of PD systems (e.g., *Spencer's cohomology*, *formal integrability*, *involution*), i.e., properties which do not depend on the choice of the coordinate system for the independent variables $x_1, \ldots, x_n$.

Let us now introduce the concepts of *extension modules* and *extension functor* which will play important roles in what follows (see, e.g., [15, 65, 109]) and in the next chapters.

If $\mathcal{F}$ is a left D-module and $R_1 \in D^{r_1 \times r_0}$, then a necessary condition for the solvability of the inhomogeneous linear system $R_1\,\eta = \zeta$ for a fixed $\zeta \in \mathcal{F}^{r_1}$ is $R_2\,\zeta = 0$, where the matrix $R_2 \in D^{r_2 \times r_1}$ is such that $\ker_D(.R_1) = D^{1 \times r_2}\,R_2$. Let us study when this necessary condition is also sufficient. We need to investigate the defect of exactness of the following complex at $\mathcal{F}^{r_1}$

$$\mathcal{F}^{r_2} \xleftarrow{R_2.} \mathcal{F}^{r_1} \xleftarrow{R_1.} \mathcal{F}^{r_0}, \tag{1.12}$$

where $R_i. : \mathcal{F}^{r_{i-1}} \longrightarrow \mathcal{F}^{r_i}$ is defined by $(R_i.)(\eta) = R_i\,\eta$ for all $\eta \in \mathcal{F}^{r_{i-1}}$ and $i = 1, 2$. Indeed, for a fixed $\zeta \in \mathcal{F}^{r_1}$, there exists $\eta \in \mathcal{F}^{r_0}$ satisfying $R_1\,\eta = \zeta$ iff $\zeta \in \mathrm{im}_{\mathcal{F}}(R_1.) = R_1\,\mathcal{F}^{r_0}$ and the necessary condition $R_2\,\zeta = 0$ (since $R_2\,R_1 = 0$) means that $\zeta \in \ker_{\mathcal{F}}(R_2.)$. Therefore, there exists $\eta \in \mathcal{F}^{r_1}$ satisfying $R_1\,\eta = \zeta$ iff the residue class of ζ in $\ker_{\mathcal{F}}(R_2.)/\mathrm{im}_{\mathcal{F}}(R_1.)$ is reduced to 0. This fact explains why the defect of exactness of the complex (1.12) at $\mathcal{F}^{r_1}$ plays an important role in mathematical systems theory. If the complex (1.12) is exact at $\mathcal{F}^{r_1}$, i.e., $\ker_{\mathcal{F}}(R_2.) = \mathrm{im}_{\mathcal{F}}(R_1.)$, then the necessary condition $R_2\,\zeta = 0$ is also sufficient. The defect of exactness $\ker_{\mathcal{F}}(R_2.)/\mathrm{im}_{\mathcal{F}}(R_1.)$ of (1.12) at $\mathcal{F}^{r_1}$ is simply denoted by $\mathrm{ext}^1_D(M, \mathcal{F})$ since a key result of homological algebra proves that it depends only on M and $\mathcal{F}$ and not on the choice of the beginning of the finite free resolution (1.6) of the left D-module M (see, e.g., [15, 65, 109]).

Using (1.6), we can define the higher extension abelian groups $\mathrm{ext}^i_D(M, \mathcal{F})$'s for $i \geq 2$ as follows. Up to abelian group isomorphism, they are defined by the defects of exactness of the following complex of abelian groups

$$\ldots \xleftarrow{R_{i+1}.} \mathcal{F}^{r_i} \xleftarrow{R_i.} \mathcal{F}^{r_{i-1}} \xleftarrow{R_{i-1}.} \ldots \xleftarrow{R_3.} \mathcal{F}^{r_2} \xleftarrow{R_2.} \mathcal{F}^{r_1} \xleftarrow{R_1.} \mathcal{F}^{r_0} \longleftarrow 0, \tag{1.13}$$

where $R_i. : \mathcal{F}^{r_{i-1}} \longrightarrow \mathcal{F}^{r_i}$ is defined by $(R_i.)(\eta) = R_i\,\eta$ for all $\eta \in \mathcal{F}^{r_{i-1}}$ and all $i \geq 1$, namely:

$$\begin{cases} \mathrm{ext}^0_D(M, \mathcal{F}) \triangleq \hom_D(M, \mathcal{F}) \cong \ker_{\mathcal{F}}(R_1.), \\ \mathrm{ext}^i_D(M, \mathcal{F}) \cong \ker_{\mathcal{F}}(R_{i+1}.)/\mathrm{im}_{\mathcal{F}}(R_i.), \quad i \geq 1. \end{cases}$$

In what follows, we shall either use the notation $\hom_D(M, \mathcal{F})$ or $\mathrm{ext}^0_D(M, \mathcal{F})$.

As for $\mathrm{ext}^1_D(M, \mathcal{F})$, a classical theorem of homological algebra proves that the $\mathrm{ext}^i_D(M, \mathcal{F})$'s depend only on the left D-modules M and $\mathcal{F}$ (up to abelian group isomorphism), i.e., they do not depend on the particular finite free resolution (1.6) of M. For more details, see [15, 65, 109].

Similarly, if D is a right noetherian ring, N a finitely generated right D-module and $\mathcal{G}$ a right D-module, then, using the finite free resolution (1.7) of N, we can define the abelian groups:

$$\begin{cases} \mathrm{ext}^0_D(N, \mathcal{G}) = \hom_D(N, \mathcal{G}) \cong \ker_{\mathcal{G}}(.S_1), \\ \mathrm{ext}^i_D(N, \mathcal{G}) \cong \ker_{\mathcal{G}}(.S_{i+1})/\mathrm{im}_{\mathcal{G}}(.S_i), \quad i \geq 1. \end{cases}$$

Example 1.2.5. Let $D = \mathbb{Q}[x]$, $R = (x\,(x-1) \quad x\,(x+1))^T$ and $M = D/(D^{1 \times 2}\,R)$ the D-module finitely presented by R. Let us compute the $\mathrm{ext}^i_D(M, D)$'s for $i \geq 0$. We first note that $M = D/(x\,(x-1), x\,(x+1)))$, where $(x\,(x-1), x\,(x+1))$ is the ideal of D generated by $x\,(x-1)$

and $x\,(x+1)$. We first need to compute a finite free resolution of M. Let us characterize $\ker_D(.R)$: $\lambda = (\lambda_1 \quad \lambda_2) \in \ker_D(.R)$ iff $\lambda_1\, x\,(x-1) + \lambda_2\, x\,(x+1) = 0$, i.e., iff $(\lambda_1\,(x-1) + \lambda_2\,(x+1))\,x = 0$, i.e., iff $\lambda_1\,(x-1) + \lambda_2\,(x+1) = 0$ since D is a domain and $x \neq 0$. As D is a greatest common divisor domain and $\gcd(x-1, x+1) = 1$, we get $\lambda_1 = d\,(x+1)$ and $\lambda_2 = -d\,(x-1)$ for all $d \in D$, i.e., $\lambda = d\,(x+1 \quad -x+1)$. Hence, if $R_1 = R$ and $R_2 = (x+1 \quad -x+1)$, then $\ker_D(.R_1) = D\,R_2$. Moreover, $\ker_D(.R_2) = 0$ since $d\,(x+1 \quad -x+1) = (0 \quad 0)$ yields $d = 0$ since D is a domain and $x+1 \neq 0$. The D-module M then admits the following finite free resolution:

$$0 \longrightarrow D \xrightarrow{.R_2} D^{1\times 2} \xrightarrow{.R_1} D \xrightarrow{\pi} M \longrightarrow 0.$$

Then, the defects of exactness of the complex $0 \longleftarrow D \xleftarrow{R_2.} D^2 \xleftarrow{R_1.} D \longleftarrow 0$ are defined by:

$$\begin{cases} \mathrm{ext}^0_D(M,D) = \hom_D(M,D) \cong \ker_D(R_1.), \\ \mathrm{ext}^1_D(M,D) \cong \ker_D(R_2.)/\mathrm{im}_D(R_1.), \\ \mathrm{ext}^2_D(M,D) \cong D/(R_2\,D^2), \\ \mathrm{ext}^i_D(M,D) = 0, \ i \geq 3. \end{cases}$$

We first note that $\ker_D(R_1.) = \{d \in D \mid R_1\,d = 0\} = 0$ since $R_1 \neq 0$ and D is a domain, which shows that $\mathrm{ext}^0_D(M,D) = 0$. Let us now compute $\ker_D(R_2.)$: $\mu = (\mu_1 \quad \mu_2)^T \in \ker_D(R_2.)$ iff $(x+1)\,\mu_1 = (x-1)\,\mu_2$, i.e., iff $\mu_1 = (x-1)\,\nu$ and $\mu_2 = (x+1)\,\nu$ for all $\nu \in D$ since D is a greatest common divisor domain and $\gcd(x+1, x-1) = 1$. Hence, if $R'_1 = (x-1 \quad x+1)^T$, then $\ker_D(R_2.) = R'_1\,D$, and thus:

$$\mathrm{ext}^1_D(M,D) \cong (R'_1\,D)/(R_1\,D).$$

We clearly have $R_1 = R'_1\,x$, which shows that $\mathrm{ext}^1_D(M,D) \neq 0$ and the residue class $\rho(R'_1)$ of R'_1 in the D-module $L \triangleq (R'_1\,D)/(R_1\,D)$ generates L, where $\rho : D\,R'_1 \longrightarrow L$ is the canonical projection onto L, and satisfies $x\,\rho(R'_1) = \rho(x\,R'_1) = \rho(R_1) = 0$. Hence, $\rho(R'_1)$ is a torsion element and thus $\mathrm{ext}^1_D(M,D)$ is a torsion D-module. Finally, since $1 \in (x+1, x-1)$, i.e., $(x+1, x-1) = D$, then $\mathrm{ext}^2_D(M,D) \cong D/(x+1, x-1) = 0$.

Example 1.2.6. If $D = \mathbb{Q}[\partial, \delta]$ is the commutative polynomial ring in ∂ and δ with coefficients in $\mathbb{Q}$, $R_1 = (\partial \quad 1-\delta)^T \in D^2$ and $M = D/(D^{1\times 2}\,R_1) = D/(D\,\partial + D\,(1-\delta))$ the D-module finitely presented by R. Then, M admits the following finite free resolution

$$0 \longrightarrow D \xrightarrow{.R_2} D^{1\times 2} \xrightarrow{.R_1} D \xrightarrow{\pi} M \longrightarrow 0,$$

where $R_2 = (1-\delta \quad -\partial) \in D^{1\times 2}$, because $\lambda = (\lambda_1 \quad \lambda_2) \in \ker_D(.R_1)$ iff $\lambda_1\,\partial + \lambda_2\,(1-\delta) = 0$, i.e., iff $\lambda_1 = \mu\,(1-\delta)$ and $\lambda_2 = -\mu\,\partial$ for all $\mu \in D$, since D is a greatest common divisor domain and $\gcd(\partial, 1-\delta) = 1$, which proves that $\lambda = \mu\,R_2$, and thus $\ker_D(.R_1) = D\,R_2$.

Let $\mathcal{F} = C^\infty(\mathbb{R})$ be endowed with the D-module structure defined by $\partial\,\eta(t) = \dot{\eta}(t)$ and $\delta\,\eta(t) = \eta(t-1)$ for all $\eta \in \mathcal{F}$. The two functional operators ∂ and δ then commute since:

$$\forall\, \eta \in \mathcal{F}, \quad \partial\,(\delta\,\eta(t)) = \partial\,(\eta(t-1)) = (\partial\,\eta)(t-1)\,\partial(t-1) = (\partial\,\eta)(t-1) = \delta\,(\partial\,\eta(t)).$$

Then, the defects of exactness of the complex $0 \longleftarrow \mathcal{F} \xleftarrow{R_2.} \mathcal{F}^2 \xleftarrow{R_1.} \mathcal{F} \longleftarrow 0$ are defined by:

$$\begin{cases} \mathrm{ext}^0_D(M,\mathcal{F}) = \hom_D(M,\mathcal{F}) \cong \ker_{\mathcal{F}}(R_1.), \\ \mathrm{ext}^1_D(M,\mathcal{F}) \cong \ker_{\mathcal{F}}(R_2.)/\mathrm{im}_{\mathcal{F}}(R_1.), \\ \mathrm{ext}^2_D(M,\mathcal{F}) \cong \mathcal{F}/(R_2\,\mathcal{F}^2), \\ \mathrm{ext}^i_D(M,\mathcal{F}) = 0, \ i \geq 3. \end{cases}$$

$\eta \in \ker_{\mathcal{F}}(R_1.)$ is equivalent to $\dot{\eta} = 0$ and $\eta(t) = \eta(t-1)$, i.e., to η is an arbitrary real constant, and thus $\ker_{\mathcal{F}}(R_1.) = \mathbb{R}$. Now, if c_1 and c_2 are two different real constants, then $(1-\delta)\, c_1 - \partial\, c_2 = 0$, i.e., $(c_1 \quad c_2)^T \in \ker_{\mathcal{F}}(R_2.)$. However, $(c_1 \quad c_2)^T \notin \operatorname{im}_{\mathcal{F}}(R_1.)$ since the first equation of the following inhomogeneous linear OD time-delay system

$$\begin{cases} \dot{\eta}(t) = c_1, \\ \eta(t) - \eta(t-1) = c_2, \end{cases}$$

gives $\eta(t) = c_1\, t + c_3$, where $c_3 \in \mathbb{R}$, and then the second one yields the contradiction $c_1 = c_2$. Thus, the D-module $\operatorname{ext}^1_D(M, \mathcal{F})$ is not reduced to 0. Finally, $R_2. : \mathcal{F}^2 \longrightarrow \mathcal{F}$ is a surjective since for all $\phi \in \mathcal{F}$, $\phi = (1-\delta)\, \zeta_1 - \partial\, \zeta_2$ where $\zeta_1 = 0$ and $\zeta_2 = -\int_{-\infty}^t \phi(s)\, ds$, i.e., $\operatorname{ext}^2_D(M, \mathcal{F}) = 0$.

Theorem 1.1.1 shows that a connection exists between $\ker_{\mathcal{F}}(R.)$ and $\hom_D(M, \mathcal{F})$. We may wonder if it still holds for the higher extension abelian groups $\operatorname{ext}^i_D(M, \mathcal{F})$'s for $i \geq 1$. If we consider (1.6), then we can introduce the following sequence of abelian group homomorphisms

$$\begin{array}{ccccccccc} \ldots & \xleftarrow{(.R_3)^\star} & \hom_D(D^{1\times r_2}, \mathcal{F}) & \xleftarrow{(.R_2)^\star} & \hom_D(D^{1\times r_1}, \mathcal{F}) & \xleftarrow{(.R_1)^\star} & \hom_D(D^{1\times r_0}, \mathcal{F}) & \longleftarrow & 0, \\ \ldots & \xleftarrow{(.R_{i+1})^\star} & \hom_D(D^{1\times r_i}, \mathcal{F}) & \xleftarrow{(.R_i)^\star} & \hom_D(D^{1\times r_{i-1}}, \mathcal{F}) & \xleftarrow{(.R_{i-1})^\star} & \hom_D(D^{1\times r_{i-2}}, \mathcal{F}) & \longleftarrow & \ldots \end{array} \tag{1.14}$$

where $(.R_i)^\star(\phi) = \phi \circ (.R_i)$ for all $\phi \in \hom_D(D^{1\times r_{i-1}}, \mathcal{F})$ and all $i \geq 1$. $R_{i+1}\, R_i = 0$ yields

$$\begin{aligned} ((.R_{i+1})^\star \circ (.R_i)^\star)(\phi) = (.R_{i+1})^\star((.R_i)^\star(\phi)) = (.R_{i+1})^\star(\phi \circ (.R_i)) = (\phi \circ (.R_i)) \circ (.R_{i+1}) \\ = \phi \circ ((.R_i) \circ (.R_{i+1})) = \phi \circ (.(R_{i+1}\, R_i)) = 0, \end{aligned}$$

for all $\phi \in \hom_D(D^{1\times r_{i-1}}, \mathcal{F})$, which proves that (1.14) is a complex of abelian groups. Now, applying Theorem 1.1.1 to $\hom_D(D^{1\times r_i}, \mathcal{F})$, i.e., with $R = (0 \ \ldots \ 0) \in D^{1\times r_i}$, we obtain $\hom_D(D^{1\times r_i}, \mathcal{F}) \cong \mathcal{F}^{r_i}$. Moreover, using Theorem 1.1.1, the abelian group homomorphism $\chi_i : \mathcal{F}^{r_i} \longrightarrow \hom_D(D^{1\times r_i}, \mathcal{F})$ defined by $\chi_i(\eta) = \phi_\eta$, where ϕ_η is defined by $\phi_\eta(\lambda) = \lambda\, \eta$ for all $\lambda \in D^{1\times r_i}$, is an isomorphism and its inverse $\chi_i^{-1} : \hom_D(D^{1\times r_i}, \mathcal{F}) \longrightarrow \mathcal{F}^{r_i}$ is defined by $\chi_i^{-1}(\phi) = (\phi(e_1) \ \ldots \ \phi(e_{r_i}))^T$, where $\{e_k\}_{k=1,\ldots,r_i}$ is the standard basis of $D^{1\times r_i}$. Hence, we get

$$(\chi_i^{-1} \circ (.R_i)^\star \circ \chi_{i-1})(\eta) = (\chi_i^{-1} \circ (.R_i)^\star)(\phi_\eta) = \chi_i^{-1} \circ \phi_\eta \circ (.R_i) = \chi_i^{-1}(\phi_\eta \circ (.R_i)) = \begin{pmatrix} e_1\, R_i\, \eta \\ \vdots \\ e_{r_i}\, R_i\, \eta \end{pmatrix},$$

for all $\eta \in \mathcal{F}^{r_{i-1}}$, which shows that $(\chi_i^{-1} \circ (.R_i)^\star \circ \chi_{i-1}) = (R_i.)$ and (1.14) is equivalent to (1.13) up to isomorphism. The complex (1.14) is said to be obtained by *applying the contravariant left exact functor* $\hom_D(\,\cdot\,, \mathcal{F})$ to the *truncated resolution* of M, namely,

$$M_\bullet \quad \ldots \xrightarrow{.R_4} D^{1\times r_3} \xrightarrow{.R_3} D^{1\times r_2} \xrightarrow{.R_2} D^{1\times r_1} \xrightarrow{.R_1} D^{1\times r_0} \longrightarrow 0, \tag{1.15}$$

i.e., the complex $M_\bullet$ obtained from (1.6) by deleting the left D-homomorphism π and the left D-module M. The truncated resolution (1.15) is exact at each position $i \geq 1$ and $H_0(M_\bullet) = M$. Hence, the complex (1.13) can be understood as the dual of (1.15) with values in the left D-module $\mathcal{F}$. Exactness is generally lost while dualizing and the defects of exactness, called *cohomologies*, are characterized by the abelian groups $\operatorname{ext}^i_D(M, \mathcal{F})$'s for $i \geq 0$.

We recall that M is a $D-E$*-bimodule* ([109]) if M is a left D-module, a right E-module and:

$$\forall\, d \in D, \quad \forall\, m \in M, \quad \forall\, e \in E, \quad (d\, m)\, e = d\, (m\, e).$$

Lemma 1.2.1 ([109]). *If M is a left (resp., right) D-module and $\mathcal{F}$ is a $D-D$-module, then $\mathrm{ext}_D^i(M, \mathcal{F})$ is a right (resp., left) D-module for all $i \in \mathbb{N}$. In particular, if D is a commutative ring, then the $\mathrm{ext}_D^i(M, \mathcal{F})$'s are D-modules.*

If M is a left (resp., right) D-module and D is the $D-D$-bimodule, then Lemma 1.2.1 shows that the $\mathrm{ext}_D^i(M, D)$'s are right (resp., left) D-modules. The next proposition gives a finer characterization when D is a noetherian domain and M a finitely generated left D-module.

Proposition 1.2.1 ([92]). *Let M be a finitely generated left (resp., right) D-module over a noetherian domain D. Then, for $i \geq 1$, the $\mathrm{ext}_D^i(M, D)$'s are either zero or finitely generated torsion right (resp., left) D-modules.*

This result explains why the D-module $\mathrm{ext}_D^1(M, D)$ obtained in Example 1.2.5 was torsion.

Let us now state a few classical results on the extension functors.

Theorem 1.2.1 ([109]). *Let $0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$ be a short exact sequence of left (resp., right) D-modules and N a left (resp., right) D-module. Then, the following long exact sequence of abelian groups holds*

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{ext}_D^0(M'', N) & \xrightarrow{g^\star} & \mathrm{ext}_D^0(M, N) & \xrightarrow{f^\star} & \mathrm{ext}_D^0(M', N) \\ & \xrightarrow{\kappa^1} & \mathrm{ext}_D^1(M'', N) & \longrightarrow & \mathrm{ext}_D^1(M, N) & \longrightarrow & \mathrm{ext}_D^1(M', N) \\ & \xrightarrow{\kappa^2} & \mathrm{ext}_D^2(M'', N) & \longrightarrow & \mathrm{ext}_D^2(M, N) & \longrightarrow & \ldots, \end{array} \tag{1.16}$$

where $f^\star$ is defined by $f^\star(\phi) = \phi \circ f$ for all $\phi \in \hom_D(M, N)$ and similarly for $g^\star$.

Roughly speaking, Theorem 1.2.1 explains why $\hom_D(\,\cdot\,, N)$ is called a *contravariant left exact functor*: the sense of the long exact sequence (1.16) is reversed while applying $\hom_D(\,\cdot\,, N)$ to the short exact sequence $0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$ and $g^\star$ is injective, namely:

$$g^\star(\psi) = \psi \circ g = 0 \quad \Rightarrow \quad \psi = 0.$$

Proposition 1.2.2 ([109]). *If M is a projective left D-module, then $\mathrm{ext}_D^i(M, N) = 0$ for all $i \geq 1$ and all left D-modules N. Similarly for right D-modules.*

From Theorem 1.2.1 and Proposition 1.2.2, we obtain the following proposition.

Proposition 1.2.3 ([109]). *Let $0 \longrightarrow Q \longrightarrow P \longrightarrow M \longrightarrow 0$ be a short exact sequence of left (resp., right) D-modules and P a projective left (resp., right) D-module. Then, for every left (resp., right) D-module N, we have:*

$$\forall\, i \geq 1, \quad \mathrm{ext}_D^{i+1}(M, N) \cong \mathrm{ext}_D^i(Q, N).$$

Let us state two useful results in module theory and homological algebra.

Proposition 1.2.4 ([109]). *If M is a projective left (resp., right) D-module, then $\hom_D(M, D)$ is a projective right (resp., left) D-module.*

Proposition 1.2.5 ([15, 65, 109]). *If $0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$ is a short exact sequence and M'' is a left (resp., right) D-module, then the short exact splits, i.e., $M \cong M' \oplus M''$.*

Let us introduce the concepts of *projective dimension* and *global dimension*.

Definition 1.2.2 ([109]). 1. A *projective resolution* of a left (resp., right) D-module M is an exact sequence of the form

$$\dots \xrightarrow{\delta_4} P_3 \xrightarrow{\delta_3} P_2 \xrightarrow{\delta_2} P_1 \xrightarrow{\delta_1} P_0 \xrightarrow{\delta_0} M \longrightarrow 0,$$

where the P_i's are projective left (resp., right) D-modules and $\delta_i \in \hom_D(P_i, P_{i-1})$ for all $i \in \mathbb{N}$. If there exists $n \in \mathbb{N}$ such that $P_m = 0$ for all $m \geq n+1$, then n is called the *length* of the projective resolution of M.

2. The *left projective dimension* of a left D-module M, denoted by $\mathrm{lpd}_D(M)$, is the minimum length of the projective resolutions of M. If no such integer exists, then $\mathrm{lpd}_D(M) = \infty$. Similarly, we can define the *right projective dimension* $\mathrm{rpd}_D(N)$ of a right D-module N.
3. The *left global dimension* (resp., *right global dimension*) of a ring D, denoted by $\mathrm{lgd}(D)$ (resp., $\mathrm{rgd}(D)$), is the supremum of $\mathrm{lpd}_D(M)$ (resp., $\mathrm{rpd}_D(N)$) for all left D-modules M (resp., all right D-modules N).
4. If the left and the right global dimension of D coincide, then the common value is denoted by $\mathrm{gld}(D)$ and called the *global dimension* of D.

The left projective dimension measures how far a left D-module M is from being projective.

Example 1.2.7. M is a projective left D-module iff $\mathrm{lpd}_D(M) = 0$. M is a quotient of two projective left D-modules, i.e., $M = P_0/\mathrm{im}\,\delta_1$, where P_0 and $\mathrm{im}\,\delta_1 \cong P_1$ are two projective left D-modules, iff $\mathrm{lpd}_D(M) \leq 1$. In particular, $\mathrm{lpd}_D(M) = 1$ if M is not a projective left D-module but M is isomorphic to the quotient of two projective left D-modules.

Let us show how to compute $\mathrm{lpd}_D(M)$ when M is a left D-module defined by a finite free resolution of finite length. We first need to introduce a result which is used to shorten the length of a finite free resolution of finite length if it is possible. Let I_q be the $q \times q$ identity matrix.

Proposition 1.2.6 ([103]). *Let M be a left D-module defined by the finite free resolution:*

$$0 \longrightarrow D^{1\times p_m} \xrightarrow{.R_m} D^{1\times p_{m-1}} \xrightarrow{.R_{m-1}} \dots \xrightarrow{.R_2} D^{1\times p_1} \xrightarrow{.R_1} D^{1\times p_0} \xrightarrow{\pi} M \longrightarrow 0. \tag{1.17}$$

1. *If $m \geq 3$ and there exists a matrix $S_m \in D^{p_{m-1}\times p_m}$ satisfying $R_m\, S_m = I_{p_m}$, then M admits the following shorter finite free resolution*

$$0 \longrightarrow D^{1\times p_{m-1}} \xrightarrow{.T_{m-1}} D^{1\times (p_{m-2}+p_m)} \xrightarrow{.T_{m-2}} D^{1\times p_{m-3}} \xrightarrow{.R_{m-3}} \dots \xrightarrow{.R_1} D^{1\times p_0} \xrightarrow{\pi} M \longrightarrow 0, \tag{1.18}$$

with the notations:

$$\begin{cases} T_{m-1} = (R_{m-1} \quad S_m) \in D^{p_{m-1}\times(p_{m-2}+p_m)}, \\ T_{m-2} = \begin{pmatrix} R_{m-2} \\ 0 \end{pmatrix} \in D^{(p_{m-2}+p_m)\times p_{m-3}}. \end{cases}$$

2. *If $m = 2$ and there exists a matrix $S_2 \in D^{p_1\times p_2}$ such that $R_2\, S_2 = I_{p_2}$, then M admits the following shorter finite free resolution*

$$0 \longrightarrow D^{1\times p_1} \xrightarrow{.T_1} D^{1\times(p_0+p_2)} \xrightarrow{\tau} M \longrightarrow 0, \tag{1.19}$$

with the notations $T_1 = (R_1 \quad S_2) \in D^{p_1\times(p_0+p_2)}$ and:

$$\begin{array}{rcl} \tau = \pi \oplus 0 : D^{1\times(p_0+p_2)} & \longrightarrow & M \\ \lambda = (\lambda_1 \quad \lambda_2) & \longmapsto & \tau(\lambda) = \pi(\lambda_1). \end{array}$$

The existence of a right inverse of a matrix can be checked by means of Gröbner basis techniques (e.g., when $D = k[x_1, \ldots, x_n]$, $A_n(k)$ and $B_n(k)$, where k is a computable field (e.g., $\mathbb{Q}$ or $\mathbb{F}_p$ for a prime p)). We first shortly explain how to compute a left inverse of a matrix.

Algorithm 1.2.2. – **Input:** A noncommutative polynomial ring D for which Buchberger's algorithm terminates for any admissible term order and a matrix $R \in D^{q\times p}$.

– **Output:** A matrix $S \in D^{p\times q}$ such that $S\,R = I_p$ if S exists and $\emptyset$ otherwise.

1. Introduce indeterminates λ_j, $j = 1, \ldots, p$ and μ_i, $i = 1, \ldots, q$, over D and define the set:
$$P = \left\{ \sum_{j=1}^{p} R_{ij}\,\lambda_j - \mu_i \mid i = 1, \ldots, q \right\}.$$
2. Compute the Gröbner basis G of P in $\bigoplus_{j=1}^{p} D\,\lambda_j \oplus \bigoplus_{i=1}^{q} D\,\mu_i$ with respect to a term order which eliminates the λ_j's.
3. Remove from G the elements which do not contain any λ_i and call H this new set.
4. Write H in the form $Q_1\,(\lambda_1 \;\ldots\; \lambda_p)^T - Q_2\,(\mu_1 \;\ldots\; \mu_q)^T$, where Q_1 and Q_2 are two matrices with entries in D.
5. If Q_1 is invertible over D, then return $S = Q_1^{-1}\,Q_2 \in D^{p\times q}$, else return $\emptyset$.

Computer algebra systems contain packages based on left Gröbner basis techniques, i.e., techniques based on computations of Gröbner bases of finitely generated left D-modules. But, they generally do not allow us to compute Gröbner bases for right D-modules (e.g., Maple).

As explained in [16], one way to handle this problem is to use the concept of *involution* of the ring D (i.e., *anti-automorphism*) ([109]), namely, a map $\theta : D \longrightarrow D$ satisfying:

$$\forall\, d_1, d_2 \in D, \quad \theta(d_1 + d_2) = \theta(d_1) + \theta(d_2), \quad \theta(d_1\,d_2) = \theta(d_2) \circ \theta(d_1), \quad \theta \circ \theta = \mathrm{id}_D.$$

If D is a commutative ring, then $\theta = \mathrm{id}_D$ is an involution. If $D = A\langle \partial_1, \ldots, \partial_n \rangle$ is a ring of PD operators with coefficients in the differential ring A, then we can define an involution θ of D by:

$$\forall\, a \in A, \quad \theta(a) = a, \quad \forall\, i = 1, \ldots, n, \quad \theta(\partial_i) = -\partial_i. \tag{1.20}$$

By extension, the involution $\theta(R)$ of a matrix $R \in D^{q\times p}$ is defined by $\theta(R) = (\theta(R_{ij}))^T \in D^{p\times q}$. If $D = A\langle \partial_1, \ldots, \partial_n \rangle$ and θ is defined by (1.20), then $\theta(R)$ corresponds to the *formal adjoint* $\widetilde{R}$ of R, i.e., the adjoint of R in the sense of the theory of distributions (see, e.g., [16, 85, 89, 66]). In what follows, if $D = A\langle \partial_1, \ldots, \partial_n \rangle$, then we shall use the standard notation $\widetilde{R}$ for $\theta(R)$.

Example 1.2.8. Let us consider matrix $R = (\partial_1 \quad \partial_2 \quad x_1\,\partial_1 + x_2\,\partial_2)$ with entries in the first Weyl algebra $D = A_2(\mathbb{Q})$. Let us compute its formal adjoint $\widetilde{R}$. If ϕ denotes a row vector of test functions, namely, a compactly supported smooth functions $\phi \in \mathcal{D}(\mathbb{R}^2)$, then the formal adjoint $\widetilde{R}$ of R can be obtained as follows:

$$\begin{aligned}
&\textstyle\int_{\mathbb{R}^2} \phi\,(\partial_1\,\eta_1 + \partial_2\,\eta_2 + (x_1\,\partial_1 + x_2\,\partial_2)\,\eta_3)\,dx_1\,dx_2 \\
&= \textstyle\int_{\mathbb{R}^2} ((-\partial_1\,\phi)\,\eta_1 + (-\partial_2\,\phi)\,\eta_2 + (-\partial_1\,(x_1\,\phi) - \partial_2\,(x_2\,\phi))\,\eta_3)\,dx_1\,dx_2, \\
&= \textstyle\int_{\mathbb{R}^2} ((-\partial_1\,\phi)\,\eta_1 + (-\partial_2\,\phi)\,\eta_2 + ((-x_1\,\partial_1 - x_2\,\partial_2 - 2)\,\phi)\,\eta_3)\,dx_1\,dx_2.
\end{aligned}$$

Hence, we get $\widetilde{R} = -(\partial_1 \quad \partial_2 \quad x_1\,\partial_1 + x_2\,\partial_2 + 2)^T \in D^2$, which can directly be found as follows:

$$\begin{aligned}
\theta(R) &= (\theta(\partial_1) \quad \theta(\partial_2) \quad \theta(x_1\,\partial_1 + x_2\,\partial_2))^T = (-\partial_1 \quad -\partial_2 \quad \theta(\partial_1)\,\theta(x_1) + \theta(\partial_2)\,\theta(x_2))^T \\
&= (-\partial_1 \quad -\partial_2 \quad -\partial_1\,x_1 - \partial_2\,x_2)^T = -(\partial_1 \quad \partial_2 \quad x_1\,\partial_1 + x_2\,\partial_2 + 2)^T.
\end{aligned}$$

If D admits an involution θ, then the search for a right inverse $T \in D^{p\times q}$ of $R \in D^{q\times p}$ can be reduced to the search for a left inverse $S \in D^{q\times p}$ of $\theta(R)$ since $S\,\theta(R) = I_q$ yields $\theta(S\,\theta(R)) = \theta^2(R)\,\theta(S) = R\,\theta(S) = \theta(I_q) = I_q$, i.e., $T = \theta(S)$.

Algorithm 1.2.3.
- **Input:** A noncommutative polynomial ring D for which Buchberger's algorithm terminates for any admissible term order and which admits an involution θ and a matrix $R \in D^{q\times p}$.
- **Output:** A matrix $T \in D^{p\times q}$ such that $R\,T = I_q$ if S exists and $\emptyset$ otherwise.

1. Compute $\theta(R) \in D^{p\times q}$.
2. Using Algorithm 1.2.2, compute a left inverse $S \in D^{q\times p}$ of $\theta(R)$ if S exists.
3. Compute $T = \theta(S) \in D^{p\times q}$.

Let us now illustrate Proposition 1.2.6 with two explicit examples.

Example 1.2.9. We consider the following time-varying linear OD system

$$\begin{cases} t^2\, y(t) = 0, \\ t\, \dot{y}(t) + 2\, y(t) = 0, \end{cases}$$

whose solution in the space of distributions $\mathcal{D}'(\mathbb{R})$ is $y = \dot{\delta}$, namely, the derivative of the Dirac distribution δ at 0. Let $D = A_1(\mathbb{Q})$ be the first Weyl algebra, $R_1 = (t^2 \quad t\,\partial + 2)^T$ and $M = D/(D^{1\times 2}\,R_1) = D/\left(D\,t^2 + D\,(t\,\partial + 2)\right)$ the left D-module finitely presented by R_1. Using Algorithm 1.2.1, a finite free resolution of M is defined by

$$0 \longrightarrow D \xrightarrow{.R_2} D^{1\times 2} \xrightarrow{.R_1} D \xrightarrow{\pi} M \longrightarrow 0,$$

where $R_2 = (\partial \quad -t) \in D^{1\times 2}$. Using Algorithm 1.2.3, we can check that $S_2 = (t \quad \partial)^T \in D^2$ is a right inverse of R_2. Using Proposition 1.2.6, M admits the following finite free resolution

$$0 \longrightarrow D^{1\times 2} \xrightarrow{.T_1} D^{1\times 2} \xrightarrow{\tau} M \longrightarrow 0, \tag{1.21}$$

with the notations:

$$T_1 = \begin{pmatrix} t^2 & t \\ t\,\partial + 2 & \partial \end{pmatrix} \in D^{2\times 2}, \quad \tau_0 = \delta_0 \oplus 0.$$

Example 1.2.10. Let us consider the first Weyl algebra $D = A_3(\mathbb{Q})$ and the matrix

$$R_1 = \frac{1}{2}\begin{pmatrix} x_2\,\partial_1 & 2\,(x_2\,\partial_2 + 1) & 2\,x_2\,\partial_3 + \partial_1 \\ -x_2\,\partial_2 - 3 & 0 & \partial_2 \\ -2\,\partial_1 - x_2\,\partial_3 & -2\,\partial_2 & -\partial_3 \end{pmatrix} \in D^{3\times 3}, \tag{1.22}$$

which defines the PD linear system $R_1\,\xi = 0$ of the infinitesimal transformations of the Lie pseudogroup defined by the contact transformations ([84]). Using Algorithm 1.2.1, the left D-module $M = D^{1\times 3}/(D^{1\times 3}\,R_1)$ admits the following finite free resolution

$$0 \longrightarrow D \xrightarrow{.R_2} D^{1\times 3} \xrightarrow{.R_1} D^{1\times 3} \xrightarrow{\pi} M \longrightarrow 0,$$

where $R_2 = (\partial_2 \quad -(\partial_1 + x_2\,\partial_3) \quad x_2\,\partial_2 + 2) \in D^{1\times 3}$. The matrix $S_2 = (-x_2 \quad 0 \quad 1)^T$ is a right inverse of R_2, and thus, using Proposition 1.2.6, we obtain the following finite free resolution

$$0 \longrightarrow D^{1\times 3} \xrightarrow{.T_1} D^{1\times 4} \xrightarrow{\tau} M \longrightarrow 0, \tag{1.23}$$

where the matrix T_1 is defined by:

$$T_1 = \frac{1}{2} \begin{pmatrix} x_2\,\partial_1 & 2\,(x_2\,\partial_2 + 1) & 2\,x_2\,\partial_3 + \partial_1 & -2\,x_2 \\ -x_2\,\partial_2 - 3 & 0 & \partial_2 & 0 \\ -2\,\partial_1 - x_2\,\partial_3 & -2\,\partial_2 & -\partial_3 & 2 \end{pmatrix} \in D^{3\times 4}. \tag{1.24}$$

We can now give an algorithm which computes the left projective dimension $\mathrm{lpd}_D(M)$ of M.

Algorithm 1.2.4.
- **Input:** A left D-module M defined by a finite free resolution of the form (1.17).
- **Output:** The left projective dimension $\mathrm{lpd}_D(M)$ of M.

1. Set $j = m$ and $T_j = R_m$.
2. Check whether or not T_j admits a right inverse S_j.
   (a) If no right inverse of T_j exists, then $\mathrm{lpd}_D(M) = j$ and stop the algorithm.
   (b) If there exists a right inverse S_j of T_j and
      i. if $j = 1$, then we have $\mathrm{lpd}_D(M) = 0$ and stop the algorithm.
      ii. if $j = 2$, then compute (1.19).
      iii. if $j \geq 3$, then compute (1.18).
3. Return to step (2) with $j \longleftarrow j - 1$.

Example 1.2.11. We consider again Example 1.2.9. We can easily check that the matrix T_1 defined in (1.21) does not admit a right inverse. Hence, using Algorithm 1.2.4, we obtain that $\mathrm{lpd}_D(M) = 1$. In particular, the left D-module M is not projective. But, the existence of the short exact sequence (1.21) shows that M can be expressed as the quotient of two finitely generated free left D-modules.

If M is a projective left D-module defined by a finite free resolution (1.17), then $\mathrm{lpd}_D(M) = 0$ and using Algorithm 1.2.4, we obtain a short exact sequence of the form

$$0 \longrightarrow D^{1\times p'} \xrightarrow{.R'} D^{1\times p'} \xrightarrow{\pi'} M \longrightarrow 0,$$

where the matrix R' admits a right inverse $S' \in D^{p'\times q'}$, i.e., $R'\,S' = I_{q'}$. If we introduce the following two left D-homomorphisms

$$\begin{array}{rcl} f : D^{1\times q'} & \longrightarrow & D^{1\times p'} \\ \lambda & \longmapsto & \lambda\,R', \end{array} \qquad \begin{array}{rcl} k : D^{1\times p'} & \longrightarrow & D^{1\times q'} \\ \mu & \longmapsto & \mu\,S', \end{array}$$

then $(k \circ f)(\lambda) = k(\lambda\,R') = \lambda\,R'\,S' = \lambda$ for all $\lambda \in D^{1\times q'}$, i.e., $k \circ f = \mathrm{id}_{D^{1\times q'}}$, which shows that the above short exact sequence splits (see 7 of Definition 1.2.1), i.e., $D^{1\times p'} \cong D^{1\times q'} \oplus M$, which proves that M is a stably free left D-module of rank $p' - q'$. We obtain the next proposition which can be traced back to Serre's work on projective modules (Serre's conjecture).

Proposition 1.2.7. *If a left D-module M admits a finite free resolution of finite length, then M is a projective left D-module iff M is a stably free left D-module.*

Example 1.2.12. We consider again Example 1.2.10. We can check that the matrix T_1 defined in (1.24) admits the following right inverse with entries in $D = A_3(\mathbb{Q})$:

$$S_1 = \left(\begin{array}{ccc} 0 & -1 & 0 \\ 1 & 0 & x_2 \\ 0 & -x_2 & 0 \\ \partial_2 & -\partial_1 - x_2\,\partial_3 & x_2\,\partial_2 + 2 \end{array} \right).$$

Using Algorithm 1.2.4, we obtain $\mathrm{lpd}_D(M) = 0$, i.e., M is a projective left D-module, and thus a stably free left D-module of rank 1 by Proposition 1.2.7. Finally, since $\mathrm{rank}_D(M) = 1$, Stafford's theorem (see 3 of Theorem 1.1.2) cannot be used to conclude that M is a free left D-module.

Let us state a classical but non-trivial result due to Auslander.

Theorem 1.2.2 ([109]). *If D is a noetherian ring, then* $\mathrm{rgd}(D) = \mathrm{lgd}(D)$*.*

Let us give global dimensions of some noetherian domains of PD operators.

Example 1.2.13. $\mathrm{gld}(A\langle\partial_1, \ldots, \partial_n\rangle) = n$, where $A = k$ is a field, $k[x_1, \ldots, x_n]$, $k(x_1, \ldots, x_n)$, $k[\![x_1, \ldots, x_n]\!]$, where k is a field of characteristic 0, or $k\{x_1, \ldots, x_n\}$, where $k = \mathbb{R}$ or $\mathbb{C}$. A ring D satisfying $\mathrm{gld}(D) = 1$ is called a *hereditary ring* (e.g., $D = A\langle\partial\rangle$, where $A = k[t]$, $k[\![t]\!]$ or $k\{t\}$). If the characteristic of k is a prime p (e.g., $k = \mathbb{F}_p$), then $\mathrm{gld}(A_n(k)) = 2\,n$ ([10, 13, 45, 66]).

Proposition 1.2.8 ([109]). $\mathrm{lgld}(D) \leq n$ *iff* $\mathrm{ext}_D^{n+1}(M, N) = 0$ *for all left D-modules M and N.*

1.3 Constructive study of module properties

> "Prenons par exemple la tâche de démontrer un théorème qui reste hypothétique (à quoi, pour certains, semblerait se réduire le travail mathématique). Je vois deux approches extrêmes pour s'y prendre. [...] On peut s'y mettre avec des pioches ou des barres à mine ou même des marteaux-piqueurs : c'est la première approche, celle du "burin" (avec ou sans marteau). L'autre est celle de la **mer**. La mer s'avance insensiblement et sans bruit, rien ne semble se casser, rien ne bouge, l'eau est si loin on l'entend à peine... Pourtant elle finit par entourer la substance rétive, celle-ci peu à peu devient une presqu'île, puis une île, puis un îlot, qui finit par être submergé à son tour, comme s'il s'était finalement dissous à dans l'océan s'étendant à perte de vue..."
>
> Alexandre Grothendieck, *Récoltes et Semailles, Réflexions et témoignage sur un passé de mathématicien.*

We are now in a position to characterize the module properties introduced in Definition 1.1.1.

Theorem 1.3.1 ([2, 16]). *Let D be a noetherian domain with a finite global dimension* $\mathrm{gld}(D)$*, $R \in D^{q \times p}$ a matrix, $M = D^{1 \times p}/(D^{1 \times q}\,R)$ the left D-module finitely presented by R and the so-called* Auslander transpose *of M, namely, the right D-module $N = D^q/(R\,D^p)$.*

1. *The following left D-isomorphism holds:*

$$t(M) \cong \mathrm{ext}_D^1(N, D). \tag{1.25}$$

2. *M is a torsion-free left D-module iff* $\mathrm{ext}_D^1(N, D) = 0$*.*

3. *We have the following long exact sequence of left D-modules,*

$$0 \longrightarrow \mathrm{ext}_D^1(N,D) \longrightarrow M \xrightarrow{\varepsilon} \hom_D(\hom_D(M,D),D) \longrightarrow \mathrm{ext}_D^2(N,D) \longrightarrow 0, \qquad (1.26)$$

where the left D-homomorphism ε is defined in 4 of Definition 1.1.1.

4. *M is reflexive iff $\mathrm{ext}_D^i(N,D)=0$ for $i=1,2$.*
5. *M is projective iff $\mathrm{ext}_D^i(N,D)=0$ for $i=1,\ldots,\mathrm{gld}(D)$.*

Theorem 1.3.1 was proved in [45] for rings of PD operators and in [96] for finitely presented modules over coherent commutative domains. See also [85, 89]. But, Theorem 1.3.1 is first due to Auslander and Bridger ([2]) and was independently found again in [16].

Remark 1.3.1. We point out that the Auslander transpose $N = D^q/(R\,D^p)$ depends only on the left D-module M up to *projective equivalence* ([109]), namely, if $M = D^{1\times p'}/(D^{1\times q'}\,R')$ is another presentation of M and $N' = D^{q'}/(R'\,D^{p'})$, then we have:

$$N \oplus D^{(p+q')} \cong N' \oplus D^{(q+p')}.$$

See the forthcoming Theorem 3.4.2 and [2, 22, 91]. If R and R' have *full row rank*, namely, $\ker_D(.R)=0$ and $\ker_D(.R')=0$, then the previous isomorphism reduces to $N \cong N'$. For a constructive version of the above isomorphism, see [22]. Since a free right D-module is projective (see Proposition 1.1.1), Proposition 1.2.2 yields $\mathrm{ext}_D^i(D^{(p+q')},D)=0$ and $\mathrm{ext}_D^i(D^{(q+p')},D)=0$ for all $i \geq 1$. Using the additivity of the extension functor (see, e.g., [15, 65, 109]), we obtain

$$\begin{array}{c} \forall\, i \geq 1, \quad \mathrm{ext}_D^i(N,D) \cong \mathrm{ext}_D^i(N,D) \oplus \mathrm{ext}_D^i(D^{(p+q')},D) \cong \mathrm{ext}_D^i(N \oplus D^{(p+q')},D) \\ \cong \\ \mathrm{ext}_D^i(N' \oplus D^{(q+p')},D) \cong \mathrm{ext}_D^i(N',D) \oplus \mathrm{ext}_D^i(D^{(q+p')},D) \cong \mathrm{ext}_D^i(N',D), \end{array}$$

$\mathrm{ext}_D^i(N,D) \cong \mathrm{ext}_D^i(N',D)$ for all $i \geq 1$, which shows that the $\mathrm{ext}_D^i(N,D)$'s for $i \geq 1$ depend only on M and not on the presentation matrix $R \in D^{q\times p}$ of the left D-module M ([2, 22, 91]).

Theorem 1.3.1 shows that the vanishing of the $\mathrm{ext}_D^i(N,D)$'s for $i \geq 1$ characterizes the module properties of the finitely left D-module M. For a commutative polynomial ring $D = k[x_1,\ldots,x_n]$ over a computable field k (e.g., $\mathbb{Q}$ or $\mathbb{F}_p$ for a prime p) or certain classes of non-commutative polynomial rings of functional operators (e.g., certain classes *Ore algebras* ([18]) or *GR-algebras* ([58])) for which Gröbner bases exist for admissible term orders, the results of Theorem 1.3.1 were implemented in the OreModules package ([16, 17]).

If D admits an involution θ, then the right D-module structure of the Auslander transpose $N = D^q/(R\,D^p)$ of the left D-module $M = D^{1\times p}/(D^{1\times q}\,R)$ can be turned into a left D-module structure by defining the so-called *adjoint* left D-module module $\widetilde{N} = D^{1\times q}/(D^{1\times p}\,\theta(R))$ of M.

Let us show how to compute $\mathrm{ext}_D^1(N,D)$ using only left Gröbner basis computations.

Algorithm 1.3.1.
- **Input:** A noncommutative polynomial ring D for which Buchberger's algorithm terminates for any admissible term order and which admits an involution θ and a matrix $R \in D^{q\times p}$.
- **Output:** Two matrices $R' \in D^{q'\times p}$ and $Q \in D^{p\times m}$ such that

$$\mathrm{ext}_D^1(N,D) \cong t(M) = (D^{1\times q'}\,R')/(D^{1\times q}\,R), \quad \ker_D(.Q) = D^{1\times q'}\,R',$$

where $N = D^q/(R\,D^p)$ is the Auslander transpose of $M = D^{1\times p}/(D^{1\times q}\,R)$.

1. Compute $\theta(R) \in D^{p\times q}$.
2. Using Algorithm 1.2.1, compute a matrix $P \in D^{m\times p}$ such that $\ker_D(.\theta(R)) = D^{1\times m}\, P$.
3. Compute $Q = \theta(P) \in D^{p\times m}$.
4. Using Algorithm 1.2.1, compute a matrix $R' \in D^{q'\times p}$ such that $\ker_D(.Q) = D^{1\times q'}\, R'$.

If $D = k[x_1, \ldots, x_n]$ is a commutative polynomial ring with coefficients in a computable field k, then we can use $\theta = \mathrm{id}_D$ in Algorithm 1.3.1. If $D = A\langle \partial_1, \ldots, \partial_n\rangle$ is a noncommutative polynomial ring of PD operators, then we can use the involution θ defined by (1.20).

Similarly, the higher extension left D-modules $\mathrm{ext}^i_D(N, D)$'s can be computed as follows:

1. Using Algorithm 1.2.1, we compute the beginning of a finite free resolution of the left D-module $\widetilde{N} = D^{1\times q}/(D^{1\times p}\, S_1)$, where $S_1 = \theta(R)$:

$$0 \longleftarrow \widetilde{N} \xleftarrow{\kappa} D^{1\times q_0} \xleftarrow{.S_1} D^{1\times q_1} \xleftarrow{.S_2} \ldots \xleftarrow{.S_{i-1}} D^{1\times q_{i-1}} \xleftarrow{.S_i} D^{1\times q_i} \xleftarrow{.S_{i+1}} \ldots \tag{1.27}$$

2. We apply the involution θ to (1.27) to get the following complex of left D-modules:

$$0 \longrightarrow D^{1\times q_0} \xrightarrow{.\theta(S_1)} D^{1\times q_1} \xrightarrow{.\theta(S_2)} \ldots \xrightarrow{.\theta(S_{i-1})} D^{1\times q_{i-1}} \xrightarrow{.\theta(S_i)} D^{1\times q_i} \xrightarrow{.\theta(S_{i+1})} \ldots$$

3. Using Algorithm 1.2.1, we compute $Q_i \in D^{q'_{i-1}\times q_i}$ such that $\ker_D(.\theta(S_{i+1})) = D^{1\times q'_{i-1}}\, Q_i$.
4. We obtain $\mathrm{ext}^i_D(N, D) \cong (D^{1\times q'_{i-1}}\, Q_i)/(D^{1\times q_{i-1}}\, \theta(S_i))$.

According to Proposition 1.2.1, the $\mathrm{ext}^i_D(N, D)$'s are either 0 or torsion left D-modules for all $i \geq 1$. If we denote by z_j the residue classes of the j^{th} row of the matrix Q_i in the left D-module $(D^{1\times q'_{i-1}}\, Q_i)/(D^{1\times q_{i-1}}\, \theta(S_i))$, then z_j is either 0 or a torsion element (i.e., there exists $d \in D\setminus\{0\}$ such that $d\, z_j = 0$). Let us now explain how to compute $\mathrm{ann}_D(z_j) = \{d \in D \mid d\, z_j = 0\}$.

To simplify the notations, we consider the output of Algorithm 1.3.1, i.e.:

$$\mathrm{ext}^1_D(N, D) \cong (D^{1\times q'}\, R')/(D^{1\times q}\, R).$$

Since $(D^{1\times q'}\, R')/(D^{1\times q}\, R)$ is a torsion left D-module, there exists $d_i \in D \setminus \{0\}$ such that $d_i\, \pi(R'_{i\bullet}) = 0$, i.e., $\pi(d_i\, R'_{i\bullet}) = 0$, which yields the existence of $\mu_i \in D^{1\times q}$ satisfying:

$$d_i\, R'_{i\bullet} = \mu_i\, R \;\Leftrightarrow\; (d_i \quad -\mu_i) \begin{pmatrix} R'_{i\bullet} \\ R \end{pmatrix} = 0.$$

Hence, we have to compute the compatibility conditions of the inhomogeneous linear systems:

$$\forall\, i = 1, \ldots, q', \quad \begin{cases} R'_{i\bullet}\, \eta = \zeta_i, \\ R\, \eta = 0, \end{cases} \quad \Rightarrow \quad d_{ij}\, \zeta_i = 0, \quad j = 1, \ldots, r_i.$$

Algorithm 1.3.2. – **Input:** A noncommutative polynomial ring D for which Buchberger's algorithm terminates for any admissible term order, $R \in D^{q\times p}$ and $R' \in D^{q'\times p}$ satisfying $D^{1\times q}\, R \subseteq D^{1\times q'}\, R'$ and such that $L = (D^{1\times q'}\, R')/(D^{1\times q}\, R)$ is a torsion left D-module.

– **Output:** A set C of generating equations satisfied by the residue class z_i of the i^{th} row $R'_{i\bullet} = (R'_{i1} \;\ldots\; R'_{ip})$ of the matrix R' in the left module $L = (D^{1\times q'}\, R')/(D^{1\times q}\, R)$.

1. Introduce the indeterminates $\eta_1, \ldots, \eta_p$ and $\zeta_1, \ldots, \zeta_q$ over D.

2. For $i = 1, \ldots, q'$, compute the Gröbner basis G_i of the following set

$$L_i = \left\{ \sum_{j=1}^{p} R'_{ij}\, \eta_j - \zeta_i \right\} \bigcup \left\{ \sum_{j=1}^{p} R_{kj}\, \eta_j \mid k = 1, \ldots, q \right\}$$

in $\bigoplus_{j=1}^{p} D\, \eta_j \oplus D\, \zeta_i$ with respect to a term order which eliminates the η_j's.

3. Return $C = \bigcup_{i=1}^{q'} (G_i \cap D\, \zeta_i)$

Let us illustrate Algorithms 1.3.1 and 1.3.2 with two explicit examples.

Example 1.3.1. Let us consider the 2-dimensional Stokes equations ([52]) defined by:

$$\begin{pmatrix} -\nu\,(\partial_x^2 + \partial_y^2) & 0 & \partial_x \\ 0 & -\nu\,(\partial_x^2 + \partial_y^2) & \partial_y \\ \partial_x & \partial_y & 0 \end{pmatrix} \begin{pmatrix} u \\ v \\ p \end{pmatrix} = 0. \tag{1.28}$$

Let $D = \mathbb{Q}(\nu)[\partial_x, \partial_y]$ be the commutative polynomial ring of PD operators with coefficients in $\mathbb{Q}(\nu)$, $R \in D^{3\times 3}$ the matrix appearing in the left-hand side of (1.28) and $M = D^{1\times 3}/(D^{1\times 3}\, R)$ the D-module finitely presented by R. Since D is a commutative ring, we can take the trivial involution $\theta = \mathrm{id}_D$, define $\theta(R) = R^T = R$ and the adjoint D-module $\widetilde{N} = D^{1\times 3}/(D^{1\times 3}\, R) = M$. Using Algorithm 1.2.1, we can easily check that $\ker_D(.R) = 0$, i.e., R has full row rank, and thus the adjoint D-module $\widetilde{N}$ admits the following finite free resolution:

$$0 \longleftarrow \widetilde{N} \xleftarrow{\pi} D^{1\times 3} \xleftarrow{.R} D^{1\times 3} \longleftarrow 0.$$

Hence, the defects of exactness of the following complex of D-modules

$$0 \longrightarrow D^{1\times 3} \xrightarrow{.R} D^{1\times 3} \longrightarrow 0$$

are $\mathrm{ext}_D^0(\widetilde{N}, D) \cong \ker_D(.R) = 0$ and $\mathrm{ext}_D^1(\widetilde{N}, D) \cong D^{1\times 3}/(D^{1\times 3}\, R) = M$. Using 1 of Theorem 1.3.1, we get $t(M) \cong \mathrm{ext}_D^1(\widetilde{N}, D) \cong M$, which shows that M is a torsion D-module. Finally, using Algorithm 1.3.2, we can decouple the system variables of (1.28) as follows

$$\begin{cases} (\partial_x^2 + \partial_y^2)^2\, u = 0, \\ (\partial_x^2 + \partial_y^2)^2\, v = 0, \\ (\partial_x^2 + \partial_y^2)\, p = 0, \end{cases} \tag{1.29}$$

i.e., $\mathrm{ann}_D(u) = \mathrm{ann}_D(v) = D\,\Delta^2$ and $\mathrm{ann}_D(p) = D\,\Delta$, where $\Delta = \partial_x^2 + \partial_y^2$.

Example 1.3.2. Let us consider the following linear PD system with polynomial coefficients

$$\begin{cases} x_3\, \partial_1\, \xi_1 - x_1\, \partial_3\, \xi_1 + x_3\, \partial_2\, \xi_2 - x_2\, \partial_3\, \xi_2 - \xi_3 = 0, \\ -\xi_1 + x_1\, \partial_2\, \xi_2 - x_2\, \partial_1\, \xi_2 + x_1\, \partial_3\, \xi_3 - x_3\, \partial_1\, \xi_3 = 0, \\ x_2\, \partial_1\, \xi_1 - x_1\, \partial_2\, \xi_1 - \xi_2 + x_2\, \partial_3\, \xi_3 - x_3\, \partial_2\, \xi_3 = 0, \end{cases} \tag{1.30}$$

which appears in the study of the Lie algebra of the special unitary group SU(2) ([9]). We consider the first Weyl algebra $D = A_3(\mathbb{Q})$ and the presentation matrix R of (1.30) defined by:

$$R = \begin{pmatrix} x_3\, \partial_1 - x_1\, \partial_3 & x_3\, \partial_2 - x_2\, \partial_3 & -1 \\ -1 & x_1\, \partial_2 - x_2\, \partial_1 & x_1\, \partial_3 - x_3\, \partial_1 \\ x_2\, \partial_1 - x_1\, \partial_2 & -1 & x_2\, \partial_3 - x_3\, \partial_2 \end{pmatrix} \in D^{3\times 3}. \tag{1.31}$$

Using the involution θ of D defined by (1.20), the formal adjoint $\widetilde{R} = \theta(R)$ of R is defined by:

$$\widetilde{R} = \left(\begin{array}{ccc} x_1\,\partial_3 - x_3\,\partial_1 & -1 & x_1\,\partial_2 - x_2\,\partial_1 \\ x_2\,\partial_3 - x_3\,\partial_2 & x_2\,\partial_1 - x_1\,\partial_2 & -1 \\ -1 & x_3\,\partial_1 - x_1\,\partial_3 & x_3\,\partial_2 - x_2\,\partial_3 \end{array}\right) \in D^{3\times 3}. \tag{1.32}$$

Let $\widetilde{N} = D^{1\times 3}/(D^{1\times 3}\,\widetilde{R})$ be the left D-module finitely presented by the matrix $\widetilde{R}$. Using Algorithm 1.2.1, we obtain the following finite free resolution of $\widetilde{N}$

$$0 \longleftarrow \widetilde{N} \xleftarrow{\kappa} D^{1\times 3} \xleftarrow{.\widetilde{R}} D^{1\times 3} \xleftarrow{.P} D \longleftarrow 0,$$

where $P = (x_2\,\partial_3 - x_3\,\partial_2 \quad x_3\,\partial_1 - x_1\,\partial_3 \quad x_1\,\partial_2 - x_2\,\partial_1)$. If $N = D^3/(R\,D^3)$ is the Auslander transpose of the left D-module $M = D^{1\times 3}/(D^{1\times 3}\,R)$, then, using Algorithm 1.3.1, the left D-modules $\mathrm{ext}^i_D(N, D)$'s, for $i = 0,\,1,\,2$, are the defects of exactness of the following complex

$$0 \longrightarrow D^{1\times 3} \xrightarrow{.R} D^{1\times 3} \xrightarrow{.Q} D \longrightarrow 0,$$

where $Q = \widetilde{P} = -P^T$, namely:

$$\left\{\begin{array}{l} \mathrm{ext}^0_D(N, D) \cong \ker_D(.R), \\ \mathrm{ext}^1_D(N, D) \cong \ker_D(.Q)/\mathrm{im}_D(.R), \\ \mathrm{ext}^2_D(N, D) \cong \mathrm{coker}_D(.Q) = D/(D^{1\times 3}\,Q), \\ \mathrm{ext}^i_D(N, D) = 0, \quad \forall\, i \geq 3. \end{array}\right.$$

Using Algorithm 1.2.1, we obtain $\ker_D(.R) = D\,(x_1\,\partial_2 - x_2\,\partial_1 \quad x_2\,\partial_3 - x_3\,\partial_2 \quad x_3\,\partial_1 - x_1\,\partial_3)$ and $\ker_D(.Q) = D^{1\times 2}\,R'$, where the matrix $R' \in D^{2\times 3}$ is defined by

$$R' = \left(\begin{array}{ccc} x_1 & x_2 & x_3 \\ \partial_1 & \partial_2 & \partial_3 \end{array}\right), \tag{1.33}$$

which yields:

$$\left\{\begin{array}{l} \mathrm{ext}^0_D(N, D) \cong D\,(x_1\,\partial_2 - x_2\,\partial_1 \quad x_2\,\partial_3 - x_3\,\partial_2 \quad x_3\,\partial_1 - x_1\,\partial_3), \\ \mathrm{ext}^1_D(N, D) \cong t(M) = (D^{1\times 2}\,R')/(D^{1\times 3}\,R), \\ \mathrm{ext}^2_D(N, D) \cong D/(D\,(x_1\,\partial_2 - x_2\,\partial_1) + D\,(x_2\,\partial_3 - x_3\,\partial_2) + D\,(x_3\,\partial_1 - x_1\,\partial_3)). \end{array}\right.$$

Let z_i be the residue class of the i^{th} row of R' in M for $i = 1,\,2$. If $\{y_j\}_{j=1,2,3}$ is the family of generators of M defined by the residue classes of the standard basis of $D^{1\times 3}$ in M, then we get:

$$\left\{\begin{array}{l} z_1 = x_1\,y_1 + x_2\,y_2 + x_3\,y_3, \\ z_2 = \partial_1\,y_1 + \partial_2\,y_2 + \partial_3\,y_3. \end{array}\right. \tag{1.34}$$

Using Algorithm 1.3.2, we obtain that the generators z_1 and z_2 of $t(M) \cong \mathrm{ext}^1_D(N, D)$ are torsion elements which satisfy the following PDEs:

$$\forall\, i = 1,\,2, \quad \left\{\begin{array}{l} (x_2\,\partial_3 - x_3\,\partial_2)\,z_i = 0, \\ (x_1\,\partial_3 - x_3\,\partial_1)\,z_i = 0, \\ (x_1\,\partial_2 - x_2\,\partial_1)\,z_i = 0. \end{array}\right. \tag{1.35}$$

Thus, the left D-module M is not torsion-free. Finally, using a Gröbner basis computation, we can check that $1 \notin D\,(x_1\,\partial_2 - x_2\,\partial_1) + D\,(x_2\,\partial_3 - x_3\,\partial_2) + D\,(x_3\,\partial_1 - x_1\,\partial_3)$, and thus the torsion left D-module $\mathrm{ext}^2_D(N, D)$ is not reduced to 0.

To check the vanishing of the left D-module $\mathrm{ext}^1_D(N, D)$, we have to check the vanishing of the left D-module $L = (D^{1\times q'}\, R')/(D^{1\times q}\, R)$. If Gröbner basis techniques can be used over the noncommutative polynomial ring D, then we can check whether or not the normal forms of the rows of the matrix R' vanish in the left D-module L, i.e., whether or not L is reduced to 0.

Let us introduce a useful lemma which gives a finite presentation of a quotient module.

Proposition 1.3.1 ([19]). *Let D be a left noetherian ring, $R \in D^{q\times p}$ and $R' \in D^{q'\times p}$ two matrices satisfying $D^{1\times q}\, R \subseteq D^{1\times q'}\, R'$, i.e., such that $R = R''\, R'$ for a certain $R'' \in D^{q\times q'}$. Moreover, let $R'_2 \in D^{r'\times q'}$ be a matrix such that $\ker_D(.R') = D^{1\times r'}\, R'_2$ and let us respectively denote by π and π' the following canonical projections:*

$$\pi : D^{1\times q'}\, R' \longrightarrow (D^{1\times q'}\, R')/(D^{1\times q}\, R), \quad \pi' : D^{1\times q'} \longrightarrow D^{1\times q'}/(D^{1\times q}\, R'' + D^{1\times r'}\, R'_2).$$

Then, the left D-homomorphism χ defined by

$$\begin{array}{rcl} \chi : D^{1\times q'}/(D^{1\times q}\, R'' + D^{1\times r'}\, R'_2) & \longrightarrow & (D^{1\times q'}\, R')/(D^{1\times q}\, R) \\ \pi'(\lambda) & \longmapsto & \pi(\lambda\, R'), \end{array} \tag{1.36}$$

is an isomorphism and its inverse χ^{-1} is defined by:

$$\begin{array}{rcl} \chi^{-1} : (D^{1\times q'}\, R')/(D^{1\times q}\, R) & \longrightarrow & D^{1\times q'}/(D^{1\times q}\, R'' + D^{1\times r'}\, R'_2) \\ \pi(\lambda\, R') & \longmapsto & \pi'(\lambda). \end{array}$$

In other words, we have the following left D-isomorphism:

$$(D^{1\times q'}\, R')/(D^{1\times q}\, R) \cong D^{1\times q'}/(D^{1\times q}\, R'' + D^{1\times r'}\, R'_2).$$

In particular, $(D^{1\times q'}\, R')/(D^{1\times q}\, R)$ is reduced to 0 iff $(R''^T \quad R_2'^T)^T$ admits a left inverse.

Example 1.3.3. We consider again Example 1.3.2. Using Proposition 1.3.1, let us compute a finite presentation of the left D-module $L = (D^{1\times 2}\, R')/(D^{1\times 3}\, R) \cong \mathrm{ext}^1_D(N, D)$. Since $\ker_D(.R') = 0$, the left D-module L admits the finite presentation $L \cong D^{1\times 2}/(D^{1\times 3}\, R'')$, where

$$R'' = \begin{pmatrix} -\partial_3 & x_3 \\ -\partial_1 & x_1 \\ -\partial_2 & x_2 \end{pmatrix} \in D^{3\times 2} \tag{1.37}$$

satisfies $R = R''\, R'$. Then, the generators z_1 and z_2 of the left D-module L satisfy the following left D-linear relations:

$$\left\{ \begin{array}{l} -\partial_3\, z_1 + x_3\, z_2 = 0, \\ -\partial_1\, z_1 + x_1\, z_2 = 0, \\ -\partial_2\, z_1 + x_2\, z_2 = 0. \end{array} \right. \tag{1.38}$$

Let us sum up some of the previous results. Let D be a noetherian domain and

$$0 \longleftarrow N \xleftarrow{\kappa} D^q \xleftarrow{R.} D^p \xleftarrow{Q.} D^m$$

the beginning of a finite free resolution of the Auslander transpose $N = D^q/(R\, D^p)$ of the left D-module $M = D^{1\times p}/(D^{1\times q}\, R)$ associated with the linear system $\ker_{\mathcal{F}}(R.)$, where $\mathcal{F}$ is a left

D-module. Applying the contravariant left exact functor $\hom_D(\cdot, D)$ to the previous exact sequence of right D-modules, we obtain the following complex of left D-modules:

$$D^{1\times q} \xrightarrow{.R} D^{1\times p} \xrightarrow{.Q} D^{1\times m}. \tag{1.39}$$

Then, 1 of Theorem 1.3.1 asserts that $\mathrm{ext}^1_D(N, D) \cong t(M) = \ker_D(.Q)/\mathrm{im}_D(.R)$. Hence, if $R' \in D^{q'\times p}$ is a matrix satisfying $\ker_D(.Q) = D^{1\times q'} R'$, then we obtain:

$$t(M) = (D^{1\times q'} R')/(D^{1\times q} R). \tag{1.40}$$

See Algorithm 1.3.1. Then, the residue classes $\{\pi(R'_{i\bullet})\}_{i=1,\ldots,q'}$ of the rows $R'_{i\bullet}$ of the matrix R' in the left D-module M define a set of generators of the torsion left D-submodule $t(M)$ of M, i.e., $t(M) = \sum_{i=1}^{q'} D\,\pi(R'_{i\bullet})$. See Algorithm 1.3.2. Applying Proposition 1.3.1 to (1.40), we get

$$t(M) \cong D^{1\times q'}/(D^{1\times q} R'' + D^{1\times q_2} R'_2), \tag{1.41}$$

where the matrices $R'' \in D^{q\times q'}$ and $R'_2 \in D^{r'\times q'}$ are respectively defined by $R = R''\,R'$ and $\ker_D(.R') = D^{1\times r'} R'_2$. Using the *third isomorphism theorem* (see, e.g., [109]), we obtain:

$$M/t(M) = [D^{1\times p}/(D^{1\times q} R)]/[(D^{1\times q'} R')/(D^{1\times q} R)] \cong D^{1\times p}/(D^{1\times q'} R'). \tag{1.42}$$

Therefore, the matrix R' returns by Algorithm 1.3.1 is a presentation matrix of the torsion-free left D-module $M/t(M)$, i.e., $M/t(M)$ admits the following finite presentation:

$$D^{1\times q'} \xrightarrow{.R'} D^{1\times p} \xrightarrow{\pi'} M/t(M) \longrightarrow 0.$$

Then, we get the following commutative exact diagram of left D-modules:

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & & & t(M) \arrow[d, "i"] & \\
 & D^{1\times q} \arrow[r, ".R"] \arrow[d, ".R''"] & D^{1\times p} \arrow[r, "\pi"] \arrow[d, equal] & M \arrow[r] \arrow[d, "\rho"] & 0 \\
D^{1\times r'} \arrow[r, ".R'_2"] & D^{1\times q'} \arrow[r, ".R'"] & D^{1\times p} \arrow[r, "\pi'"] & M/t(M) \arrow[r] \arrow[d] & 0. \\
 & & & 0 &
\end{tikzcd} \tag{1.43}$$

Since $\ker_D(.Q) = D^{1\times q'} R'$, the exact sequence $D^{1\times q'} \xrightarrow{.R'} D^{1\times p} \xrightarrow{.Q} D^{1\times m}$ holds, which yields:

$$M/t(M) \cong D^{1\times p}/(D^{1\times q'} R') = D^{1\times p}/\ker_D(.Q) = \mathrm{coim}_D(.Q) \cong \mathrm{im}_D(.Q) \cong D^{1\times p}\,Q.$$

Let $\phi : M/t(M) \longrightarrow D^{1\times p}\,Q$ be the left D-isomorphism defined by $\phi(\pi'(\lambda)) = \lambda\,Q$ for all $\lambda \in D^{1\times p}$. It is a well-defined left D-homomorphism since $\pi'(\lambda) = \pi'(\lambda')$ yields $\lambda = \lambda' + \mu'\,R'$ for a certain $\mu' \in D^{1\times q'}$, and thus $\phi(\pi'(\lambda)) = \lambda\,Q = \lambda'\,Q + \mu'\,R'\,Q = \lambda'\,Q = \phi(\pi'(\lambda'))$. Then, we have the following commutative exact diagram of left D-modules

$$\begin{tikzcd}
 & & 0 & \\
D^{1\times q'} \arrow[r, ".R'"] & D^{1\times p} \arrow[r, ".Q"] & D^{1\times p}\,Q \arrow[r] \arrow[u] & 0 \\
D^{1\times q'} \arrow[r, ".R'"] \arrow[u, equal] & D^{1\times p} \arrow[r, "\pi'"] \arrow[u, equal] & M/t(M) \arrow[r] \arrow[u, "\phi"] & 0, \\
 & & 0 \arrow[u] &
\end{tikzcd}$$

and $\phi(M/t(M)) = D^{1\times p}\,Q$, i.e., every element $m' = \pi'(\lambda)$ of $M/t(M)$ is in a one-to-one correspondence with the element $\phi(m') = \lambda\,Q$. Equivalently, every $m' = \pi(\lambda') \in M/t(M)$ is such that $m' = \phi^{-1}(\lambda'\,Q)$. The matrix Q is called a *parametrization* of the torsion-free left D-module $M/t(M)$ since, up to the isomorphism ϕ, the elements of $M/t(M)$ are parametrized by Q.

Example 1.3.4. We consider again Example 1.3.2. We obtain:

$$M/t(M) \cong D^{1\times 3}/(D^{1\times 2}\,R') \cong D^{1\times 3}\,Q = D\,(x_1\,\partial_2 - x_2\,\partial_1) + D\,(x_2\,\partial_3 - x_3\,\partial_2) + D\,(x_3\,\partial_1 - x_1\,\partial_3).$$

Since $M/t(M) \cong D^{1\times 3}\,Q \subseteq D$ and D is a torsion-free left D-module, we find again that $M/t(M)$ is a torsion-free left D-module and, up to isomorphism, $M/t(M)$ is parametrized by Q.

Example 1.3.5. Let $D = \mathbb{Q}[\partial_1, \partial_2, \partial_3]$, $R = (\partial_1 \quad \partial_2 \quad \partial_3) \in D^{1\times 3}$ be the divergence operator in $\mathbb{R}^3$ and $M = D^{1\times 3}/(D\,R)$ the left D-module finitely presented by R and associated with the linear PD system $\ker_{\mathcal{F}}(R.) = \{\eta \in \mathcal{F}^3 \mid R\,\eta = \vec{\nabla}\,.\,\eta = 0\}$, where $\mathcal{F}$ is a D-module (e.g., $\mathcal{F} = C^\infty(\mathbb{R}^3)$). Let us study the module properties of M. Let us first introduce the Auslander transpose $N = D/(R\,D^3)$ of M. Since D is a commutative ring, $N = D/(D^{1\times 3}\,R^T) = \widetilde{N}$, where $\theta = \mathrm{id}_D$. Let now us compute the D-modules $\mathrm{ext}^i_D(N, D)$ for $0 \leq i \leq 3$. We first note that $R^T = R_1$, where R_1 is the matrix introduced in Example 1.2.3. Using Example 1.2.3, the D-module N admits the following finite free resolution

$$0 \longrightarrow D \xrightarrow{.R_3} D^{1\times 3} \xrightarrow{.R_2} D^{1\times 3} \xrightarrow{.R_1} D \xrightarrow{\kappa} N \longrightarrow 0, \tag{1.44}$$

where R_2 is defined by (1.10) and $R_3 = R$. The D-modules $\mathrm{ext}^i_D(N, D)$'s are then the defects of exactness of the following complex of D-modules:

$$0 \longleftarrow D \xleftarrow{.R_3^T} D^{1\times 3} \xleftarrow{.R_2^T} D^{1\times 3} \xleftarrow{.R_1^T} D \longleftarrow 0.$$

Since $R_3^T = R^T = R_1$, $R_2^T = -R_2$ and $R_1^T = R$, using the long exact sequence (1.44), we obtain:

$$\mathrm{ext}^0_D(N, D) = 0 \quad \mathrm{ext}^1_D(N, D) = 0, \quad \mathrm{ext}^2_D(N, D) = 0, \quad \mathrm{ext}^3_D(N, D) = D/(D^{1\times 3}\,R_3^T) = M.$$

Using Theorem 1.3.1, we obtain that M is a reflexive but not projective D-module.

Example 1.3.6. Let us consider the first set of *Maxwell equations* ([51, 84]), namely,

$$\begin{cases} \dfrac{\partial \vec{B}}{\partial t} + \vec{\nabla} \wedge \vec{E} = \vec{0}, \\ \vec{\nabla}\,.\,\vec{B} = 0, \end{cases} \tag{1.45}$$

where $\vec{B}$ (resp., $\vec{E}$) denotes the *magnetic* (resp., *electric*) *field*. For the notations, see Example 1.2.3. Let us consider the commutative polynomial ring $D = \mathbb{Q}[\partial_t, \partial_1, \partial_2, \partial_3]$ of PD operators with rational constant coefficients, the presentation matrix R_1 of (1.45), namely,

$$R_1 = \begin{pmatrix} \partial_t & 0 & 0 & 0 & -\partial_3 & \partial_2 \\ 0 & \partial_t & 0 & \partial_3 & 0 & -\partial_1 \\ 0 & 0 & \partial_t & -\partial_2 & \partial_1 & 0 \\ \partial_1 & \partial_2 & \partial_3 & 0 & 0 & 0 \end{pmatrix} \in D^{4\times 6},$$

and the finitely presented D-module $M = D^{1\times 6}/(D^{1\times 4}\,R_1)$. Using Algorithm 1.2.1, we obtain that the D-module M admits the following finite free resolution

$$0 \longrightarrow D \xrightarrow{.R_2} D^{1\times 4} \xrightarrow{.R_1} D^{1\times 6} \xrightarrow{\pi} M \longrightarrow 0, \tag{1.46}$$

where the matrix $R_2 = (\partial_1 \quad \partial_2 \quad \partial_3 \quad -\partial_t) \in D^{1\times 4}$ defines the compatibility conditions

$$\vec{\nabla}\,.\,\vec{\gamma}_1 - \frac{\partial \gamma_2}{\partial t} = 0 \tag{1.47}$$

of the inhomogeneous linear PD system:

$$\begin{cases} \dfrac{\partial \vec{B}}{\partial t} + \vec{\nabla} \wedge \vec{E} = \vec{\gamma}_1, \\ \vec{\nabla}\,.\,\vec{B} = \gamma_2. \end{cases}$$

Let us study the module properties of M. The formal adjoint $\widetilde{R_1}$ of R_1 can be obtained by contracting (1.45) by a vector and by integrating the result by parts:

$$\begin{array}{c} \vec{C}\,.\left(\dfrac{\partial \vec{B}}{\partial t} + \vec{\nabla} \wedge \vec{E}\right) + G\,\left(\vec{\nabla}\,.\,\vec{B}\right) \\ = -\dfrac{\partial \vec{C}}{\partial t}\,.\,\vec{B} + \left(\vec{\nabla} \wedge \vec{C}\right)\,.\,\vec{E} - \left(\vec{\nabla}\,G\right)\,.\,\vec{B} + \dfrac{\partial}{\partial t}\left(\vec{C}\,.\,\vec{B}\right) + \vec{\nabla}\,.\,\left(-\vec{C} \wedge \vec{E}\right) + \vec{\nabla}\,.\,\left(G\,\vec{B}\right) \end{array} \tag{1.48}$$

The last three terms can be written as $(\partial_t \quad \partial_1 \quad \partial_2 \quad \partial_3)\,.\,\left(\vec{C}\,.\,\vec{B} \quad (G\,\vec{B} - \vec{C} \wedge \vec{E})^T\right)^T$, i.e., under a divergence form in space-time, a fact showing that the adjoint D-module $\widetilde{N} = D^{1\times 4}/(D^{1\times 6}\,\widetilde{R_1})$ is defined by the following linear PD system:

$$\begin{cases} -\dfrac{\partial \vec{C}}{\partial t} - \vec{\nabla}\,G = \vec{0}, \\ \vec{\nabla} \wedge \vec{C} = 0. \end{cases} \tag{1.49}$$

The compatibility conditions of the inhomogeneous linear PD system

$$\begin{cases} -\dfrac{\partial \vec{C}}{\partial t} - \vec{\nabla}\,G = \vec{F}, \\ \vec{\nabla} \wedge \vec{C} = \vec{D}, \end{cases} \tag{1.50}$$

are obtained by eliminating $\vec{C}$ and G from (1.50) and we get

$$\begin{cases} \dfrac{\partial \vec{D}}{\partial t} + \vec{\nabla} \wedge \vec{F} = \vec{0}, \\ \vec{\nabla}\,.\,\vec{D} = 0, \end{cases} \tag{1.51}$$

which has exactly the same form as (1.45). Moreover, we can easily check that the compatibility conditions of the following inhomogeneous PD linear system

$$\begin{cases} \dfrac{\partial \vec{D}}{\partial t} + \vec{\nabla} \wedge \vec{F} = \vec{J}, \\ \vec{\nabla}\,.\,\vec{D} = I, \end{cases}$$

are defined by

$$\vec{\nabla}\,.\,\vec{J} - \frac{\partial I}{\partial t} = 0,$$

which has the same form as (1.47). Hence, we obtain the following finite free resolution of $\widetilde{N}$

$$0 \longleftarrow \widetilde{N} \longleftarrow D^{1\times 4} \xleftarrow{.\widetilde{R_1}} D^{1\times 6} \xleftarrow{.\widetilde{R_0}} D^{1\times 4} \xleftarrow{.\widetilde{R_{-1}}} D \longleftarrow 0,$$

where the matrices $\widetilde{R_1}$, $\widetilde{R_0}$ and $\widetilde{R_{-1}}$ are defined by:

$$\widetilde{R_1} = \begin{pmatrix} -\partial_t & 0 & 0 & -\partial_1 \\ 0 & -\partial_t & 0 & -\partial_2 \\ 0 & 0 & -\partial_t & -\partial_3 \\ 0 & -\partial_3 & \partial_2 & 0 \\ \partial_3 & 0 & -\partial_1 & 0 \\ -\partial_2 & \partial_1 & 0 & 0 \end{pmatrix}, \quad \widetilde{R_0} = R_1, \quad \widetilde{R_{-1}} = R_2.$$

Up to isomorphism, the $\mathrm{ext}^i_D(\widetilde{N}, D)$'s are defined by the defects of exactness of the complex:

$$0 \longrightarrow D^{1\times 4} \xrightarrow{.R_1} D^{1\times 6} \xrightarrow{.R_0} D^{1\times 4} \xrightarrow{.R_{-1}} D \longrightarrow 0.$$

Moreover, we can easily check that

$$\left\{ \begin{array}{l} -\vec{\nabla}\,\xi = \vec{A}, \\ \dfrac{\partial \xi}{\partial t} = V, \end{array} \right. \quad \Rightarrow \quad \left\{ \begin{array}{l} \vec{\nabla} \wedge \vec{A} = \vec{0}, \\ -\dfrac{\partial \vec{A}}{\partial t} - \vec{\nabla}\, V = \vec{0}, \end{array} \right. \quad \left\{ \begin{array}{l} \vec{\nabla} \wedge \vec{A} = \vec{B}, \\ -\dfrac{\partial \vec{A}}{\partial t} - \vec{\nabla}\, V = \vec{E}, \end{array} \right. \quad \Rightarrow \quad (1.45), \qquad (1.52)$$

where "$a \Rightarrow b$" means "b generates the compatibility conditions of a", which proves that we have $\mathrm{ext}^i_D(\widetilde{N}, D) = 0$ for $i = 1, 2$, and the first set Maxwell equations (1.45) generates a reflexive D-module M by 4 of Theorem 1.3.1. Finally, we have $\mathrm{ext}^3_D(\widetilde{N}, D) \cong D/(\partial_1, \partial_2, \partial_3, \partial_t) \neq 0$ since $1 \notin (\partial_1, \partial_2, \partial_3, \partial_t)$, which proves that M is not a projective D-module by 5 of Theorem 1.3.1.

If M is a torsion left module over a domain D, then for every $m \in M$, there exists $d \in D \setminus \{0\}$ such that $d\, m = 0$. If $f \in \hom_D(M, D)$, then $d\, f(m) = f(d\, m) = f(0) = 0$ and, since $f(m) \in D$ and D is a domain, then $f(m) = 0$, i.e., $f = 0$ and $\hom_D(M, D) = 0$. If M is a finitely generated left module over a noetherian domain D, then the converse of this result is true. Indeed, if $\hom_D(M, D) = 0$, then $\hom_D(\hom_D(M, D), D) = 0$ and using 1 and 2 of Theorem 1.3.1, $M = \ker \varepsilon \cong \mathrm{ext}^1_D(N, D) \cong t(M)$, which shows that M is a torsion left D-module.

Corollary 1.3.1 ([16]). *Let M be a finitely generated left module over a noetherian domain D. Then, M is a torsion left D-module iff $\hom_D(M, D) = 0$. Similarly for right D-modules.*

Example 1.3.7. Let us consider again Example 1.3.1, i.e., the $D = \mathbb{Q}(\nu)[\partial_x, \partial_y]$-module $M = D^{1\times 3}/(D^{1\times 3}\, R)$, where the matrix R is defined by (1.28). Since $\ker_D(.R) = 0$, M admits the finite free resolution $0 \longrightarrow D^{1\times 3} \xrightarrow{.R} D^{1\times 3} \xrightarrow{\pi} M \longrightarrow 0$. Applying Theorem 1.1.1 to M, we get $\hom_D(M, D) \cong \ker_D(R.)$. Since D is a commutative ring, $R^T = R$ and $\ker_D(.R) = 0$, $\ker_D(R.) \cong \ker_D(.R^T) = \ker_D(.R) = 0$, i.e., $\hom_D(M, D) = 0$ and we find again that M is a torsion D-module by Corollary 1.3.1 (see Example 1.3.1).

A straightforward consequence of Theorem 1.3.1 is the following corollary.

Corollary 1.3.2 ([16, 89]). *Let D be a noetherian domain with a finite global dimension $\mathrm{gld}(D) = n$. Moreover, let $M = D^{1\times p}/(D^{1\times q}\, R)$ be the left D-module finitely presented by the matrix $R \in D^{q\times p}$. If we set $Q_1 = R$, $p_1 = p$ and $p_0 = q$, then we have the following results:*

1. *M is a torsion-free left D-module iff there exists a matrix $Q_2 \in D^{p_1 \times p_2}$ such that the following exact sequence of left D-modules holds:*

$$D^{1\times p_0} \xrightarrow{.Q_1} D^{1\times p_1} \xrightarrow{.Q_2} D^{1\times p_2}.$$

2. *M is a reflexive left D-module iff there exist two matrices $Q_2 \in D^{p_1\times p_2}$ and $Q_3 \in D^{p_2\times p_3}$ such that the following exact sequence of left D-modules holds:*

$$D^{1\times p_0} \xrightarrow{.Q_1} D^{1\times p_1} \xrightarrow{.Q_2} D^{1\times p_2} \xrightarrow{.Q_3} D^{1\times p_3}.$$

3. *M is a projective left D-module iff there exist n matrices $Q_i \in D^{p_{i-1}\times p_i}$, $i = 2, \dots, n+1$, such that the following long exact sequence of left D-modules holds:*

$$D^{1\times p_0} \xrightarrow{.Q_1} D^{1\times p_1} \xrightarrow{.Q_2} D^{1\times p_2} \xrightarrow{.Q_3} D^{1\times p_3} \xrightarrow{.Q_4} \dots \xrightarrow{.Q_n} D^{1\times p_n} \xrightarrow{.Q_{n+1}} D^{1\times p_{n+1}}. \quad (1.53)$$

Corollary 1.3.2 gives necessary and sufficient conditions for a left D-module M to be embedded into an exact sequence of finite free left D-modules (*inverse problem of the syzygy module computation*).

Let us give a classical characterization of projectivity which is sometimes simpler to test than 5 of Theorem 1.3.1 (for more constructive results on projective modules, see [64]).

Proposition 1.3.2 (see, e.g., [64, 87])**.** *Let $M = D^{1\times p}/(D^{1\times q}\, R)$ be a left D-module finitely presented by a matrix $R \in D^{q\times p}$. Then, the following equivalent conditions hold:*

1. *M is a projective left D-module.*
2. *R admits a* generalized inverse*, namely, there exists a matrix $S \in D^{p\times q}$ such that:*

$$R\, S\, R = R.$$

3. *There exists an idempotent matrix $\Pi \in D^{p\times p}$, namely, $\Pi^2 = \Pi$, presenting M, namely:*

$$M = D^{1\times p}/(D^{1\times p}\, \Pi).$$

Let us explain how to use Algorithm 1.2.3 to compute generalized inverses ([87]).

Algorithm 1.3.3. – **Input:** A noncommutative polynomial ring D for which Buchberger's algorithm terminates for any admissible term order and which admits an involution θ and a left D-module M defined by the following finite free resolution of finite length

$$0 \longrightarrow D^{1\times p_m} \xrightarrow{.R_m} D^{1\times p_{m-1}} \xrightarrow{.R_{m-1}} \dots \xrightarrow{.R_3} D^{1\times p_2} \xrightarrow{.R_2} D^{1\times p_1} \xrightarrow{.R_1} D^{1\times p_0} \xrightarrow{\pi} M \longrightarrow 0,$$

with the notations $R_1 = R$, $p_0 = p$ and $p_1 = q$.

– **Output:** A matrix $S \in D^{p\times q}$ such that $R\, S\, R = R$ if S exists and $\emptyset$ otherwise.

1. Compute a right inverse $S_m \in D^{p_{m-1}\times p_m}$ of R_m if it exists and set $S = S_m$ and $i = m-1$. If no such matrix exists, stop the algorithm with $S = \emptyset$.
2. While $i > 0$, do:
   (a) Compute $F_i = I_{p_i} - \theta(R_{i+1})\, \theta(S_{i+1}) \in D^{p_i \times p_i}$.
   (b) Compute a matrix $L_i \in D^{p_i \times p_{i-1}}$ such that $F_i = L_i\, \theta(R_i)$ if it exists by checking that the normal forms of the rows of F_i are reduced to 0 with respect to a Gröbner basis of $D^{1\times p_{i-1}}\, \theta(R_i)$. If such a matrix does not exist, stop the algorithm with $S = \emptyset$.

(c) Compute $S_i = \theta(L_i) \in D^{p_{i-1} \times p_i}$, set $S = S_i$ and return to 2 with $i \longleftarrow i-1$.

3. Return S.

Example 1.3.8. Let $D = A_1(\mathbb{Q})$ be the first Weyl algebra and $M = D^{1\times 2}/(D^{1\times 2}\, R)$ the left D-module finitely presented by the following matrix:

$$R = \begin{pmatrix} -t^2 & t\,\partial - 1 \\ -(t\,\partial + 2) & \partial^2 \end{pmatrix} \in D^{2\times 2}.$$

Using Algorithms 1.2.2 and 1.2.3, we can check that R does not admit a left and a right inverse. Using Algorithm 1.3.3, let us check whether or not R admits a generalized inverse. Using Algorithm 1.2.1, we first compute a finite free resolution of M:

$$0 \longrightarrow D \xrightarrow{.R_2} D^{1\times 2} \xrightarrow{.R} D^{1\times 2} \xrightarrow{\pi} M \longrightarrow 0, \quad R_2 = (\partial \quad -t).$$

Applying Algorithm 1.2.3 to R_2 with the involution θ of D defined by (1.20), we obtain that R_2 admits the right inverse $S_2 = (t \quad \partial)^T$ and:

$$F_1 = I_2 - \theta(R_2)\,\theta(S_2) = \begin{pmatrix} 2 + t\,\partial & -\partial^2 \\ t^2 & -t\,\partial + 1 \end{pmatrix}.$$

Using a Gröbner basis computation, we can check that $F_1 = L_1\,\theta(R)$, where:

$$L_1 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

The matrix $S = \theta(L_1) = L_1$ then satisfies $S_2\,R_2 + R\,S = I_2$ and, by post-multiplying the last identity by R and using $R_2\,R = 0$, we obtain $R\,S\,R = R$, which proves that S is a generalized inverse of R over D and M is a projective left D-module by 2 of Proposition 1.3.2. Since M admits a finite free resolution, Proposition 1.2.7 proves that M is a stably free left D-module of rank 1. Finally, if $\Pi = S\,R$, then $\Pi^2 = S\,(R\,S\,R) = S\,R = \Pi$ and we clearly have $D^{1\times 2}\,\Pi = D^{1\times 2}\,R$, which proves that $M = D^{1\times 2}/(D^{1\times 2}\,\Pi)$.

If M is a stably free left D-module of rank l, then there exist two non-negative integers r and s such that $M \oplus D^{1\times s} \cong D^{1\times r}$ and $l = r - s$. If $\phi : M \oplus D^{1\times s} \longrightarrow D^{1\times r}$ is a left D-isomorphism and $i_2 : D^{1\times s} \longrightarrow M \oplus D^{1\times s}$ the canonical injection, then the split short exact sequence holds $0 \longrightarrow D^{1\times s} \xrightarrow{\phi\circ i_2} D^{1\times r} \xrightarrow{\gamma} M \longrightarrow 0$. In the standard bases of $D^{1\times s}$ and $D^{1\times r}$, the left D-homomorphism $\phi\circ i_2 : D^{1\times s} \longrightarrow D^{1\times r}$ is defined by $(\phi\circ i_2)(\lambda) = \lambda\,T$ for all $\lambda \in D^{1\times s}$, where $T \in D^{s\times t}$ is a matrix admitting a right inverse (see the comment after Example 1.2.11). Therefore, the above split exact sequence becomes the following one:

$$0 \longrightarrow D^{1\times s} \xrightarrow{.T} D^{1\times r} \xrightarrow{\gamma} M \longrightarrow 0. \tag{1.54}$$

Conversely, if M is defined by the split exact sequence (1.54), then $D^{1\times r} \cong D^{1\times s} \oplus M$, which proves that M is a stably free left D-module of rank $r - s$. The matrix T can be computed by means of Algorithm 1.2.4 if the left D-module M admits a finite free resolution of finite length since we then have $\mathrm{lpd}_D(M) = 0$.

Corollary 1.3.3 ([29, 103]). *If $R \in D^{q\times p}$ has full row rank, i.e., $\ker_D(.R) = 0$, then the following equivalent assertions hold:*

1. *$M = D^{1\times p}/(D^{1\times q}\, R)$ is a stably free left D-module.*
2. *R admits a right inverse, i.e., there exists $S \in D^{p\times q}$ such that $R\,S = I_q$.*
3. *The Auslander transpose right D-module $N = D^q/(R\,D^p) \cong \mathrm{ext}^1_D(M, D)$ of M vanishes.*

Algorithm 1.2.3 can be used to check whether or not a left D-module M finitely presented by a full row rank matrix R is stably free.

Example 1.3.9. In Example 1.2.10, we proved $M = D^{1\times 3}/(D^{1\times 3}\, R) \cong D^{1\times 4}/(D^{1\times 3}\, T_1)$, where $D = A_3(\mathbb{Q})$ and the matrices R and T_1 are respectively defined by (1.22) and (1.24). Moreover, it was shown that the matrix T_1 admitted the left inverse S_1 defined in Example 1.2.12, which proves that M is a stably free left D-module of rank 1 (see also Example 1.2.12).

1.4 Parametrizations of linear systems

> "Pure mathematics and physics are becoming ever more closely connected, though their methods remain different. One may describe the situation by saying that the mathematician plays a game in which he himself invents the rules while the physicist plays a game in which the rules are provided by Nature, but as time goes on it becomes increasingly evident that the rules which the mathematician finds interesting are the same as those which Nature has chosen. It is difficult to predict what the result of all this will be. Possibly, the two subjects will ultimately unify, every branch of pure mathematics then having its physical application, its importance in physics being proportional to its interest in mathematics."
>
> Paul Dirac, *The Relation between Mathematics and Physics*, Proceedings of the Royal Society of Edinburgh, LIX, 1939, p. 22.

Let us show how the parametrizations of a torsion-free left D-module $M = D^{1\times p}/(D^{1\times q}\, R)$ can be used to parametrize the solution space $\ker_{\mathcal{F}}(R.)$. If $L = D^{1\times m}/(D^{1\times p}\, Q)$ is the left D-module finitely presented by the parametrization Q of the torsion-free left D-module M and $\mathcal{F}$ a left D-module, then applying the contravariant functor $\hom_D(\,\cdot\,, \mathcal{F})$ to the truncated finite free resolution (1.39) of L, i.e., $D^{1\times q} \xrightarrow{.R} D^{1\times p} \xrightarrow{.Q} D^{1\times m} \longrightarrow 0$, we obtain the following complex:

$$\mathcal{F}^q \xleftarrow{R.} \mathcal{F}^p \xleftarrow{Q.} \mathcal{F}^m.$$

Therefore, $\mathrm{ext}^1_D(L, \mathcal{F}) \cong \ker_{\mathcal{F}}(R.)/\mathrm{im}_{\mathcal{F}}(Q.)$ defines the obstruction for an element η of the linear system $\ker_{\mathcal{F}}(R.)$, i.e., for $\eta \in \mathcal{F}^p$ satisfying $R\,\eta = 0$, to belong to $\mathrm{im}_{\mathcal{F}}(Q.)$, i.e., to be of the form $\eta = Q\,\xi$ for a certain $\xi \in \mathcal{F}^m$. Hence, $\mathrm{ext}^1_D(L, \mathcal{F})$ defines the obstruction for the the linear system $\ker_{\mathcal{F}}(R.)$ to be parametrized by the matrix Q, i.e., to have the form $\ker_{\mathcal{F}}(R.) = Q\,\mathcal{F}^m$.

Let us study the dual statement of Proposition 1.2.2, i.e., when $\mathrm{ext}^i_D(\,\cdot\,, \mathcal{F}) = 0$ for all $i \geq 1$.

Definition 1.4.1 ([109])**.** A left D-module $\mathcal{F}$ is called *injective* if $\mathrm{ext}^i_D(M, \mathcal{F}) = 0$ for all left D-modules M and all $i \geq 1$.

Example 1.4.1. Example 1.2.6 shows that the $\mathbb{Q}[\partial, \delta]$-module $C^\infty(\mathbb{R})$ is not injective.

The next theorem gives a characterization of injective modules over a noetherian ring.

Theorem 1.4.1 ([109])**.** *(Baer's criterion) Let D be a left noetherian ring. Then, a left D-module $\mathcal{F}$ is injective iff for every $q \geq 1$ and every $R \in D^q$, the linear system $R\,\eta = \zeta$ admits a solution $\eta \in \mathcal{F}$, for all $\zeta \in \mathcal{F}^q$ satisfying the compatibility conditions of $R\,\eta = \zeta$, namely, $R_2\,\zeta = 0$, where $\ker_D(.R) = D^{1\times r}\, R_2$.*

Let us give a few interesting examples of injective modules.

Example 1.4.2. If Ω is an open convex subset of $\mathbb{R}^n$, then the space $C^\infty(\Omega)$ (resp., $\mathcal{D}'(\Omega)$, $\mathcal{S}'(\Omega)$, $\mathcal{A}(\Omega)$, $\mathcal{B}(\Omega)$) of smooth functions (resp., distributions, temperate distributions, real analytic functions, hyperfunctions) on Ω is an injective $D = k[\partial_1, \ldots, \partial_n]$-module, where $k = \mathbb{R}$ or $\mathbb{C}$ ([67, 78]). If $\mathcal{G}$ denotes the set of all functions that are smooth on $\mathbb{R}$ except for a finite number of points, then $\mathcal{G}$ is an injective left $B_1(k)$-module, where $k = \mathbb{R}$ or $\mathbb{C}$ ([121]). Finally, if I is an open interval of $\mathbb{R}$ and $A = \mathbb{C}(t) \cap \mathcal{A}(I)$ the ring of rational functions which are analytic on I, and $D = A\langle \partial \rangle$ the ring of OD operators with coefficients in A, then the left D-module $\mathcal{B}(I)$ of *Sato's hyperfunctions* on I ([46]) is injective ([34]).

Let us now explain the main interest of the concept of injective left D-module in mathematical systems. If M is a left D-module admitting a finite free resolution of the form

$$\ldots \xrightarrow{.R_4} D^{1\times p_3} \xrightarrow{.R_3} D^{1\times p_2} \xrightarrow{.R_2} D^{1\times p_1} \xrightarrow{.R_1} D^{1\times p_0} \xrightarrow{\pi} M \longrightarrow 0,$$

then applying the functor $\hom_D(\,\cdot\,, \mathcal{F})$ to the previous exact sequence and using $\operatorname{ext}^i_D(\,\cdot\,, \mathcal{F}) = 0$ for all $i \geq 1$ and Theorem 1.1.1, we obtain the following exact sequence of abelian groups:

$$\ldots \xleftarrow{R_4.} \mathcal{F}^{p_3} \xleftarrow{.R_3} \mathcal{F}^{p_2} \xleftarrow{R_2.} \mathcal{F}^{p_1} \xleftarrow{R_1.} \mathcal{F}^{p_0} \longleftarrow \hom_D(M, \mathcal{F}) \longleftarrow 0.$$

Hence, $\ker_{\mathcal{F}}(R_{i+1}.) = R_i\, \mathcal{F}^{p_{i-1}}$ for all $i \geq 1$. We say that the contravariant functor $\hom_D(\,\cdot\,, \mathcal{F})$ is *exact*, i.e., transforms exact sequences of left D-modules into exact sequences of abelian groups.

If $\mathcal{F}$ is an injective left D-module, then the results of Corollary 1.3.2 can be dualized to get the following system-theoretic interpretations of the module properties in terms of the existence of a chain of parametrizations.

Corollary 1.4.1 ([16])**.** *Let D be a noetherian domain with a finite global dimension* $\operatorname{gld}(D) = n$*, $R \in D^{q\times p}$, $M = D^{1\times p}/(D^{1\times q}\, R)$ the left D-module finitely presented by R and $\mathcal{F}$ an injective left D-module. If we set $Q_1 = R$, $p_1 = p$ and $p_0 = q$, then we have the following results:*

1. *If M is a torsion-free left D-module, then there exists a matrix $Q_2 \in D^{p_1\times p_2}$ such that the following exact sequence of abelian groups holds*

$$\mathcal{F}^{p_0} \xleftarrow{Q_1.} \mathcal{F}^{p_1} \xleftarrow{Q_2.} \mathcal{F}^{p_2},$$

i.e., $\ker_{\mathcal{F}}(Q_1.) = Q_2\, \mathcal{F}^{p_2}$, and Q_2 is called a parametrization *of the linear system* $\ker_{\mathcal{F}}(Q_1.)$.

2. *If M is a reflexive left D-module, then there exist $Q_2 \in D^{p_1\times p_2}$ and $Q_3 \in D^{p_2\times p_3}$ such that the following exact sequence of abelian groups holds*

$$\mathcal{F}^{p_0} \xleftarrow{Q_1.} \mathcal{F}^{p_1} \xleftarrow{Q_2.} \mathcal{F}^{p_2} \xleftarrow{Q_3.} \mathcal{F}^{p_3},$$

i.e., $\ker_{\mathcal{F}}(Q_1.) = Q_2\, \mathcal{F}^{p_2}$ and $\ker_{\mathcal{F}}(Q_2.) = Q_3\, \mathcal{F}^{p_3}$.

3. *If M is a projective left D-module, then there exist n matrices $Q_i \in D^{p_{i-1}\times p_i}$ for all $i = 2, \ldots, n+1$ such that the following exact sequence of abelian groups holds*

$$\mathcal{F}^{p_0} \xleftarrow{Q_1.} \mathcal{F}^{p_1} \xleftarrow{Q_2.} \mathcal{F}^{p_2} \xleftarrow{Q_3.} \mathcal{F}^{p_3} \xleftarrow{Q_4.} \ldots \xleftarrow{Q_n.} \mathcal{F}^{p_n} \xleftarrow{Q_{n+1}.} \mathcal{F}^{p_{n+1}}, \tag{1.55}$$

i.e., $\ker_{\mathcal{F}}(Q_i.) = Q_{i+1}\, \mathcal{F}^{p_{i+1}}$ for $i = 1, \ldots, n$.

Remark 1.4.1. If the left D-module M is projective and admits a finite free resolution of finite length, then (1.55) does not need the assumption that the left D-module $\mathcal{F}$ is injective, i.e., it holds for all left D-modules $\mathcal{F}$. This result comes from the fact that Algorithm 1.3.3 proves that the long exact sequence (1.53) *splits*, namely, there exist $n+1$ matrices $S_i \in D^{p_i \times p_{i-1}}$ such that:

$$\forall\, i = 1, \ldots, n, \quad S_i\, Q_i + Q_{i+1}\, S_{i+1} = I_{p_i}.$$

Then, the complex (1.55), i.e., $Q_{i+1}\, \mathcal{F}^{p_{i+1}} \subseteq \ker_{\mathcal{F}}(Q_i.)$ for all $i \geq 1$, is exact for all left D-modules $\mathcal{F}$ since $\eta \in \ker_{\mathcal{F}}(Q_i.)$ yields $\eta = S_i\, Q_i\, \eta + Q_{i+1}\, S_{i+1}\, \eta = Q_{i+1}\, (S_{i+1}\, \eta) \in Q_{i+1}\, \mathcal{F}^{p_{i+1}}$, i.e., $\ker_{\mathcal{F}}(Q_i.) = Q_{i+1}\, \mathcal{F}^{p_{i+1}}$ for all $i \geq 1$.

Remark 1.4.2. The converse of the results of Corollary 1.4.1 holds if we assume that $\mathcal{F}$ is a so-called *injective cogenerator* left D-module, namely, if $\mathcal{F}$ is an injective left D-module and a *cogenerator* left D-module, namely, for every left D-module M and every nonzero $m \in M$, there exists $f \in \hom_D(M, \mathcal{F})$ such that $f(m) \neq 0$. If $\mathcal{F}$ is a cogenerator left D-module and $M \neq 0$, then $\ker_{\mathcal{F}}(R.) \cong \hom_D(M, \mathcal{F}) \neq 0$. We can prove that an injective cogenerator left (resp., right) D-module always exists (see, e.g., [109]). For instance, if Ω is an open convex subset of $\mathbb{R}^n$ and $k = \mathbb{R}$ or $\mathbb{C}$, then $C^\infty(\Omega)$ and $\mathcal{D}'(\Omega)$ are two injective cogenerator $D = k[\partial_1, \ldots, \partial_n]$-modules ([78]). Similarly, the left $B_1(k)$-module $\mathcal{G}$ defined in Example 1.4.2 is injective cogenerator ([121]). Roughly speaking, the injective cogenerator condition on $\mathcal{F}$ plays the same role as the condition of algebraically closed base field in classical algebraic geometry.

Example 1.4.3. If Ω is an open convex subset of $\mathbb{R}^3$, $k = \mathbb{R}$ or $\mathbb{C}$, and $\mathcal{F} = C^\infty(\Omega)$, $\mathcal{D}'(\Omega)$, $\mathcal{S}'(\Omega)$, $\mathcal{A}(\Omega)$ or $\mathcal{B}(\Omega)$, then Example 1.4.2 shows that $\mathcal{F}$ is an injective $D = k[\partial_1, \partial_2, \partial_3]$-module. Example 1.3.5 and Corollary 1.4.1 then prove the exactness of the following complex:

$$0 \longleftarrow \mathcal{F} \xleftarrow{R_3.} \mathcal{F}^3 \xleftarrow{R_2.} \mathcal{F}^3 \xleftarrow{R_1.} \mathcal{F} \longleftarrow \hom_D(M, \mathcal{F}) \longleftarrow 0.$$

We find again the well-known result in mathematical physics that the divergence operator in $\mathbb{R}^3$ is parametrized by the curl operator, i.e., $\ker_{\mathcal{F}}(R_3.) = R_2\, \mathcal{F}^3$, and the curl operator is parametrized by the gradient operator, i.e., $\ker_{\mathcal{F}}(R_2.) = R_1\, \mathcal{F}$, when $\mathcal{F} = C^\infty(\Omega)$ and Ω is an open convex subset of $\mathbb{R}^n$.

Example 1.4.4. If Ω is an open convex subset of $\mathbb{R}^4$ and $\mathcal{F}$ is an injective $D = \mathbb{R}[\partial_t, \partial_1, \partial_2, \partial_3]$-module (e.g., $C^\infty(\Omega)$, $\mathcal{D}'(\Omega)$ or $\mathcal{S}'(\Omega)$ by Example 1.4.2), then using Corollary 1.4.1 and Example 1.3.6, the first set of Maxwell equation (1.45) is parametrized by

$$\left\{ \begin{array}{l} \vec{B} = \vec{\nabla} \wedge \vec{A}, \\ \vec{E} = -\dfrac{\partial \vec{A}}{\partial t} - \vec{\nabla}\, V, \end{array} \right. \tag{1.56}$$

where $(\vec{A}, V) \in \mathcal{F}^4$ is called the *quadri-potential* of (1.45), i.e., $\ker_{\mathcal{F}}(R_1.) = R_0\, \mathcal{F}^4$. The quadri-potential $(\vec{A}, V)$ is not uniquely defined since the right-hand side of (1.56) is parametrized by

$$\left\{ \begin{array}{l} \vec{A} = -\vec{\nabla}\, \xi, \\ V = \dfrac{\partial \xi}{\partial t}, \end{array} \right.$$

i.e., $\ker_{\mathcal{F}}(R_0.) = R_{-1}\, \mathcal{F}$ (see (1.52)). Hence, for any $\xi \in \mathcal{F}$, the following *gauge transformation*

$$\vec{A} \longmapsto \vec{A} - \vec{\nabla}\, \xi, \quad V \longmapsto V + \frac{\partial \xi}{\partial t},$$

gives the same fields $\vec{E}$ and $\vec{B}$. This degree of freedom in the choice of the quadri-potential is used in *gauge theory* (e.g., gauge fixing condition, Lorenz gauge, Coulomb gauge) ([51, 83, 84]).

Let us generalize the concept of the *rank* of a finitely generated module M over a noetherian domain D given in 1 and 2 of Definition 1.1.1.

Definition 1.4.2. If D is a noetherian domain and M is a finitely generated left D-module, then the *rank* of M, denoted by $\operatorname{rank}_D(M)$, is the maximal rank of free left D-modules F contained in M, i.e., the maximal rank of free left D-modules F such that the following short exact sequence

$$0 \longrightarrow F \xrightarrow{i} M \xrightarrow{\varpi} T \longrightarrow 0$$

holds, where $T = M/F$ is a torsion left D-module.

Remark 1.4.3. The rank of a finitely generated left module M over a noetherian domain D can also be defined as $\operatorname{rank}_D(M) = \dim_K(K \otimes_D M)$, where K is the *division ring of fractions* of D (*Ore localization*) and $\otimes$ the *tensor product*. For more details, see, e.g., [45, 54, 71].

Let us state an extension of the so-called *Euler-Poincaré characteristic*.

Proposition 1.4.1 ([71, 109]). *If D is a noetherian domain and M', M and M'' are three finitely generated left D-modules, then the short exact sequence $0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$ yields the following equality:*

$$\operatorname{rank}_D(M) = \operatorname{rank}_D(M') + \operatorname{rank}_D(M'').$$

A similar result holds for short exact sequence of right D-module.

Using Proposition 1.4.1 and splicing a long exact sequence into a sequence of short exact sequences, we can show that the alternative sum of the rank of the modules composing this long exact sequence is 0. Hence, if M admits the following finite free resolution of finite length

$$0 \longrightarrow D^{1\times p_m} \xrightarrow{.R_m} D^{1\times p_{m-1}} \xrightarrow{.R_{m-1}} \dots \xrightarrow{.R_3} D^{1\times p_2} \xrightarrow{.R_2} D^{1\times p_1} \xrightarrow{.R_1} D^{1\times p_0} \xrightarrow{\pi} M \longrightarrow 0,$$

then, using Proposition 1.4.1 and 1 of Definition 1.1.1, we obtain:

$$\operatorname{rank}_D(M) = \sum_{i=0}^{m} (-1)^i \operatorname{rank}_D(D^{1\times p_i}) = \sum_{i=0}^{m} (-1)^i \, p_i. \tag{1.57}$$

Example 1.4.5. If M is a stably free left D-module of rank l, then there exist two non-negative integers r and s such that $M \oplus D^{1\times s} \cong D^{1\times r}$ and $l = r - s$. Therefore, the split exact sequence (1.54) holds. Using Proposition 1.4.1 or (1.57), we find again that $\operatorname{rank}_D(M) = r - s$.

Example 1.4.6. Using Example 1.2.3 and the finite free resolution (1.9) of the $D = \mathbb{Q}[\partial_1, \partial_2, \partial_3]$-module $M = D/(D^{1\times 3} R_1)$, where $R_1 = (\partial_1 \quad \partial_2 \quad \partial_3)^T$ is the gradient operator in $\mathbb{R}^3$, we obtain $\operatorname{rank}_D(M) = 1 - 3 + 3 - 1 = 0$. In particular, using Definition 1.4.2, the trivial exact sequence $D^{1\times 0} = 0 \longrightarrow M \longrightarrow T = M \longrightarrow 0$ holds, and thus M is a torsion D-module.

Similarly, if $M_2 = D^{1\times 3}/(D^{1\times 3} R_2)$, where R_2 is the matrix of PD operators defining the curl operator (see (1.10)), then the exact sequence (1.9) yields the following one:

$$0 \longrightarrow D \xrightarrow{.R_3} D^{1\times 3} \xrightarrow{.R_2} D^{1\times 3} \xrightarrow{\pi_2} M_2 \longrightarrow 0.$$

Then, using (1.57), we obtain $\operatorname{rank}_D(M_2) = 3 - 3 + 1 = 1$.

Finally, if $M_3 = D^{1\times 3}/(D\, R_1^T)$ is the D-module defining the divergence operator in $\mathbb{R}^3$, then the exact sequence (1.9) yields the finite presentation $0 \longrightarrow D \xrightarrow{.R_3} D^{1\times 3} \xrightarrow{\pi_3} M_3 \longrightarrow 0$ of M_3, and (1.57) yields $\operatorname{rank}_D(M_3) = 3 - 1 = 2$.

In Example 1.4.3, the divergence operator in $\mathbb{R}^3$ was proved to be parametrized by means of 3 arbitrary functions also called *potentials*. However, Example 1.4.6 shows that the rank of the D-module M_3 associated with the divergence operator is 2. Hence, we can ask whether or not there exists a parametrization of the divergence operator containing only two potentials. This remark leads to the concept of *minimal parametrization* of a torsion-free left D-module.

Definition 1.4.3 ([16, 88])**.** Let $M = D^{1\times p}/(D^{1\times q}\, R)$ be a torsion-free left D-module. A matrix $Q \in D^{p\times m}$ is called a *minimal parametrization* of M if Q is a parametrization of M, i.e., $\ker_D(.Q) = D^{1\times q}\, R$, such that the left D-module $L = D^{1\times m}/(D^{1\times p}\, Q)$ is either zero or torsion.

Equivalently, the matrix Q is a minimal parametrization of the torsion-free left D-module $M = D^{1\times p}/(D^{1\times q}\, R)$ if we have the following exact sequence of left D-modules

$$D^{1\times q} \xrightarrow{.R} D^{1\times p} \xrightarrow{.Q} D^{1\times m} \xrightarrow{\sigma} L \longrightarrow 0, \tag{1.58}$$

where L is either 0 or a torsion left D-module. Let us prove $\mathrm{rank}_D(M) = m$. We first note that

$$M = D^{1\times p}/(D^{1\times q}\, R) = D^{1\times p}/\ker_D(.Q) = \mathrm{coim}_D(.Q) \cong \mathrm{im}_D(.Q) = D^{1\times p}\, Q,$$

and thus $\mathrm{rank}_D(M) = \mathrm{rank}_D(D^{1\times p}\, Q)$. Then, (1.58) yields the short exact sequence

$$0 \longrightarrow D^{1\times p}\, Q \xrightarrow{i} D^{1\times m} \xrightarrow{\sigma} L \longrightarrow 0$$

and Proposition 1.4.1 yields $\mathrm{rank}_D(L) = m - \mathrm{rank}_D(D^{1\times p}\, Q) = m - \mathrm{rank}_D(M)$, and thus, $m = \mathrm{rank}_D(M)$ since $\mathrm{rank}_D(L) = 0$ because L is a torsion left D-module.

Let us state a result which proves the existence of minimal parametrizations.

Theorem 1.4.2 ([16, 88])**.** *Let D be a noetherian domain, $R \in D^{q\times p}$ and $M = D^{1\times p}/(D^{1\times q}\, R)$ a torsion-free left D-module. Then, there exists a minimal parametrization of M.*

Minimal parametrizations of a finitely presented torsion-free left D-module M can be obtained as explained in the following algorithm.

Algorithm 1.4.1.
- **Input:** A noetherian domain D and a matrix $R \in D^{q\times p}$ defining a torsion-free left D-module $M = D^{1\times p}/(D^{1\times q}\, R)$.
- **Output:** A matrix $Q \in D^{p\times m}$ defining a minimal parametrization of M.

1. Compute a matrix $P \in D^{p\times l}$ such that $\ker_D(R.) = P\, D^l$.
2. Select $m = \mathrm{rank}_D(M)$ right D-linearly independent column vectors of P and form a matrix Q with them.

If the ring D admits an involution θ, then, using Algorithm 1.2.1, we can compute a matrix $U \in D^{l\times p}$ such that $\ker_D(.\theta(R)) = D^{1\times l}\, U$, select m left D-linearly independent rows of U and form a matrix $V \in D^{m\times p}$ with them to get the minimal parametrization $Q = \theta(V) \in D^{p\times m}$ of the torsion-free left D-module $M = D^{1\times p}/(D^{1\times q}\, R)$ of rank m. The condition that the rows of V are left D-linearly independent, i.e., $\ker_D(.V) = 0$, can be checked by Algorithm 1.2.1.

Example 1.4.7. We consider again Example 1.4.6. Since the $D = \mathbb{Q}[\partial_1, \partial_2, \partial_3]$-module M_3 defined by the divergence operator in $\mathbb{R}^3$ is reflexive of rank 2 (see Examples 1.3.5 and 1.4.6), we can obtain a minimal parametrization of M_3 by transposing the matrix formed by selecting two D-linearly independent rows of the matrix R_2^T, i.e., by considering two D-linearly independent columns of the parametrization R_2 of M_3. Hence, the matrix Q_1 (resp., Q_2 and Q_3) defined by removing the first (resp., second, third) column of the non-minimal parametrization R_2 of M is a minimal parametrization of M. If Ω is an open convex subset of $\mathbb{R}^3$ and $\mathcal{F} = C^\infty(\Omega)$, $\mathcal{D}'(\Omega)$ or $\mathcal{S}'(\Omega)$, then applying the contravariant exact functor $\hom_D(\,\cdot\,, \mathcal{F})$ to the exact sequence

$$D \xrightarrow{.R_3} D^{1\times 3} \xrightarrow{.Q_i} D^{1\times 2} \xrightarrow{\sigma_i} L_i \longrightarrow 0, \quad i = 1,\, 2,\, 3,$$

we obtain the following exact sequence of D-modules

$$\mathcal{F} \xleftarrow{R_3 .} \mathcal{F}^3 \xleftarrow{Q_i .} \mathcal{F}^2 \longleftarrow \hom_D(L_i, \mathcal{F}) \longleftarrow 0, \quad i = 1,\, 2,\, 3,$$

which proves that the linear PD system $\ker_{\mathcal{F}}(R_3 .) = \{\eta \in \mathcal{F}^3 \mid R_3\, \eta = \vec{\nabla} \,.\, \eta = 0\}$ admits the following minimal parametrizations:

$$\left\{\begin{array}{l} \eta_1 = -\partial_3\, \xi_2 + \partial_2\, \xi_3 \\ \eta_2 = -\partial_1\, \xi_3 \\ \eta_3 = \partial_1\, \xi_2, \end{array}\right. \quad \left\{\begin{array}{l} \eta_1 = \partial_2\, \xi_3, \\ \eta_2 = \partial_3\, \xi_1 - \partial_1\, \xi_3, \\ \eta_3 = -\partial_2\, \xi_1, \end{array}\right. \quad \left\{\begin{array}{l} \eta_1 = -\partial_3\, \xi_2, \\ \eta_2 = \partial_3\, \xi_1, \\ \eta_3 = -\partial_2\, \xi_1 + \partial_1\, \xi_2. \end{array}\right. \quad \forall\, \xi_1,\, \xi_2,\, \xi_3 \in \mathcal{F}.$$

Equivalently, a minimal parametrization of $\ker_{\mathcal{F}}(R_3 .)$ can be obtained by setting one of the arbitrary potentials ξ_i's to 0 in the non-minimal parametrization R_2 of $\ker_{\mathcal{F}}(R_3 .)$ ([88]).

Example 1.4.8. We consider again the first set of Maxwell equations (1.45) (see Example 1.3.6). Applying (1.57) to the finite free resolution of finite length (1.46) of the $D = \mathbb{Q}[\partial_t, \partial_1, \partial_2, \partial_3]$-module $M = D^{1\times 6}/(D^{1\times 4}\, R_1)$, we get $\mathrm{rank}_D(M) = 6-4+1 = 3$. Therefore, the torsion-free D-module M admits minimal parametrizations defined by matrices $Q_i \in D^{6\times 3}$ formed by selecting three D-linearly independent columns of the matrix $R_0 = \widetilde{R_1}$ defined in Example 1.3.6. For instance, we obtain the following four minimal parametrizations of (1.45):

$$\left\{\begin{array}{l} -\partial_t\, A_1 - \partial_1\, V = E_1, \\ -\partial_t\, A_2 - \partial_2\, V = E_2, \\ -\partial_3\, V = E_3, \\ -\partial_3\, A_2 = B_1, \\ \partial_3\, A_1 = B_2, \\ -\partial_2\, A_1 + \partial_1\, A_2 = B_3, \end{array}\right. \quad \left\{\begin{array}{l} -\partial_t\, A_1 - \partial_1\, V = E_1, \\ -\partial_2\, V = E_2, \\ -\partial_t\, A_3 - \partial_3\, V = E_3, \\ \partial_2\, A_3 = B_1, \\ \partial_3\, A_1 - \partial_1\, A_3 = B_2, \\ -\partial_2\, A_1 = B_3, \end{array}\right. \quad \left\{\begin{array}{l} -\partial_1\, V = E_1, \\ -\partial_t\, A_2 - \partial_2\, V = E_2, \\ -\partial_t\, A_3 - \partial_3\, V = E_3, \\ -\partial_3\, A_2 + \partial_2\, A_3 = B_1, \\ -\partial_1\, A_3 = B_2, \\ \partial_1\, A_2 = B_3, \end{array}\right. \quad \left\{\begin{array}{l} -\dfrac{\partial\, \vec{A}}{\partial t} = \vec{E}, \\ \vec{\nabla} \wedge \vec{A} = \vec{B}. \end{array}\right.$$

Example 1.4.9. We quote pages 15-17 of [116]: "The necessary and sufficient conditions, that the six strain components can be derived from three single-valued functions as given in

$$\begin{array}{lll} \varepsilon_x = \dfrac{\partial u}{\partial x}, & \varepsilon_y = \dfrac{\partial v}{\partial y}, & \varepsilon_z = \dfrac{\partial w}{\partial z}, \\ \gamma_{yz} = \dfrac{\partial w}{\partial y} + \dfrac{\partial v}{\partial z}, & \gamma_{zx} = \dfrac{\partial u}{\partial z} + \dfrac{\partial w}{\partial x}, & \gamma_{xy} = \dfrac{\partial v}{\partial x} + \dfrac{\partial u}{\partial y}, \end{array} \tag{1.59}$$

are called *the conditions of compatibility.* It is shown in Refs. 1 through 5, for example, that the conditions of compatibility are given in a matrix form as,

$$[R] = \left[\begin{array}{ccc} R_x & U_z & U_y \\ U_z & R_y & U_x \\ U_y & U_x & R_z \end{array}\right] = 0,$$

$$\begin{aligned}
R_x &= \frac{\partial^2 \varepsilon_z}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial z^2} - \frac{\partial^2 \gamma_{yz}}{\partial y\, \partial z}, \quad & U_x &= -\frac{\partial^2 \varepsilon_x}{\partial y\, \partial z} + \frac{1}{2}\frac{\partial}{\partial x}\left(-\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right),\\
R_y &= \frac{\partial^2 \varepsilon_x}{\partial z^2} + \frac{\partial^2 \varepsilon_z}{\partial x^2} - \frac{\partial^2 \gamma_{zx}}{\partial z\, \partial x}, \quad & U_y &= -\frac{\partial^2 \varepsilon_y}{\partial z\, \partial x} + \frac{1}{2}\frac{\partial}{\partial y}\left(\frac{\partial \gamma_{yz}}{\partial x} - \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right),\\
R_z &= \frac{\partial^2 \varepsilon_y}{\partial x^2} + \frac{\partial^2 \varepsilon_x}{\partial y^2} - \frac{\partial^2 \gamma_{xy}}{\partial x\, \partial y}, \quad & U_z &= -\frac{\partial^2 \varepsilon_z}{\partial x\, \partial y} + \frac{1}{2}\frac{\partial}{\partial z}\left(\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} - \frac{\partial \gamma_{xy}}{\partial z}\right).
\end{aligned} \tag{1.60}$$

[$\cdots$] We know from Eqs. (1.4) that when the body forces are absent, the equations of equilibrium can be written as:

$$\begin{aligned}
&\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = 0,\\
&\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} = 0,\\
&\frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} = 0.
\end{aligned} \tag{1.61}$$

These equations are satisfied identically when stress components are expressed in terms of either Maxwell's stress functions χ_1, χ_2 and χ_3 defined by

$$\begin{aligned}
\sigma_x &= \frac{\partial^2 \chi_3}{\partial y^2} + \frac{\partial^2 \chi_2}{\partial z^2}, \quad & \tau_{yz} &= -\frac{\partial^2 \chi_1}{\partial y\, \partial z},\\
\sigma_y &= \frac{\partial^2 \chi_1}{\partial z^2} + \frac{\partial^2 \chi_3}{\partial x^2}, \quad & \tau_{zx} &= -\frac{\partial^2 \chi_2}{\partial z\, \partial x},\\
\sigma_z &= \frac{\partial^2 \chi_2}{\partial x^2} + \frac{\partial^2 \chi_1}{\partial y^2}, \quad & \tau_{xy} &= -\frac{\partial^2 \chi_3}{\partial x\, \partial y},
\end{aligned} \tag{1.62}$$

or Morera's stress functions ψ_1, ψ_3 and ψ_3 defined by

$$\begin{aligned}
\sigma_x &= \frac{\partial^2 \psi_1}{\partial y\, \partial z}, \quad & \tau_{yz} &= -\frac{1}{2}\frac{\partial}{\partial x}\left(-\frac{\partial \psi_1}{\partial x} + \frac{\partial \psi_2}{\partial y} + \frac{\partial \psi_3}{\partial z}\right),\\
\sigma_y &= \frac{\partial^2 \psi_2}{\partial z\, \partial x}, \quad & \tau_{zx} &= -\frac{1}{2}\frac{\partial}{\partial y}\left(\frac{\partial \psi_1}{\partial x} - \frac{\partial \psi_2}{\partial y} + \frac{\partial \psi_3}{\partial z}\right),\\
\sigma_z &= \frac{\partial^2 \psi_3}{\partial x\, \partial y}, \quad & \tau_{xy} &= -\frac{1}{2}\frac{\partial}{\partial z}\left(\frac{\partial \psi_1}{\partial x} + \frac{\partial \psi_2}{\partial y} - \frac{\partial \psi_3}{\partial z}\right).
\end{aligned} \tag{1.63}$$

It is interesting to note that, when these two kinds of stress functions are combined such that

$$\sigma_x = \frac{\partial^2 \chi_3}{\partial y^2} + \frac{\partial^2 \chi_2}{\partial z^2} - \frac{\partial^2 \psi_1}{\partial y\, \partial z}, \ldots, \quad \tau_{yz} = -\frac{\partial^2 \chi_1}{\partial y\, \partial z} + \frac{1}{2}\frac{\partial}{\partial x}\left(-\frac{\partial \psi_1}{\partial x} + \frac{\partial \psi_2}{\partial y} + \frac{\partial \psi_3}{\partial z}\right), \ldots, \tag{1.64}$$

the expressions (1.60) and (1.64) have similar forms."

Using the concept of minimal parametrizations, let us explain the last sentence and particularly the relation between (1.60), (1.64), Maxwell's stress functions and Morera's stress functions. Let $D = \mathbb{Q}[\partial_x, \partial_y, \partial_z]$ be the ring of PD operators with rational constant coefficients and $N = D^{1\times 3}/(D^{1\times 6}\, P)$ the D-module finitely presented by the matrix P defined by:

$$P = \begin{pmatrix} \partial_x & 0 & 0 \\ 0 & \partial_y & 0 \\ 0 & 0 & \partial_z \\ 0 & \partial_z & \partial_y \\ \partial_z & 0 & \partial_x \\ \partial_y & \partial_x & 0 \end{pmatrix} \in D^{6\times 3}.$$

Using Algorithm 1.2.1, we can check that the D-module N admits the finite free resolution:

$$0 \longrightarrow D^{1\times 3} \xrightarrow{.R} D^{1\times 6} \xrightarrow{.Q} D^{1\times 6} \xrightarrow{.P} D^{1\times 3} \xrightarrow{\pi} N \longrightarrow 0, \tag{1.65}$$

$$Q = \begin{pmatrix} 0 & \partial_z^2 & \partial_y^2 & -\partial_y\,\partial_z & 0 & 0 \\ \partial_z^2 & 0 & \partial_x^2 & 0 & -\partial_x\,\partial_z & 0 \\ \partial_y^2 & \partial_x^2 & 0 & 0 & 0 & -\partial_x\,\partial_y \\ -\partial_y\,\partial_z & 0 & 0 & -\frac{1}{2}\,\partial_x^2 & \frac{1}{2}\,\partial_x\,\partial_y & \frac{1}{2}\,\partial_x\,\partial_z \\ 0 & -\partial_x\,\partial_z & 0 & \frac{1}{2}\,\partial_x\,\partial_y & -\frac{1}{2}\,\partial_y^2 & \frac{1}{2}\,\partial_y\,\partial_z \\ 0 & 0 & -\partial_x\,\partial_y & \frac{1}{2}\,\partial_x\,\partial_z & \frac{1}{2}\,\partial_y\,\partial_z & -\frac{1}{2}\,\partial_z^2 \end{pmatrix} \in D^{6\times 6},$$

$$R = \begin{pmatrix} \partial_x & 0 & 0 & 0 & \partial_z & \partial_y \\ 0 & \partial_y & 0 & \partial_z & 0 & \partial_x \\ 0 & 0 & \partial_z & \partial_y & \partial_x & 0 \end{pmatrix} \in D^{3\times 6}.$$

Let Ω be an open convex subset of $\mathbb{R}^3$ and $\mathcal{F} = C^\infty(\Omega)$ (resp., $\mathcal{D}'(\Omega)$, $\mathcal{S}'(\mathbb{R}^3)$). Applying the exact functor $\hom_D(\,\cdot\,,\mathcal{F})$ to the exact sequence (1.65), we obtain the following exact sequence:

$$0 \longleftarrow \mathcal{F}^3 \xleftarrow{R.} \mathcal{F}^6 \xleftarrow{Q.} \mathcal{F}^6 \xleftarrow{P.} \mathcal{F}^3 \longleftarrow \ker_{\mathcal{F}}(P.) \longleftarrow 0.$$

The PD operator $P. : \mathcal{F}^6 \longrightarrow \mathcal{F}^3$ is defined by (1.59) and corresponds to the *Killing operator* $\xi \longmapsto \frac{1}{2}\,\mathcal{L}_\xi(\omega) = (\varepsilon \quad \frac{1}{2}\,\gamma)$, where $\xi = u\,\partial_x + v\,\partial_y + w\,\partial_z$ is a displacement of $\mathbb{R}^3$ and ω the euclidean metric of $\mathbb{R}^3$, namely, $\omega_{ij} = 1$ for $i = j$ and 0 otherwise $(i, j = 1,\, 2,\, 3)$ ([53, 83, 84]). The PD operator $Q. : \mathcal{F}^6 \longrightarrow \mathcal{F}^4$ defines the compatibility conditions (1.60) of $P. : \mathcal{F}^6 \longrightarrow \mathcal{F}^3$. These compatibility conditions are called the *Saint-Venant compatibility conditions*.

Let us now consider the Auslander transpose D-module $M = D^{1\times 6}/(D^{1\times 3}\, P^T)$ of the D-module $N = D^{1\times 3}/(D^{1\times 6}\, P)$. M is associated with (1.61). Let us study the properties of M. According to Theorem 1.3.1, we need to compute the D-modules $\mathrm{ext}^i_D(N, D)$'s for $i = 1,\, 2,\, 3$, namely, the defects of exactness of the following complex of D-modules:

$$0 \longleftarrow D^{1\times 3} \xleftarrow{.R^T} D^{1\times 6} \xleftarrow{.Q^T} D^{1\times 6} \xleftarrow{.P^T} D^{1\times 3} \longleftarrow 0. \tag{1.66}$$

We can check that $\mathrm{ext}^1_D(N, D) = 0$, $\mathrm{ext}^2_D(N, D) = 0$ and $\mathrm{ext}^3_D(N, D) = D^{1\times 3}/(D^{1\times 6}\, R^T) \neq 0$, which proves that M is a reflexive but not a projective D-module. Moreover, we obtain that Q^T (resp., R^T) defines a parametrization of M (resp., $D^{1\times 6}/(D^{1\times 6}\, Q^T)$). Moreover, applying the exact functor $\hom_D(\,\cdot\,,\mathcal{F})$ to (1.66), we obtain the following exact sequence:

$$0 \longrightarrow \ker_{\mathcal{F}}(R^T.) \longrightarrow \mathcal{F}^3 \xrightarrow{R^T.} \mathcal{F}^6 \xrightarrow{Q^T.} \mathcal{F}^6 \xrightarrow{P^T.} \mathcal{F}^3 \longrightarrow 0.$$

Thus, the PD operator $Q^T. : (\chi \quad \psi) \longmapsto (\sigma \quad \tau)$ is a parametrization of the stress tensor (1.61) by means of 6 arbitrary functions $\chi \in \mathcal{F}^3$ and $\psi \in \mathcal{F}^3$, i.e., $\ker_{\mathcal{F}}(P.^T) = Q^T \mathcal{F}^6$. We point out that this parametrization is exactly the PD operator defined by (1.64).

Finally, since P^T has full row rank, $\mathrm{rank}_D(M) = 6 - 3 = 3$. Hence, (1.64) does not define a minimal parametrization of (1.61). However, according to Theorem 1.4.2, the torsion-free D-module M can be embedded into a free D-module of rank 3, which, by exact duality, yields minimal parametrizations of $\ker_{\mathcal{F}}(P.^T)$ depending on three arbitrary potentials of $\mathcal{F}$. Minimal parametrizations can be obtained by setting 3 of the 6 arbitrary functions $\chi \in \mathcal{F}^3$ and $\psi \in \mathcal{F}^3$ to 0. Taking $\psi = 0$ (resp., $\chi = 0$), we obtain the Maxwell's (resp., Morera's) parametrization (1.62) (resp., (1.63)) of the stress tensor (1.61). These results mathematically explain Washizu's last sentence.

1.5 Quillen-Suslin theorem and Stafford's theorems

Let us now characterize when a finitely presented left D-module M is free.

If $M = D^{1\times p}/(D^{1\times q} R)$ is a free left D-module of rank m, then there exists a left D-isomorphism $\psi : M \longrightarrow D^{1\times m}$, which yields the following exact sequence:

$$D^{1\times q} \xrightarrow{.R} D^{1\times p} \xrightarrow{\psi\circ\pi} D^{1\times m} \longrightarrow 0.$$

Writing the left D-homomorphism $\psi \circ \pi : D^{1\times p} \longrightarrow D^{1\times m}$ in the standard bases of $D^{1\times p}$ and $D^{1\times m}$, there exists a matrix $Q \in D^{p\times m}$ such that the following short exact sequence holds:

$$0 \longrightarrow D^{1\times q} R \longrightarrow D^{1\times p} \xrightarrow{.Q} D^{1\times m} \longrightarrow 0. \qquad (1.67)$$

Since $D^{1\times m}$ is a projective left D-module, this short exact sequence splits by Proposition 1.2.5, i.e., there exists $T \in D^{m\times p}$ such that the left D-homomorphism $.T : D^{1\times m} \longrightarrow D^{1\times p}$ satisfies $(.Q) \circ (.T) = .(T\,Q) = .I_m$, i.e., $T\,Q = I_m$. Hence, the minimal parametrization Q of M admits a left inverse. The converse of this result is clearly true since then $D^{1\times p} Q = D^{1\times m}$ and

$$M = D^{1\times p}/(D^{1\times q} R) = D^{1\times p}/\ker_D(.Q) \cong D^{1\times p} Q = D^{1\times m},$$

which proves that M is a free left D-module of rank m. We obtain the following result.

Proposition 1.5.1 ([29, 103]). *The finitely presented left D-module $M = D^{1\times p}/(D^{1\times q} R)$ is free of rank m iff there exist two matrices $Q \in D^{p\times m}$ and $T \in D^{m\times p}$ satisfying:*

$$\ker_D(.Q) = D^{1\times q} R, \quad T\,Q = I_m.$$

Then, $\{\pi(T_{k\bullet})\}_{k=1,\ldots,m}$ is a basis of the free left D-module M of rank m, where $T_{k\bullet}$ denotes the k^{th} row of the matrix T.

The matrix Q defined in Proposition 1.5.1 is called an *injective parametrization* of the free left D-module M of rank m since, with the notation $z_k = \pi(T_{k\bullet})$ for all $k = 1, \ldots, m$, we have

$$\forall\, j = 1, \ldots, p, \quad y_j = \sum_{k=1}^{m} Q_{jk}\, z_k, \quad \forall\, k = 1, \ldots, m, \quad z_k = \sum_{j=1}^{p} T_{kj}\, y_j,$$

where $y_j = \pi(f_j)$ for $j = 1, \ldots, p$ and $\{f_j\}_{j=1,\ldots,p}$ is the standard basis of $D^{1\times p}$ (see Section 1.1).

Example 1.5.1. We consider again Example 1.2.10. Using Algorithm 1.4.1, we can prove that the left $D = A_3(\mathbb{Q})$-module $M = D^{1\times 3}/(D^{1\times 3}\, R_1)$ admits the following minimal parametrization

$$Q_1 = \begin{pmatrix} -\partial_2 \\ \partial_1 + x_2\,\partial_3 \\ -x_2\,\partial_2 - 2 \end{pmatrix},$$

i.e., $M \cong D^{1\times 3}\, Q_1$ and $L = D/(D^{1\times 3}\, Q_1)$ is a torsion left D-module. Using Algorithm 1.2.2, we can check that the matrix Q_1 admits the left inverse $T_1 = \frac{1}{2}\,(x_2 \quad 0 \quad -1)$, which yields $M \cong D^{1\times 3}\, Q_1 \cong D$ and proves that M is a free left D-module of rank 1. The matrix Q_1 is an injective parametrization of the free left D-module M of rank 1. Finally, if $\{f_j\}_{j=1,2,3}$ is the standard basis of the free left D-module $D^{1\times 3}$, $\pi : D^{1\times 3} \longrightarrow M$ the canonical projection onto M and $\{y_j\}_{j=1,2,3}$ the family of generators of M defined by $y_j = \pi(f_j)$, then the residue class z of T_1 in M, namely, $z = \frac{1}{2}\,(x_2\, y_1 - y_3)$, is a basis of M, and we have:

$$\begin{cases} y_1 = -\partial_2\, z, \\ y_2 = (x_2\,\partial_3 + \partial_1)\, z, \\ y_3 = -(x_2\,\partial_2 + 2)\, z. \end{cases}$$

Corollary 1.5.1 ([29, 103]). *If $M = D^{1\times p}/(D^{1\times q}\, R)$ is a free left D-module of rank m and Q an injective parametrization of M, i.e., $\ker_D(.Q) = D^{1\times q}\, R$, which admits a left inverse $T \in D^{m\times p}$, i.e., $T\, Q = I_m$, then Q defines an injective parametrization of the linear system $\ker_{\mathcal{F}}(R.)$ for all left D-modules $\mathcal{F}$, i.e., $\ker_{\mathcal{F}}(R.) = Q\,\mathcal{F}^m$ and $Q\,\xi = \eta$ implies $\xi = T\,\eta$.*

If R has full row rank, i.e., $\ker_D(.R) = 0$, then the split exact sequence (1.67) becomes

$$0 \longrightarrow D^{1\times q} \underset{\overleftarrow{\;.S\;}}{\xrightarrow{.R}} D^{1\times p} \underset{\overleftarrow{\;.T\;}}{\xrightarrow{.Q}} D^{1\times m} \longrightarrow 0,$$

(see 7 of Definition 1.2.1), i.e., $p = q + m$ by Proposition 1.4.1 and the following identities hold:

$$\begin{pmatrix} R \\ T \end{pmatrix} (S \quad Q) = \begin{pmatrix} I_q & 0 \\ 0 & I_m \end{pmatrix} = I_{q+m}, \quad (S \quad Q) \begin{pmatrix} R \\ T \end{pmatrix} = I_p. \tag{1.68}$$

Definition 1.5.1. Let $\mathrm{GL}_p(D) \triangleq \{U \in D^{p\times p} \mid \exists\, V \in D^{p\times p} : \; U\,V = V\,U = I_p\}$ be the *general linear group* of D of index p. An element $U \in \mathrm{GL}_p(D)$ is called a *unimodular matrix*.

If $\ker_D(.R) = 0$, then the previous result proves that $M = D^{1\times p}/(D^{1\times q}\, R)$ is free of rank $p - q$ iff R can be completed to a unimodular matrix

$$V = \begin{pmatrix} R \\ T \end{pmatrix} \in \mathrm{GL}_p(D),$$

or equivalently, if there exists $U = V^{-1} \in \mathrm{GL}_p(D)$ such that $R\,U = (I_q \quad 0)$. Then, the following commutative exact diagram of left D-modules holds:

$$\begin{tikzcd} 0 \arrow[r] & D^{1\times q} \arrow[r, ".R"] \arrow[d, equal] & D^{1\times p} \arrow[r, ".Q"] \arrow[d, equal] \arrow[l, bend right, ".S"'] & D^{1\times (p-q)} \arrow[r] \arrow[l, bend right, ".T"'] & 0 \\ 0 \arrow[r] & D^{1\times q} \arrow[r, ".R"] & D^{1\times p} \arrow[r, "\pi"] & M \arrow[r] \arrow[u, "\psi"'] & 0. \end{tikzcd}$$

Corollary 1.5.2. *Let $R \in D^{q \times p}$ be a full row rank matrix, i.e., $\ker_D(.R) = 0$. Then, the left D-module $M = D^{1 \times p}/(D^{1 \times q} R)$ is a free left D-module of rank $p-q$ iff there exists $U \in \mathrm{GL}_p(D)$ such that:*

$$R\,U = (I_q \quad 0). \tag{1.69}$$

If we write $U = (S \quad Q)$, where $S \in D^{p \times q}$ and $Q \in D^{p \times (p-q)}$, then

$$\begin{array}{rcl} \psi : M & \longrightarrow & D^{1 \times (p-q)} \\ \pi(\lambda) & \longmapsto & \lambda\, Q, \end{array}$$

is a left D-isomorphism and its inverse $\psi^{-1} : D^{1 \times (p-q)} \longrightarrow M$ is defined by $\psi^{-1}(\mu) = \pi(\mu\, T)$ for all $\mu \in D^{1 \times (p-q)}$, where the matrix $T \in D^{(p-q) \times p}$ is defined by:

$$U^{-1} = \left(\begin{array}{c} R \\ T \end{array}\right) \in D^{p \times p}.$$

Then, $M \cong D^{1 \times p}\, Q = D^{1 \times (p-q)}$ and the matrix Q is an injective parametrization of M. Finally, $\{\pi(T_{k \bullet})\}_{k=1,\ldots,p-q}$ is a basis of the free left D-module M of rank $p-q$.

Contrary to the linear algebra, the computation of bases of a finitely generated free left D-module is generally a difficult issue in module theory. We shortly study particular situations.

If D is a principal left ideal domain (e.g., $D = \mathbb{Z}$, $k[x]$, where k is a field, $K\langle \partial \rangle$, where K is a differential field such that $k(t)$ or $k\{t\}[t^{-1}]$)) and $R \in D^{q \times p}$ a matrix admitting a right inverse, then computing the so-called *Jacobson normal form* of R (generalization of *Smith normal form*) (see, e.g., [25, 43, 49]), we obtain two matrices $F \in \mathrm{GL}_q(D)$ and $G \in \mathrm{GL}_p(D)$ satisfying:

$$R = F\,(I_q \quad 0)\, G.$$

If $m = p - q$, $G = (G_1^T \quad G_2^T)^T$, where $G_1 \in D^{q \times p}$, $G_2 \in D^{m \times p}$ and $G^{-1} = (H_1 \quad H_2)$, where $H_1 \in D^{p \times q}$, $H_2 \in D^{p \times m}$, then we obtain $R = F\, G_1$, i.e., $G_1 = F^{-1}\, R$, and

$$\begin{array}{ll} \left(\begin{array}{c} F^{-1}\, R \\ G_2 \end{array}\right) G^{-1} = I_p & \Rightarrow \left(\begin{array}{cc} F^{-1} & 0 \\ 0 & I_r \end{array}\right) \left(\begin{array}{c} R \\ G_2 \end{array}\right) G^{-1} = I_p, \\ \Rightarrow \left(\begin{array}{c} R \\ G_2 \end{array}\right) G^{-1} \left(\begin{array}{cc} F^{-1} & 0 \\ 0 & I_r \end{array}\right) = I_p & \Rightarrow \left(\begin{array}{c} R \\ G_2 \end{array}\right) (H_1\, F^{-1} \quad H_2) = I_p, \end{array}$$

which shows that we can take $U = (H_1\, F^{-1} \quad H_2) \in \mathrm{GL}_p(D)$ and $T = G_2$ in Corollary 1.5.2. The computation of Jacobson normal forms was implemented in the Jacobson package ([25]).

The results obtained in Section 1.3 can be used to check whether or not a finitely presented $D = k[x_1, \ldots, x_n]$-module, where k is a field, is projective, i.e., free by the Quillen-Suslin theorem (see 2 of Theorem 1.1.2). However, the explicit computation of a basis generally requires tricky methods. Known constructive proofs of the Quillen-Suslin theorem are based on the next theorem which allows one to compute a matrix $U \in \mathrm{GL}_p(D)$ satisfying (1.69) by an induction on the number of the variables x_i's.

Theorem 1.5.1 ([107, 114])**.** *Let k be a field, $D = k[x_1, \ldots, x_n]$ and $R \in D^{q \times p}$ a matrix which admits a right inverse. Then, for every $a_n \in k$, there exists a matrix $U \in \mathrm{GL}_p(D)$ satisfying:*

$$R(x_1, \ldots, x_n)\, U(x_1, \ldots, x_n) = R(x_1, \ldots, x_{n-1}, a_n). \tag{1.70}$$

Hence, for all $a_1, \ldots, a_n \in k$, there exists $V \in \mathrm{GL}_p(D)$ such that:

$$R(x_1, \ldots, x_n)\, V(x_1, \ldots, x_n) = R(a_1, \ldots, a_n).$$

The constructive proofs of Theorem 1.5.1 are rather involved but are generally based on three main steps: Noether's normalization processes, computation of local bases (e.g., Horrock's theorem) and the patching of the local solutions to get a global basis. See, e.g., [30, 55, 61, 62, 64]. See the QuillenSuslin ([29]) package for an implementation of Theorem 1.5.1 and for the computation of bases and injective parametrizations of free $D = k[x_1, \ldots, x_n]$-module.

Let us state an interesting system-theoretic interpretation of Theorem 1.5.1.

Corollary 1.5.3 ([29]). *Let k be a field, $D = k[x_1, \ldots, x_n]$, $R \in D^{q \times p}$ a full row rank matrix, i.e., $\ker_D(.R) = 0$, and $\mathcal{F}$ a D-module. If the D-module $M = D^{1\times p}/(D^{1\times q}\, R)$ is free, then we have the following D-isomorphisms*

$$\begin{array}{rcl} \chi : \ker_{\mathcal{F}}(R(\bullet, a_n).) & \longrightarrow & \ker_{\mathcal{F}}(R(\bullet, x_n).) \\ \zeta & \longmapsto & \eta = U\,\zeta, \end{array} \qquad \begin{array}{rcl} \chi^{-1} : \ker_{\mathcal{F}}(R(\bullet, x_n).) & \longrightarrow & \ker_{\mathcal{F}}(R(\bullet, a_n).) \\ \eta & \longmapsto & \zeta = U^{-1}\,\eta, \end{array}$$

where $a_n \in k$ and $U \in \mathrm{GL}_p(D)$ satisfies (1.70). Hence, the elements of $\ker_{\mathcal{F}}(R(\bullet, x_n).)$ and $\ker_{\mathcal{F}}(R(\bullet, a_n).)$ are in a one-to-one correspondence. More generally, the linear system $\ker_{\mathcal{F}}(R.)$ is D-isomorphic to the linear system obtained by setting all but one variables x_i's to $a_i \in k$ (e.g., $a_i = 0$) (resp., all the variables x_i's to $a_i \in k$) in the presentation matrix R.

Example 1.5.2. Let us consider the following linear OD time-delay system ([73]):

$$\left\{ \begin{array}{l} \dot{y}_1(t) - y_1(t-h) + 2\,y_1(t) + 2\,y_2(t) - 2\,u(t-h) = 0, \\ \dot{y}_1(t) + \dot{y}_2(t) - \dot{u}(t-h) - u(t) = 0. \end{array} \right. \tag{1.71}$$

Let $D = \mathbb{Q}[\partial, \delta]$ be the commutative polynomial ring of OD time-delay operators with rational constant coefficients (i.e., $\partial\, y(t) = \dot{y}(t)$, $\delta\, y(t) = y(t-h)$) and the presentation matrix of (1.71):

$$R = \left(\begin{array}{ccc} \partial - \delta + 2 & 2 & -2\,\delta \\ \partial & \partial & -\partial\,\delta - 1 \end{array} \right) \in D^{2\times 3}. \tag{1.72}$$

Using Algorithm 1.2.2, we can check that R admits a right inverse S defined by:

$$S = \frac{1}{2} \left(\begin{array}{cc} 0 & 0 \\ \partial\,\delta + 2 & -2\,\delta \\ \partial & -2 \end{array} \right) \in D^{3\times 2}.$$

Then, using 2 of Corollary 1.3.3, the D-module $M = D^{1\times 3}/(D^{1\times 2}\, R)$ is projective, i.e., free by the Quillen-Suslin theorem (see 2 of Theorem 1.1.2). Applying Theorem 1.5.1 to the matrix R and $a_2 = 0$, the linear OD time-delay system (1.71) is equivalent to the linear OD system obtained by setting δ to 0 in the presentation matrix R, i.e., (1.71) is equivalent to:

$$\left\{ \begin{array}{l} \dot{z}_1(t) + 2\,z_1(t) + 2\,z_2(t) = 0, \\ \dot{z}_1(t) + \dot{z}_2(t) - v(t) = 0. \end{array} \right. \tag{1.73}$$

Applying a constructive version of the Quillen-Suslin theorem to R, we obtain that a transformation which bijectively maps the trajectories of (1.71) to the ones of (1.73) is defined by:

$$\left\{ \begin{array}{l} y_1(t) = z_1(t), \\ y_2(t) = \frac{1}{2}\,(\dot{z}_1(t-2\,h) + z_1(t-h)) + z_2(t) + v(t-h), \\ u(t) = \frac{1}{2}\,\dot{z}_1(t-h) + v(t), \end{array} \right. \Leftrightarrow \left\{ \begin{array}{l} z_1(t) = y_1(t), \\ z_2(t) = -\frac{1}{2}\,y_1(t-h) + y_2(t) - u(t-h), \\ v(t) = -\frac{1}{2}\,\dot{y}_1(t-h) + u(t). \end{array} \right. \tag{1.74}$$

Applying again Theorem 1.5.1 to (1.73), we obtain that the linear OD system (1.73) is equivalent to the purely algebraic system obtained by setting to δ and ∂ to 0 in R, namely:

$$\left\{\begin{array}{l} 2\,x_1(t)+2\,x_2(t)=0,\\ -w(t)=0. \end{array}\right. \tag{1.75}$$

Applying a constructive version of the Quillen-Suslin theorem to $R(\partial,0)$, we get that a transformation which bijectively maps the trajectories of (1.73) to the ones of (1.75) is defined by:

$$\left\{\begin{array}{l} z_1(t)=x_1(t),\\ z_2(t)=x_2(t)-\frac{1}{2}\,\dot{x}_1(t),\\ v(t)=w(t)-\frac{1}{2}\,\ddot{x}_1(t)+\dot{x}_1(t)+\dot{x}_2(t), \end{array}\right. \Leftrightarrow \left\{\begin{array}{l} x_1(t)=z_1(t),\\ x_2(t)=z_2(t)+\frac{1}{2}\,\dot{z}_1(t),\\ w(t)=v(t)+\dot{z}_1(t)+\dot{z}_2(t). \end{array}\right. \tag{1.76}$$

Composing the invertible transformations (1.74) and (1.76), we obtain a one-to-one correspondence between the solutions of (1.71) and (1.75). The solutions of (1.71) (resp., (1.73)) are parametrized by means of (1.74) (resp., (1.76)), where z_1, z_2 and v (resp., x_1, x_2 and w) satisfy (1.73) (resp., (1.75)). Solving the algebraic system (1.75), we obtain $x_2=-x_1$ and $w=0$ and substituting these values into the first system of (1.76) and then the result into the first transformation of (1.74), we find that the injective parametrization of (1.71) is defined by:

$$\forall\, x_1\in\mathcal{F},\quad \left\{\begin{array}{l} y_1(t)=x_1(t),\\ y_2(t)=-\frac{1}{2}\,(\ddot{x}_1(t-h)-\dot{x}_1(t-2\,h)+\dot{x}_1(t)-x_1(t-h)+2\,x_1(t)),\\ u(t)=\frac{1}{2}\,(\dot{x}_1(t-h)-\ddot{x}_1(t)). \end{array}\right.$$

An OD time-delay system $\ker_{\mathcal{F}}(R.)$ which defines a free D-module $M=D^{1\times q}/(D^{1\times q}\,R)$ is called *flat* and a basis of M corresponds to a *flat output* of $\ker_{\mathcal{F}}(R.)$ ([33, 73]). For more details, see 6 of the forthcoming Definition 1.6.1. The *motion planning problem* in control theory can easily be achieved for flat systems (see, e.g., [32, 73, 74, 75, 76, 79]). Corollary 1.5.3 shows that every linear OD time-delay system is equivalent to the flat (i.e., controllable) linear OD system obtained by setting all the time-delay operators to 1, i.e., to the corresponding controllable linear OD system without time-delays ([29]).

The following generalization of Quillen-Suslin theorem was proposed by Lin and Bose in [60].

Lin-Bose's problems: Let k be a field, $D=k[x_1,\ldots,x_n]$, $R\in D^{q\times p}$ a full row rank matrix such that the ideal of D generated by the $q\times q$-minors $\{m_i\}_{i=1,\ldots,r}$ of R satisfies $(m_1,\ldots,m_r)=(d)$, where d is the greatest common divisor of the $q\times q$ minors of the matrix R.

1. Find two matrices $R'\in D^{q\times p}$ and $R''\in D^{q\times q}$ such that $R=R''\,R'$, $\det(R'')=d$ and $R'\in D^{q\times p}$ admits a right inverse.
2. Find a matrix $T\in D^{(p-q)\times p}$ such that $\det((R^T\quad T^T)^T)=d$.

1 and 2 were shown to be equivalent in [60].

In [29], we proved that the output of the next algorithm returns the matrix R' defined in 1 and R'' can then be found by means of a factorization using Gröbner basis techniques.

Algorithm 1.5.1. – **Input:** A commutative polynomial ring $D=k[x_1,\ldots,x_n]$ over a computable field k, a full row rank matrix $R\in D^{q\times p}$ and the finitely presented D-module $M=D^{1\times p}/(D^{1\times q}\,R)$ such that $M/t(M)$ is a free D-module.
– **Output:** A full row rank matrix $R'\in D^{q\times p}$ satisfying $M/t(M)=D^{1\times p}/(D^{1\times q}\,R')$.

1. Using Algorithm 1.3.1, compute a matrix $Q\in D^{q'\times p}$ satisfying $M/t(M)\cong D^{1\times p}/(D^{1\times q'}\,Q)$.

2. Using Algorithm 1.2.1, compute a matrix $Q_2 \in D^{q_2' \times q'}$ satisfying $\ker_D(.Q) = D^{1\times q_2'}\, Q_2$.
3. If $\ker_D(.Q) = 0$, i.e., if Q has full row rank, then stop the algorithm with $R' = Q$.
4. Using a constructive version of the Quillen-Suslin, compute a basis of the free D-module $L = D^{1\times q'}/(D^{1\times q_2'}\, Q_2) \cong D^{1\times q'}\, Q$. We obtain a full row rank matrix $B \in D^{q\times q'}$ such that $\{\pi_2(B_{i\bullet})\}_{i=1,\ldots,q}$ is a basis of free D-module L, where $\pi_2 : D^{1\times q'} \longrightarrow L$ is the canonical projection onto L and $B_{i\bullet}$ is the i^{th} row of B.
5. Return the full row rank matrix $R' = B\, Q \in D^{q\times p}$.

Algorithm 1.5.1 was implemented in the QuillenSuslin package ([29]).

The next algorithm solves the second problem as explained in [29].

Algorithm 1.5.2. – **Input:** A commutative polynomial ring $D = k[x_1, \ldots, x_n]$ over a computable field k, a full row rank matrix $R \in D^{q\times p}$ such that the ideal of D generated by the $q \times q$-minors $\{m_i\}_{i=1,\ldots,r}$ of R satisfies $(m_1, \ldots, m_r) = (d)$, where d is the greatest common divisor of the $q \times q$-minors of R.

– **Output:** A matrix $T \in D^{(p-q)\times p}$ satisfying $\det((R^T \quad T^T)^T) = d$.

1. Using Algorithm 1.3.1, compute a matrix $Q \in D^{q'\times p}$ satisfying $M/t(M) \cong D^{1\times p}/(D^{1\times q'}\, Q)$.
2. Using a constructive version of the Quillen-Suslin, compute a basis of the free D-module $M/t(M) = D^{1\times p}/(D^{1\times q'}\, Q)$. We obtain a full row rank matrix $T \in D^{(p-q)\times p}$ such that $\{\pi'(T_{i\bullet})\}_{i=1,\ldots,p-q}$ is a basis of the free D-module $M/t(M)$, where $\pi' : D^{1\times p} \longrightarrow M/t(M)$ is the canonical projection onto $M/t(M)$ and $T_{i\bullet}$ is the i^{th} row of T.
3. Return the matrix $U = (R^T \quad T^T)^T$.

Algorithm 1.5.2 is also implemented in the QuillenSuslin package ([29]).

Example 1.5.3. Let us consider the OD time-delay model of a flexible rod with a force applied on one end studied in [74]:

$$\begin{cases} \dot{y}_1(t) - \dot{y}_2(t-1) - u(t) = 0, \\ 2\, \dot{y}_1(t-1) - \dot{y}_2(t) - \dot{y}_2(t-2) = 0. \end{cases}$$

Let $D = \mathbb{Q}[\partial, \delta]$ be the commutative polynomial ring of OD time-delay operators (i.e., $\partial\, y(t) = \dot{y}(t)$, $\delta\, y(t) = y(t-h)$) and the D-module $M = D^{1\times 3}/(D^{1\times 2}\, R)$ finitely presented by:

$$R = \begin{pmatrix} \partial & -\partial\,\delta & -1 \\ 2\,\partial\,\delta & -\partial\,(1+\delta^2) & 0 \end{pmatrix} \in D^{2\times 3}. \tag{1.77}$$

Using Algorithm 1.3.1, we obtain that the matrix Q is defined by

$$Q = \begin{pmatrix} -2\,\delta & \delta^2+1 & 0 \\ -\partial & \partial\,\delta & 1 \\ \partial\,\delta & -\partial & \delta \end{pmatrix} \in D^{3\times 3}$$

satisfies $M/t(M) = D^{1\times 3}/(D^{1\times 3}\, Q)$ and $t(M) \cong (D^{1\times 3}\, Q)/(D^{1\times 2}\, R)$. Reducing the rows of Q with respect to $D^{1\times 2}\, R$, we obtain that the only non-trivial torsion element of M is defined by

$$m = -2\,\delta\, y_1 + (\delta^2+1)\, y_2, \quad \partial\, m = 0,$$

where y_1, y_2 and y_3 are the residue classes of the standard basis $\{f_j\}_{j=1,2,3}$ of $D^{1\times 3}$ in M. Hence, we get $t(M) = D\, m$. Using Algorithm 1.2.1, the full row rank matrix $Q_2 = (\partial \quad -\delta \quad 1)$ satisfies

$\ker_D(.Q) = D\,Q_2$. Then, we have to compute a basis of the free D-module $L = D^{1\times 3}/(D\,Q_2)$. Using a constructive version of the Quillen-Suslin theorem (e.g., the QuillenSuslin package), we obtain the split exact sequence

$$0 \longrightarrow D \underset{\xleftarrow{.S_2}}{\xrightarrow{.Q_2}} D^{1\times 2} \underset{\xleftarrow{.B}}{\xrightarrow{.P_2}} D \longrightarrow 0$$

with the following notations:

$$S_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad P_2 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \\ \partial & \delta \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

In particular, we have $D^{1\times 3}\,Q = D^{1\times 2}\,R'$, where the full row rank matrix R' is defined by:

$$R' = B\,Q = \begin{pmatrix} 2\,\delta & -\delta^2 - 1 & 0 \\ -\partial & \partial\,\delta & 1 \end{pmatrix}.$$

Then, we get the factorization $R = R''\,R'$, where the matrix $R'' \in D^{2\times 2}$ is defined by:

$$R'' = \begin{pmatrix} 0 & -1 \\ \partial & 0 \end{pmatrix}.$$

We can check that $\det R'' = \partial$, where ∂ is the greatest common divisor of the 2×2 minors of R (i.e., $\mathrm{ann}_D(m)$), which solves the first problem. Let us now study the second one. We have to compute a basis of the free D-module $M/t(M)$ defined by the following finite free resolution:

$$0 \longrightarrow D \xrightarrow{.Q_2} D^{1\times 3} \xrightarrow{.Q} D^{1\times 3} \xrightarrow{\pi'} M/t(M) \longrightarrow 0,$$

Using Algorithm 1.2.4, $M/t(M)$ admits the following shortest free resolution

$$0 \longrightarrow D^{1\times 3} \xrightarrow{.Q'} D^{1\times 4} \xrightarrow{\pi' \oplus 0} M/t(M) \longrightarrow 0,$$

where $Q' = (Q^T \quad S_2^T)^T$. Now, applying a constructive version of the Quillen-Suslin theorem to the matrix Q' using, e.g., the QuillenSuslin package, we find that a basis of the free D-module $M/t(M)$ is defined by $(\pi' \oplus 0)(T')$, where $T' = (1 \quad \delta/2 \quad 0 \quad 0)$. Hence, if T is the matrix defined by the first three entries of T', then $U = (R^T \quad T^T)^T$ satisfies $\det U = \partial$.

For more applications of the Quillen-Suslin theorem in mathematical systems theory (e.g., computation of (weakly) doubly coprime factorizations of rational transfer matrices ([96])), see [29] and the QuillenSuslin package. See also Chapters 3 and 4.

Let us now explain the main ideas of the constructive proof of Stafford's theorem (see 3 of Theorem 1.1.2) obtained in [103] and implemented in the Stafford package ([103]).

We first need to introduce a well-known result due to Stafford ([110]) on the efficient generation of ideals of the Weyl algebras $A_n(k)$ and $B_n(k)$, when k is a field of characteristic 0.

Theorem 1.5.2 ([110]). *Let k be a field of characteristic 0 and $D = A_n(k)$ or $B_n(k)$. If $v_1, v_2, v_3 \in D$, then there exist a_1, a_2 of D such that the left ideal $I = D\,v_1 + D\,v_2 + D\,v_3$ of D can be generated as follows:*

$$I = D\,(v_1 + a_1\,v_3) + D\,(v_2 + a_2\,v_3).$$

Thus, every left ideal of D can be generated by two elements of D. Similarly for right ideals.

Example 1.5.4. Let us consider $D = A_3(\mathbb{Q})$ and the left ideal $I = D\,(\partial_1 + x_3) + D\,\partial_2 + D\,\partial_3$ of D. We can check the identity $(\partial_2 + \partial_3)\,(\partial_1 + x_3) - (\partial_1 + x_3)\,(\partial_2 + \partial_3) = 1$, which yields

$$\left\{ \begin{array}{l} \partial_2 = (\partial_2\,(\partial_2 + \partial_3))\,(\partial_1 + x_3) - (\partial_2\,(\partial_1 + x_3))\,(\partial_2 + \partial_3), \\ \partial_3 = (\partial_3\,(\partial_2 + \partial_3))\,(\partial_1 + x_3) - (\partial_3\,(\partial_1 + x_3))\,(\partial_2 + \partial_3), \end{array} \right.$$

and shows that I can be generated by $\partial_1 + x_3$ and $\partial_2 + \partial_3$, i.e., $I = D\,(\partial_1 + x_3) + D\,(\partial_2 + \partial_3)$.

If we now consider the left ideal $J = D\,\partial_1 + D\,\partial_2 + D\,\partial_3$ of D defined by the gradient operator in $\mathbb{R}^3$, then J satisfies $J = D\,\partial_1 + D\,(\partial_2 + x_1\,\partial_3)$ since we have:

$$\left\{ \begin{array}{l} \partial_2 = x_1\,(\partial_2 + x_1\,\partial_3)\,\partial_1 + (-x_1\,\partial_1 + 1)\,(\partial_2 + x_1\,\partial_3), \\ \partial_3 = -(\partial_2 + x_1\,\partial_3)\,\partial_1 + \partial_1\,(\partial_2 + x_1\,\partial_3). \end{array} \right.$$

Two constructive algorithms of Theorem 1.5.2 were developed by Hillebrand and Schmale on the one hand ([40]) and by Leykin on the other hand ([57]). Both strategies were implemented in the Stafford package ([103]).

Let us introduce a few more definitions.

Definition 1.5.2.
1. The *elementary group* $\mathrm{EL}_m(D)$ is the subgroup of $\mathrm{GL}_m(D)$ generated by all matrices of the form $I_m + r\,E_{ij}$, where $r \in D$, $i \neq j$ and E_{ij} is the matrix defined by 1 at the position (i, j) and 0 else.
2. A column vector $v = (v_1 \; \ldots \; v_m)^T \in D^m$ is called *unimodular* if it admits a left inverse, i.e., if there exists $w = (w_1 \; \ldots \; w_m) \in D^{1\times m}$ such that $w\,v = \sum_{i=1}^m w_i\,v_i = 1$. The set of unimodular column vectors of D^m is denoted by $\mathrm{U}_m(D)$.

Example 1.5.5. Upper and lower triangular matrices with 1 on the diagonal belong to the elementary group ([71]).

Proposition 1.5.2 ([103]). *If k is a field of characteristic 0, $D = A_n(k)$ or $B_n(k)$, $m \geq 3$ and $v \in \mathrm{U}_m(D)$, then there exists a matrix $E \in \mathrm{E}_m(D)$ satisfying:*

$$E\,v = (1\;0\;\ldots\;0)^T.$$

More precisely, let $a_1, a_2 \in D$ be such that $D\,v_1 + D\,v_2 + D\,v_m = D\,(v_1 + a_1\,v_m) + D\,(v_2 + a_2\,v_m)$, and $d_1, \ldots, d_{m-1} \in D$ satisfying the Bézout identity $\sum_{i=1}^{m-1} d_i\,v_i' = 1$, with the following notations:

$$v_1' = v_1 + a_1\,v_m, \quad v_2' = v_2 + a_2\,v_m, \quad \forall\, i \geq 3, \quad v_i' = v_i.$$

If $v_i'' = (v_1' - 1 - v_m)\,d_i$, for all $i = 1, \ldots, m-1$, and

$$E_1 = \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 & a_1 \\ 0 & 1 & 0 & \ldots & 0 & a_2 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & 0 \\ 0 & 0 & 0 & \ldots & 0 & 1 \end{pmatrix} \in \mathrm{E}_m(D), \quad E_2 = \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 1 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & 0 \\ v''_1 & v''_2 & v''_3 & \ldots & v''_{m-1} & 1 \end{pmatrix} \in \mathrm{E}_m(D),$$

$$E_3 = \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 & -1 \\ 0 & 1 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & 0 \\ 0 & 0 & 0 & \ldots & 0 & 1 \end{pmatrix} \in \mathrm{E}_m(D), \quad E_4 = \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 & 0 \\ -v'_2 & 1 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -v'_{m-1} & 0 & 0 & \ldots & 1 & 0 \\ -v'_1+1 & 0 & 0 & \ldots & 0 & 1 \end{pmatrix} \in \mathrm{E}_m(D),$$

then we have $(E_4\, E_3\, E_2\, E_1)\, v = (1\; 0\; \ldots\; 0)^T$.

Proposition 1.5.2 can be used to handle Gaussian elimination on the columns of the formal adjoint $\widetilde{R}$ of R. For more details, see [103]. We have the following algorithm ([103]).

Algorithm 1.5.3. – **Input:** $D = A_n(k)$ or $B_n(k)$, where k is a computable field of characteristic 0, a matrix $R \in D^{q\times p}$ which admits a right inverse $S \in D^{p\times q}$ and $p - q \geq 2$.

– **Output:** Two matrices $Q \in D^{p\times(p-q)}$ and $T \in D^{(p-q)\times p}$ satisfying $T\,Q = I_{p-q}$ and $\{\pi(T_{i\bullet})\}_{i=1,\ldots,p-q}$ is a basis of the free left D-module $M = D^{1\times p}/(D^{1\times q}\,R)$ of rank $p-q$, where $T_{i\bullet}$ is the i^{th} row of T and $\pi : D^{1\times p} \longrightarrow M$ the canonical projection onto M.

1. Compute $\widetilde{R} = \theta(R) \in D^{p\times q}$ and set $i = 1$, $V = \widetilde{R}$ and $U = I_p$.
2. Denote by $V_i \in D^{p-i+1}$ the column vector formed by taking the last $p-i+1$ elements of the i^{th} column of V.
3. Applying Proposition 1.5.2 to V_i, compute $F_i \in \mathrm{E}_{p-i+1}(D)$ such that $F_i\, V_i = (1\; 0\; \ldots\; 0)^T$.
4. Define the matrix $G_i = \begin{pmatrix} I_{i-1} & 0 \\ 0 & F_i \end{pmatrix} \in \mathrm{E}_p(D)$ where $G_1 = F_1$.
5. If $i < q$, then return to 2 with $V \longleftarrow G_i\, V$, $U \longleftarrow G_i\, U$ and $i \longleftarrow i+1$.
6. Define $G = G_q\, U$ and the matrix P formed by selecting the last $p-q$ rows of G.
7. Define $Q = \theta(P) \in D^{p\times(p-q)}$ and compute a left inverse $T \in D^{(p-q)\times p}$ of Q.

Algorithm 1.5.3 is inspired by a result of [63, 64] obtained for commutative rings.

Example 1.5.6. Let us consider the first Weyl algebra $D = A_1(\mathbb{Q})$, the following matrices

$$R = \begin{pmatrix} 0 & \partial & 0 & -1 \\ \partial & 0 & -t & 0 \end{pmatrix} \in D^{2\times 4}, \quad S = \begin{pmatrix} 0 & 0 & 0 & -1 \\ t & 0 & \partial & 0 \end{pmatrix}^T \in D^{4\times 2}, \tag{1.78}$$

and the left D-module $M = D^{1\times 4}/(D^{1\times 2}\,R)$. We can easily check that S is a right inverse of R. Therefore, M is a stably free left D-module and $\mathrm{rank}_D(M) = 2$. 3 of Theorem 1.1.2 then shows that M is free left D-module of rank 2. Using Algorithm 1.5.3, let us compute a basis of M.

Let us first compute the formal adjoint $\widetilde{R}$ of R:

$$\widetilde{R} = \begin{pmatrix} 0 & -\partial & 0 & -1 \\ -\partial & 0 & -t & 0 \end{pmatrix}^T \in D^{4\times 2}.$$

Let us now consider the first column $v_1 = (0 \quad -\partial \quad 0 \quad -1)^T$ of $\widetilde{R}$. The vector $v_1' = (1 \quad -\partial \quad 0)^T$ is unimodular since $w' = (1 \quad 0 \quad 0)$ is a left inverse of v_1'. Then, we can take $a_1 = -1$, $a_2 = 0$, $d_1 = 1$, $d_2 = 0$ in Proposition 1.5.2. Applying Proposition 1.5.2 to v_1, we get:

$$E_1 = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix},$$
$$E_3 = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad E_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \partial & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

In particular, we have:

$$G_1 = E_4\, E_3\, E_2\, E_1 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & -\partial \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \in \mathrm{E}_4(D), \quad G_1\, \widetilde{R} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & -t \\ 0 & -\partial \end{pmatrix}.$$

Let us now consider the subcolumn $v_2 = (0 \quad -t \quad -\partial)^T$ of the second column of matrix $G_1\, \widetilde{R}$. We can easily check that $v_2' = (-\partial \quad -t)^T$ has a left inverse defined by $w_2' = (t \quad -\partial)$. Hence, taking $a_1 = 1$, $a_2 = 0$, $d_1 = -t$ and $d_2 = -\partial$ in Proposition 1.5.2, we get:

$$E_1' = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \; E_2' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -t & \partial & 1 \end{pmatrix}, \; E_3' = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \; E_4' = \begin{pmatrix} 1 & 0 & 0 \\ t & 1 & 0 \\ \partial+1 & 0 & 1 \end{pmatrix}.$$

Then, we have:

$$F_2 = E_4'\, E_3'\, E_2'\, E_1' = \begin{pmatrix} 1+t & -\partial & t \\ t\,(t+1) & -t\,\partial+1 & t^2 \\ t\,\partial+\partial+2 & -\partial^2 & t\,\partial+2 \end{pmatrix} \in \mathrm{E}_4(D), \quad F_2\, v_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Let us define the following matrices:

$$G_2 = \begin{pmatrix} 1 & 0 \\ 0 & F_2 \end{pmatrix}, \quad G = G_2\, G_1 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ t & t+1 & -\partial & -(t+1)\,\partial \\ t^2 & t\,(t+1) & -t\,\partial+1 & -t\,(t+1)\,\partial \\ t\,\partial+2 & (t+1)\,\partial+2 & -\partial^2 & -\left((t+1)\,\partial+2\right)\,\partial \end{pmatrix}.$$

Then, we have $G\,\widetilde{R} = (I_2 \quad 0)^T$. Finally, if we consider the following two matrices

$$Q = \left(\begin{array}{cc} t^2 & -t\,\partial + 1 \\ t^2 + t & -(t+1)\,\partial + 1 \\ t\,\partial + 2 & -\partial^2 \\ t\,(t+1)\,\partial + 2\,t + 1 & -(t+1)\,\partial^2 \end{array} \right), \quad T = \left(\begin{array}{cccc} 0 & 0 & t+1 & -1 \\ t+1 & -t & 0 & 0 \end{array} \right), \tag{1.79}$$

where Q is formed by taking the last two columns of the formal adjoint $\widetilde{G}$ of G and T is a left inverse of Q, then a basis of M is defined by $\{\pi((0 \quad 0 \quad t+1 \quad -1)),\ \pi((t+1 \quad -t \quad 0 \quad 0))\}$, where $\pi : D^{1\times 4} \longrightarrow M$ is the canonical projection onto M.

Let us consider a left D-module $\mathcal{F}$ (e.g., $\mathcal{F} = C^\infty(\mathbb{R}_+)$) and the linear system $\ker_{\mathcal{F}}(R.)$. Using the matrix Q defined by (1.79), we obtain the following parametrization of $\ker_{\mathcal{F}}(R.)$:

$$\left\{ \begin{array}{l} \dot{x}_2(t) - u_2(t) = 0, \\ \dot{x}_1(t) - t\,u_1(t) = 0, \end{array} \right. \quad \Leftrightarrow \quad \left\{ \begin{array}{l} x_1(t) = t^2\,\xi_1(t) - t\,\dot{\xi}_2(t) + \xi_2(t), \\ x_2(t) = t\,(t+1)\,\xi_1(t) - (t+1)\,\dot{\xi}_2(t) + \xi_2(t), \\ u_1(t) = t\,\dot{\xi}_1(t) + 2\,\xi_1(t) - \ddot{\xi}_2(t), \\ u_2(t) = t\,(t+1)\,\dot{\xi}_1(t) + (2\,t+1)\,\xi_1(t) - (1+t)\,\ddot{\xi}_2(t). \end{array} \right. \tag{1.80}$$

Finally, since $T\,Q = I_2$, (1.80) is an injective parametrization of $\ker_{\mathcal{F}}(R.)$, i.e.:

$$\left(\begin{array}{c} \xi_1(t) \\ \xi_2(t) \end{array} \right) = T \left(\begin{array}{c} x_1(t) \\ x_2(t) \\ u_1(t) \\ u_2(t) \end{array} \right) \quad \Leftrightarrow \quad \left\{ \begin{array}{l} \xi_1(t) = (t+1)\,u_1(t) - u_2(t), \\ \xi_2(t) = (t+1)\,x_1(t) - t\,x_2(t). \end{array} \right. \tag{1.81}$$

In control theory, the OD system $\ker_{\mathcal{F}}(R.)$ is called a *differentially flat system* and the basis (1.81) of the free left D-module M corresponds to a (non-singular) flat output of $\ker_{\mathcal{F}}(R.)$ ([32]).

For PD examples, see [103] and the library of examples of the STAFFORD package.

Let us now study the case of stably free left D-module of rank 1.

Proposition 1.5.3 ([103]). *Let $D = A_n(\mathbb{Q})$ or $B_n(\mathbb{Q})$ be a Weyl algebra and M a stably free left D-module of rank 1. If $Q \in D^p$ is a minimal parametrization of M, then M is a free left D-module of rank 1 iff the left ideal $D^{1\times p}\,Q$ of D admits a reduced Gröbner defined by only one element P of D. If so, then the column vector $Q\,P^{-1} \in D^p$ defines an injective parametrization of the free left D-module M and the residue class in M of a left inverse $T \in D^{1\times p}$ of the column vector $Q\,P^{-1}$ defines a basis of the free left D-module M of rank 1.*

Example 1.5.7. Let us consider the time-varying linear OD system $\dot{x}(t) = t^k\,u(t)$, $k \in \mathbb{N}$, and let $D = A_1(\mathbb{Q})$, $R_k = (\partial \quad -t^k)$ and $M_k = D^{1\times 2}/(D\,R_k)$. Since R_k has full row rank, according to Corollary 1.3.3, M_k is stably free iff the left D-module $\widetilde{N} = D^{1\times q}/(D^{1\times p}\,\widetilde{R_k})$, where $\widetilde{R_k} = \left(-\partial \quad -t^k\right)^T$ is the formal adjoint of R_k, is reduced to zero:

$$\left\{ \begin{array}{l} -\dot{\lambda} = 0, \\ -t^k\,\lambda = 0, \end{array} \right. \Rightarrow\ t^k\,\dot{\lambda} + k\,t^{k-1}\,\lambda = 0\ \Rightarrow\ t^{k-1}\,\lambda = 0\ \Rightarrow\ \ldots\ \Rightarrow\ \lambda = 0\ \Rightarrow\ \widetilde{N} = 0.$$

Hence, for all $k \in \mathbb{N}$, the left D-module M_k is stably free of rank 1. Using Algorithm 1.4.1, the torsion-free left D-module M_k admits the following minimal parametrization:

$$0 \longrightarrow D \xrightarrow{.R_k} D^{1\times 2} \xrightarrow{.Q_k} D \xrightarrow{\sigma_k} D/(D^{1\times 2}\,Q_k) \longrightarrow 0, \quad Q_k = \left(\begin{array}{c} t^{k+1} \\ t\,\partial + k + 1 \end{array} \right).$$

Therefore, we get $M_k = D^{1\times 2}/(D\, R_k) \cong D^{1\times 2}\, Q_k = D\, t^{k+1} + D\, (t\, \partial + k + 1)$, showing that M_k is isomorphic to the left ideal I_k of D generated by t^{k+1} and $t\, \partial + k + 1$. Since D is a domain, we obtain that M_k is a free left D-module iff I_k is a principal left ideal of D. However, we can prove that t^{k+1} and $t\, \partial + k + 1$ form a reduced Gröbner basis of I_k iff $k \geq 1$, and thus M_k is a stably free but not free left D-module when $k \geq 1$ (see also [103]). For $k = 0$, we have $I_0 = D\, t + D\, (t\, \partial + 1) = D\, t$ because $\partial\, t = t\, \partial + 1$. Hence, I_0 is a principal left ideal of D and thus M_0 is a free left D-module. Using $(t\, \partial + 1)\, t^{-1} = \partial$, we obtain that an injective parametrization of M_0 is defined by $Q_0\, t^{-1} = (1 \quad \partial)^T$. To conclude, the time-varying linear OD system $\dot{x}(t) = t^k\, u(t)$ is flat in a neighbourhood of $t = 0$ iff $k = 0$ and, for $k \geq 1$, the singularity at $t = 0$ of its injective parametrization $u(t) = t^{-k}\, \dot{x}(t)$ over $B_1(\mathbb{Q})$ cannot be removed.

If M is a stably free left $D = A_1(k)$-module M which is not free, then $B_1(k) \otimes_D M$ is a torsion-free left $B_1(k)$-module, and thus a free one by 1 of Theorem 1.1.2 ($B_1(k)$ is a principal left ideal domain). Hence, the obstructions for M to be free come from irremovable singularities.

The next proposition generalizes a remark of Malgrange ([69]) on a result of [70].

Proposition 1.5.4 ([103])**.** *Let $R \in D^{q\times p}$ be a matrix which admits a right inverse $S \in D^{p\times q}$, the stably free left D-module $M = D^{1\times p}/(D^{1\times q}\, R)$ and $\pi : D^{1\times p} \longrightarrow M$ the canonical projection. If $R' = (R \quad 0) \in D^{q\times(p+q)}$, then we have the following split exact sequence*

$$0 \longrightarrow D^{1\times q} \underset{\xleftarrow{.S'}}{\xrightarrow{.R'}} D^{1\times(p+q)} \underset{\xleftarrow{.T'}}{\xrightarrow{.Q'}} D^{1\times p} \longrightarrow 0, \tag{1.82}$$

with the notations:

$$S' = \begin{pmatrix} S \\ -I_q \end{pmatrix} \in D^{(p+q)\times q}, \quad T' = (I_p \quad S) \in D^{p\times(p+q)}, \quad Q' = \begin{pmatrix} I_p - S\, R \\ R \end{pmatrix} \in D^{(p+q)\times p}.$$

Hence, we have $M \oplus D^{1\times q} \cong D^{1\times p}$, i.e., $M \oplus D^{1\times q}$ is a free left D-module with a basis defined by $\{\kappa(T'_{i\bullet})\}_{i=1,\ldots,p}$, where $T'_{i\bullet}$ denotes the i^{th} row of T' and $\kappa : D^{1\times(p+q)} \longrightarrow D^{1\times(p+q)}/(D^{1\times q}\, R')$ is the left D-homomorphism defined by $\kappa((\lambda_1 \ \ldots \ \lambda_{p+q})) = (\pi(\lambda_1 \ \ldots \ \lambda_p) \ \ \lambda_{p+1} \ \ldots \ \lambda_{p+q})$.

We have the following system-theoretic interpretation of Proposition 1.5.4.

Corollary 1.5.4 ([103])**.** *With the notations of Proposition 1.5.4, if $\mathcal{F}$ is a left D-module, then:*

$$\ker_{\mathcal{F}}(R'.) = \left\{(\eta^T \quad \zeta^T)^T \in \mathcal{F}^{(p+q)} \mid R\, \eta = 0\right\} = Q'\, \mathcal{F}^p.$$

Moreover, for all $\zeta \in \mathcal{F}^q$ and all $\eta \in \ker_{\mathcal{F}}(R.)$, there exists a unique $\xi = \eta + S\, \zeta \in \mathcal{F}^p$ such that:

$$\begin{cases} \eta = (I_p - S\, R)\, \xi, \\ \zeta = R\, \xi. \end{cases}$$

Finally, the linear system $\ker_{\mathcal{F}}(R'.) = \ker_{\mathcal{F}}(R.) \oplus \mathcal{F}^q$ projects onto the linear system $\ker_{\mathcal{F}}(R.)$ under the canonical projection $\rho : \mathcal{F}^{(p+q)} \longrightarrow \mathcal{F}^p$ defined by $\rho((\eta^T \quad \zeta^T)^T) = \eta^T$.

If $D = A_1(k)$, then Corollary 1.5.4 can be interpreted as the *blowing-up* of the singularities: embedding the linear system $\ker_{\mathcal{F}}(R.) \subseteq \mathcal{F}^p$ into a larger space $\mathcal{F}^{(p+q)}$, the new system $\ker_{\mathcal{F}}(R'.) = \ker_{\mathcal{F}}(R.) \oplus \mathcal{F}^q$ has no more singularities, i.e., it is free. The situation is similar to the blowing-up in algebraic geometry ([27]).

Example 1.5.8. Let us consider again Example 1.5.7 and particularly the stably free but not free left $D = A_1(\mathbb{Q})$-module $M = D^{1\times 2}/(D\,R)$ of rank 1, the matrix $R = (\partial \quad -t)$, which is associated with the time-varying linear system $\dot{x}(t) - t\,u(t) = 0$. If $\mathcal{F}$ is a left D-module, then using Algorithm 1.3.1, we obtain the following parametrization of $\ker_{\mathcal{F}}(R.)$:

$$\forall\, \xi_1,\, \xi_2 \in \mathcal{F}, \quad \left\{ \begin{array}{l} x(t) = -t\,\dot{\xi}_1(t) + \xi_1(t) + t^2\,\xi_2(t), \\ u(t) = -\ddot{\xi}_1(t) + t\,\dot{\xi}_2(t) + 2\,\xi_2(t). \end{array} \right.$$

But, we cannot express the potentials ξ_1 and ξ_2 in terms of x, u and their derivatives, i.e., this parametrization is not injective since it would imply that $\mathrm{rank}_D(M)$ is 2 whereas it is 1.

The left $B_1(\mathbb{Q})$-module $B_1(\mathbb{Q}) \otimes_D M \cong B_1(\mathbb{Q})^{1\times 2}/(B_1(\mathbb{Q})\,R)$ is free and the corresponding system $\ker_{\mathcal{G}}(R.)$, where $\mathcal{G}$ is any left $B_1(\mathbb{Q})$-module, admits the injective parametrization:

$$\forall\, \psi \in \mathcal{G}, \quad \left\{ \begin{array}{l} x(t) = \psi(t), \\ u(t) = \dfrac{1}{t}\,\dot{\psi}(t). \end{array} \right.$$

The fact that M is not a free left D-module means that we cannot remove the singularity at $t = 0$. However, if $R' = (R \quad 0) \in D^{1\times 3}$, Corollary 1.5.4 shows that the linear OD system

$$\ker_{\mathcal{F}}(R'.) = \{(x \quad u \quad v)^T \in \mathcal{F}^3 \mid \dot{x}(t) - t\,u(t) = 0\}$$

admits an injective parametrization defined by the matrix $Q' = ((I_2 - S\,R)^T \quad R^T)^T \in D^{3\times 2}$

$$\left\{ \begin{array}{l} \dot{x}(t) - t\,u(t) = 0, \\ v \in \mathcal{F}, \end{array} \right. \quad \Leftrightarrow \quad \left\{ \begin{array}{l} x(t) = -t\,\dot{\varphi}_1(t) + \varphi_1(t) + t^2\,\varphi_2(t), \\ u(t) = -\ddot{\varphi}_1(t) + t\,\dot{\varphi}_2(t) + 2\,\varphi_2(t), \\ v(t) = \dot{\varphi}_1(t) - t\,\varphi_2(t), \end{array} \right.$$

where $\varphi_1(t) = x(t) + t\,v(t)$ and $\varphi_2(t) = u(t) + \dot{v}(t)$. Hence, Corollary 1.5.4 allows us to "blow up" the singularity at $t = 0$ and the non-flat linear system $\ker_{\mathcal{F}}(R.)$ is the projection of the flat behaviour $\ker_{\mathcal{F}}(R'.) = \ker_{\mathcal{F}}(R.) \oplus \mathcal{F} \cong \mathcal{F}^2$ under the following canonical projection:

$$\begin{array}{rcl} \rho : \mathcal{F}^3 & \longrightarrow & \mathcal{F}^2 \\ (x \quad u \quad v)^T & \longmapsto & (x \quad u)^T. \end{array}$$

Let us now show how the previous results on Stafford's theorem can be extended to the case of $D = A\langle\partial\rangle$, where $A = k[\![t]\!]$ and k is a field of characteristic 0, or $k\{t\}$ and $k = \mathbb{R}$ or $\mathbb{C}$.

Theorem 1.5.3 ([106]). *If $A = k[\![t]\!]$ and k is a field of characteristic 0, or $A = k\{t\}$ and $k = \mathbb{R}$ or $\mathbb{C}$, $D = A\langle\partial\rangle$ and v_1, v_2, $v_3 \in D$, then there exist two elements a_1, $a_2 \in D$ such that the left ideal $I = D\,v_1 + D\,v_2 + D\,v_3$ can also be generated as follows:*

$$I = D\,(v_1 + a_1\,v_3) + D\,(v_2 + a_2\,v_3).$$

In particular, every left ideal of the ring $D = A\langle\partial\rangle$, where A is defined in Theorem 1.5.3, can be generated by two elements ([35, 66]).

Proposition 1.5.2 can also be extended to the ring of OD operators $D = A\langle\partial\rangle$ for the differential rings A introduced in Theorem 1.5.3. Let us give an explicit example.

Example 1.5.9. If $D = \mathbb{R}\{t\}\langle\partial\rangle$ and $v = (0 \quad \sin(t) \quad \partial)^T$, then v admits a left inverse since bringing the OD linear system $v\,y = 0$, i.e.,

$$\begin{cases} \Phi_1 = 0, \\ \Phi_2 = \sin(t)\,y, \\ \Phi_3 = \partial\,y, \end{cases}$$

to formal integrability, we successively obtain $\partial\,\Phi_2 - \sin(t)\,\Phi_3 = \cos(t)\,y$ and:

$$\sin(t)\,\Phi_2 + \cos(t)\,(\partial\,\Phi_2 - \sin(t)\,\Phi_3) = y.$$

Hence, the column vector v admits the left inverse $w = (0 \quad \cos(t)\,\partial + \sin(t) \quad -\cos(t)\,\sin(t))$ and $D\,0 + D\,\sin(t) + D\,\partial = D$. Taking $a_1 = 1$ and $a_2 = 0$, we get $I = D\,(0 + \partial) + D\,\sin(t)$ and thus $v_1' = \partial$, $v_2' = \sin(t)$, $d_1 = -\cos(t)\,\sin(t)$, $d_2 = \cos(t)\,\partial + \sin(t)$, $v_1'' = \cos(t)\,\sin(t)$, $v_2'' = -\cos(t)\,\partial - \sin(t)$. Then, we can define the following four matrices:

$$E_1 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \cos(t)\,\sin(t) & -\cos(t)\,\partial - \sin(t) & 1 \end{pmatrix},$$

$$E_3 = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad E_4 = \begin{pmatrix} 1 & 0 & 0 \\ -\sin(t) & 1 & 0 \\ -\partial + 1 & 0 & 1 \end{pmatrix}.$$

Hence, the matrix $E = E_4\,E_3\,E_2\,E_1 \in \mathrm{E}_3(D)$ defined by

$$E = \begin{pmatrix} 1 - \cos(t)\,\sin(t) & \cos(t)\,\partial + \sin(t) & -\cos(t)\,\sin(t) \\ \sin(t)\,(\cos(t)\,\sin(t) - 1) & -\cos(t)\,(\sin(t)\,\partial - \cos(t)) & \sin^2(t)\,\cos(t) \\ (\cos(t)\,\sin(t) - 1)\,\partial + 2\,\cos^2(t) & -\cos(t)\,(\partial^2 + 1) & \cos(t)\,(\sin(t)\,\partial + 2\,\cos(t)) \end{pmatrix},$$

satisfies $E\,v = (1 \;\; 0 \;\; 0)^T$. Finally, we check that $E^{-1} \in D^{3\times 3}$, i.e., $E \in \mathrm{GL}_3(D)$, since:

$$E^{-1} = \begin{pmatrix} 0 & -\cos(t)\,\partial - \sin(t) & \cos(t)\,\sin(t) \\ \sin(t) & 1 & 0 \\ \partial & \cos(t)\,\partial + \sin(t) & 1 - \cos(t)\,\sin(t) \end{pmatrix}.$$

Theorem 1.5.4 ([106]). *If $A = k[\![t]\!]$ and k is a field of characteristic 0, or $A = k\{t\}$ and $k = \mathbb{R}$ or $\mathbb{C}$, then every finitely generated projective left $D = A\langle\partial\rangle$-module M of rank at least 2 is free.*

We can use Algorithm 1.5.3 to compute bases of free left $A\langle\partial\rangle$-module M of rank at least 2.

Example 1.5.10. Let us consider the following time-varying linear OD system:

$$\begin{cases} \dot{x}_2(t) - u_2(t) = 0, \\ \dot{x}_1(t) - \sin(t)\,u_1(t) = 0. \end{cases} \tag{1.83}$$

We can easily check that (1.83) admits the following injective parametrization:

$$\begin{cases} u_1(t) = \dfrac{\dot{x}_1(t)}{\sin(t)}, \\ u_2(t) = \dot{x}_2(t). \end{cases} \tag{1.84}$$

This injective parametrization is singular at $t = 0$ since $\sin(t)^{-1} = t^{-1} + t/6 + O(t^2)$ and thus $\{x_1, x_2\}$ is a basis of the free $E = \mathbb{R}\{t\}[t^{-1}]\langle\partial\rangle$-module $L = E^{1\times 4}/(E^{1\times 2}\, R)$ of rank 2, where R is the system matrix of (1.83) defined by:

$$R = \begin{pmatrix} 0 & \partial & 0 & -1 \\ \partial & 0 & -\sin(t) & 0 \end{pmatrix}.$$

This result can be checked again by means of the computation of a Jacobson normal form of the matrix R over the principal left ideal domain $E = \mathbb{R}\{t\}[t^{-1}]\langle\partial\rangle$ (see, e.g., [25]), namely,

$$\begin{pmatrix} -1 & 0 \\ 0 & -\sin(t)^{-1} \end{pmatrix} R \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & \sin(t)^{-1}\,\partial \\ 1 & 0 & \partial & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \tag{1.85}$$

and by considering the last two columns of third matrix of (1.85).

Let us now study whether or not (1.83) admits a non-singular injective parametrization at $t = 0$. To do that, we consider the left $D = \mathbb{R}\{t\}\langle\partial\rangle$-module $M = D^{1\times 4}/(D^{1\times 2}\, R)$ finitely presented by R. Since R has full row rank, $\mathrm{rank}_D(M) = 2$, and R admits the right inverse:

$$S = \begin{pmatrix} 0 & \cos(t)\,\sin(t) \\ 0 & 0 \\ 0 & \cos(t)\,\partial - 2\,\sin(t) \\ -1 & 0 \end{pmatrix} \in D^{4\times 2}.$$

Therefore, the left D-module M is stably free of rank 2 and thus free by Theorem 1.5.4. Let us compute a basis of M. Applying Algorithm 1.5.3 to the first column $\widetilde{R}_{\bullet 1} = (0 \quad -\partial \quad 0 \quad -1)^T$ of the formal adjoint $\widetilde{R}$ of R, i.e.,

$$\widetilde{R} = \begin{pmatrix} 0 & -\partial \\ -\partial & 0 \\ 0 & -\sin(t) \\ -1 & 0 \end{pmatrix} \in D^{4\times 2},$$

we can take $a_1 = 1$ and $a_2 = 0$ since $D\,0 + D\,(-\partial) + D\,(-1) = D\,(0-1) + D\,(-\partial)$, i.e., $v_1' = -1$, $v_2' = -\partial$ and $v_3' = 0$, and thus $d_1 = -1$, $d_2 = 0$, $d_3 = 0$, $v_1'' = 1$, $v_2'' = 0$ and $v_3'' = 0$, and we define the following matrices:

$$E_1 = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \; E_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}, \; E_3 = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \; E_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \partial & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{pmatrix}.$$

Then, we have:

$$F_1 = E_4\,E_3\,E_2\,E_1 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & -\partial \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \in \mathrm{E}_4(D), \quad F_1\,\widetilde{R} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & -\sin(t) \\ 0 & -\partial \end{pmatrix}.$$

We now apply again Algorithm 1.5.3 to the vector $(0 \quad -\sin(t) \quad -\partial)^T$. Up to a sign, this was already done in Example 1.5.9. Therefore, we obtain that the matrix $F_2 = -E$ satisfies $F_2\,(0 \quad -\sin(t) \quad -\partial)^T = (1 \quad 0 \quad 0)^T$, where E is defined in Example 1.5.9. Then, the matrix $G_2 = \mathrm{diag}(1, F_2)\,F_1 \in \mathrm{E}_4(D)$ is such that $G_2\,\widetilde{R} = (I_2^T \quad 0^T)^T$ and thus $R\,V = (I_2 \quad 0)$, where the matrix $V = \widetilde{G_2} \in \mathrm{E}_4(D)$ is defined by:

$$V = \begin{pmatrix} 0 & \cos(t)\,\sin(t) & -\cos(t)\,\sin^2(t) & \cos(t)\,\sin(t)\,\partial - 1 \\ 0 & -1+\cos(t)\,\sin(t) & -\sin(t)\,(\cos(t)\,\sin(t)-1) & (\cos(t)\,\sin(t)-1)\,\partial - 1 \\ 0 & \cos(t)\,\partial - 2\,\sin(t) & -\cos(t)\,\sin(t)\,\partial - 3\,\cos^2(t)+1 & (\cos(t)\,\partial - 2\,\sin(t))\,\partial \\ -1 & (\cos(t)\,\sin(t)-1)\,\partial + 2\,\cos^2(t) - 1 & (\sin(t)-\cos(t)+\cos^3(t))\,\partial - 3\,\cos^2(t)\,\sin(t)+\sin(t)+\cos(t) & (\cos(t)\,\sin(t)-1)\,\partial^2 - 2\,\sin^2(t)\,\partial \end{pmatrix}.$$

The matrix Q formed by the last two columns of V defines an injective parametrization of (1.83), i.e., $\ker_{\mathcal{F}}(R.) = Q\,\mathcal{F}^2$ for all left D-modules $\mathcal{F}$, and $T\,Q = I_2$, where the matrix $T \in D^{2\times 4}$ is defined by $V^{-1} = (R^T \quad T^T)^T$ where:

$$V^{-1} = \begin{pmatrix} 0 & \partial & 0 & -1 \\ \partial & 0 & -\sin(t) & 0 \\ \cos(t)\,\partial - 2\,\sin(t) & -\cos(t)\,\partial + 2\,\sin(t) & -1 & 0 \\ -1+\cos(t)\,\sin(t) & -\cos(t)\,\sin(t) & 0 & 0 \end{pmatrix} \in D^{4\times 4}.$$

Finally, the residue classes of the two rows $T_{1\bullet}$ and $T_{2\bullet}$ of T in the D-module M, namely

$$\begin{cases} z_1 = (\cos(t)\,\partial - 2\,\sin(t))\,x_1 + (-\cos(t)\,\partial + 2\,\sin(t))\,x_2 - u_1, \\ z_2 = (-1+\cos(t)\,\sin(t))\,x_1 - \cos(t)\,\sin(t)\,x_2, \end{cases} \tag{1.86}$$

defines a basis $\{z_1, z_2\}$ of the free left D-module M of rank 2 and:

$$(x_1 \quad x_2 \quad u_1 \quad u_2)^T = Q\,(z_1 \quad z_2)^T.$$

Within the language of control theory ([32]), the linear system (1.83) is differentially flat and it admits the non-singular flat outputs (1.86) and the non-singular injective parametrization $\ker_{\mathcal{F}}(R.) = Q\,\mathcal{F}^2$.

The computation of bases of free modules will play an important role in Chapters 3 and 4.

1.6 Applications to multidimensional control theory

We shortly explain recent applications of the constructive algebraic analysis to control theory. For more results, see [16, 17, 25, 29, 31, 33, 73, 78, 80, 88, 92, 103, 104, 117, 120, 121].

Definition 1.6.1. Let D be a noetherian domain, $R \in D^{q\times p}$, $\mathcal{F}$ an injective cogenerator left D-module and $\ker_{\mathcal{F}}(R.) = \{\eta \in \mathcal{F}^p \mid R\,\eta = 0\}$ the linear system defined by R and $\mathcal{F}$. Then, we have the following definitions:

1. An *observable* of $\ker_{\mathcal{F}}(R.)$ is a left D-linear combination of the system variables η_i's. An observable $\psi(\eta)$ is *autonomous* if it satisfies a non-trivial equation over D, namely, $d\,\psi(\eta) = 0$ for some $d \in D \setminus \{0\}$. An observable is said to be *free* if it is not autonomous.

2. The linear system $\ker_{\mathcal{F}}(R.)$ is *autonomous* if every observable of $\ker_{\mathcal{F}}(R.)$ is autonomous.
3. The linear system $\ker_{\mathcal{F}}(R.)$ is *autonomous-free* if every observable of $\ker_{\mathcal{F}}(R.)$ is free.
4. The linear system $\ker_{\mathcal{F}}(R.)$ is *parametrizable* if there exists a matrix $Q \in D^{p\times m}$ such that $\ker_{\mathcal{F}}(R.) = Q\,\mathcal{F}^m$, i.e., for every $\eta \in \ker_{\mathcal{F}}(R.)$, there exists $\xi \in \mathcal{F}^m$ satisfying that $\eta = Q\,\xi$. The matrix Q is then called a (*potential-like*) *parametrization* of $\ker_{\mathcal{F}}(R.)$ and ξ a *potential*.
5. Let $R = (R_1 \quad R_2)$ be a partition of the matrix R and
$$\ker_{\mathcal{F}}(R.) = \{\eta = (\eta_1^T \quad \eta_2^T)^T \in \mathcal{F}^p \mid R_1\,\eta_1 + R_2\,\eta_2 = 0\}$$
the corresponding linear system. Then, η_1 is said to be *observable from* η_2 if η_1 is uniquely determined by η_2 in the sense that $\zeta = (\zeta_1^T \quad \eta_2^T)^T \in \ker_{\mathcal{F}}(R.)$ implies that $\zeta_1 = \eta_1$ or, equivalently, $R_1\,(\zeta_1 - \eta_1) = 0$ yields $\zeta_1 = \eta_1$.
6. The linear system $\ker_{\mathcal{F}}(R.)$ is *flat* if it admits an *injective parametrization*, namely, there exists a parametrization $Q \in D^{p\times m}$ of $\ker_{\mathcal{F}}(R.)$ which has a left inverse $T \in D^{m\times p}$, i.e., $T\,Q = I_m$. In other words, $\ker_{\mathcal{F}}(R.)$ is flat if it is parametrizable and every component ξ_i of the corresponding potential ξ is an observable of the system. The potential ξ is then called a *flat output* of $\ker_{\mathcal{F}}(R.)$.

The concepts of *observables* and *autonomous or free observables* were first introduced in [84]. For the introduction of the concept of *parametrizable systems* in the literature of mathematical systems theory, see [32, 84]. Moreover, *flat systems* were first introduced in [32]. The concept of observables of a linear system defined in 1 of Definition 1.6.1 and borrowed from quantum mechanics, must not be confused with the concept of an observable variable defined in 5 of Definition 1.6.1. Finally, within the behavioural approach (see, e.g., [81, 78, 80, 92, 117, 120]), a parametrization of a linear system is called *an image representation* and a flat system is a behaviour admitting *an observable image representation*. In the light of the algebraic analysis framework, it appears that the terminology developed by different communities should be unified.

We give module-theoretic characterizations of the system properties defined in Definition 1.6.1.

Theorem 1.6.1 ([16]). *Let D be a noetherian domain, $R \in D^{q\times p}$, $\mathcal{F}$ an injective cogenerator left D-module, $\ker_{\mathcal{F}}(R.) = \{\eta \in \mathcal{F}^p \,|\, R\,\eta = 0\}$ the linear system defined by R and $\mathcal{F}$ and $M = D^{1\times p}/(D^{1\times q}\,R)$ the left D-module finitely presented by R. Then, we have:*

1. *The observables of $\ker_{\mathcal{F}}(R.)$ are in a one-to-one correspondence with the elements of M.*
2. *The autonomous elements of $\ker_{\mathcal{F}}(R.)$ are in a one-to-one correspondence with the torsion elements of M.*
3. *The linear system $\ker_{\mathcal{F}}(R.)$ is autonomous iff the left D-module M is torsion.*
4. *The linear system $\ker_{\mathcal{F}}(R.)$ is autonomous-free iff the left D-module M is torsion-free.*
5. *The linear system $\ker_{\mathcal{F}}(R.)$ is parametrizable iff the left D-module M is torsion-free. Then, any parametrization $Q \in D^{p\times m}$ of M, i.e., $M \cong D^{1\times p}\,Q$, defines a parametrization of the system $\ker_{\mathcal{F}}(R.)$.*
6. *The linear system $\ker_{\mathcal{F}}(R.)$ is flat iff M is a free left D-module. Then, the bases of M are in a one-to-one correspondence with the flat outputs of $\ker_{\mathcal{F}}(R.)$.*
7. *If $R = (R_1 \quad R_2)$ denotes a partition of R, where $R_1 \in D^{q\times p_1}$ and $R_2 \in D^{q\times p_2}$, and $\ker_{\mathcal{F}}(R.) = \{\eta = (\eta_1^T \quad \eta_2^T)^T \in \mathcal{F}^p \mid R_1\,\eta_1 + R_2\,\eta_2 = 0\}$ the corresponding system, then, η_1 is observable from η_2 iff we have $M_1 = D^{1\times p_1}/(D^{1\times q}\,R_1) = 0$, i.e., iff R_1 admits a left inverse $S_1 \in D^{p_1\times q}$, i.e., $S_1\,R_1 = I_{p_1}$.*

We recall the concept of *controllability* for state-space linear OD systems due to Kalman.

Definition 1.6.2 ([44]). Let $D = \mathbb{R}[\partial]$ be the commutative polynomial ring of OD operators, $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $R = (\partial\, I_n - A \quad - B) \in D^{n \times (n+m)}$ and $\mathcal{F}$ a D-module. Then, the linear system $\ker_{\mathcal{F}}(R.)$ is said to be *controllable* if the state x of the system can be transferred from any initial state $x(0) = x_0$ to any given terminate state $x_T \in \mathbb{R}^n$ at any time $T \geq 0$, i.e., there exists an input $u : [0, T] \longrightarrow \mathbb{R}^m$ such that $x(T) = x_T$.

In mathematical systems theory, the following results are nowadays very classical.

Proposition 1.6.1 ([43, 44, 81]). *Let $D = \mathbb{R}[\partial]$ be the commutative ring of OD operators, $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $R = (\partial\, I_n - A \quad - B) \in D^{n \times (n+m)}$, and $\mathcal{F} = C^{\infty}(\mathbb{R}_+)$. Then, we have:*

1. $\ker_{\mathcal{F}}(R.)$ *is controllable iff* $\operatorname{rank}_{\mathbb{R}}(B \quad A\,B \quad A^2\,B \ \ldots \ A^{n-1}\,B) = n$.
2. $\ker_{\mathcal{F}}(R.)$ *is controllable iff R admits a right inverse $S \in D^{p \times q}$, i.e., $R\,S = I_q$.*

Example 1.6.1. Let $D = \mathbb{R}[\partial]$ be the principal ideal domain of OD operators, the matrices $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$, the presentation matrix $R = (\partial\, I_n - A \quad - B) \in D^{n \times (n+m)}$ and the finitely presented D-module $M = D^{1 \times (n+m)}/(D^{1 \times n}\, R)$. If x_i (resp., u_i) is the residue class of the i^{th} vector of the standard basis of $D^{1 \times (n+m)}$ in M for $i = 1, \ldots, n$ (resp., $i = n+1, \ldots, m$), then the family of generators $\{x_1, \ldots, x_n, u_1, \ldots, u_m\}$ of M satisfies the following D-linear relations

$$\partial\, x_i = \sum_{j=1}^{n} A_{ij}\, x_j + \sum_{k=1}^{m} B_{ik}\, u_k, \quad i = 1, \ldots, n,$$

i.e., $\dot{x} = A\,x + B\,u$, where $x = (x_1 \ \ldots \ x_n)^T$ and $u = (u_1 \ \ldots \ u_m)^T$. If $\mathcal{F}$ is a D-module (e.g., $\mathcal{F} = C^{\infty}(\mathbb{R}_+)$), then we have:

$$\hom_D(M, \mathcal{F}) \cong \ker_{\mathcal{F}}(R.) = \{(x^T \quad u^T)^T \in \mathcal{F}^{(n+m)} \mid \dot{x} = A\,x + B\,u\}.$$

Since D is a principal ideal domain, the D-module M is torsion-free iff M is free (see 1 of Theorem 1.1.2). Since R has full row rank, using Corollary 1.3.3, the D-module M is torsion-free iff $N = D^n/(R\, D^{(n+m)}) = 0$, i.e., iff the adjoint D-module $\widetilde{N} = D^{1 \times n}/(D^{1 \times (n+m)}\, \widetilde{R}) = 0$, where $\widetilde{R} = (-\partial\, I_n - A^T \quad - B^T)^T \in D^{(n+m) \times n}$. If we denote by λ_j the residue class of the j^{th} vector of the standard basis of $D^{1 \times n}$ in $\widetilde{N}$, then the family of generators $\{\lambda_j\}_{j=1,\ldots,n}$ satisfies

$$\begin{cases} \mu_1 \triangleq \partial\, \lambda + A^T\, \lambda = 0, \\ \mu_2 \triangleq B^T\, \lambda = 0. \end{cases} \tag{1.87}$$

In the literature of control theory, (1.87) is called the *dual system*. (1.87) is generally not formally integrable since (1.87) contains a first order and a zero order ODE, i.e., (1.87) is generally not a Gröbner basis of $D^{1 \times (n+m)}\, \widetilde{R}$. Hence, applying ∂ to the zero order equation, we get that $B^T\, \partial\, \lambda = 0$ and taking into account $\partial\, \lambda = -A^T\, \lambda$, we obtain the new zero order equation $B^T\, A^T\, \lambda = 0$. Repeating again the same process and using the Cayley-Hamilton theorem saying that $A^n = \sum_{i=0}^{n-1} \alpha_i\, A^i$, for some α_i's belonging to $\mathbb{R}$, we obtain the formally integrable system

$$(1.87) \quad \Leftrightarrow \quad \begin{cases} \mu_1 = \partial\, \lambda + A^T\, \lambda = 0, \\ \begin{pmatrix} X_0 \\ X_1 \\ \vdots \\ X_{n-1} \end{pmatrix} = \begin{pmatrix} B^T \\ B^T\, A^T \\ \vdots \\ B^T\, (A^T)^{n-1} \end{pmatrix} \lambda = 0, \end{cases}$$

where the elements X_i's are defined by:

$$\left\{ \begin{array}{l} X_0 = \mu_2, \\ X_i = \sum_{j=1}^{i} B^T\,(A^T)^{i-j}\,(-\partial)^{j-1}\,\mu_1 + (-1)^i\,\partial^i\,\mu_2, \quad i = 1, \ldots, n-1. \end{array} \right.$$

Then, (1.87) is reduced to 0, i.e., M is a torsion-free D-module, iff:

$$\mathrm{rank}_{\mathbb{R}}(B \quad A\,B \quad A^2\,B \ \ldots \ A^{n-1}\,B) = n. \tag{1.88}$$

Hence, $\ker_{\mathcal{F}}(R.)$ is controllable iff the D-module M is torsion-free, i.e., using 4 of Theorem 1.6.1, iff $\ker_{\mathcal{F}}(R.)$ is autonomous-free ([31, 84]). The previous result can be interpreted as the observability test for the dual system (1.87). Now, according to 2 of Corollary 1.3.3, M is a stably free D-module iff the matrix R admits a right inverse $S \in D^{(n+m)\times n}$, i.e., $R\,S = I_n$, or equivalently, iff $\partial\,I_n - A$ and B are left-coprime. If the rank condition (1.88) is satisfied, then there exists a matrix $C = (C_0 \ \ldots \ C_{n-1}) \in \mathbb{R}^{n\times(m\,n)}$ such that $C\,(B \quad A\,B \quad A^2\,B \ \ldots \ A^{n-1}\,B)^T = I_n$. Then, we have $\lambda = C_0\,X_0 + \ldots + C_{n-1}\,X_{n-1}$ and if $\Delta = (1 \quad -\partial \quad \partial^2 \ \ldots \ (-\partial)^{n-1})^T$, then we get $\lambda = C\,B^T\,H(A^T)\,\Delta\,\mu_1 + C\,\Delta\,\mu_2$, where the matrix H is defined by:

$$\forall\, L \in \mathbb{R}^{n\times n}, \quad H(L) = \left(\begin{array}{cccccc} 0 & 0 & 0 & 0 & 0 & 0 \\ I_n & 0 & 0 & 0 & 0 & 0 \\ L & I_n & 0 & 0 & 0 & 0 \\ L^2 & L & I_n & 0 & 0 & 0 \\ L^3 & L^2 & L & I_n & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ L^{n-2} & L^{n-3} & L^{n-4} & \ldots & I_n & 0 \end{array} \right).$$

Moreover, if $U = C\,B^T\,H(A^T)\,\Delta \ \ V = C\,\Delta$, then $\lambda = U\,\mu_1 + V\,\mu_2$, which yields the Bézout identity $U\,(\partial\,I_n + A^T) + V\,B^T = I_n$. Applying the involution θ of D defined by (1.20) to this Bézout identity, we get $(\partial\,I_n - A)\,X - B\,Y = I_n$, where:

$$X = -\theta(U) = -\sum_{k=0}^{n-2} \left(\sum_{l=k+1}^{n-1} A^{l-k-1}\,B\,C_l^T \right) \partial^k, \quad Y = -\theta(V) = -\sum_{k=0}^{n-1} C_k^T\,\partial^k.$$

Now, a non-minimal parametrization of $\ker_{\mathcal{F}}(R.)$ can be obtained by applying the involution θ to the compatibility conditions of $\widetilde{R}\,\lambda = \mu$ (see Algorithm 1.4.1). These compatibility conditions are obtained by substituting $\lambda = U\,\mu_1 + V\,\mu_2$ into $\widetilde{R}\,\lambda = \mu$ to get:

$$\left(\begin{array}{cc} (\partial\,I_n + A^T)\,U - I_n & (\partial\,I_n + A^T)\,V \\ B^T\,U & B^T\,V - I_m \end{array} \right) \left(\begin{array}{c} \mu_1 \\ \mu_2 \end{array} \right) = 0. \tag{1.89}$$

Hence, we obtain the following non-injective parametrization of $\ker_{\mathcal{F}}(R.)$:

$$\forall\, \xi \in \mathcal{F}^{(n+m)}, \quad \left(\begin{array}{c} x \\ u \end{array} \right) = \left(\begin{array}{cc} X\,(\partial\,I_n - A) - I_n & -X\,B \\ Y\,(\partial\,I_n - A) & -Y\,B - I_m \end{array} \right) \xi.$$

Minimal parametrizations of $\ker_{\mathcal{F}}(R.)$ can be obtained by setting to zero n components of the potential ξ. For instance, considering $\xi = (0 \quad -\chi^T)^T$, where $\chi \in \mathcal{F}^m$, we obtain:

$$\forall\, \chi \in \mathcal{F}^m, \quad \left(\begin{array}{c} x \\ u \end{array} \right) = \left(\begin{array}{c} X\,B \\ Y\,B + I_m \end{array} \right) \chi.$$

Now, if the linear system $\dot{x} = A\,x + B\,u$ is not controllable, then, in control theory ([43, 44, 81]), it is well-known that there exists $P \in \mathrm{GL}_n(\mathbb{R})$ such that the transformation $\overline{x} = P\,x$ defines an equivalent system $\dot{\overline{x}} = (P\,A\,P^{-1})\,\overline{x} + (P\,B)\,u$ of the form

$$\begin{cases} \dot{\overline{x}}_1 = \overline{A}_{11}\,\overline{x}_1 + \overline{A}_{12}\,\overline{x}_2 + \overline{B}_1\,u, \\ \dot{\overline{x}}_2 = \overline{A}_{22}\,\overline{x}_2, \end{cases} \tag{1.90}$$

with the notations $\overline{A} = P\,A\,P^{-1}$ and $\overline{B} = P\,B$ ([44]). (1.90) is called the *Kalman's decomposition* of $\dot{x} = A\,x + B\,u$. The dimension of the vector $\overline{x}_2$ is $l = n - \mathrm{rank}_{\mathbb{R}}(B \quad A\,B \quad A^2\,B \ \ldots \ A^{n-1}\,B)$. Clearly, the invertible transformation $\overline{x} = P\,x$ is only a change of generators of the D-module M from $\{x_1, \ldots, x_n, u_1, \ldots, u_m\}$ to $\{\overline{x}_1, \ldots, \overline{x}_n, u_1, \ldots, u_m\}$. Hence, (1.90) is only another presentation of the D-module M. In (1.90), we can easily see that all the components $\overline{x}_{2i}$'s of $\overline{x}_2$ satisfy $\det(\partial\,I_l - \overline{A}_{22})\,\overline{x}_{2i} = 0$, $i = 1, \ldots, l$, i.e., define torsion elements of M, and thus, autonomous elements of $\ker_{\mathcal{F}}(R.)$. Finally, using the following integration by parts

$$\lambda^T\,(\dot{x} - A\,x - B\,u) = -x^T\,(\dot{\lambda} + A^T\,\lambda) - u^T\,(B^T\,\lambda) + \frac{d}{dt}\,(\lambda^T\,x),$$

we can easily compute first integrals of motion of $\ker_{\mathcal{F}}(R.)$. Indeed, if $\eta = (x^T \quad u^T)^T \in \ker_{\mathcal{F}}(R.)$ and $\overline{\lambda}$ is the general solution of the adjoint system

$$\begin{cases} \dot{\lambda} + A^T\,\lambda = 0, \\ B^T\,\lambda = 0, \end{cases}$$

which, by assumption, is non-trivial, then $\Phi = \overline{\lambda}\,x = \sum_{i=1}^n \overline{\lambda}_i\,x_i$ is a first integral, i.e., $\dot{\Phi} = 0$.

Definition 1.6.2 was generalized by Willems for general time-invariant OD systems.

Definition 1.6.3 ([81]). Let $D = \mathbb{R}[\partial]$ be the commutative polynomial ring of OD operators, $R \in D^{q \times p}$ a full row rank matrix and $\mathcal{F}$ a D-module. Then, $\ker_{\mathcal{F}}(R.)$ is *controllable* if for all $T \geq 0$ and for all η_p and $\eta_f \in \ker_{\mathcal{F}}(R.)$, there exists $\eta \in \ker_{\mathcal{F}}(R.)$ such that:

$$\begin{cases} \eta_{|\,]-\infty,0]} = \eta_{p\,|\,]-\infty,0]}, \\ \eta_{|\,[T,+\infty[} = \eta_{f\,|\,[T,+\infty[}. \end{cases} \tag{1.91}$$

According to Definition 1.6.3, a time-invariant linear system $\ker_{\mathcal{F}}(R.)$ is controllable if it can switch from any arbitrary pasted trajectory η_p of $\ker_{\mathcal{F}}(R.)$ to any arbitrary future trajectory η_f in a given time T by means of a third trajectory η of $\ker_{\mathcal{F}}(R.)$. See Figure 1.1.

Example 1.6.2. Let $D = \mathbb{R}[\partial]$ be the commutative polynomial ring of OD operators, $R \in D^{q \times p}$ a full row rank matrix (e.g., $R = (R_1 \quad -R_2)$, where $R_1 \in D^{q \times q}$, $\det R_1 \neq 0$, $R_2 \in D^{q \times p}$) and $M = D^{1 \times p}/(D^{1 \times q}\,R)$ the D-module finitely presented by R. Using 1 of Theorem 1.1.2, M is a torsion-free D-module iff M is free. According to Corollary 1.5.2, the D-module M is free iff the matrix R can be embedded in $V \in \mathrm{GL}_p(D)$, i.e., iff there exist three matrices $S \in D^{p \times q}$, $Q \in D^{p \times (p-q)}$ and $T \in D^{(p-q) \times p}$ such that the following two Bézout identities hold

$$\begin{pmatrix} R \\ T \end{pmatrix} (S \quad Q) = \begin{pmatrix} I_q & 0 \\ 0 & I_{p-q} \end{pmatrix} = I_p, \quad (S \quad Q) \begin{pmatrix} R \\ T \end{pmatrix} = I_p,$$

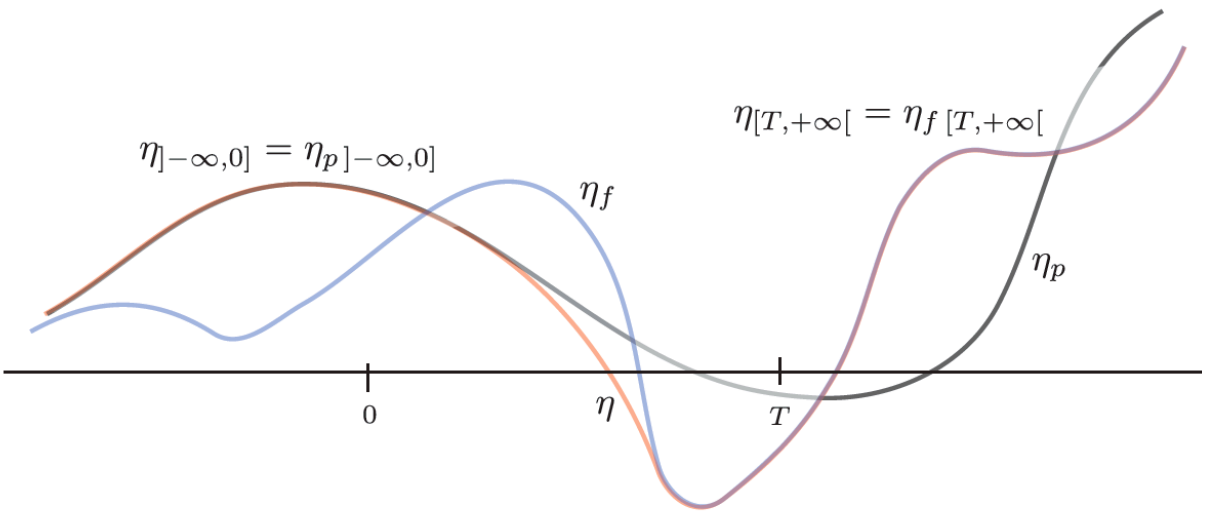

Figure 1.1: Controllability à la Willems

which are equivalent to the following split exact sequence:

$$0 \longrightarrow D^{1\times q} \underset{\overset{.S}{\longleftarrow}}{\overset{.R}{\longrightarrow}} D^{1\times p} \underset{\overset{.T}{\longleftarrow}}{\overset{.Q}{\longrightarrow}} D^{1\times (p-q)} \longrightarrow 0.$$

If $\mathcal{F}$ is a left D-module (e.g., $\mathcal{F} = C^{\infty}(\mathbb{R}_+)$), then applying the functor $\hom_D(\,\cdot\,, \mathcal{F})$ to the above split exact sequence, we obtain the following split exact sequence

$$0 \longleftarrow \mathcal{F}^{q} \underset{\overset{S.}{\longrightarrow}}{\overset{R.}{\longleftarrow}} \mathcal{F}^{p} \underset{\overset{T.}{\longrightarrow}}{\overset{Q.}{\longleftarrow}} \mathcal{F}^{(p-q)} \longleftarrow 0.$$

which shows that Q is an injective parametrization of the flat linear OD system $\ker_{\mathcal{F}}(R.)$, i.e., $\ker_{\mathcal{F}}(R.) = Q\,\mathcal{F}^{(p-q)}$ and $T\,Q = I_{(p-q)}$. The injective parametrization $\eta = Q\,\xi$ of $R\,\eta = 0$ is called the *controller form* and $\xi = T\,\eta$ the *generalized state* of the linear system $\ker_{\mathcal{F}}(R.)$ (see [43]). We note that the generalized state ξ is observable from η (see 5 of Definition 1.6.1).

The generalized state ξ of $\ker_{\mathcal{F}}(R.)$ can be used to find again Willems' approach to controllability. Indeed, we can define $\xi_p = T\,\eta_p$ and $\xi_f = T\,\eta_f$. Now, if $\mathcal{F} = C^{\infty}(\mathbb{R})$, then, using the partition of unity on the compact subset $[0, T]$ of $\mathbb{R}$, we can find $\xi \in \mathcal{F}^{(p-q)}$ satisfying that $\xi_{|\,]-\infty,0]} = \xi_{p\,|\,]-\infty,0]}$ and $\xi_{|\,[T,+\infty[} = \xi_{f\,|\,[T,+\infty[}$. Then, $\eta = Q\,\xi$ satisfies (1.91), which shows that a free D-module M defines a controllable linear OD system $\ker_{\mathcal{F}}(R.)$.

Finally, since D is a principal ideal domain, the full row rank matrix $R \in D^{q\times p}$ admits a Smith normal form, namely, there exist two matrices $V \in \mathrm{GL}_q(D)$ and $W \in \mathrm{GL}_p(D)$ such that $V\,R\,W = \mathrm{diag}(d_1, \ldots, d_q)$, where $d_i \in D \setminus \{0\}$ and $d_i \,|\, d_{i+1}$ for $i = 1, \ldots, q$. Now, let $M' = D^{1\times p'}/(D^{1\times q'}\,R')$ be the D-module finitely presented by the matrix $R' = \mathrm{diag}(\alpha_1, \ldots, \alpha_q)$ and $\pi' : D^{1\times p} \longrightarrow M'$ the canonical projection onto M'. We can easily check that the D-homomorphism $f : M \longrightarrow M'$ defined by $f(\pi(\lambda)) = \pi'(\lambda\,W)$ is an isomorphism (see Chapter 3), and thus $M' \cong M$. If $\{e_i\}_{i=1,\ldots,q}$ is the standard basis of $D^{1\times q}$, then we have:

$$M' = D^{1\times p}/\left(\bigoplus_{i=1}^{q} D\,d_i\,e_i\right) \cong \bigoplus_{i=1}^{q} D/(D\,d_i) \oplus D^{1\times (p-q)} \;\Rightarrow\; \ker_{\mathcal{F}}(R.) \cong \bigoplus_{i=1}^{q} \ker_{\mathcal{F}}(d_i.) \oplus \mathcal{F}^{(p-q)}.$$

Hence, if $M \cong M'$ is not a free D-module, then one the d_i's is a non-invertible element of D and defines a torsion element corresponding to the non-trivial cyclic D-module $D/(D\, d_i)$. Then, $\ker_{\mathcal{F}}(d_i.)$ is clearly non-controllable and so is $\ker_{\mathcal{F}}(R.)$, which finally proves that a linear OD system $\ker_{\mathcal{F}}(R.)$ is controllable iff M is a free D-module, i.e., iff M is a torsion-free D-module.

Proposition 1.6.2 ([31, 84, 88])**.** *Let $D = \mathbb{R}[\partial]$ be the commutative polynomial ring of OD operators, $M = D^{1\times p}/(D^{1\times q}\, R)$ the D-module finitely presented by a full row rank matrix R and $\mathcal{F} = C^{\infty}(\mathbb{R})$. Then, the linear system $\ker_{\mathcal{F}}(R.)$ is controllable iff the D-module M is torsion-free.*

Pillai and Shankar have extended Willems' definition of controllability and Proposition 1.6.2 to the case of underdetermined linear PD systems with constant coefficients ([80]).

Theorem 1.6.2 ([80])**.** *Let $D = \mathbb{R}[\partial_1, \ldots, \partial_n]$ be the commutative polynomial ring of PD operators, $R \in D^{q\times p}$, $\mathcal{F} = C^{\infty}(\Omega)$, where Ω is an open convex subset of $\mathbb{R}^n$, $M = D^{1\times p}/(D^{1\times q}\, R)$ the D-module finitely presented by R. Then, the following two assertions are equivalent:*

1. $\ker_{\mathcal{F}}(R.)$ *is* controllable *in the sense that, for all η_1 and $\eta_2 \in \ker_{\mathcal{F}}(R.)$ and all open subsets U_1 and U_2 of Ω such that their closures $\overline{U_1}$ and $\overline{U_2}$ do not intersect (i.e., $\overline{U_1} \cap \overline{U_2} = \emptyset$), there exists $\eta \in \ker_{\mathcal{F}}(R.)$ which coincides with η_1 on U_1 and with η_2 in U_2.*
2. *The D-module $M = D^{1\times p}/(D^{1\times q}\, R)$ is torsion-free.*

The next theorem, due to Malgrange and Komatsu, shows how closely the algebraic and analytic properties of linear PD systems with constant coefficients are interlinked.

Theorem 1.6.3 ([48, 68])**.** *Let $D = \mathbb{R}[\partial_1, \ldots, \partial_n]$, $R \in D^{q\times p}$ and $M = D^{1\times p}/(D^{1\times q}\, R)$ be the D-module finitely by R. Then, the following assertions are equivalent:*

1. $\mathrm{ext}^1_D(M, D) = 0$.
2. *For all bounded open convex subset Ω of $\mathbb{R}^n$, the restriction D-homomorphism is surjective:*
$$\Gamma_{\Omega} : \hom_D(M, C^{\infty}(\mathbb{R}^n)) \longrightarrow \hom_D(M, C^{\infty}(\mathbb{R}^n \setminus \Omega)).$$
3. *For all bounded open convex subset Ω of $\mathbb{R}^n$, the restriction D-homomorphism is surjective:*
$$\Gamma'_{\Omega} : \hom_D(M, \mathcal{D}'(\mathbb{R}^n)) \longrightarrow \hom_D(M, \mathcal{D}'(\mathbb{R}^n \setminus \Omega)).$$

According to Theorem 1.1.1, the D-homomorphism Γ_{Ω} is equivalent to the D-homomorphism:

$$\begin{array}{rcl} \gamma_{\Omega} : \ker_{C^{\infty}(\mathbb{R}^n)}(R.) & \longrightarrow & \ker_{C^{\infty}(\mathbb{R}^n \setminus \Omega)}(R.) \\ \eta & \longmapsto & \eta_{|\, \mathbb{R}^n \setminus \Omega}. \end{array} \tag{1.92}$$

Example 1.6.3. Let $M = D^{1\times 3}/(D\, R)$ be the $D = \mathbb{R}[\partial_1, \partial_2, \partial_3]$-module finitely presented by the divergence operator $R = (\partial_1 \quad \partial_2 \quad \partial_3)$ in $\mathbb{R}^3$. The Auslander transposed D-module $N = D/(R\, D^3) = D/(D^{1\times 3}\, R^T)$ of M is to the D-module defined by the gradient operator:

$$\left\{ \begin{array}{l} \partial_1\, \lambda = 0, \\ \partial_2\, \lambda = 0, \\ \partial_3\, \lambda = 0. \end{array} \right.$$

Let Ω be a bounded convex open subset of $\mathbb{R}^3$. Then, $\hom_D(N, C^{\infty}(\mathbb{R}^3 \setminus \Omega))$ is the D-module formed by constant functions defined over the small open neighbourhood of $\mathbb{R}^3 \setminus \Omega$. Then, the restriction map γ_{Ω} defined by (1.92) is clearly surjective. Then, we find again that the D-module M defining the divergence operator is torsion-free (see Example 1.3.5).

Definition 1.6.4. Using the previous notations, the linear PD system $\hom_D(M, C^\infty(\mathbb{R}^n))$ (resp., $\hom_D(M, \mathcal{D}'(\mathbb{R}^n))$) is said to be *extendable* if it satisfies 2 (resp., 3) of Theorem 1.6.3.

We obtain the following corollary of Theorems 1.6.3 and 1.3.1.

Corollary 1.6.1 ([99]). *With the previous notations, the following conditions are equivalent:*

1. *The linear PD system* $\ker_{C^\infty(\mathbb{R}^n)}(R.)$ *is controllable.*
2. *The linear PD system* $\ker_{C^\infty(\mathbb{R}^n)}(\widetilde{R}.)$ *is extendable.*
3. *The linear PD system* $\ker_{\mathcal{D}'(\mathbb{R}^n)}(\widetilde{R}.)$ *is extendable.*
4. $M = D^{1\times p}/(D^{1\times q}\, R)$ *is a torsion-free* D*-module.*

Example 1.6.4. Example 1.6.3 shows that the system formed by the smooth solutions of the divergence operator in $\mathbb{R}^3$ is controllable in the sense of 1 of Theorem 1.6.2.

If R has full row rank, then $\mathrm{ext}^1_D(M, D) \cong N = D^q/(R\, D^p)$ is the Auslander transpose of $M = D^{1\times p}/(D^{1\times q}\, R)$. Corollary 1.3.3 shows that M is a stably free, and thus, a free D-module by the Quillen-Suslin theorem (see 2 of Theorem 1.1.2), iff $\mathrm{ext}^1_D(M, D) \cong N = 0$.

Corollary 1.6.2 ([99]). *Let* $D = \mathbb{R}[\partial_1, \ldots, \partial_n]$ *and* $M = D^{1\times p}/(D^{1\times q}\, R)$ *be the* D*-module finitely presented by a full row rank matrix* $R \in D^{q\times p}$*. Then, the conditions are equivalent:*

1. *The* D*-module* M *is a free* D*-module.*
2. *The linear PD system* $\ker_{C^\infty(\mathbb{R}^n)}(R.)$ *is extendable.*
3. *The linear PD system* $\ker_{\mathcal{D}'(\mathbb{R}^n)}(R.)$ *is extendable.*
4. *The linear PD system* $\ker_{C^\infty(\mathbb{R}^n)}(R.)$ *is flat.*
5. *The linear PD system* $\ker_{\mathcal{D}'(\mathbb{R}^n)}(R.)$ *is flat.*

Corollary 1.6.2 extends the above results obtained for time-invariant linear OD systems.

Let $D = A\langle\partial_1, \ldots, \partial_n\rangle$ be a ring of PD operators with coefficients in a differential ring A, $R \in D^{q\times p}$ a matrix of PD operators of order r, $\mathcal{F}$ an injective left D-module and $\ker_{\mathcal{F}}(R.)$ the linear PD system defined by R and $\mathcal{F}$. Let us introduce the *quadratic Lagrangian function*

$$L(\eta) = \frac{1}{2}\, \eta_r^T\, L\, \eta_r, \tag{1.93}$$

where $\eta = (\eta_1 \ \ldots \ \eta_p)^T$, $\partial^\alpha \eta_k = \partial_1^{\alpha_1} \ldots \partial_n^{\alpha_n} \eta_k$, where $\alpha = (\alpha_1 \ \ldots \ \alpha_n)^T \in \mathbb{N}^n$ is a multi-index of length $|\alpha| = \alpha_1 + \ldots + \alpha_n$, $\eta_r = (\partial^\alpha \eta_k, |\alpha| = 0, \ldots, r)^T_{k=1,\ldots,p}$ and L a symmetric matrix with entries in A, i.e., $L^{k,l}_{\alpha,\beta} = L^{l,k}_{\beta,\alpha}$ for all k, $l = 1, \ldots, p$ and for all α, $\beta \in \mathbb{N}^n$ such that $|\alpha| = 0, \ldots, r$ and $|\beta| = 0, \ldots, r$. Let us study the problem of extremizing the following *Lagrangian functional*

$$I = \int_\Omega \frac{1}{2}\, \eta_r^T\, L\, \eta_r\, dx, \quad \eta \in \ker_{\mathcal{F}}(R.),$$

under the differential constraint formed by the linear PD system $\ker_{\mathcal{F}}(R.)$. The first variation of the Lagrangian density is

$$\delta L(\eta) = \sum_{|\alpha|=0,\ldots,r,\, k=1,\ldots,p} \pi^k_\alpha\, \delta(\partial^\alpha \eta_k), \quad \pi^k_\alpha(\eta) = \frac{\partial L(\eta)}{\partial(\partial^\alpha \eta_k)} = \sum_{|\beta|=0,\ldots,r,\, i=1,\ldots,p} L^{k,i}_{\alpha,\beta}\, \partial^\beta \eta_i,$$

where $\delta(\partial^\alpha \eta_k)$ denotes the variation of $\partial^\alpha \eta_k$. Let us introduce the following PD operator:

$$\begin{array}{rcl} \mathcal{B} : \mathcal{F}^p & \longrightarrow & \mathcal{F}^p \\ \eta & \longmapsto & \left(\sum_{|\alpha|=0,\dots,r} (-1)^{|\alpha|} \, \partial^\alpha \, \pi_\alpha^k\right)_{k=1,\dots,p}. \end{array} \tag{1.94}$$

Using the symmetry of L, namely, $L_{\alpha,\beta}^{k,i} = L_{\beta,\alpha}^{i,k}$, we can prove that $\widetilde{\mathcal{B}} = \mathcal{B}$ ([93]), where $\widetilde{\mathcal{B}}$ is the formal adjoint of $\mathcal{B}$. If $\lambda \in \mathcal{F}^q$ is a *Lagrange multiplier*, using the following identity

$$\lambda^T \, R \, \eta = \eta^T \, \widetilde{R} \, \lambda + \mathrm{div}(\Phi(\lambda, \eta)), \tag{1.95}$$

where Φ is a vector of bilinear forms in λ, η and their derivatives and $\mathrm{div} = (\partial_1 \, \dots \, \partial_n)$ is the divergent operator in $\mathbb{R}^n$ (see, e.g., [66, 85]), then we get

$$\delta \int_\Omega (L(\eta) - \lambda^T \, R \, \eta) \, dx = \int_\Omega (\delta \eta)^T \, (\mathcal{B} \, \eta - \widetilde{R} \, \lambda) \, dx + \int_\Omega \mathrm{div}(\Phi(\lambda, \delta\eta)) \, dx,$$

which proves that a necessary condition for the existence of an extremum of the previous variational problem is $\mathcal{B} \, \eta - \widetilde{R} \, \lambda = 0$, where $\eta \in \ker_{\mathcal{F}}(R.)$. We obtain the following result.

Proposition 1.6.3 ([93]). *If $\mathcal{F}$ an injective left D-module, then a necessary condition for the existence of $\eta \in \ker_{\mathcal{F}}(R.)$ which extremizes the Lagrangian functional (1.93) is*

$$\left\{ \begin{array}{l} R \, \eta = 0, \\ \mathcal{B} \, \eta - \widetilde{R} \, \lambda = 0, \end{array} \right. \tag{1.96}$$

where λ is a Lagrangian multiplier, $\widetilde{R}$ the formal adjoint of R and $\mathcal{B}$ is defined by (1.94).

Moreover, if $\widetilde{Q} \in D^{p \times m}$ is a matrix defining the compatibility conditions of the inhomogeneous linear system $\widetilde{R} \, \lambda = \mu$, i.e., $\ker_D(.\widetilde{R}) = D^{1 \times m} \, \widetilde{Q}$, then (1.96) is equivalent to

$$\left\{ \begin{array}{l} R \, \eta = 0, \\ (\widetilde{Q} \circ \mathcal{B}) \, \eta = 0, \end{array} \right. \tag{1.97}$$

where $\circ$ denotes the composition of differential operators. Finally, we have the following diagram of exact sequences:

$$\begin{array}{ccccc} & & \mathcal{F}^p & \xrightarrow{R.} & \mathcal{F}^q \\ & & \downarrow \mathcal{B}. & & \\ \mathcal{F}^m & \xleftarrow{\widetilde{Q}.} & \mathcal{F}^p & \xleftarrow{\widetilde{R}.} & \mathcal{F}^q. \end{array}$$

Example 1.6.5. Let us extremize the following Lagrangian functional

$$I = \int_{t_0}^{t_1} \frac{1}{2} \, (x \quad u)^T \begin{pmatrix} L_1 & 0 \\ 0 & L_2 \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix} dt + \frac{1}{2} \, x(t_1)^T \, S \, x(t_1),$$

where L_1 (resp., L_2, S) is a positive definite (resp., semi-definite) symmetric real matrix and x and u satisfy the linear system $\dot{x} = A \, x + B \, u$ and $x(t_0) = x_0$ (see Example 1.6.1). We then get:

$$\begin{array}{rcl} \mathcal{B} : \mathcal{F}^{n+m} & \longrightarrow & \mathcal{F}^{n+m} \\ \begin{pmatrix} x \\ u \end{pmatrix} & \longmapsto & \begin{pmatrix} L_1 & 0 \\ 0 & L_2 \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix} = \begin{pmatrix} L_1 \, x \\ L_2 \, u \end{pmatrix}. \end{array}$$

Using Proposition 1.6.3, the optimal system (1.96) is defined by:

$$\left\{\begin{array}{ll} \dot{x} - A\,x - B\,u = 0, & x(t_0) = x_0, \\ \dot{\lambda} + A^T \lambda + L_1\,x = 0, & \lambda(t_1) = S\,x(t_1), \\ L_2\,u + B^T\,\lambda = 0. & \end{array}\right. \tag{1.98}$$

For instance, let $I = \int_0^T \frac{1}{2}\,(x(t)^2 + u(t)^2)\,dt$, where x and u satisfy the linear OD system:

$$\dot{x}(t) + x(t) - u(t) = 0, \quad x(0) = x_0. \tag{1.99}$$

Using the integration by parts $\lambda\,(\dot{x}+x-u) = (-\dot{\lambda}+\lambda)\,x - \lambda\,u + \frac{d}{dt}\,(\lambda\,x)$, we get $\widetilde{R} = (-\partial + 1 \quad -1)^T$. Moreover, computing the first variation of I, namely,

$$\delta I = \int_0^T (x(t)\,\delta x(t) + u(t)\,\delta u(t))\,dt = \int_0^T (\delta x(t) \quad \delta u(t)) \begin{pmatrix} x(t) \\ u(t) \end{pmatrix} dt,$$

we obtain $\mathcal{B} = I_2$. Therefore, the optimal system (1.96) is defined by:

$$\left\{\begin{array}{ll} \dot{x}(t) + x(t) - u(t) = 0, & x(0) = x_0, \\ \dot{\lambda}(t) - \lambda(t) + x(t) = 0, & \lambda(T) = 0, \\ \lambda + u = 0. & \end{array}\right.$$

Since $\widetilde{R}$ clearly defines an injective operator, the linear OD system $\dot{x}(t) + x(t) - u(t) = 0$ is controllable. For more details, see Example 1.6.1. Hence, substituting $\lambda = -u$ in the previous optimal system, we obtain that (1.97) is defined by:

$$\left\{\begin{array}{ll} \dot{x}(t) + x(t) - u(t) = 0, & x(0) = x_0, \\ \dot{u}(t) - u(t) - x(t) = 0, & u(T) = 0. \end{array}\right.$$

Example 1.6.6. Let us consider the electromagnetism Lagrangian functional

$$\int \frac{1}{2} \left(\epsilon_0 \parallel \vec{E} \parallel^2 - \frac{1}{\mu_0} \parallel \vec{B} \parallel^2 \right) dt\,dx_1\,dx_2\,dx_3,$$

where ϵ_0 is the *dielectric constant* and μ_0 is the *magnetic constant*, under the differential constraint formed by the first set of Maxwell equations (see Example 1.3.6):

$$\left\{\begin{array}{l} \dfrac{\partial \vec{B}}{\partial t} + \vec{\nabla} \wedge \vec{E} = \vec{0}, \\ \vec{\nabla}\,.\,\vec{B} = 0. \end{array}\right. \tag{1.100}$$

Varying the Lagrangian functional, we obtain that $\mathcal{B}$ is defined by:

$$\begin{array}{ccc} \mathcal{F}^6 & \xrightarrow{\mathcal{B}.} & \mathcal{F}^6 \\ \begin{pmatrix} \vec{B} \\ \vec{E} \end{pmatrix} & \longmapsto & \begin{pmatrix} -\dfrac{1}{\mu_0}\,\vec{B} \\ \epsilon_0\,\vec{E} \end{pmatrix}. \end{array}$$

Using (1.49), we obtain that the optimal system (1.96) is defined by:

$$\begin{cases} \dfrac{\partial \vec{B}}{\partial t} + \vec{\nabla} \wedge \vec{E} = \vec{0}, \\ \vec{\nabla} \,.\, \vec{B} = 0, \\ -\dfrac{1}{\mu_0}\, \vec{B} = -\dfrac{\partial \vec{C}}{\partial t} - \vec{\nabla}\, G, \\ \epsilon_0\, \vec{E} = \vec{\nabla} \wedge \vec{C}. \end{cases}$$

If $\widetilde{Q}$ is the compatibility conditions (1.51) of the formal adjoint of the first set of Maxwell equations (1.100) (see Example 1.3.6), then the PD operator $\widetilde{Q} \circ \mathcal{B} : \mathcal{F}^6 \longrightarrow \mathcal{F}^4$ is defined by

$$(\vec{B}, \vec{E}) \longmapsto \begin{cases} \dfrac{1}{\mu_0}\, \vec{\nabla} \wedge \vec{B} - \epsilon_0\, \dfrac{\partial \vec{E}}{\partial t} = \vec{\jmath}, \\ \epsilon_0\, \vec{\nabla} \,.\, \vec{E} = \rho, \end{cases}$$

where $\vec{\jmath}$ (resp., ρ) is the density of current (resp., charge) and corresponds to the second set of Maxwell equations for the *electromagnetism induction* $\vec{D} = \epsilon_0\, \vec{E}$ and $\vec{H} = \vec{B}/\mu_0$. Hence, using (1.50), we obtain that the optimal system (1.97) is defined by

$$\begin{cases} \dfrac{\partial \vec{B}}{\partial t} + \vec{\nabla} \wedge \vec{E} = \vec{0}, \\ \vec{\nabla} \,.\, \vec{B} = 0, \\ \dfrac{1}{\mu_0}\, \vec{\nabla} \wedge \vec{B} - \epsilon_0\, \dfrac{\partial \vec{E}}{\partial t} = \vec{0}, \\ \epsilon_0\, \vec{\nabla} \,.\, \vec{E} = 0, \end{cases} \tag{1.101}$$

which is the complete set of Maxwell equations in vacuum. Using Algorithms 1.3.1 and 1.3.2, we can prove that the finitely presented $D = \mathbb{Q}(\epsilon_0, \mu_0)[\partial_t, \partial_1, \partial_2, \partial_3]$-module associated with (1.101) is torsion and the components of the fields $\vec{B}$ and $\vec{E}$ satisfy the following wave equations

$$\forall\, i = 1,\, 2,\, 3, \quad \left(\frac{1}{c_0^2}\, \partial_t^2 - \Delta\right) E_i = 0, \quad \left(\frac{1}{c_0^2}\, \partial_t^2 - \Delta\right) B_i = 0,$$

where $\Delta = \partial_1^2 + \partial_2^2 + \partial_3^2$ is the Laplacian operator and $c_0^2 = 1/(\epsilon_0\, \mu_0)$, i.e., the fields $\vec{B}$ and $\vec{E}$ are space-time waves. A modern formulation of the previous results uses the rewriting of the Maxwell equations in terms of differential forms (2-forms) on space-time and the *Hodge duality*.

According to Corollary 1.3.3, if the matrix R has full row rank, then the left D-module $M = D^{1\times p}/(D^{1\times q}\, R)$ is stably free iff there exists a matrix $S \in D^{p\times q}$ satisfying $R\, S = I_q$. Then, we have $\widetilde{S}\, \widetilde{R} = I_q$, where $\widetilde{S}$ is the formal adjoint of S. In this case, pre-multiplying the last equation of (1.96) by $\widetilde{S}$, we obtain $\lambda = (\widetilde{S} \circ \mathcal{B})\, \eta$.

Proposition 1.6.4 ([93]). *Let us suppose that the matrix $R \in D^{q\times p}$ has full row rank and $M = D^{1\times p}/(D^{1\times q}\, R)$ is a stably free left D-module. Then, from (1.96), we obtain $\lambda = (\widetilde{S} \circ \mathcal{B})\, \eta$, where $\widetilde{S} \in D^{q\times p}$ is a left inverse of $\widetilde{R}$. Hence, the Lagrange multiplier λ can be observed from the system variables η in the sense of 5 of Definition 1.6.1.*

Using (1.95) and (1.96), if $\eta \in \ker_{\mathcal{F}}(R.)$, then

$$\eta^T \mathcal{B}\, \eta = \eta^T \widetilde{R}\, \lambda = \lambda^T R\, \eta - \operatorname{div}(\Phi(\lambda, \eta)) = -\operatorname{div}(\Phi(\lambda, \eta)),$$

and thus we get:

$$I = \int_{\Omega} \frac{1}{2}\, \eta^T \mathcal{B}\, \eta\, dx = -\frac{1}{2} \int_{\Omega} \operatorname{div}(\Phi(\lambda, \eta))\, dx = -\frac{1}{2} \int_{\partial\Omega} \Phi(\lambda, \eta)\, d\gamma.$$

Using Example 1.6.1, every controllable time-invariant linear OD system satisfies the hypotheses of Proposition 1.6.4. Hence, if $n = 1$, then we obtain:

$$\begin{aligned} I = \int_0^T \frac{1}{2}\, \eta^T \mathcal{B}\, \eta\, dt &= \frac{1}{2}\, (\Phi(\lambda(0), \eta(0)) - \Phi(\lambda(T), \eta(T))) \\ &= \frac{1}{2}\, (\Phi((\widetilde{S} \circ \mathcal{B}\, \eta)(0), \eta(0)) - \Phi((\widetilde{S} \circ \mathcal{B}\, \eta)(T), \eta(T))). \end{aligned} \tag{1.102}$$

Now, let us suppose that the linear system $\ker_{\mathcal{F}}(R.)$ is parametrizable, i.e., the left D-module $M = D^{1\times p}/(D^{1\times q}\, R)$ is torsion-free. Then, there exists a matrix $Q \in D^{p\times m}$ satisfying that $\ker_{\mathcal{F}}(R.) = Q\, \mathcal{F}^m$. Substituting $\eta = Q\, \xi$ into the Lagrangian I, the previous variational problem becomes a variational problem without differential constraint, which can be solved by computing the corresponding Euler-Lagrange equations. Let us illustrate this idea.

Example 1.6.7. We consider again Example 1.6.5. Using Algorithm 1.3.1, we can easily check that the linear OD system (1.99) is parametrizable and an injective parametrization of (1.99) is:

$$\begin{cases} \xi(t) = x(t), \\ \dot{\xi}(t) + \xi(t) = u(t). \end{cases}$$

Substituting the previous parametrization into I, the previous optimization problem is then equivalent to extremizing the following Lagrangian functional

$$I = \int_0^T \frac{1}{2}\, (\xi(t)^2 + (\dot{\xi}(t) + \xi(t))^2)\, dt$$

under the only algebraic constraint $\xi(0) = x_0$. We can easily check that we have

$$\delta I = \int_0^T (-\ddot{\xi}(t) + 2\, \xi(t))\, \delta\xi(t)\, dt + [(\dot{\xi}(t) + \xi(t))\, \delta\xi(t)]_0^T,$$

and thus, the optimal system is equivalent to the following OD linear system:

$$\begin{cases} \ddot{\xi}(t) - 2\, \xi(t) = 0, \quad \xi(0) = x_0, \quad \dot{\xi}(T) + \xi(T) = 0, \\ \xi(t) = x(t), \\ \dot{\xi}(t) + \xi(t) = u(t). \end{cases} \tag{1.103}$$

Integrating (1.103) and eliminating x_0 between x and u, the optimal controller is defined by:

$$u(t) = \left(\sqrt{2}\, \coth \omega - 1\right)^{-1} x(t), \quad \omega = \sqrt{2}\, (t - T), \quad \coth \omega = \frac{e^{\omega} + e^{-\omega}}{e^{\omega} - e^{-\omega}}.$$

Finally, using Example 1.6.5, the bilinear form Φ is defined by $\Phi(\lambda, \eta) = \lambda\, x$, which, using (1.102), yields $I = \frac{1}{2}\, (\lambda(0)\, x_0 - \lambda(T)\, x(T)) = \frac{1}{2}\, \lambda(0)\, x_0$ because $\lambda(T) = 0$. Finally, using $\lambda = -u$ (see Example 1.6.5), the extremum value of the Lagrangian functional is then:

$$I = \frac{1}{2}\, \left(\sqrt{2}\, \coth \omega_0 + 1\right)^{-1} x_0^2, \quad \omega_0 = \sqrt{2}\, T.$$

Corollary 1.6.3 ([93]). *With the previous hypotheses and notations, let us suppose that the linear PD system* $\ker_{\mathcal{F}}(R.)$ *is parametrized by a matrix* $Q \in D^{p\times m}$*, i.e.,* $\ker_{\mathcal{F}}(R.) = Q\,\mathcal{F}^m$*. Then, a necessary condition for the existence of an extremum of the Lagrangian functional*

$$I = \int \frac{1}{2}\,\eta_r^T\, L\,\eta_r\, dx_1\, dx_2 \,\ldots\, dx_n,$$

where $\eta \in \ker_{\mathcal{F}}(R.)$ *and* L *is a symmetric matrix with entries in* A*, is given by*

$$\begin{cases} \mathcal{A}\,\xi = 0, \\ \eta = Q\,\xi. \end{cases} \tag{1.104}$$

where $\mathcal{A} : \mathcal{F}^m \longrightarrow \mathcal{F}^m$ *is the self-adjoint PD operator defined by* $\mathcal{A} = \widetilde{Q} \circ \mathcal{B} \circ Q$*, i.e.,* $\widetilde{\mathcal{A}} = \mathcal{A}$*. Finally, we have the following* twisted exact diagram*:*

$$\begin{array}{ccccc} \mathcal{F}^m & \xrightarrow{Q.} & \mathcal{F}^p & \xrightarrow{R.} & \mathcal{F}^q \\ \downarrow \mathcal{A}. & & \downarrow \mathcal{B}. & & \\ \mathcal{F}^m & \xleftarrow{\widetilde{Q}.} & \mathcal{F}^p & \xleftarrow{\widetilde{R}.} & \mathcal{F}^q. \end{array} \tag{1.105}$$

Example 1.6.8. Let $D = \mathbb{R}[\partial]$, $R \in D^{q\times p}$ and $\mathcal{F} = C^\infty(\mathbb{R}_+)$. Using Proposition 1.6.2, the linear OD system $\ker_{\mathcal{F}}(R.)$ is controllable iff the D-module $M = D^{1\times p}/(D^{1\times q}\,R)$ is a torsion-free. If so, then there exists a matrix $Q \in D^{p\times m}$ satisfying $\ker_{\mathcal{F}}(R.) = Q\,\mathcal{F}^m$. $I f L$ is a symmetric real matrix, then Corollary 1.6.3 shows the optimal system which extremizes $\int_0^{+\infty} \frac{1}{2}\,\eta^T(t)\,L\,\eta(t)\,dt$ is defined by:

$$\begin{cases} \eta = Q\,\xi, \\ \mathcal{A}\,\xi = (\widetilde{Q} \circ L \circ Q)\,\xi = 0. \end{cases}$$

If $\delta = \det(\mathcal{A})$, then $\delta(\partial) = \det(\mathcal{A}(\partial)^T) = \det(\mathcal{A}(-\partial)) = \delta(-\partial)$, and thus the eigenvalues of the dynamics of $\mathcal{A}\,\xi = 0$ are symmetric with respect to the real axis, which leads to the importance concept of *spectral factorization* $\mathcal{A} = \widetilde{\mathcal{D}} \circ \mathcal{D}$ in optimal control problems (see, e.g., [49] and the references therein).

We now show how Corollary 1.6.3 can be applied to the case of the Maxwell equations.

Example 1.6.9. We consider again Example 1.6.6. In Example 1.4.4, we proved that the first set of Maxwell equations (1.45) were parametrized by means of the quadri-potential $(\vec{A}, V)$:

$$\begin{cases} \vec{\nabla} \wedge \vec{A} = \vec{B}, \\ -\dfrac{\partial \vec{A}}{\partial t} - \vec{\nabla}\, V = \vec{E}, \end{cases} \quad \Leftrightarrow \quad \begin{cases} \dfrac{\partial \vec{B}}{\partial t} + \vec{\nabla} \wedge \vec{E} = \vec{0}, \\ \vec{\nabla}\,.\,\vec{B} = 0. \end{cases}$$

The PD operator $\mathcal{A} : \mathcal{F}^4 \longrightarrow \mathcal{F}^4$ is obtained by substituting the previous parametrization into the last two equations of (1.101) and by using the relation $\vec{\nabla} \wedge \vec{\nabla} \wedge \vec{A} = \vec{\nabla}\,(\vec{\nabla}\,.\,\vec{A}) - \Delta\,\vec{A}$. If $c_0^2 = 1/(\epsilon_0\,\mu_0)$, where c_0 is the speed of light in vacuum, then we obtain:

$$(\vec{A}, V) \longmapsto \begin{cases} \dfrac{1}{\mu_0}\left(\dfrac{1}{c_0^2}\,\dfrac{\partial^2 \vec{A}}{\partial t^2} - \Delta\,\vec{A} + \vec{\nabla}\left(\vec{\nabla}\,.\,\vec{A} + \dfrac{1}{c_0^2}\,\dfrac{\partial V}{\partial t}\right)\right) = \vec{\jmath}, \\ \epsilon_0\left(\dfrac{1}{c_0^2}\,\dfrac{\partial^2 V}{\partial t^2} - \Delta\,V - \dfrac{\partial}{\partial t}\left(\vec{\nabla}\,.\,\vec{A} + \dfrac{1}{c_0^2}\,\dfrac{\partial V}{\partial t}\right)\right) = \rho. \end{cases}$$

Then, using to Corollary 1.6.3, the optimal system can be rewritten as (1.104), i.e.:

$$\left\{\begin{array}{l} \dfrac{1}{c_0^2}\,\dfrac{\partial^2 \vec{A}}{\partial t^2} - \Delta\,\vec{A} + \vec{\nabla}\left(\vec{\nabla}\,.\,\vec{A} + \dfrac{1}{c_0^2}\,\dfrac{\partial V}{\partial t}\right) = 0,\\ \dfrac{1}{c_0^2}\,\dfrac{\partial^2 V}{\partial t^2} - \Delta\,V - \dfrac{\partial}{\partial t}\left(\vec{\nabla}\,.\,\vec{A} + \dfrac{1}{c_0^2}\,\dfrac{\partial V}{\partial t}\right) = 0,\\ \vec{\nabla}\wedge\vec{A} = \vec{B},\\ -\dfrac{\partial \vec{A}}{\partial t} - \vec{\nabla}\,V = \vec{E}. \end{array}\right. \tag{1.106}$$

In electromagnetism, the previous equations are generally simplified as follows

$$\left\{\begin{array}{l} \dfrac{1}{c_0^2}\,\dfrac{\partial^2 \vec{A}}{\partial t^2} - \Delta\,\vec{A} = 0,\\ \dfrac{1}{c_0^2}\,\dfrac{\partial^2 V}{\partial t^2} - \Delta\,V = 0,\\ \vec{\nabla}\wedge\vec{A} = \vec{B},\\ -\dfrac{\partial \vec{A}}{\partial t} - \vec{\nabla}\,V = \vec{E}, \end{array}\right. \tag{1.107}$$

by fixing the so-called *Lorenz gauge* defined by $\vec{\nabla}\,.\,\vec{A} + \dfrac{1}{c_0^2}\,\dfrac{\partial V}{\partial t} = 0$. This result shows that each component of the quadri-potential $(\vec{A}, V)$ is a space-time wave. The use of the Lorenz gauge can be explained by the fact that the quadri-potential $(\vec{A}, V)$ is not uniquely defined since:

$$\left\{\begin{array}{l} -\vec{\nabla}\,\xi = \vec{A},\\ \dfrac{\partial \xi}{\partial t} = V, \end{array}\right. \quad \Leftrightarrow \quad \left\{\begin{array}{l} \vec{\nabla}\wedge\vec{A} = \vec{0},\\ -\dfrac{\partial \vec{A}}{\partial t} - \vec{\nabla}\,V = \vec{0}. \end{array}\right.$$

See Example 1.4.4. Hence, if we consider the new potential $(\vec{A}_\star, V_\star) = \left(\vec{A} + \vec{\nabla}\,\xi, V - \partial_t\,\xi\right)$ instead of $(\vec{A}, V)$, where ξ is an arbitrary function of $\mathcal{F} = C^\infty(\Omega)$ and Ω is an open convex subset of $\mathbb{R}^4$, then we can easily check that (1.106) is unchanged but $(\vec{A}, V)$ is replaced by $(\vec{A}_\star, V_\star)$. Moreover, since $\mathcal{F}$ is an injective $D = \mathbb{Q}(\epsilon_0, \mu_0)[\partial_t, \partial_1, \partial_2, \partial_3]$-module, there always exists $\xi \in \mathcal{F}$ satisfying the following inhomogeneous PDE

$$\frac{1}{c_0^2}\,\frac{\partial^2 \xi}{\partial t^2} - \Delta\,\xi = \vec{\nabla}\,.\,\vec{A} + \frac{1}{c_0^2}\,\frac{\partial V}{\partial t},$$

so that the new quadri-potential $(\vec{A}_\star, V_\star)$ satisfies the Lorenz gauge.

Finally, we have the following corollary of Proposition 1.6.3.

Corollary 1.6.4 ([93]). *With the previous hypotheses and notations, if the PD operator $\mathcal{B}$ defined by (1.94) is invertible, then the optimal system (1.97) can be rewritten only in terms of the new variable $\mu = \mathcal{B}\,\eta$ as follows:*

$$\left\{\begin{array}{l} (R \circ \mathcal{B}^{-1})\,\mu = 0,\\ \widetilde{Q}\,\mu = 0. \end{array}\right. \tag{1.108}$$

Moreover, the optimal system (1.96) is equivalent to the following linear PD system

$$\left\{\begin{array}{l} \mathcal{C}\,\lambda = 0, \\ \eta = (\mathcal{B}^{-1} \circ \widetilde{R})\,\lambda, \end{array}\right. \tag{1.109}$$

where the PD operator $\mathcal{C} : \mathcal{F}^q \longrightarrow \mathcal{F}^q$ *is defined by* $\mathcal{C} = R \circ \mathcal{B}^{-1} \circ \widetilde{R}$*:*

$$\begin{array}{ccc} \mathcal{F}^p & \xrightarrow{R.} & \mathcal{F}^q \\ \uparrow \mathcal{B}^{-1} & & \uparrow \mathcal{C} \\ \mathcal{F}^p & \xleftarrow{\widetilde{R}.} & \mathcal{F}^q. \end{array}$$

Example 1.6.10. We consider again Example 1.6.5 where the matrix L_2 is a now supposed to be positive definite. Hence, the operator $\mathcal{B}$ is invertible and $\mathcal{B}^{-1}$ is defined by:

$$\left(\begin{array}{c} x \\ u \end{array}\right) = \mathcal{B}^{-1} \left(\begin{array}{c} \mu_1 \\ \mu_2 \end{array}\right) = \left(\begin{array}{cc} L_1^{-1} & 0 \\ 0 & L_2^{-1} \end{array}\right) \left(\begin{array}{c} \mu_1 \\ \mu_2 \end{array}\right) = \left(\begin{array}{c} L_1^{-1}\,\mu_1 \\ L_2^{-1}\,\mu_2 \end{array}\right). \tag{1.110}$$

According to Corollary 1.6.4, the optimal system (1.98) is equivalent to (1.109), i.e.:

$$\left\{\begin{array}{l} -L_1^{-1}\,\ddot{\lambda} + (A\,L_1^{-1} - L_1^{-1}\,A^T)\,\dot{\lambda} + (A\,L_1^{-1}\,A^T + B\,L_2^{-1}\,B^T)\,\lambda = 0, \\ x = -L_1^{-1}\,(\dot{\lambda} + A^T\,\lambda), \\ u = -L_2^{-1}\,B^T\,\lambda, \\ S\,L_1^{-1}\,(\dot{\lambda}(t_1) + A^T\,\lambda(t_1)) + \lambda(t_1) = 0, \\ \dot{\lambda}(t_0) + A^T\,\lambda(t_0) + L_1\,x_0 = 0. \end{array}\right.$$

For instance, if we consider again the second half of Example 1.6.5, where $L_1 = L_2 = 1$, $A = -1$, $B = 1$, $S = 0$, $t_0 = 0$ and $t_1 = T$, then (1.109) is defined by:

$$\left\{\begin{array}{l} \ddot{\lambda}(t) - 2\,\lambda(t) = 0, \\ x(t) = -\dot{\lambda}(t) + \lambda(t), \\ u(t) = -\lambda(t), \\ \lambda(T) = 0, \\ \dot{\lambda}(0) - \lambda(0) + x_0 = 0. \end{array}\right.$$

The previous results also apply to linear elasticity. Let us consider again Example 1.4.9.

Example 1.6.11. For an isotropic material, the stress-strain relations are defined by

$$\begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{zx} \\ \tau_{xy} \end{pmatrix} = \mathcal{B} \begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{zx} \\ \gamma_{xy} \end{pmatrix}, \quad \mathcal{B} = G \begin{pmatrix} \frac{2(1-\nu)}{1-2\nu} & \frac{2\nu}{1-2\nu} & \frac{2\nu}{1-2\nu} & 0 & 0 & 0 \\ \frac{2\nu}{1-2\nu} & \frac{2(1-\nu)}{1-2\nu} & \frac{2\nu}{1-2\nu} & 0 & 0 & 0 \\ \frac{2\nu}{1-2\nu} & \frac{2\nu}{1-2\nu} & \frac{2(1-\nu)}{1-2\nu} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

where ν is the *Poisson's ratio* and G the *modulus of rigidity*. The linear operator $\mathcal{B}$ is invertible and its inverse $\mathcal{B}^{-1}$ is defined by

$$\begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{zx} \\ \gamma_{xy} \end{pmatrix} = \mathcal{B}^{-1} \begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{zx} \\ \tau_{xy} \end{pmatrix}, \quad \mathcal{B}^{-1} = \begin{pmatrix} \frac{1}{E} & -\frac{\nu}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ -\frac{\nu}{E} & \frac{1}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ -\frac{\nu}{E} & -\frac{\nu}{E} & \frac{1}{E} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G} \end{pmatrix},$$

where E is *Young's modulus* defined by $E = 2\,G\,(1+\nu)$. Using the constitutive law $\mathcal{B}$, the notations and the results of Example 1.4.9 and $\widetilde{P} = -P^T$, $\widetilde{Q} = Q^T$ and $\widetilde{R} = -R^T$, we obtain the following twisted exact diagram

$$\begin{array}{ccccccccccccc} 0 \longrightarrow & \ker_{\mathcal{F}}(P.) & \longrightarrow & \mathcal{F}^3 & \xrightarrow{P.} & \mathcal{F}^6 & \xrightarrow{Q.} & \mathcal{F}^6 & \xrightarrow{R.} & \mathcal{F}^3 & \longrightarrow & 0 & \\ & & & \downarrow \mathcal{A}. & & \downarrow \mathcal{B}. & & \uparrow \mathcal{C}. & & \uparrow \mathcal{D}. & & & \\ & 0 & \longleftarrow & \mathcal{F}^3 & \xleftarrow{\widetilde{P}.} & \mathcal{F}^6 & \xleftarrow{\widetilde{Q}.} & \mathcal{F}^6 & \xleftarrow{\widetilde{R}.} & \mathcal{F}^3 & \longleftarrow & \ker_{\mathcal{F}}(\widetilde{R}.) & \longleftarrow 0, \end{array}$$

where $\mathcal{A} = \widetilde{P} \circ \mathcal{B} \circ P$, $\mathcal{C} = Q \circ \mathcal{B}^{-1} \circ \widetilde{Q}$ and $\mathcal{D} = R \circ \mathcal{C} \circ \widetilde{R} = 0$. More precisely, if $\Delta = \partial_x^2 + \partial_y^2 + \partial_z^2$, then the PD operator $\mathcal{A}$ is defined by:

$$-\frac{G}{(1-2\nu)} \begin{pmatrix} (1-2\nu)\,\Delta + \partial_x^2 & \partial_x\,\partial_y & \partial_x\,\partial_z \\ \partial_x\,\partial_y & (1-2\nu)\,\Delta + \partial_y^2 & \partial_y\,\partial_z \\ \partial_x\,\partial_z & \partial_y\,\partial_z & (1-2\nu)\,\Delta + \partial_z^2 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = 0.$$

In other words, we have $\mathcal{A} = -G\left(\Delta\, I_3 + \frac{1}{(1-2\nu)}\,\mathrm{grad}\,\mathrm{div}\right)$, where $\mathrm{div} = (\partial_x \quad \partial_y \quad \partial_z) = \mathrm{grad}^T$, or $\mathcal{A} = -(\mu\,\Delta\, I_3 + (\lambda+\mu)\,\mathrm{grad}\,\mathrm{div})$, whenever λ and μ are the two *Lamé constants* defined by:

$$\lambda = \frac{E\,\nu}{(1-2\nu)\,(1+\nu)}, \quad \mu = \frac{E}{2\,(1+\nu)} = G.$$

If $\xi = (u \quad v \quad w)^T$ is the displacement and $f = (f_1 \quad f_2 \quad f_3)$ the density of forces acting on the continuous medium, then the PD operator $\mathcal{A}\,\xi = f$ is usually called the *Lamé-Navier operator*.

Let us now explain how the *Lamé-Navier equations* appear the theory of elasticity. The equation of equilibrium is defined by $\widetilde{P}\,\sigma = f$, where $\sigma = (\sigma_x \quad \sigma_y \quad \sigma_z \quad \tau_{yz} \quad \tau_{zx} \quad \tau_{xy})^T$. If there is no density of forces, i.e., $f = 0$, then according to Proposition 1.6.3 and Corollary 1.6.3, the extremization of the energy of deformation defined by the following Lagrangian functional

$$\int \frac{1}{2}\,\epsilon^T\,\mathcal{B}\,\epsilon\,dx\,dy\,dz, \quad \epsilon = (\varepsilon_x \quad \varepsilon_y \quad \varepsilon_z \quad \gamma_{yz} \quad \gamma_{zx} \quad \gamma_{xy})^T,$$

under the PD constraint $Q\,\epsilon = 0$ gives the following equivalent linear PD systems:

$$\left\{\begin{array}{l} Q\,\epsilon = 0, \\ \mathcal{B}\,\epsilon - \widetilde{Q}\,\lambda = 0, \end{array}\right. \quad \Leftrightarrow \quad \left\{\begin{array}{l} Q\,\epsilon = 0, \\ (\widetilde{P} \circ \mathcal{B})\,\epsilon = 0, \end{array}\right. \quad \Leftrightarrow \quad \left\{\begin{array}{l} \mathcal{A}\,\xi = 0, \\ \epsilon = P\,\xi. \end{array}\right. \tag{1.111}$$

Using Algorithms 1.3.1 and 1.3.2, we can prove that the $D = \mathbb{Q}(G,\nu)[\partial_x, \partial_y, \partial_z]$-module associated with the PD operator $\mathcal{A}$ is torsion and the components u, v and w of the displacement ξ satisfy $\Delta^2\,u = 0$, $\Delta^2\,v = 0$ and $\Delta^2\,w = 0$, i.e., u, v and w are three biharmonic functions.

Since the constitution law $\mathcal{B}$ is invertible, the second system in the above chain of equivalences shows that the optimal system (1.111) can be expressed only in terms of the stress tensor $\sigma \triangleq (\sigma_x \quad \sigma_y \quad \sigma_z \quad \tau_{yz} \quad \tau_{zx} \quad \tau_{xy}) = \mathcal{B}\,\epsilon$ as follows:

$$\left\{\begin{array}{l} (Q \circ \mathcal{B}^{-1})\,\sigma = 0, \\ \widetilde{P}\,\sigma = 0. \end{array}\right. \tag{1.112}$$

In the forthcoming Example 3.4.2, we shall prove that (1.112) is equivalent to:

$$\left\{\begin{array}{l}
\Delta\,\sigma_x + \dfrac{1}{(1+\nu)}\,\dfrac{\partial^2}{\partial x^2}\,(\sigma_x + \sigma_y + \sigma_z) = 0, \\[2mm]
\Delta\,\sigma_y + \dfrac{1}{(1+\nu)}\,\dfrac{\partial^2}{\partial y^2}\,(\sigma_x + \sigma_y + \sigma_z) = 0, \\[2mm]
\Delta\,\sigma_z + \dfrac{1}{(1+\nu)}\,\dfrac{\partial^2}{\partial z^2}\,(\sigma_x + \sigma_y + \sigma_z) = 0, \\[2mm]
\Delta\,\tau_{yz} + \dfrac{1}{(1+\nu)}\,\dfrac{\partial^2}{\partial y\,\partial z}\,(\sigma_x + \sigma_y + \sigma_z) = 0, \\[2mm]
\Delta\,\tau_{zx} + \dfrac{1}{(1+\nu)}\,\dfrac{\partial^2}{\partial z\,\partial x}\,(\sigma_x + \sigma_y + \sigma_z) = 0, \\[2mm]
\Delta\,\tau_{xy} + \dfrac{1}{(1+\nu)}\,\dfrac{\partial^2}{\partial x\,\partial y}\,(\sigma_x + \sigma_y + \sigma_z) = 0, \\[2mm]
\dfrac{\partial \sigma_x}{\partial x} + \dfrac{\partial \tau_{zx}}{\partial z} + \dfrac{\partial \tau_{xy}}{\partial y} = 0, \\[2mm]
\dfrac{\partial \sigma_y}{\partial y} + \dfrac{\partial \tau_{yz}}{\partial z} + \dfrac{\partial \tau_{xy}}{\partial x} = 0, \\[2mm]
\dfrac{\partial \sigma_z}{\partial z} + \dfrac{\partial \tau_{yz}}{\partial y} + \dfrac{\partial \tau_{zx}}{\partial x} = 0.
\end{array}\right. \tag{1.113}$$

The first six equations of (1.113) are called the *Beltrami-Michell equations* and the last three ones are the *equilibrium equations*. Using Algorithms 1.3.1 and 1.3.2, we can prove that the

D-module associated with (1.112) is torsion and each component σ_i of σ satisfies $\Delta^2\,\sigma_i = 0$ for $i = 1, \ldots, 6$. Hence, we have $\Delta^2\,\sigma = 0$ and, since $\sigma = \mathcal{B}\,\epsilon$ and $\mathcal{B}$ is invertible, we also get $\Delta^2\,\epsilon = 0$, i.e., both the strain and stress tensors are biharmonic tensors.

Substituting the parametrization $\sigma = \widetilde{Q}\,\lambda$ of the linear PD system $\ker_{\mathcal{F}}(\widetilde{P}.)$ in (1.112), we obtain the following linear PD system depending only on the Lagrangian multiplier λ:

$$\left\{ \begin{array}{l} \mathcal{C}\,\lambda = 0, \\ \epsilon = (\mathcal{B}^{-1} \circ \widetilde{Q})\,\lambda. \end{array} \right. \tag{1.114}$$

See Corollary 1.6.4. Using again Algorithms 1.3.1 and 1.3.2, we can prove that the D-module associated with the PD operator $\mathcal{C}$ is torsion and the components λ_i's of λ satisfy $\Delta^2\,\lambda_i = 0$ for $i = 1, \ldots, 6$, i.e., the components of λ are also biharmonic functions.

Finally, (1.114) can be simplified by considering a minimal parametrization of the equilibrium system $\ker_{\mathcal{F}}(\widetilde{P}.)$ such as Maxwell's or Morera's parametrization (see Example 1.4.9):

1. If we consider Maxwell's parametrization (1.62) of (1.61) obtained by selecting the first three columns of the formal adjoint $\widetilde{Q}$ of Q defined in Example 1.4.9, namely,

$$\widetilde{Q}_1 = \begin{pmatrix} 0 & \partial_z^2 & \partial_y^2 \\ \partial_z^2 & 0 & \partial_x^2 \\ \partial_y^2 & \partial_x^2 & 0 \\ -\partial_y\,\partial_z & 0 & 0 \\ 0 & -\partial_x\,\partial_z & 0 \\ 0 & 0 & -\partial_x\,\partial_y \end{pmatrix},$$

i.e., $\sigma = \widetilde{Q_1}\,\chi$ and χ is Maxwell's stress function, then we obtain the twisted exact diagram

$$\begin{tikzcd} 0 \arrow[r] & \ker_{\mathcal{F}}(P.) \arrow[r] & \mathcal{F}^3 \arrow[r, "P."] \arrow[d, "\mathcal{A}."] & \mathcal{F}^6 \arrow[r, "Q."] & \mathcal{F}^6 \arrow[r, "R."] & \mathcal{F}^6 \arrow[r] & 0 \\ & 0 & \mathcal{F}^3 \arrow[l] & \mathcal{F}^6 \arrow[l, "\widetilde{P}."'] \arrow[u, "\mathcal{B}^{-1}."] & \mathcal{F}^3 \arrow[l, "\widetilde{Q_1}."'] \arrow[u, "\mathcal{C}_1."] & \ker_{\mathcal{F}}(\widetilde{Q_1}.) \arrow[l] \arrow[u, "\mathcal{D}_1."] & 0, \arrow[l] \end{tikzcd}$$

where $\mathcal{C}_1 = Q \circ \mathcal{B}^{-1} \circ \widetilde{Q_1}$ and $\mathcal{D}_1 = 0$. Then, (1.112) is equivalent to the following system:

$$\left\{ \begin{array}{l} \mathcal{C}_1\,\chi = 0, \\ \epsilon = (\mathcal{B}^{-1} \circ \widetilde{Q_1})\,\chi. \end{array} \right.$$

2. If we now consider Morera's parametrization (1.63) of (1.61) obtained by selecting the last three columns of the formal adjoint $\widetilde{Q}$ of Q defined in Example 1.4.9, namely,

$$\widetilde{Q}_2 = \begin{pmatrix} -\partial_y\,\partial_z & 0 & 0 \\ 0 & -\partial_x\,\partial_z & 0 \\ 0 & 0 & -\partial_x\,\partial_y \\ -\frac{1}{2}\,\partial_x^2 & \frac{1}{2}\,\partial_x\,\partial_y & \frac{1}{2}\,\partial_x\,\partial_z \\ \frac{1}{2}\,\partial_x\,\partial_y & -\frac{1}{2}\,\partial_y^2 & \frac{1}{2}\,\partial_y\,\partial_z \\ \frac{1}{2}\,\partial_x\,\partial_z & \frac{1}{2}\,\partial_y\,\partial_z & -\frac{1}{2}\,\partial_z^2 \end{pmatrix},$$

i.e., $\sigma = \widetilde{Q_2}\, \psi$ and ψ is Morera's stress function, then we obtain the twisted exact diagram

$$\begin{array}{ccccccccccc}
0 \longrightarrow & \ker_{\mathcal{F}}(P.) & \longrightarrow & \mathcal{F}^3 & \xrightarrow{P.} & \mathcal{F}^6 & \xrightarrow{Q.} & \mathcal{F}^6 & \xrightarrow{R.} & \mathcal{F}^6 & \longrightarrow 0 \\
& & & \downarrow \mathcal{A}. & & \uparrow \mathcal{B}^{-1}. & & \uparrow \mathcal{C}_2. & & \uparrow \mathcal{D}_2. & \\
& 0 & \longleftarrow & \mathcal{F}^3 & \xleftarrow{\widetilde{P}.} & \mathcal{F}^6 & \xleftarrow{\widetilde{Q_2}.} & \mathcal{F}^3 & \longleftarrow & \ker_{\mathcal{F}}(\widetilde{Q_2}.) & \longleftarrow 0,
\end{array}$$

where $\mathcal{C}_2 = Q \circ \mathcal{B}^{-1} \circ \widetilde{Q_2}$ and $\mathcal{D}_2 = 0$. Then, (1.112) is equivalent to the following system:

$$\left\{ \begin{array}{l} \mathcal{C}_2\, \psi = 0, \\ \epsilon = (\mathcal{B}^{-1} \circ \widetilde{Q_2})\, \psi. \end{array} \right.$$

Finally, for more results, details and examples on constructive algebraic analysis and its applications to mathematical systems theory and mathematical physics, see [100].

Chapter 2

Monge parametrizations and purity filtration

"La structure d'une chose n'est nullement une chose que nous puissions "inventer". Nous pouvons seulement la mettre à jour patiemment, humblement en faire connaissance, la "**découvrir**". S'il y a inventivité dans ce travail, et s'il nous arrive de faire œuvre de forgeron ou d'infatigable bâtisseur, ce n'est nullement pour "façonner", ou pour "bâtir", des "structures". Celles-ci ne nous ont nullement attendus pour être, et pour être exactement ce qu'elles sont ! Mais c'est pour **exprimer**, le plus fidèlement que nous le pouvons, ces choses que nous sommes en train de découvrir et de sonder, et cette structure réticente à se livrer, que nous essayons à tâtons, et par un langage encore balbutiant peut-être, à cerner. Ainsi sommes-nous amenés à constamment "**inventer**" **le langage** apte à exprimer de plus en plus finement la structure intime de la chose mathématique, et à "construire" à l'aide de ce langage, au fur et à mesure et de toutes pièces, les "théories" qui sont censées rendre compte de ce qui a été appréhendé et vu. Il y a là un mouvement de va-et-vient continuel, ininterrompu, entre **l'appréhension** des choses, et **l'expression** de ce qui est appréhendé, par un langage qui s'affine et se re-crée au fil du travail, sous la constante pression du besoin immédiat".

Alexandre Grothendieck, *Récoltes et Semailles, Réflexions et témoignage sur un passé de mathématicien.*

2.1 Baer's extensions and Baer's isomorphism

In Chapter 1, we showed how to compute $\mathrm{ext}^1_D(M, D)$, whenever M was a finitely presented left or right D-module. In this section, we study the abelian group $\mathrm{ext}^1_D(M, N)$, when M and N are two finitely presented left D-modules. Moreover, we explain Baer's interpretation of the elements of $\mathrm{ext}^1_D(M, N)$ in terms of equivalence classes of short exact sequences of the form

$$0 \longrightarrow N \xrightarrow{f} E \xrightarrow{g} M \longrightarrow 0$$

for a certain equivalence relation. In particular, we explicitly parametrize all the possible left D-modules E. The results developed in this section will be abundantly used in the next sections. They are important techniques for the study of mathematical systems theory.

We first introduce the concept of *Baer extensions*. For more details, see, e.g., [15, 27, 65, 109].

Definition 2.1.1. 1. Let M and N be two left D-modules. An *extension of N by M* is a short exact sequence e of left D-modules of the form:

$$e : 0 \longrightarrow N \xrightarrow{f} E \xrightarrow{g} M \longrightarrow 0. \tag{2.1}$$

2. Two extensions of N by M, $e_i : 0 \longrightarrow N \xrightarrow{f_i} E_i \xrightarrow{g_i} M \longrightarrow 0$ for $i = 1, 2$, are said to be *equivalent* and denoted by $e_1 \sim e_2$ if there exists a left D-homomorphism $\phi : E_1 \longrightarrow E_2$ such that the following commutative exact diagram holds

$$\begin{array}{ccccccc} 0 \longrightarrow & N & \xrightarrow{f_1} & E_1 & \xrightarrow{g_1} & M & \longrightarrow 0 \\ & \| & & \downarrow \phi & & \| & \\ 0 \longrightarrow & N & \xrightarrow{f_2} & E_2 & \xrightarrow{g_2} & M & \longrightarrow 0, \end{array}$$

i.e., such that $f_2 = \phi \circ f_1$ and $g_1 = g_2 \circ \phi$.

3. We denote by $[e]$ the equivalence class of the extension e for the equivalence relation $\sim$. The set of all equivalence classes of extensions of N by M is denoted by $\mathrm{e}_D(M, N)$.

Remark 2.1.1. Applying the *snake lemma* to the commutative exact diagram defined in 2 of Definition 2.1.1 (see e.g., [15, 27, 65, 109]), we obtain that the left D-homomorphism ϕ defined in 2 of Definition 2.1.1 is necessarily an isomorphism. Hence, we can easily check that $\sim$ is an equivalence relation (see 3 of Definition 2.1.1).

We point out that two extensions of N by M, $e_i : 0 \longrightarrow N \xrightarrow{f_i} E_i \xrightarrow{g_i} M \longrightarrow 0$ for $i = 1, 2$, where $E_1 \cong E_2$ are not necessarily equivalent because if $\phi : E_1 \longrightarrow E_2$ is a left D-isomorphism, then the conditions $f_2 = \phi \circ f_1$ and $g_1 = g_2 \circ \phi$ are not necessarily satisfied.

Let us illustrate Definition 2.1.1 with a simple but important example.

Example 2.1.1. Let us consider an extension e of N by M defining the split short exact sequence (1.8) where $M' = N$, $M = E$ and $M'' = M$ (see 7 of Definition 1.2.1). Then, we have the following commutative exact diagram

$$\begin{array}{ccccccc} 0 \longrightarrow & N & \xrightarrow{f} & E & \xrightarrow{g} & M & \longrightarrow 0 \\ & \| & & \downarrow (k, g) & & \| & \\ 0 \longrightarrow & N & \xrightarrow{i_1} & N \oplus M & \xrightarrow{p_2} & M & \longrightarrow 0, \end{array}$$

with the following notations:

$$\begin{array}{ccccccc} e' : 0 \longrightarrow & N & \xrightarrow{i_1} & N \oplus M & \xrightarrow{p_2} & M & \longrightarrow 0. \\ & n & \longmapsto & (n, 0) & & & \\ & & & (n, m) & \longmapsto & m & \end{array}$$

We obtain that the extensions e and e' of N by M are equivalent, i.e., $[e] = [e'] \in \mathrm{e}_D(M, N)$.

Let us introduce the concept of *Baer sum* of two extensions.

Definition 2.1.2 ([15]). Let $e_i : 0 \longrightarrow N \xrightarrow{f_i} E_i \xrightarrow{g_i} M \longrightarrow 0$ for $i = 1, 2$ be two extensions of N by M and let us define the following two left D-homomorphisms:

$$\begin{array}{rcl} -f_1 \oplus f_2 : N & \longrightarrow & E_1 \oplus E_2 \\ n & \longmapsto & (-f_1(n),\, f_2(n)) \end{array} \qquad \begin{array}{rcl} (g_1, -g_2) : E_1 \oplus E_2 & \longrightarrow & M \\ (a_1,\, a_2) & \longmapsto & g_1(a_1) - g_2(a_2). \end{array}$$

Then, the *Baer sum* of the extensions e_1 and e_2, denoted by $e_1 + e_2$, is defined by the left D-module $E_3 = \ker(g_1,\, -g_2)/\mathrm{im}\,(-f_1 \oplus f_2)$, i.e., by the equivalence class of the following extension

$$\begin{array}{ccccccc} 0 \longrightarrow & N & \xrightarrow{f_3} & E_3 & \xrightarrow{g_3} & M & \longrightarrow 0, \\ & n & \longmapsto & \varpi((f_1(n),\, 0)) = \varpi((0,\, f_2(n))) & & & \\ & & & \varpi((a_1,\, a_2)) & \longmapsto & g_1(a_1) = g_2(a_2) & \end{array}$$

where $\varpi : \ker(g_1,\, -g_2) \longrightarrow E_3$ is the canonical projection onto E_3.

We note that E_3 is exactly the defect of exactness of the following complex at $E_1 \oplus E_2$:

$$0 \longrightarrow N \xrightarrow{-f_1 \oplus f_2} E_1 \oplus E_2 \xrightarrow{(g_1,\, -g_2)} M \longrightarrow 0.$$

The Baer sum can also be defined using the concepts of *pullback* and *pushout* ([27, 109]).

The following classical result on extensions can be traced back to Baer's work [3].

Theorem 2.1.1 ([15, 65, 109])**.** *The set* $\mathrm{e}_D(M, N)$ *equipped with the Baer sum forms an abelian group: the equivalence class of the split short exact sequence (1.8) defines the zero element of* $\mathrm{e}_D(M, N)$ *and the inverse of the equivalence class* $[e]$ *of (2.1) is defined by the equivalence class of the following equivalent extensions:*

$$0 \longrightarrow N \xrightarrow{-f} E \xrightarrow{g} M \longrightarrow 0, \qquad 0 \longrightarrow N \xrightarrow{f} E \xrightarrow{-g} M \longrightarrow 0.$$

The next theorem is an important result of homological algebra.

Theorem 2.1.2 ([65, 109])**.** *Let M and N be two left D-modules. Then, the abelian groups* $\mathrm{ext}^1_D(M, N)$ *and* $\mathrm{e}_D(M, N)$ *are isomorphic, i.e.:*

$$\mathrm{e}_D(M, N) \cong \mathrm{ext}^1_D(M, N).$$

Similarly for right D-modules.

We note that Theorem 2.1.2 explains the etymology of the name of the bifunctor $\mathrm{ext}^1_D(\,\cdot\,,\,\cdot\,)$.

Similar interpretations of the $\mathrm{ext}^i_D(M, N)$'s for $i \geq 2$ can be found in [118] (see also [27]).

In what follows, we shall assume that D is a noetherian domain.

Let us explicitly characterize the abelian group $\mathrm{ext}^1_D(M, N)$ for two finitely presented left D-modules $M = D^{1\times p}/(D^{1\times q}\, R)$ and $N = D^{1\times s}/(D^{1\times t}\, S)$. We first consider the beginning of a finite free resolution of the left D-module M:

$$D^{1\times r} \xrightarrow{.R_2} D^{1\times q} \xrightarrow{.R} D^{1\times p} \xrightarrow{\pi} M \longrightarrow 0. \tag{2.2}$$

Applying the contravariant left exact functor $\hom_D(\,\cdot\,, N)$ to the exact sequence (2.2), we get the following complex of abelian groups (see Section 1.2)

$$N^r \xleftarrow{R_2.} N^q \xleftarrow{R.} N^p \longleftarrow \hom_D(M, N) \longleftarrow 0, \tag{2.3}$$

where $(R_i.)(\eta) = R_i\, \eta$ for $i = 1, 2$. In particular, we have:

$$\mathrm{ext}^1_D(M, N) \cong \ker_N(R_2.)/\mathrm{im}_N(R.).$$

We recall that the abelian group $\mathrm{ext}^1_D(M,N)$ characterizes the obstructions for the existence of $\xi \in N^p$ satisfying the inhomogeneous linear system $R\,\xi = \zeta$, where ζ is a fixed element of N^q verifying the compatibility conditions $R_2\,\zeta = 0$. Hence, the vanishing of $\mathrm{ext}^1_D(M,N)$ implies that $R_2\,\zeta = 0$ is a necessary and sufficient condition for the existence of $\xi \in N^p$ satisfying:

$$R\,\xi = \zeta.$$

Let us explicitly characterize $\mathrm{ext}^1_D(M,N)$. If we consider a finite presentation of N

$$D^{1\times t} \xrightarrow{.S} D^{1\times s} \xrightarrow{\delta} N \longrightarrow 0, \tag{2.4}$$

then, taking the direct sum of m copies of (2.4), we obtain the following exact sequence

$$D^{m\times t} \xrightarrow{.S} D^{m\times s} \xrightarrow{\mathrm{id}_m\otimes\delta} N^m \longrightarrow 0, \tag{2.5}$$

where $(\mathrm{id}_m \otimes \delta)(\Lambda) = (\delta(\Lambda_{1\bullet}) \ \ldots \ \delta(\Lambda_{m\bullet}))^T$ for all $\Lambda \in D^{m\times s}$. We say that (2.5) is obtained by applying the *covariant exact functor* $D^m \otimes_D \cdot$ ([15, 65, 109]) to (2.4). This functor is exact since D^m is a free right D-module (and thus, a *flat* right D-module) ([54, 109]). Then, combining (2.3) and (2.5), we get the following commutative diagram of abelian groups with exact columns:

$$\begin{array}{ccccc}
0 & & 0 & & 0\\
\uparrow & & \uparrow & & \uparrow\\
N^r & \xleftarrow{R_2.} & N^q & \xleftarrow{R.} & N^p\\
\uparrow \mathrm{id}_r\otimes\delta & & \uparrow \mathrm{id}_q\otimes\delta & & \uparrow \mathrm{id}_p\otimes\delta\\
D^{r\times s} & \xleftarrow{R_2.} & D^{q\times s} & \xleftarrow{R.} & D^{p\times s}\\
\uparrow .S & & \uparrow .S & & \uparrow .S\\
D^{r\times t} & \xleftarrow{R_2.} & D^{q\times t} & \xleftarrow{R.} & D^{p\times t}.
\end{array} \tag{2.6}$$

Indeed, for every $\Lambda \in D^{q\times s}$, we have

$$\begin{aligned}
R_2(\mathrm{id}_q\otimes\delta)(\Lambda)) = R_2 \begin{pmatrix} \delta(\Lambda_{1\bullet}) \\ \vdots \\ \delta(\Lambda_{q\bullet}) \end{pmatrix} &= \begin{pmatrix} \sum_{j=1}^q (R_2)_{1j}\,\delta(\Lambda_{j\bullet}) \\ \vdots \\ \sum_{j=1}^q (R_2)_{rj}\,\delta(\Lambda_{j\bullet}) \end{pmatrix} = \begin{pmatrix} \delta\left(\sum_{j=1}^q (R_2)_{1j}\,\Lambda_{j\bullet}\right) \\ \vdots \\ \delta\left(\sum_{j=1}^q (R_2)_{rj}\,\Lambda_{j\bullet}\right) \end{pmatrix}\\
&= (\mathrm{id}_r \otimes \delta)(R_2\,\Lambda),
\end{aligned}$$

i.e., we have $(R_2.)\circ(\mathrm{id}_q\otimes\delta) = (\mathrm{id}_r\otimes\delta)\circ(R_2.)$. Similarly, we have $(R.)\circ(\mathrm{id}_p\otimes\delta) = (\mathrm{id}_q\otimes\delta)\circ(R.)$. Now, for every $\Gamma \in D^{q\times t}$, $(R_2. \circ .S)(\Gamma) = R_2\,(\Gamma\,S) = R_2\,\Gamma\,S = (R_2\,\Gamma)\,S = (.S \circ R_2.)(\Gamma)$, which shows that $R_2. \circ .S = .S \circ R_2.$. Similarly, we have $R. \circ .S = .S \circ R.$, which proves that (2.6) is a commutative diagram whose columns are exact.

We can now use the commutative diagram (2.6) to characterize the following abelian groups:

$$\begin{aligned}
\ker_N(R_2.) &= \{(\mathrm{id}_q\otimes\delta)(A) \in N^q \mid A \in D^{q\times s}:\ R_2\,((\mathrm{id}_q\otimes\delta)(A)) = 0\},\\
\mathrm{im}_N(R.) &= \{(\mathrm{id}_q\otimes\delta)(A) \in N^q \mid A \in D^{q\times s} : \exists\, X \in D^{p\times s},\ (\mathrm{id}_q\otimes\delta)(A) = R\,((\mathrm{id}_p\otimes\delta)(X))\}.
\end{aligned}$$

Since the columns of (2.6) are exact sequences of left D-modules, we get:

$$R_2((\mathrm{id}_q\otimes\delta)(A)) = (\mathrm{id}_r\otimes\delta)(R_2\,A) = 0 \ \Leftrightarrow\ \exists\, B \in D^{r\times t}:\ R_2\,A = B\,S.$$

$$\begin{aligned}
(\mathrm{id}_q\otimes\delta)(A) = R\,((\mathrm{id}_p\otimes\delta)(X)) = (\mathrm{id}_q\otimes\delta)(R\,X) &\Leftrightarrow\ (\mathrm{id}_q\otimes\delta)(A - R\,X) = 0\\
&\Leftrightarrow\ \exists\, Y \in D^{q\times t}:\ A = R\,B + Y\,S.
\end{aligned}$$

Lemma 2.1.1. *With the previous notations, we have:*

$$\begin{aligned}\ker_N(R_2.) &= \{(\mathrm{id}_q \otimes \delta)(A) \in N^q \mid A \in D^{q\times s} : \exists\, B \in D^{r\times t},\ R_2\, A = B\, S\},\\ \mathrm{im}_N(R.) &= \{(\mathrm{id}_q \otimes \delta)(A) \in N^q \mid A \in D^{q\times s} :\ \exists\, X \in D^{p\times s},\ \exists\, Y \in D^{q\times t},\ A = R\, X + Y\, S\}\\ &= (R\, D^{p\times s} + D^{q\times t}\, S)/(D^{q\times t}\, S).\end{aligned}$$

If we introduce the following abelian group

$$\Omega = \{A \in D^{q\times s} \mid \exists\, B \in D^{r\times t} :\ R_2\, A = B\, S\}, \tag{2.7}$$

then we have the following isomorphism of abelian groups

$$\begin{array}{rcl}\mathrm{ext}^1_D(M,N) \cong \ker_N(R_2.)/\mathrm{im}_N(R.) & \xrightarrow{\upsilon} & \Omega/(R\, D^{p\times s} + D^{q\times t}\, S),\\ \rho((\mathrm{id}_q \otimes \delta)(A)) & \longmapsto & \epsilon(A),\end{array} \tag{2.8}$$

where $A \in \Omega$, $\rho : \ker_N(R_2.) \longrightarrow \ker_N(R_2.)/\mathrm{im}_N(R.)$ *and* $\epsilon : \Omega \longrightarrow \Omega/(R\, D^{p\times s} + D^{q\times t}\, S)$ *are the respective canonical projections.*

The proof of Lemma 2.1.1 is just a straightforward application of the classical *third isomorphism theorem* in module theory (see, e.g., [109]), namely

$$\begin{aligned}\mathrm{ext}^1_D(M,N) \cong \ker_N(R_2.)/\mathrm{im}_N(R.) &= [\Omega/(D^{q\times t}\, S)]/[(R\, D^{p\times s} + D^{q\times t}\, S)/(D^{q\times t}\, S)]\\ &\cong \Omega/(R\, D^{p\times s} + D^{q\times t}\, S),\end{aligned}$$

for all finitely presented left D-modules $M = D^{1\times p}/(D^{1\times q}\, R)$ and $N = D^{1\times s}/(D^{1\times t}\, S)$.

Remark 2.1.2. If $\ker_D(.R) = 0$, i.e., $R_2 = 0$, then Lemma 2.1.1 yields $\Omega = D^{q\times s}$.

In [104, 105], we explicitly characterized the isomorphism $\mathrm{e}_D(M,N) \cong \Omega/(R\, D^{p\times s} + D^{q\times t}\, S)$ and obtained the next theorem which exhibits a representative of each equivalence class of Baer's extensions of N by M in terms of $\epsilon(A) \in \Omega/(R\, D^{p\times s} + D^{q\times t}\, S)$.

Theorem 2.1.3 ([104, 105])**.** *Let* $M = D^{1\times p}/(D^{1\times q}\, R)$ *and* $N = D^{1\times s}/(D^{1\times t}\, S)$ *be two finitely presented left* D*-modules and* $R_2 \in D^{r\times q}$ *satisfying* $\ker_D(.R) = D^{1\times r}\, R_2$*. Then, every equivalence class of extensions of* N *by* M *is defined by the following extension of* N *by* M

$$e:\ 0 \longrightarrow N \xrightarrow{\alpha} E \xrightarrow{\beta} M \longrightarrow 0, \tag{2.9}$$

where the left D*-module* E *is defined by*

$$D^{1\times(q+t)} \xrightarrow{.Q} D^{1\times(p+s)} \xrightarrow{\varrho} E \longrightarrow 0, \quad Q = \begin{pmatrix} R & -A \\ 0 & S \end{pmatrix} \in D^{(q+t)\times(p+s)}, \tag{2.10}$$

A *is a certain element of the abelian group* $\Omega = \{A \in D^{q\times s} \mid \exists\, B \in D^{r\times t} : R_2\, A = B\, S\}$ *and*

$$\begin{array}{rclcrcl}\alpha : N & \longrightarrow & E & \qquad & \beta : E & \longrightarrow & M\\ \delta(\mu) & \longmapsto & \varrho(\mu\,(0 \quad I_s)), & & \varrho(\lambda) & \longmapsto & \pi(\lambda\,(I_p \quad 0)^T),\end{array}$$

where $\pi : D^{1\times p} \longrightarrow M$ *(resp.,* $\delta : D^{1\times s} \longrightarrow N$, $\varrho : D^{1\times(p+s)} \longrightarrow E$*) is the canonical projection onto* E *(resp.,* N, E*).*

The equivalence class $[e]$ *depends only on the residue class* $\epsilon(A)$ *of* $A \in \Omega$ *in the abelian group* $\Omega/(R\, D^{p\times s} + D^{q\times t}\, S) = \upsilon(\mathrm{ext}^1_D(M,N))$*, where* υ *is the* $\mathbb{Z}$*-isomorphism defined by (2.8).*

Theorem 2.1.3 will be illustrated in what follows. Let us characterize the matrices $A \in \Omega$ defining the left D-module E defined in Theorem 2.1.3.

Corollary 2.1.1 ([104]). *With the notations of Theorem 2.1.3, if we consider an extension of* $N = D^{1\times s}/(D^{1\times t}\, S)$ *by* $M = D^{1\times p}/(D^{1\times q}\, R)$ *defined by*

$$0 \longrightarrow N \xrightarrow{u} F \xrightarrow{v} M \longrightarrow 0, \tag{2.11}$$

and if $\{f_j\}_{j=1,\dots,p}$ *is the standard basis of* $D^{1\times p}$, $y_j = \pi(f_j)$ *for all* $j = 1, \dots, p$, $z_j \in F$ *any pre-image of* y_j *under* v, *then* $\sum_{j=1}^{p} R_{ij}\, z_j \in \operatorname{im} u$ *for all* $i = 1, \dots, q$, *and, since* u *is injective, there exists a unique* $n_i \in N$ *satisfying* $u(n_i) = \sum_{j=1}^{p} R_{ij}\, z_j$. *If we consider any pre-image* $a_i \in D^{1\times s}$ *of* n_i *under* δ, *i.e.,* $n_i = \delta(a_i)$ *for all* $i = 1, \dots, q$, *then the extension (2.11) of* N *by* M *belongs to the same equivalence class of (2.9), where the left* D*-module* E *is defined by (2.10) with:*

$$A = \begin{pmatrix} a_1 \\ \vdots \\ a_q \end{pmatrix} \in D^{q\times s}.$$

Equivalently, we have the following commutative exact diagram

$$\begin{array}{ccccccccc} & & D^{1\times q} & \xrightarrow{.R} & D^{1\times p} & \xrightarrow{\pi} & M & \longrightarrow 0 \\ & & \downarrow \phi & & \downarrow \psi & & \| & \\ 0 \longrightarrow & & N & \xrightarrow{u} & F & \xrightarrow{v} & M & \longrightarrow 0, \end{array}$$

where the left D*-homomorphisms* ψ *and* ϕ *are respectively defined by*

$$\begin{array}{rcl} \psi : D^{1\times p} & \longrightarrow & F \\ f_j & \longmapsto & z_j, \; j = 1, \dots, p, \end{array} \qquad \begin{array}{rcl} \phi : D^{1\times q} & \longrightarrow & N \\ e_i & \longmapsto & n_i = \delta(a_i), \; i = 1, \dots, q, \end{array}$$

and $\{e_i\}_{i=1,\dots,q}$ *is the standard basis of* $D^{1\times q}$.

Remark 2.1.3. With the notations of Corollary 2.1.1, if $\lambda \in \ker_D(.R)$, then using the commutative exact diagram of Corollary 2.1.1, we get $u(\phi(\lambda)) = \psi(\lambda\, R) = \psi(0) = 0$, and thus $\phi(\lambda) = 0$ since u is injective. Therefore, $\phi \in \hom_D(D^{1\times q}, N)$ yields a unique $\bar{\phi} \in \hom_D(D^{1\times q}\, R, N)$ defined by $\bar{\phi}(e_i\, R) = n_i$ for all $i = 1, \dots, q$. Applying the contravariant exact functor $\hom_D(\,\cdot\,, N)$ to the short exact sequence $0 \longrightarrow D^{1\times q}\, R \xrightarrow{j} D^{1\times p} \xrightarrow{\pi} M \longrightarrow 0$ and using $\operatorname{ext}_D^1(D^{1\times p}, N) = 0$ since $D^{1\times p}$ is a projective left D-module (see Propositions 1.1.1 and 1.2.2), Theorem 1.2.1 yields the following exact sequence of abelian groups:

$$0 \longrightarrow \hom_D(M, N) \longrightarrow \hom_D(D^{1\times p}, N) \longrightarrow \hom_D(D^{1\times q}\, R, N) \xrightarrow{\kappa^1} \operatorname{ext}_D^1(M, N) \longrightarrow 0.$$

Hence, $\bar{\phi} \in \hom_D(D^{1\times q}\, R, N)$ defines a unique $\kappa^1(\bar{\phi}) \in \operatorname{ext}_D^1(M, N) \cong e_D(M, N)$ and (2.11).

Let now compute $\operatorname{ext}_D^1(M, N)$ for a commutative ring D. In this particular case, $\operatorname{ext}_D^1(M, N)$ inherits a D-module structure since $\ker_N(R_2.)$ and $\operatorname{im}_N(R.)$ are then both D-modules. Moreover, if D is a noetherian ring, then the D-module $\operatorname{ext}_D^1(M, N)$ can be characterized by means of generators and relations. To do that, we first recall the definition of the *Kronecker product*.

Definition 2.1.3. The *Kronecker product* of $U \in D^{n\times m}$ and $V \in D^{q\times p}$ is defined by:

$$U \otimes V \triangleq (U_{ij}\, V) = \begin{pmatrix} U_{11}\, V & U_{12}\, V & \dots & U_{1m}\, V \\ U_{21}\, V & U_{22}\, V & \dots & U_{2m}\, V \\ \vdots & \vdots & \vdots & \vdots \\ U_{n1}\, V & U_{n2}\, V & \dots & U_{nm}\, V \end{pmatrix} \in D^{n\,q\times m\,p}.$$

The next lemma on Kronecker products is classical for a commutative ring D (see [109]).

Lemma 2.1.2. *Let D be a commutative ring and $U \in D^{a\times b}$, $V \in D^{b\times c}$, $W \in D^{c\times d}$. Then*

$$\mathrm{row}(U\,V\,W) = \mathrm{row}(V)\,(U^T \otimes W),$$

with the notation $\mathrm{row}(V) = (V_{1\bullet} \dots V_{b\bullet})$ and where $V_{i\bullet}$ denotes the i^{th} row of the matrix V.

If D is a commutative ring, using Lemma 2.1.2, then we have:

$$\left\{\begin{array}{l} \mathrm{row}(R_2\,A) = \mathrm{row}(R_2\,A\,I_s) = \mathrm{row}(A)\,(R_2^T \otimes I_s), \\ \mathrm{row}(B\,S) = \mathrm{row}(I_p\,B\,S) = \mathrm{row}(B)\,(I_p \otimes S), \end{array}\right.$$

$$\Rightarrow\ R_2\,A = B\,S \ \Leftrightarrow\ (\mathrm{row}(A) \quad -\mathrm{row}(B)) \left(\begin{array}{c} R_2^T \otimes I_s \\ I_r \otimes S \end{array}\right) = 0.$$

Moreover, an element $A \in R\,D^{p\times s} + D^{q\times t}\,S$ can be written as $A = R\,X + Y\,S$ where $X \in D^{p\times s}$ and $Y \in D^{q\times s}$ and, using the Kronecker product, we then get:

$$\left\{\begin{array}{l} \mathrm{row}(R\,X) = \mathrm{row}(R\,X\,I_s) = \mathrm{row}(X)\,(R^T \otimes I_s), \\ \mathrm{row}(Y\,S) = \mathrm{row}(I_q\,Y\,S) = \mathrm{row}(Y)\,(I_q \otimes S), \end{array}\right.$$

$$\Rightarrow\ \mathrm{row}(A) = (\mathrm{row}(X) \quad \mathrm{row}(Y)) \left(\begin{array}{c} R^T \otimes I_s \\ I_q \otimes S \end{array}\right).$$

Let us denote by:

$$L = \left(\begin{array}{c} R^T \otimes I_s \\ I_q \otimes S \end{array}\right) \in D^{(p\,s+q\,t)\times q\,s}, \quad P = \left(\begin{array}{c} R_2^T \otimes I_s \\ I_r \otimes S \end{array}\right) \in D^{(q\;s+r\,t)\times r\,s}. \tag{2.12}$$

If D is a noetherian ring, then $\ker_D(.P)$ is a finitely generated D-module, and thus there exists a matrix $(T \quad -U) \in D^{u\times(q\,s+r\,t)}$, where $T \in D^{u\times q\,s}$ and $U \in D^{u\times r\,t}$, such that:

$$\ker_D(.P) = D^{1\times u}\,(T \quad -U).$$

Hence, the D-module $\Omega/(R\,D^{p\times s} + D^{q\times t}\,S)$ can be rewritten as the following D-module:

$$J = (D^{1\times u}\,T)/(D^{1\times(p\,s+q\,t)}\,L). \tag{2.13}$$

Let us now find a finite presentation of the D-module J defined by (2.13). The inclusion $D^{1\times(p\,s+q\,t)}\,L \subseteq D^{1\times u}\,T$ yields the existence of a matrix $F \in D^{(p\,s+q\,t)\times u}$ satisfying $L = F\,T$. Denoting by $V \in D^{v\times u}$ a matrix satisfying $\ker_D(.T) = D^{1\times v}\,V$, then Proposition 1.3.1 yields:

$$J \cong J_1 = D^{1\times u}/\left(D^{1\times((p\,s+q\,t)+v)}\left(\begin{array}{c} F \\ V \end{array}\right)\right). \tag{2.14}$$

If $D = k[x_1, \dots, x_n]$ is a polynomial ring over a computable field k (e.g., $k = \mathbb{Q}$ or $\mathbb{F}_p$ for a prime p), then using Gröbner basis techniques, we can explicitly describe the D-module J and thus the D-module $\mathrm{ext}_D^1(M, N)$ in terms of generators and relations. In particular, using (2.14), $J_1 = 0$, i.e., $J \cong \mathrm{ext}_D^1(M, N) = 0$, iff the matrix $(F^T \quad V^T)^T$ admits a left inverse, which can be tested by means of Algorithm 1.2.2.

Let us sum up the previous results in the following algorithm.

Algorithm 2.1.1. – **Input:** Two matrices $R \in D^{q \times p}$ and $S \in D^{t \times s}$ with entries in a commutative polynomial ring $D = k[x_1, \ldots, x_n]$ over computable field k and which define two finitely presented D-modules $M = D^{1 \times p}/(D^{1 \times q} R)$ and $N = D^{1 \times s}/(D^{1 \times t} S)$.

– **Output:** A matrix $X \in D^{((p\,s+q\,t)+v) \times u}$ presenting the following D-module:

$$J_1 = D^{1 \times u}/(D^{1 \times ((p\,s+q\,t)+v)} X) \cong \Omega/(R\, D^{p \times s} + D^{q \times t} S).$$

1. Compute a matrix $R_2 \in D^{r \times q}$ satisfying $\ker_D(.R) = D^{1 \times r} R_2$.
2. If R has full row rank, i.e., $R_2 = 0$, then return the matrix:

$$X = \begin{pmatrix} R^T \otimes I_s \\ I_q \otimes S \end{pmatrix} \in D^{(p\,s+q\,t) \times q\,s}.$$

   Otherwise, compute the matrices L and P defined by:

$$L = \begin{pmatrix} R^T \otimes I_s \\ I_q \otimes S \end{pmatrix} \in D^{(p\,s+q\,t) \times q\,s}, \quad P = \begin{pmatrix} R_2^T \otimes I_s \\ I_r \otimes S \end{pmatrix} \in D^{(q\ s+r\,t) \times r\,s}.$$

3. Compute a matrix $(T \quad -U)$ such that $\ker_D(.P) = D^{1 \times u} (T \quad -U)$, where $T \in D^{u \times q\,s}$ and $U \in D^{u \times r\,t}$.
4. Compute a matrix $F \in D^{(p\,s+q\,t) \times u}$ such that $L = F\,T$.
5. Compute a matrix $V \in D^{v \times u}$ satisfying $\ker_D(.T) = D^{1 \times v} V$.
6. Return the matrix $X = (F^T \quad V^T)^T$.

For an implementation of Algorithm 2.1.1, see `homalg` ([4]) and OreMorphisms ([20]).

Example 2.1.2. Let us consider the commutative polynomial ring $D = \mathbb{Q}[x_1, x_2]$, the matrices

$$R = \begin{pmatrix} x_1 & 0 \\ x_2 & x_1 \\ 0 & x_2 \end{pmatrix} \in D^{3 \times 2}, \quad S = (x_1 - x_2) \in D,$$

and the finitely presented D-module $M = D^{1 \times 2}/(D^{1 \times 3} R)$ and $N = D/(x_1 - x_2) \cong \mathbb{Q}[x_1]$. Following Algorithm 2.1.1, let us compute the D-module $\mathrm{ext}^1_D(M, N)$. We first obtain that the matrix $R_2 = (x_2^2 \quad -x_1\,x_2 \quad x_1^2)$ is such that $\ker_D(.R) = D\,R_2$. Hence, we get $p = 2$, $q = 3$, $r = 1$, $s = 1$, $t = 1$ and the matrices L and P are defined by:

$$L = \begin{pmatrix} x_1 & x_2 & 0 \\ 0 & x_1 & x_2 \\ x_1 - x_2 & 0 & 0 \\ 0 & x_1 - x_2 & 0 \\ 0 & 0 & x_1 - x_2 \end{pmatrix} \in D^{5 \times 3}, \quad P = \begin{pmatrix} x_2^2 \\ -x_1\,x_2 \\ x_1^2 \\ x_1 - x_2 \end{pmatrix} \in D^4.$$

Computing the syzygy D-module of $D^{1 \times 4} P$, we obtain $\ker_D(.P) = D^{1 \times 4} (T \quad -U)$, where:

$$T = \begin{pmatrix} 1 & 1 & 0 \\ x_1 & x_2 & 0 \\ 0 & -1 & -1 \\ 0 & x_1 & x_2 \end{pmatrix} \in D^{4 \times 3}, \quad U = -\begin{pmatrix} x_2 \\ 0 \\ x_1 \\ 0 \end{pmatrix} \in D^4.$$

Using Lemma 2.1.1, we have $\mathrm{ext}^1_D(M,N) \cong \Omega/(R\,D^2 + D^3\,S)$, where the abelian group Ω is defined by $\Omega = \{A \in D^3 \mid \exists\, B \in D: \; R_2\,A = B\,S\}$. Using (2.13), $J = (D^{1\times 4}\,T)/(D^{1\times 5}\,L)$. Moreover, we have $L = F\,T$ and $\ker_D(.T) = D\,V$, where:

$$F = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -x_2 & 1 & 0 & 0 \\ 0 & 0 & x_2 & 1 \\ 0 & 0 & -x_1 & -1 \end{pmatrix} \in D^{5\times 4}, \quad V = (x_1 \quad -1 \quad -x_2 \quad -1) \in D^{1\times 4}.$$

Using (2.14), if $X = (F^T \quad V^T)^T \in D^{6\times 4}$ then $J_1 = D^{1\times 4}/(D^{1\times 6}\,X) \cong J$. Let $\{e_i\}_{i=1,\ldots,4}$ be the standard basis of $D^{1\times 4}$ and $\sigma : D^{1\times 4} \longrightarrow J_1$ the canonical projection. Using Algorithms 1.3.1 and 1.3.2, we can check J_1 is a torsion D-module and:

$$\begin{cases} x_1\,\sigma(e_i) = 0, \quad i = 1, 3, \\ x_2\,\sigma(e_i) = 0, \quad i = 1, 3, \\ \sigma(e_i) = 0, \quad i = 2, 4. \end{cases}$$

Using the D-isomorphism (1.36) defined in Proposition 1.3.1, we finally obtain that the residue classes of the first and third rows of T in J generate the torsion D-module J, i.e., the residue classes $\epsilon((1 \quad 1 \quad 0)^T)$ and $\epsilon((0 \quad -1 \quad -1)^T)$ generate the D-module $\Omega/(R\,D^2 + D^3\,S)$ or, in other words, using (2.8), $\rho((\delta(1) \quad \delta(1) \quad \delta(0))^T)$ and $\rho((\delta(0) \quad -\delta(1) \quad -\delta(1))^T)$ generate the torsion D-module $\mathrm{ext}^1_D(M,N)$. In particular, we have:

$$R_2 \begin{pmatrix} \delta(1) \\ \delta(1) \\ \delta(0) \end{pmatrix} = (x_2^2 - x_1\,x_2)\,\delta(1) = \delta(x_2\,(x_2 - x_1)) = 0, \quad \begin{pmatrix} \delta(1) \\ \delta(1) \\ \delta(0) \end{pmatrix} \notin \mathrm{im}_N(R.),$$

$$R_2 \begin{pmatrix} \delta(0) \\ -\delta(1) \\ -\delta(1) \end{pmatrix} = (x_1\,x_2 - x_1^2)\,\delta(1) = \delta(x_1\,(x_1 - x_2)) = 0, \quad \begin{pmatrix} \delta(0) \\ -\delta(1) \\ -\delta(1) \end{pmatrix} \notin \mathrm{im}_N(R.).$$

Contrary to the case of a commutative ring D, $\mathrm{ext}^1_D(M,N)$ has generally no left or right D-module structure when D is a noncommutative ring. It is generally only an abelian group and a k-vector space when D is a k-algebra and k a field (see, e.g., [109]). If M and N are two *holonomic* left modules (see the forthcoming Definition 2.3.6) over the ring $D = A\langle \partial_1, \ldots, \partial_n\rangle$ of PD operators with coefficients in $A = k[x_1,\ldots,x_n]$, $k[\![x_1,\ldots,x_n]\!]$, where k is a field of characteristic 0, $\mathbb{R}\{x_1,\ldots,x_n\}$ or $\mathbb{C}\{x_1,\ldots,x_n\}$, then $\mathrm{ext}^1_D(M,N)$ is a finite-dimensional k-vector space (see [10, 11]). Hence, a basis of the finite k-vector space $\mathrm{ext}^1_D(M,N)$ can be computed using, for instance, the algorithms developed in [77, 115]. Unfortunately, contrary to what happens in the study of special functions and in combinatorics ([18]), most of the classical linear systems of PD equations studied in mathematical physics and engineering sciences do not define holonomic differential modules. In this case, we can only obtain a filtration of Ω by computing the matrices $A \in \Omega$ formed by PD operators of fixed order and degree/valuation. But, we cannot generally check whether or not $\epsilon(\Omega)$ is reduced to 0 in $\Omega/(R\,D^{p\times s} + D^{q\times t}\,S) \cong \mathrm{ext}^1_D(M,N)$.

Example 2.1.3. Let us consider a noncommutative ring D (e.g., $A_n(k)$ or $B_n(k)$), two elements R and S of D and the finitely presented left D-modules $M = D/(D\,R)$ and $N = D/(D\,S)$. Using Lemma 2.1.1, we get $\mathrm{ext}^1_D(D/(D\,R), D/(D\,S)) \cong D/(R\,D + D\,S)$. Hence, $\mathrm{ext}^1_D(M,N) = 0$ iff there exists X and $Y \in D$ satisfying the identity $R\,X + Y\,S = 1$.

2.2 Monge parametrizations

> "J'espère [que ces résultats] pourront contribuer à appeler l'attention de quelques jeunes mathématiciens sur un sujet difficile et bien peu étudié", E. Goursat, [36], p. 250.

In Chapter 1, we studied when a linear system $\ker_{\mathcal{F}}(R.)$ could be parametrized by means of potentials, namely, by arbitrary functions of all the independent variables. In other words, we studied the existence of a matrix $Q \in D^{p\times m}$ such that $\ker_{\mathcal{F}}(R.) = Q\,\mathcal{F}^m$. When $\mathcal{F}$ is a rich enough functional space (i.e., an injective (cogenerator) left D-module), the obstructions for the existence of a parametrization of the linear system $\ker_{\mathcal{F}}(R.)$ are given by the torsion elements of the left D-module $M = D^{1\times p}/(D^{1\times q}\,R)$ finitely presented by the system matrix $R \in D^{q\times p}$. If M admits non-trivial torsion elements, namely, elements $m \in M \setminus \{0\}$ satisfying $d\,m = 0$ for a certain $d \in D \setminus \{0\}$, then we can wonder if the concept of a potential-like parametrization can be generalized. In this section, we study the so-called *Monge parametrization* obtained by glueing the parametrization of the parametrizable linear subsystem $\ker_{\mathcal{F}}(R'.)$ of $\ker_{\mathcal{F}}(R.)$, where $M/t(M) = D^{1\times p}/(D^{1\times q'}\,R')$, with the integration of the torsion elements, i.e., with the elements of $\hom_D(t(M), \mathcal{F})$. This new kind of parametrizations, called *Monge parametrizations*, allows us to parametrize $\ker_{\mathcal{F}}(R.)$ by means of a certain number of potentials but also by a certain number of arbitrary functions in fewer independent variables (e.g., arbitrary constants). This problem was first studied by Monge in [72] for nonlinear OD systems (the so-called *Monge problem*).

> "Le problème de Monge à une variable indépendante dans le sens le plus large, consiste à intégrer explicitement un système de k $(k \leq n-1)$ équations de Monge
>
> $$F_i(x_1, x_2, \ldots, x_{n+1}; dx_1, dx_2, \ldots, dx_{n+1}) = 0, \quad (i = 1, 2, \ldots, k)$$
>
> les F étant des fonctions homogènes par rapport à $dx_1, dx_2, \ldots, dx_{n+1}$.
>
> Par intégration explicite nous entendons celle où l'on exprime les variables x par des fonctions déterminées d'un paramètre, de $n-k$ fonctions arbitraires de ce paramètre et de leurs dérivées jusqu'à celle d'un certain ordre, pouvant contenir aussi un nombre fini de constantes arbitraires", P. Zervos, [119], p. 1.

We first give an application of Theorem 2.1.3 to the parametrization of all the equivalence classes of extensions of $t(M)$ by $M/t(M)$, when M is a finitely presented left D-module.

Let $R \in D^{q\times p}$ be a matrix with entries in a noetherian domain D and let us consider the finitely presented left D-module $M = D^{1\times p}/(D^{1\times q}\,R)$. Computing the left D-module $\mathrm{ext}^1_D(N, D)$, where $N = D^q/(R\,D^p)$ is the Auslander transpose of M, we get a matrix $R' \in D^{q'\times p}$ satisfying:

$$\begin{cases} t(M) = (D^{1\times q'}\,R')/(D^{1\times q}\,R), \\ M/t(M) \cong D^{1\times p}/(D^{1\times q'}\,R'). \end{cases} \tag{2.15}$$

See (1.40) and (1.42). We denote by $\pi : D^{1\times p} \longrightarrow M$ (resp., $\pi' : D^{1\times p} \longrightarrow M/t(M)$) the canonical projection onto M (resp., $M/t(M)$). Using the following canonical short exact sequence

$$0 \longrightarrow t(M) \xrightarrow{i} M \xrightarrow{\rho} M/t(M) \longrightarrow 0, \tag{2.16}$$

we have $\pi' = \rho \circ \pi$, where ρ is the canonical projection $M \longrightarrow M/t(M)$. See the commutative exact diagram (1.43). Using Proposition 1.3.1, let us find an explicit finite presentation for the torsion left D-submodule $t(M)$ of M (see also (1.41)). If $R'' \in D^{q\times q'}$ and $R'_2 \in D^{r'\times q'}$ are respectively defined by $R = R'' R'$ and $\ker_D(.R') = D^{1\times r'} R'_2$, then applying Proposition 1.3.1 to the left D-module $t(M)$, we obtain the following left D-isomorphism

$$\begin{array}{rcl} \chi : T \triangleq D^{1\times q'}/(D^{1\times q} R'' + D^{1\times r'} R'_2) & \longrightarrow & t(M) \\ \delta(\nu) & \longmapsto & \pi(\nu R'), \end{array} \tag{2.17}$$

where $\delta : D^{1\times q'} \longrightarrow T$ is the canonical projection onto T, i.e., $t(M) \cong T$. For more details, see (1.41). The left D-module $t(M)$ then admits the following finite presentation

$$D^{1\times(q+r')} \xrightarrow{.(R''^T \quad R'^T_2)^T} D^{1\times q'} \xrightarrow{\chi\circ\delta} t(M) \longrightarrow 0,$$

where the left D-homomorphism $\chi\circ\delta$ is defined by:

$$\begin{array}{rcl} \chi\circ\delta : D^{1\times q'} & \longrightarrow & t(M) \\ \nu & \longmapsto & \pi(\nu R'). \end{array}$$

Hence, we obtain the following straightforward corollary of Theorem 2.1.3.

Corollary 2.2.1 ([104, 105])**.** *With the previous notations, an extension of $t(M)$ by $M/t(M)$*

$$e : 0 \longrightarrow t(M) \xrightarrow{\alpha} E \xrightarrow{\beta} M/t(M) \longrightarrow 0 \tag{2.18}$$

is defined by the left D-module $E = D^{1\times(p+q')}/(D^{1\times(q'+q+r')} P_A)$, where

$$P_A = \begin{pmatrix} R' & -A \\ 0 & R'' \\ 0 & R'_2 \end{pmatrix} \in D^{(q'+q+r')\times(p+q')}, \tag{2.19}$$

and A is an element of the abelian group Ω defined by:

$$\Omega = \left\{ A \in D^{q'\times q'} \mid \exists\, B \in D^{r'\times(q+r')} : \; R'_2 A = B \begin{pmatrix} R'' \\ R'_2 \end{pmatrix} \right\}. \tag{2.20}$$

Moreover, the equivalence classes of the extensions of $t(M)$ by $M/t(M)$ depend only on the residue classes $\epsilon(A)$ of $A \in \Omega$ in the following abelian group

$$\Omega/\left(R' D^{p\times q'} + D^{q'\times(q+r')} \begin{pmatrix} R'' \\ R'_2 \end{pmatrix} \right) = \upsilon(\mathrm{ext}^1_D(M/t(M), t(M))), \tag{2.21}$$

where υ is the isomorphism defined by (2.8).

Example 2.2.1. Let $M = D^{1\times 2}/(D^{1\times 2} R)$ be the left $D = A_2(\mathbb{Q})$-module finitely presented by:

$$R = \begin{pmatrix} x_1\,\partial_1 + 1 & x_2\,\partial_1 \\ x_1\,\partial_2 & x_2\,\partial_2 + 1 \end{pmatrix} \in D^{2\times 2}.$$

Using Algorithm 1.3.1, we obtain that $R' = (x_1 \quad x_2)$ and $Q = (-x_2 \quad x_1)^T$ satisfy:

$$t(M) = (D\,R')/(D^{1\times 2} R), \quad M/t(M) \cong D^{1\times 2}/(D\,R') \cong D^{1\times 2} Q = D\,x_1 + D\,x_2.$$

Moreover, using Proposition 1.3.1, we get $t(M) \cong T = D/(D\,\partial_1 + D\,\partial_2)$. If $I = D\,x_1 + D\,x_2$, then the short exact sequence (2.16) yields the short exact sequence $0 \longrightarrow T \xrightarrow{j} M \xrightarrow{p} I \longrightarrow 0$.

Since the left ideal I of D admits the finite free resolution $0 \longrightarrow D \xrightarrow{.R'} D^{1\times 2} \xrightarrow{.Q} I \longrightarrow 0$, then $\ker_D(.R') = 0$, i.e., $R'_2 = 0$, and Remark 2.1.2 shows that $\Omega = D$ and (2.8) yields:

$$\begin{aligned}\operatorname{ext}^1_D(M/t(M), t(M)) \cong \operatorname{ext}^1_D(I, T) \cong D/\left(D^{1\times 2}\begin{pmatrix}\partial_1\\ \partial_2\end{pmatrix} + (x_1 \quad x_2)\, D^2\right)\\ = D/(D\,\partial_1 + D\,\partial_2 + x_1\,D + x_2\,D).\end{aligned}$$

Then, $\operatorname{ext}^1_D(M/t(M), t(M))$ is reduced to 0 iff $1 \in D\,\partial_1 + D\,\partial_2 + x_1\,D + x_2\,D$, i.e., iff there exist $d_1, d_2, d_3, d_4 \in D$ satisfying $d_1\,\partial_1 + d_2\,\partial_2 + x_1\,d_3 + x_2\,d_4 = 1$, i.e., $1 - x_1\,d_3 - x_2\,d_4 \in D\,\partial_1 + D\,\partial_2$, which shows that we can always assume that $d_3,\ d_4 \in k[x_1, x_2]$ and yields $1 - x_1\,d_3 - x_2\,d_4 = 0$. This equation is impossible since $(0,\ 0)$ is a common zero of x_1 and x_2, which proves that the abelian group $\operatorname{ext}^1_D(M/t(M), t(M))$ is not reduced to 0. Finally, since $R'' = (\partial_1 \quad \partial_2)^T$, Corollary 2.2.1 shows that every extension of $t(M)$ by $M/t(M)$ can be defined by the short exact sequence (2.18), where the left D-module $E = D^{1\times 3}/(D^{1\times 3}\,P_A)$ is finitely presented by

$$P_A = \begin{pmatrix} x_1 & x_2 & -A\\ 0 & 0 & \partial_1\\ 0 & 0 & \partial_2\end{pmatrix},$$

and $A \in \Omega = D$ is any representative of the residue class $\epsilon(A) \in D/(D\,\partial_1 + D\,\partial_2 + x_1\,D + x_2\,D)$. In particular, we can always choose $A \in k[x_1, x_2]$.

Example 2.2.2. If we redo Example 2.2.1 with the following new matrix

$$R = \begin{pmatrix}\partial_1^2 & \partial_1\,\partial_2\\ \partial_1\,\partial_2 & \partial_2^2\end{pmatrix} \in D^{2\times 2},$$

then we obtain $R' = (\partial_1 \quad \partial_2)$, $Q = (-\partial_2 \quad \partial_1)^T$, $t(M) = (D\,R')/(D^{1\times 2}\,R) \cong D/(D\,\partial_1 + D\,\partial_2)$ and $M/t(M) \cong D^{1\times 2}/(D\,R') \cong D^{1\times 2}\,Q = D\,\partial_1 + D\,\partial_2$, where $M = D^{1\times 2}/(D^{1\times 2}\,R)$ is the left $D = A_2(\mathbb{Q})$-module finitely presented by R. Then, Remark 2.1.2 and (2.8) yield $\Omega = D$ and:

$$\operatorname{ext}^1_D(M/t(M), t(M)) \cong D/\left(D^{1\times 2}\begin{pmatrix}\partial_1\\ \partial_2\end{pmatrix} + (\partial_1 \quad \partial_2)\, D^2\right) = D/(D\,\partial_1 + D\,\partial_2 + \partial_1\,D + \partial_2\,D).$$

In this case, we have $\operatorname{ext}^1_D(M/t(M), t(M)) = 0$ since the following identity holds:

$$1 = \partial_1\,x_1 - x_1\,\partial_1 \in D\,\partial_1 + D\,\partial_2 + \partial_1\,D + \partial_2\,D.$$

Then, Theorem 2.1.2 shows that the only equivalence class of extensions of $t(M)$ by $M/t(M)$ is trivial one, namely, $E \cong t(M) \oplus M/t(M)$, i.e., the one defined by (2.18), where the left D-module $E = D^{1\times 3}/(D^{1\times 3}\,P_0)$ is finitely presented by the following block-diagonal matrix:

$$P_0 = \begin{pmatrix}\partial_1 & \partial_2 & 0\\ 0 & 0 & \partial_1\\ 0 & 0 & \partial_2\end{pmatrix}.$$

Corollary 2.2.1 gives a parametrization of all the equivalence classes of extensions of $t(M)$ by $M/t(M)$. In particular, the left D-module M defines the extension (2.16) of $t(M)$ by $M/t(M)$.

Hence, there exists a matrix $A \in \Omega$ such that $E = D^{1\times(p+q')}/(D^{1\times(q'+q+r')}\,P_A) \cong M$. Using (1.43) and (2.17), we can easily check that the following commutative exact diagram holds

$$\begin{array}{ccccccc} & D^{1\times q'} & \xrightarrow{.R'} & D^{1\times p} & \xrightarrow{\pi'} & M/t(M) & \longrightarrow 0\\ & \downarrow \phi & & \downarrow \pi & & \| & \\ 0 \longrightarrow & T & \xrightarrow{i\circ\chi^{-1}} & M & \xrightarrow{\rho} & M/t(M) & \longrightarrow 0, \end{array}$$

where $\phi : D^{1\times q'} \longrightarrow T$ is defined by $\phi(h_k) = \delta(h_k) = \pi(h_k\,R')$ for $k = 1,\ldots,q'$ and $\{h_k\}_{k=1,\ldots,q'}$ is the standard basis of $D^{1\times q'}$. Hence, using Corollary 2.1.1, we can take $A = I_{q'}$ in (2.19).

Theorem 2.2.1 ([104, 105]). *Let $R \in D^{q\times p}$, $R' \in D^{q'\times p}$, $R'' \in D^{q\times q'}$ and $R'_2 \in D^{r'\times q'}$ be four matrices satisfying $M = D^{1\times p}/(D^{1\times q}\,R)$, $M/t(M) = D^{1\times p}/(D^{1\times q'}\,R')$, $R = R''\,R'$ and $\ker_D(.R') = D^{1\times r'}\,R'_2$. Moreover, let $E = D^{1\times(p+q')}/(D^{1\times(q'+q+r')}\,P)$ be the left D-module finitely presented by the matrix P defined by*

$$P = \begin{pmatrix} R' & -I_{q'} \\ 0 & R'' \\ 0 & R'_2 \end{pmatrix} \in D^{(q'+q+r')\times(p+q')}, \tag{2.22}$$

and $\varrho : D^{1\times(p+q')} \longrightarrow E$ (resp., $\pi : D^{1\times p} \longrightarrow M$) the canonical projection onto E (resp., M).

1. *If $U = (I_p \quad 0) \in D^{p\times(p+q')}$, then we have the following left D-isomorphism*

$$\begin{array}{rcl} M & \longrightarrow & E = D^{1\times(p+q')}/(D^{1\times(q'+q+r')}\,P) \\ \pi(\lambda) & \longmapsto & \varrho(\lambda\,U), \end{array}$$

   i.e., $M \cong E$.

2. *The following two extensions of $t(M)$ by $M/t(M)$ defined by*

$$0 \longrightarrow t(M) \xrightarrow{i} M \xrightarrow{\rho} M/t(M) \longrightarrow 0, \quad 0 \longrightarrow t(M) \xrightarrow{\alpha} E \xrightarrow{\beta} M/t(M) \longrightarrow 0,$$

   belong to the same equivalence class in the abelian group $\mathrm{e}_D(M/t(M), t(M))$.

3. *For every left D-module $\mathcal{F}$, $\ker_{\mathcal{F}}(R.) \cong \hom_D(M,\mathcal{F}) \cong \hom_D(E,\mathcal{F}) \cong \ker_{\mathcal{F}}(P.)$, i.e.*

$$R\,\eta = 0 \quad \Leftrightarrow \quad \left\{ \begin{array}{l} R'\,\zeta - \theta = 0, \\ R''\,\theta = 0, \\ R'_2\,\theta = 0, \end{array} \right. \tag{2.23}$$

   and we have the following invertible transformations:

$$\begin{array}{rcl} \gamma : \ker_{\mathcal{F}}(P.) & \longrightarrow & \ker_{\mathcal{F}}(R.) \\ \begin{pmatrix} \zeta \\ \theta \end{pmatrix} & \longmapsto & \eta = U \begin{pmatrix} \zeta \\ \theta \end{pmatrix} = \zeta, \end{array} \qquad \begin{array}{rcl} \gamma^{-1} : \ker_{\mathcal{F}}(R.) & \longrightarrow & \ker_{\mathcal{F}}(P.) \\ \eta & \longmapsto & \begin{pmatrix} \zeta \\ \theta \end{pmatrix} = \begin{pmatrix} I_p \\ R' \end{pmatrix} \eta. \end{array}$$

We point out that the presentation matrix P of the left D-module E is block-triangular.

Theorem 2.2.1 can be used to parametrize the linear system $\ker_{\mathcal{F}}(R.)$. Indeed, (2.23) shows that the linear system $\ker_{\mathcal{F}}(R.)$ can be integrated in cascade: we first integrate the linear system

$$\left\{ \begin{array}{l} R''\,\theta = 0, \\ R'_2\,\theta = 0, \end{array} \right. \tag{2.24}$$

and then solve the inhomogeneous linear system $R'\,\eta = \theta$. Hence, η is the sum of a particular solution $\eta^\star \in \mathcal{F}^p$ of $R'\,\eta = \theta$ and of the general solution of the homogenous linear system $R'\,\eta = 0$. Since the torsion-free left D-module $M/t(M) = D^{1\times p}/(D^{1\times q'}\,R')$, 1 of Corollary 1.3.2 shows that $M/t(M)$ admits a parametrization, i.e., there exists $Q \in D^{p\times m}$ such that $M/t(M) \cong D^{1\times p}\,Q$. If $\mathcal{F}$ is an injective left D-module, then 1 of Corollary 1.4.1 proves that $\ker_{\mathcal{F}}(R'.) = Q\,\mathcal{F}^m$, i.e., every element $\eta \in \ker_{\mathcal{F}}(R'.)$ has the form $\eta = Q\,\xi$ for a certain $\xi \in \mathcal{F}^m$. Therefore, the elements of $\ker_{\mathcal{F}}(R.)$ can be parametrized as follows:

$$\forall\, \xi \in \mathcal{F}^m, \quad \eta = \eta^\star + Q\,\xi. \tag{2.25}$$

The parametrization (2.25) is called a *Monge parametrization* of the linear system $\ker_{\mathcal{F}}(R.)$.

If we consider an injective left D-module $\mathcal{F}$ and apply the exact functor $\hom_D(\,\cdot\,, \mathcal{F})$ to the commutative exact diagram (1.43), then we get the following commutative exact diagram

$$\begin{tikzcd}
& & & 0 & \\
& & & \hom_D(t(M), \mathcal{F}) \arrow[u] & \\
& \mathcal{F}^q & \mathcal{F}^p \arrow[l, "R."'] & \ker_{\mathcal{F}}(R.) \arrow[l] \arrow[u] & 0 \arrow[l] \\
\mathcal{F}^{r'} & \mathcal{F}^{q'} \arrow[l, "R'_2."'] \arrow[u, "R''."'] & \mathcal{F}^p \arrow[l, "R'."'] \arrow[u, equal] & \ker_{\mathcal{F}}(R'.) \arrow[l] \arrow[u] & 0, \arrow[l] \\
& & & 0 \arrow[u] &
\end{tikzcd}$$

where $\hom_D(t(M), \mathcal{F}) \cong \hom_D(T, \mathcal{F}) \cong \ker_{\mathcal{F}}((R''^T \;\; R_2'^T)^T.)$ and $\ker_{\mathcal{F}}(R'.) = Q\,\mathcal{F}^m$. Hence, the above remark can be found again by an easy chase in the previous commutative exact diagram.

Algorithm 2.2.1. – **Input:** A matrix $R \in D^{q\times p}$ over a noetherian domain D for which Buchberger's algorithm terminates for admissible term orders and $\mathcal{F}$ a left D-module.

– **Output:** A non-empty affine subset of elements of $\ker_{\mathcal{F}}(R.)$.

1. Applying Algorithm 1.3.1 to the left D-module $M = D^{1\times p}/(D^{1\times q}\,R)$, compute two matrices $R' \in D^{q'\times p}$ and $Q \in D^{p\times m}$ such that:
$$M/t(M) = D^{1\times p}/(D^{1\times q'}\,R'), \quad \ker_D(.Q) = D^{1\times q'}\,R'.$$
2. Factorize R by R' to get a matrix $R'' \in D^{q\times q'}$ satisfying $R = R''\,R'$.
3. Compute a matrix $R_2' \in D^{r'\times q'}$ satisfying $\ker_D(.R') = D^{1\times r'}\,R_2'$.
4. Find the $\mathcal{F}$-solutions of the linear system (2.24), i.e.:
$$\begin{cases} R''\,\theta = 0, \\ R_2'\,\theta = 0. \end{cases}$$
If $\mathcal{F}$ is a cogenerator left D-module, then a solution of the previous system always exists.
5. Find a particular solution $\eta^\star \in \mathcal{F}^p$ of the linear inhomogeneous system $R'\,\eta = \theta$, where θ is a general solution of (2.24). If $\mathcal{F}$ is an injective left D-module, then such a particular solution $\eta^\star$ always exists.
6. For all $\xi \in \mathcal{F}^m$, the element $\eta = \eta^\star + Q\,\xi$ belongs to $\ker_{\mathcal{F}}(R.)$.

Example 2.2.3. We consider the linear PD system $\vec{\nabla}\,(\vec{\nabla}\,.\,\vec{v}) = \vec{0}$ appearing in mathematical physics, where $\vec{\nabla} = (\partial_1 \quad \partial_2 \quad \partial_3)^T$ (see Example 1.2.3), namely:

$$\begin{cases} \partial_1\,(\partial_1\,v_1 + \partial_2\,v_2 + \partial_3\,v_3) = 0, \\ \partial_2\,(\partial_1\,v_1 + \partial_2\,v_2 + \partial_3\,v_3) = 0, \\ \partial_3\,(\partial_1\,v_1 + \partial_2\,v_2 + \partial_3\,v_3) = 0. \end{cases} \tag{2.26}$$

For instance, in acoustic, the speed $\vec{v}$ satisfies the PD linear system $\partial_t\,\vec{v}/c^2 - \vec{\nabla}\,(\vec{\nabla}\,.\,\vec{v}) = \vec{0}$, where c denotes the speed of sound ([52]). Hence, if we want to compute the stationary solutions, then we have to solve the linear PD system $\vec{\nabla}\,(\vec{\nabla}\,.\,\vec{v}) = \vec{0}$.

Let us parametrize all the $\mathcal{F} = C^\infty(\mathbb{R}^3)$-solutions of (2.26). Let $D = \mathbb{Q}[\partial_1, \partial_2, \partial_3]$ be the ring of PD operators with rational constant coefficients and $M = D^{1\times 3}/(D^{1\times 3}\,R)$ the D-module finitely presented by the presentation matrix $R \in D^{3\times 3}$ of (2.26). Using Algorithm 1.3.1 and (1.40), we obtain that the matrices $R' = (\partial_1 \quad \partial_2 \quad \partial_3) \in D^{1\times 3}$ and $R'' = (\partial_1 \quad \partial_2 \quad \partial_3)^T \in D^3$ satisfy $M/t(M) = D^{1\times 3}/(D\,R')$, $\ker_D(.R') = 0$ and $t(M) = (D\,R')/(D^{1\times 3}\,R) \cong D/(D^{1\times 3}\,R'')$. Then, Theorem 2.2.1 shows that $\ker_{\mathcal{F}}(R.) \cong \ker_{\mathcal{F}}(P.)$, where P is defined by (2.22), i.e.:

$$\begin{cases} \partial_1\,v_1 + \partial_2\,v_2 + \partial_3\,v_3 - \theta = 0, \\ \partial_1\,\theta = 0, \\ \partial_2\,\theta = 0, \\ \partial_3\,\theta = 0. \end{cases}$$

Then, θ is a constant $C \in \mathbb{R}$ and we have to parametrize all the $\mathcal{F} = C^\infty(\mathbb{R}^3)$-solutions of the inhomogeneous linear PD system $\vec{\nabla}\,.\,\vec{v} = C$. We can easily check that a particular solution of this inhomogeneous system is $\vec{v}^\star = (C\,x_1 \quad 0 \quad 0)^T$. A more symmetric particular solution is $\vec{v}^\star = \frac{C}{3}\,(x_1 \quad x_2 \quad x_3)^T$. Since the smooth solutions of the divergence operator in $\mathbb{R}^3$ are parametrized by the curl operator (see Example 1.4.3), all $\mathcal{F}$-solutions of (2.26) are of the form:

$$\forall\, C \in \mathbb{R}, \quad \forall\, \vec{\xi} \in \mathcal{F}^3, \quad \vec{v} = \vec{v}^\star + \vec{\nabla} \wedge \vec{\xi} = \begin{pmatrix} \frac{1}{3}\,C\,x_1 + \partial_2\,\xi_2 - \partial_3\,\xi_3 \\ \frac{1}{3}\,C\,x_2 + \partial_3\,\xi_1 - \partial_1\,\xi_3 \\ \frac{1}{3}\,C\,x_3 - \partial_2\,\xi_1 + \partial_1\,\xi_2 \end{pmatrix}.$$

Example 2.2.4. Let us consider a model of the motion of a fluid in a one-dimensional tank studied in [79] and defined by the following system of OD time-delay equations

$$\begin{cases} \dot{y}_1(t) - \dot{y}_2(t - 2\,h) + \alpha\,\ddot{y}_3(t - h) = 0, \\ \dot{y}_1(t - 2\,h) - \dot{y}_2(t) + \alpha\,\ddot{y}_3(t - h) = 0, \end{cases} \tag{2.27}$$

where h is a positive real number. Let $D = \mathbb{Q}(\alpha)[\partial, \delta]$ be the commutative polynomial ring of OD time-delay operators with rational constant coefficients (i.e., $\partial\,y(t) = \dot{y}(t)$, $\delta\,y(t) = y(t-h)$),

$$R = \begin{pmatrix} \partial & -\partial\,\delta^2 & \alpha\,\partial^2\,\delta \\ \partial\,\delta^2 & -\partial & \alpha\,\partial^2\,\delta \end{pmatrix} \in D^{2\times 3},$$

the presentation matrix of (2.27) and the D-module $M = D^{1\times 3}/(D^{1\times 2}\,R)$ finitely presented by R. Using Algorithm 1.3.1 and (1.40), we obtain that the following matrices

$$R' = \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1-\delta^2 & \alpha\,\partial\,\delta \end{pmatrix}, \quad Q = \begin{pmatrix} -\alpha\,\partial\,\delta \\ \alpha\,\partial\,\delta \\ 1+\delta^2 \end{pmatrix}, \quad R'' = \begin{pmatrix} \partial & \partial \\ \partial\,\delta^2 & \partial \end{pmatrix},$$

satisfy $M/t(M) = D^{1\times 3}/(D^{1\times 2}\,R')$, $\ker_D(.Q) = D^{1\times 3}\,R'$, $R = R''\,R'$, $\ker_D(R'.) = 0$ and $t(M) = (D^{1\times 2}\,R')/(D^{1\times 2}\,R) \cong D^{1\times 2}/(D^{1\times 2}\,R'')$. Let us find a Monge parametrization of $\ker_{\mathcal{F}}(R.)$, where $\mathcal{F}$ is an injective D-module. In order to do that, we first need to compute $\ker_{\mathcal{F}}(R''.)$, i.e.,

$$\left\{\begin{array}{l} \dot{\theta}_1(t) + \dot{\theta}_2(t) = 0, \\ \dot{\theta}_1(t - 2\,h) + \dot{\theta}_2(t) = 0, \end{array}\right. \quad \Leftrightarrow \quad \left\{\begin{array}{l} \theta_1(t) = \psi(t) + \dfrac{(c_1 - c_2)}{2\,h}\,t, \\ \theta_2(t) = -\psi(t) + c_1 - \dfrac{(c_1 - c_2)}{2\,h}\,t, \end{array}\right.$$

where c_1 and c_2 are two arbitrary real constants and ψ is an arbitrary $2\,h$-periodic of $\mathcal{F}$.

Then, we have to solve the inhomogeneous system $R'\,\eta = \theta$, namely:

$$\left\{\begin{array}{l} y_1(t) + y_2(t) = \psi(t) + \dfrac{(c_1 - c_2)}{2\,h}\,t, \\ -y_2(t) - y_2(t - 2\,h) + \alpha\,\dot{y}_3(t - h) = -\psi(t) + c_1 - \dfrac{(c_1 - c_2)}{2\,h}\,t. \end{array}\right. \tag{2.28}$$

We can easily check that a particular solution of (2.28) is defined by:

$$\left\{\begin{array}{l} y_1(t) = \dfrac{1}{2}\left(\psi(t) + \dfrac{(c_1 - c_2)}{2\,h}\,t + \dfrac{(c_1 + c_2)}{2}\right), \\ y_2(t) = \dfrac{1}{2}\left(\psi(t) + \dfrac{(c_1 - c_2)}{2\,h}\,t - \dfrac{(c_1 + c_2)}{2}\right), \\ y_3(t) = 0. \end{array}\right.$$

Finally, using $\ker_{\mathcal{F}}(R'.) = Q\,\mathcal{F}$, (2.25) shows that every element of $\ker_{\mathcal{F}}(R.)$ has the form

$$\left\{\begin{array}{l} y_1(t) = \dfrac{1}{2}\,(\psi(t) + C_1\,t + C_2) - \alpha\,\dot{\xi}(t - h), \\ y_2(t) = \dfrac{1}{2}\,(\psi(t) + C_1\,t - C_2) + \alpha\,\dot{\xi}(t - h), \\ y_3(t) = \xi(t) + \xi(t - 2\,h), \end{array}\right.$$

where C_1 and C_2 are two arbitrary real constants, ψ an arbitrary $2\,h$-periodic function of $\mathcal{F}$ and ξ an arbitrary function of $\mathcal{F}$. We find again a parametrization of (2.27) obtained in [79].

Let us explain how the search for a particular solution $\eta^\star$ of the inhomogeneous linear system $R'\,\eta = \theta$ can be simplified in certain cases by means of a "method of variation of constants".

Theorem 2.2.1 and Corollary 2.2.1 show that $E = D^{1\times(p+q')}/(D^{1\times(q'+q+r')}\,P_A) \cong M$, where the matrix P_A is defined by (2.19) for all matrices $A \in \Omega$ belonging to the same equivalence class as $\epsilon(I_{m'})$ in the abelian group $\Omega/\left(R'\,D^{p\times q'} + D^{q'\times q}\,R'' + D^{q'\times r'}\,R'_2\right)$, i.e., for all matrices

$$A = I_{q'} - R'\,X - Y\,R'' - Z\,L'_2,$$

where $X \in D^{p\times q'}$, $Y \in D^{q'\times q}$ and $Z \in D^{q'\times r'}$ are arbitrarily matrices. Taking $A = 0$, the block-diagonal form of P_0 shows that the left D-module F finitely presented by the matrix P_0 defines the trivial extension of $t(M)$ by $M/t(M)$, i.e., $F \cong t(M) \oplus M/t(M)$. Hence, the canonical short exact sequence (2.16) splits iff $\epsilon(I_{m'}) = \epsilon(0)$, i.e., iff there exist three matrices $X \in D^{p\times q'}$, $Y \in D^{q'\times q}$ and $Z \in D^{q'\times r'}$ satisfying $R'\,X + Y\,R'' + Z\,R'_2 = I_{q'}$.

Proposition 2.2.1 ([101, 104, 105]). *Let $R \in D^{q \times p}$, $R' \in D^{q' \times p}$ and $R'_2 \in D^{r' \times q'}$ be three matrices such that $M = D^{1 \times p}/(D^{1 \times q}\, R)$, $M/t(M) \cong D^{1 \times p}/(D^{1 \times q'}\, R')$ and $\ker_D(.R') = D^{1 \times r'}\, R'_2$. Then, the canonical short exact sequence*

$$0 \longrightarrow t(M) \xrightarrow{i} M \xrightarrow{\rho} M/t(M) \longrightarrow 0 \tag{2.29}$$

splits, i.e., $M \cong t(M) \oplus M/t(M)$, iff there exist $X \in D^{p \times q'}$, $Y \in D^{q' \times q}$ and $Z \in D^{q' \times r'}$ satisfying

$$R'\, X + Y\, R'' + Z\, R'_2 = I_{q'}. \tag{2.30}$$

or equivalently, if there exist two matrices $X \in D^{p \times q'}$ and $Y \in D^{q' \times q}$ satisfying:

$$R' - R'\, X\, R' = Y\, R. \tag{2.31}$$

Then, the following left D-homomorphism

$$\begin{array}{rcl} \sigma : M/t(M) & \longrightarrow & M \\ \pi'(\lambda) & \longmapsto & \pi(\lambda\,(I_p - X\, R')), \end{array}$$

where $\pi : D^{1 \times p} \longrightarrow M$ and $\pi' : D^{1 \times p} \longrightarrow M/t(M)$ are respectively the projections onto M and $M/t(M)$), is a right inverse of the canonical projection $\rho : M \longrightarrow M/t(M)$ onto $M/t(M)$, i.e.:

$$\rho \circ \sigma = \mathrm{id}_{M/t(M)}.$$

Let us explain why (2.30) is equivalent to (2.31). Post-multiplying (2.30) by R' and using the relations $R = R''\, R'$ and $R'_2\, R' = 0$, we get (2.31). Conversely, using $R = R''\, R'$, (2.31) yields $(I_{q'} - R'\, X - Y\, R'')\, R' = 0$, i.e., $D^{1 \times q'}\,(I_{q'} - R'\, X - Y\, R'') \subseteq \ker_D(.R') = D^{1 \times r'}\, R'_2$, and thus there exists $Z \in D^{q' \times r'}$ such that $I_{q'} - R'\, X - Y\, R'' = Z\, R'_2$, i.e., we get (2.30).

Remark 2.2.1. If D is a commutative polynomial ring, using Kronecker products, then we get:

$$(2.30) \quad \Leftrightarrow \quad \mathrm{row}(I_{q'}) = (\mathrm{row}(X) \quad \mathrm{row}(Y) \quad \mathrm{row}(Z)) \begin{pmatrix} R'^T \otimes I_{q'} \\ I_{q'} \otimes R'' \\ I_{q'} \otimes R'_2 \end{pmatrix}.$$

Then, the existence of the matrices X, Y and Z satisfying (2.30) is reduced to checking whether or not $\mathrm{row}(I_{q'})$ belongs to the D-module generated by the rows of the last matrix. If so, then the computation of the normal form of $\mathrm{row}(I_{q'})$ with respect to a Gröbner basis of the matrix defined in the above equation gives matrices X, Y and Z satisfying (2.30).

If $M \cong t(M) \oplus M/t(M)$, then we can use (2.30) to obtain a particular solution $\eta^\star \in \mathcal{F}^p$ of the inhomogeneous linear system $R'\, \eta = \theta$. Indeed, post-multiplying (2.30) by θ, we get

$$\theta = R'\,(X\, \theta) + Y\,(R''\, \theta) + Z\,(R'_2\, \theta) = R'\,(X\, \theta),$$

since $\theta \in \mathcal{F}^{q'}$ satisfies (2.24). Therefore, $\eta^\star = X\, \theta$ is a particular solution of $R'\, \eta = \theta$ and thus every $\eta \in \ker_{\mathcal{F}}(R.)$ has the form

$$\eta = X\, \theta + Q\, \xi,$$

for all $\xi \in \mathcal{F}^m$ and θ satisfying (2.24). Hence, the elements of the linear system $\ker_{\mathcal{F}}(R.)$ are parametrized by those of the linear system (2.24) and arbitrary elements ξ of $\mathcal{F}^m$.

Corollary 2.2.2 ([101]). *Let $M = D^{1\times p}/(D^{1\times q}\,R)$ be a finitely presented left D-module and let us suppose that the canonical short exact sequence (2.29) splits, where $M/t(M) = D^{1\times p}/(D^{1\times q'}\,R')$. Moreover, let $\mathcal{F}$ be an injective left D-module. Then, every element η of $\ker_{\mathcal{F}}(R.)$ has the form*

$$\eta = X\,\theta + Q\,\xi,$$

where $\theta \in \mathcal{F}^{q'}$ is a solution of (2.24), ξ an arbitrary element of $\mathcal{F}^m$ and the matrix $X \in D^{p\times q'}$ (resp., $Q \in D^{p\times m}$) satisfies (2.30) (resp., $\ker_D(.Q) = D^{1\times p}\,R'$).

Example 2.2.5. Let us consider the another model of the motion of a fluid in a one-dimensional tank studied in [26] and defined by the following system of OD time-delay equations

$$\begin{cases} y_1(t-2\,h) + y_2(t) - 2\,\dot{y}_3(t-h) = 0, \\ y_1(t) + y_2(t-2\,h) - 2\,\dot{y}_3(t-h) = 0, \end{cases} \tag{2.32}$$

where h is a positive real number. Let $D = \mathbb{Q}[\partial, \delta]$ be the commutative polynomial ring of OD time-delay operators with rational constant coefficients (i.e., $\partial\,y(t) = \dot{y}(t)$, $\delta\,y(t) = y(t-h)$),

$$R = \begin{pmatrix} \delta^2 & 1 & -2\,\partial\,\delta \\ 1 & \delta^2 & -2\,\partial\,\delta \end{pmatrix} \in D^{2\times 3}, \tag{2.33}$$

and the D-module $M = D^{1\times 3}/(D^{1\times 2}\,R)$. Using Algorithm 1.3.1, we obtain that the matrices

$$R' = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1+\delta^2 & -2\,\partial\,\delta \end{pmatrix}, \quad Q = \begin{pmatrix} 2\,\delta\,\partial \\ 2\,\delta\,\partial \\ 1+\delta^2 \end{pmatrix}, \quad R'' = \begin{pmatrix} \delta^2 & 1 \\ 1 & 1 \end{pmatrix},$$

satisfy $M/t(M) = D^{1\times 3}/(D^{1\times 2}\,R')$, $\ker_D(.Q) = D^{1\times 3}\,R'$, $R = R''\,R'$, $\ker_D(R'.) = 0$ and $t(M) = (D^{1\times 2}\,R')/(D^{1\times 2}\,R) \cong D^{1\times 2}/(D^{1\times 2}\,R'')$. Let us find a Monge parametrization of $\ker_{\mathcal{F}}(R.)$, where $\mathcal{F}$ is an injective D-module. In order to do that, we first need to compute $\ker_{\mathcal{F}}(R''.)$, i.e.,

$$\begin{cases} \delta^2\,\theta_1 + \theta_2 = 0, \\ \theta_1 + \theta_2 = 0, \end{cases} \quad \Leftrightarrow \quad \begin{cases} \theta_2 = -\theta_1, \\ \delta^2\,\theta_1 - \theta_1 = 0, \end{cases}$$

which shows that θ_1 is a $2\,h$-periodic function of $\mathcal{F}$. Then, we have to find a particular solution $\eta^\star \in \mathcal{F}^3$ satisfying $R'\,\eta = \theta$. Using Remark 2.2.1, we can check that the following matrices

$$X = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 0 \end{pmatrix}, \quad Y = \frac{1}{2}\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}.$$

satisfy (2.31). Then, Corollary 2.2.2 shows that (2.32) is parametrized by

$$\begin{cases} y_1(t) = \frac{1}{2}\,\theta_1(t) + 2\,\dot{\xi}(t-h), \\ y_2(t) = -\frac{1}{2}\,\theta_1(t) + 2\,\dot{\xi}(t-h), \\ y_3(t) = \xi(t) + \xi(t-2\,h), \end{cases}$$

where ξ (resp., θ_1) is an arbitrary function (resp., $2\,h$-periodic function) of $\mathcal{F}$ (see also [26]).

If $M/t(M)$ is a projective left D-module, then Proposition 1.2.5 proves that the canonical short exact sequence (2.29) splits. We note that combining Proposition 1.2.2 and Theorem 2.1.2, we get $\mathrm{e}_D(M/t(M), t(M)) \cong \mathrm{ext}^1_D(M/t(M), t(M)) = 0$, which proves again that (2.29) is a split short exact sequence. Moreover, Proposition 1.3.2 proves that the presentation matrix R' of the left D-module $M/t(M) = D^{1\times p}/(D^{1\times q'}\, R')$ admits a generalized inverse, namely, there exists a matrix $X \in D^{p\times q'}$ satisfying $R'\, X\, R' = R'$. Hence, if $M/t(M)$ is a projective left D-module, then (2.31) holds with $Y = 0$, and the hypothesis of Corollary 2.2.2 is fulfilled.

Corollary 2.2.3. *Let $M = D^{1\times p}/(D^{1\times q}\, R)$ be a left D-module such that the torsion-free left D-module $M/t(M) = D^{1\times p}/(D^{1\times q'}\, R')$ is projective and $X \in D^{p\times q'}$ a generalized inverse of the matrix R'. If $\mathcal{F}$ is an injective left D-module, then every element η of $\ker_{\mathcal{F}}(R.)$ has the form*

$$\eta = X\,\theta + Q\,\xi, \tag{2.34}$$

where $\theta \in \mathcal{F}^{q'}$ is a solution of (2.24) and ξ an arbitrary element of $\mathcal{F}^m$.

Example 2.2.6. Let us consider the commutative polynomial algebra $D = \mathbb{Q}[\partial, \delta]$ of OD time-delay operators (i.e., $\partial\, y(t) = \dot{y}(t)$, $\delta\, y(t) = y(t-h)$, where $h \in \mathbb{R}_+$) and the following matrix

$$R = \begin{pmatrix} \partial & -\partial\,\delta & -1 \\ 2\,\partial\,\delta & -\partial\,(1+\delta^2) & 0 \end{pmatrix} \in D^{2\times 3},$$

which describes the torsion of a flexible rod with a force applied on one end studied in [74]:

$$\begin{cases} \dot{y}_1(t) - \dot{y}_2(t-h) - y_3(t) = 0, \\ 2\,\dot{y}_1(t-h) - \dot{y}_2(t) - \dot{y}_2(t-2\,h) = 0. \end{cases} \tag{2.35}$$

Using Algorithm 1.3.1, we can prove that the D-module $M = D^{1\times 3}/(D^{1\times 2}\, R)$ admits non-trivial torsion elements and $t(M) = (D^{1\times 3}\, R')/(D^{1\times 2}\, R)$ and $M/t(M) \cong D^{1\times 3}/(D^{1\times 3}\, R')$, where:

$$R' = \begin{pmatrix} -2\,\delta & 1+\delta^2 & 0 \\ -\partial & \partial\,\delta & 1 \\ \partial\,\delta & -\partial & \delta \end{pmatrix} \in D^{3\times 3}.$$

Moreover, we have $R = R''\, R'$ and $\ker_D(.R') = D\, R'_2$, where

$$R'' = \begin{pmatrix} 0 & -1 & 0 \\ 0 & -\delta & 1 \end{pmatrix}, \quad R'_2 = (\partial \quad -\delta \quad 1),$$

and the matrix $Q = \left(1+\delta^2 \quad 2\,\delta \quad (1-\delta^2)\,\partial\right)^T$ is such that $\ker_D(.Q) = D^{1\times 3}\, R'$. Moreover, using Algorithm 1.3.3, we can check that R' admits a generalized inverse X defined by

$$X = \frac{1}{2} \begin{pmatrix} \delta & 0 & 0 \\ 2 & 0 & 0 \\ -\partial\,\delta & 2 & 0 \end{pmatrix} \in D^{3\times 3},$$

which shows that the D-module $M/t(M)$ is projective by 2 of Proposition 1.3.2. Now, (2.24) is the following linear OD time-delay system:

$$\begin{cases} -\theta_2 = 0, \\ -\delta\,\theta_2 + \theta_3 = 0, \\ \partial\,\theta_1 - \delta\,\theta_2 + \theta_3 = 0, \end{cases} \quad \Leftrightarrow \quad \begin{cases} \partial\,\theta_1 = 0, \\ \theta_2 = 0, \\ \theta_3 = 0, \end{cases} \quad \Leftrightarrow \quad \begin{cases} \theta_1 = c \in \mathbb{R}, \\ \theta_2 = 0, \\ \theta_3 = 0. \end{cases}$$

Then, Corollary 2.2.3 shows that (2.35) admits the following Monge parametrization

$$\begin{cases} y_1(t) = \frac{1}{2}\,c + \xi(t) + \xi(t-2\,h), \\ y_2(t) = c + 2\,\xi(t-h), \\ y_3(t) = \dot{\xi}(t) - \dot{\xi}(t-2\,h), \end{cases}$$

where c is an arbitrary constant and ξ an arbitrary function of $\mathcal{F}$.

If $D = A\langle\partial\rangle$, where $A = k[t]$ or $k[\![t]\!]$ and k is a field of characteristic 0 or $A = k\{t\}$ and $k = \mathbb{R}$ or $\mathbb{C}$, then Example 1.2.13 shows that $\mathrm{gld}(D) = 1$, i.e., D is a hereditary ring. Thus, Theorem 1.3.1 proves that the torsion-free left D-module $M/t(M) = D^{1\times p}/(D^{1\times q'}\,R')$ is projective, and thus Corollary 2.2.3 holds for all finitely presented left D-modules M.

Now, if the matrix $R' \in D^{q'\times p}$ in Corollary 2.2.3 has full row rank and the left D-module $M/t(M) = D^{1\times p}/(D^{1\times q'}\,R')$ is free, then Corollary 1.5.2 shows that there exists $U \in \mathrm{GL}_p(D)$ such that $R'\,U = (I_{q'} \quad 0)$. If we write $U = (X \quad Q)$, where $X \in D^{p\times q'}$ and $Q \in D^{p\times(p-q')}$, then (2.34) becomes $\eta = U\,(\theta^T \quad \xi^T)^T$ (see also (1.68)). Using 1 of Theorem 1.1.2, this result holds when $D = K[\partial]$ and K is a differential field such as a field k, $k(t)$, $k[\![t]\!][t^{-1}]$ or $k\{t\}[t^{-1}]$, where $k = \mathbb{R}$ or $\mathbb{C}$, since the torsion-free left D-module $M/t(M)$ is then free.

In this section, we proved that a Monge parametrization of the linear system $\ker_{\mathcal{F}}(R.)$ could be obtained by glueing the parametrization of its parametrizable linear subsystem $\ker_{\mathcal{F}}(R'.)$ with the elements of $\hom_D(t(M),\mathcal{F})$ (which are the obstructions for $\ker_{\mathcal{F}}(R.)$ to admit a potential-like parametrization). This result, based on the system equivalence (2.23), generalizes 1 of Corollary 1.4.1. In Section 2.4, we shall show that Theorem 2.2.1 and (2.23) are just the first steps to more precise characterizations of M and $\ker_{\mathcal{F}}(R.)$ based on the concept of *purity filtration* of the left D-module M ([10, 11]). In particular, we shall give an equivalent block-triangular form of the linear system (2.24) which is more suitable for its closed-form integration (if it exits) (see 4 of Algorithm 2.2.1) and for the study of the structural properties of (2.24).

Finally, let us shortly explain one application of the Monge parametrization to the study of variational problems and optimal control problems. Substituting a Monge parametrization $\eta^\star + Q\,\xi$ of $\ker_{\mathcal{F}}(R.)$ in (1.96) instead of a potential-like parametrization $\eta = Q\,\xi$ as it was done in Corollary 1.6.3, we then obtain the following generalization of Corollary 1.6.3.

Theorem 2.2.2 ([102]). *Let $D = A\langle\partial_1,\dots,\partial_n\rangle$ be a ring of PD operators with coefficients in a differential ring A, $R \in D^{q\times p}$ a matrix of PD operators of order r, $\mathcal{F}$ an injective left D-module and $\ker_{\mathcal{F}}(R.)$ a linear PD system. Let us consider a Monge parametrization of $\ker_{\mathcal{F}}(R.)$:*

$$\forall\,\xi \in \mathcal{F}^k, \quad \eta = \eta^\star + Q\,\xi.$$

Then, a necessary condition for the existence of an extremum of the Lagrangian functional

$$I = \int \frac{1}{2}\,\eta_r^T\,L\,\eta_r\,dx, \quad \eta \in \ker_{\mathcal{F}}(R.),$$

where L is a symmetric matrix with entries in A, is defined by

$$\forall\,\xi \in \mathcal{F}^k, \quad \begin{cases} \eta = \eta^\star + Q\,\xi, \\ \mathcal{A}\,\xi + (\widetilde{Q}\circ\mathcal{B})\,\eta^\star = 0, \end{cases} \tag{2.36}$$

where $\mathcal{A} = \widetilde{Q}\circ\mathcal{B}\circ Q$ is defined as in Corollary 1.6.3.

Example 2.2.7. Let us consider the following quadratic optimal problem

$$I = \int_0^T \frac{1}{2}\,(x_1^2(t) + x_2^2(t) + u^2(t))\,dt, \tag{2.37}$$

under the differential constraint defined by the state-space linear OD system:

$$\dot{x}_1 = x_2 + u, \quad \dot{x}_2 = x_1 + u, \quad x_1(0) = x_1^0, \quad x_2(0) = x_2^0. \tag{2.38}$$

Let us choose $\mathcal{F} = C^\infty(\mathbb{R}_+)$. We can easily check that (2.38) is not controllable but stabilizable (namely, for every autonomous element τ of $\ker_{\mathcal{F}}(R.)$, we have $\lim_{t\to+\infty}\tau(t) = 0$). By Corollary 2.2.2, the $\mathcal{F}$-solutions of (2.38) are parametrized by:

$$\forall\, \xi \in \mathcal{F}, \quad \left\{\begin{array}{l} x_1(t) = (x_1^0 - x_2^0)\,e^{-t} + \xi(t), \\ x_2(t) = \xi(t), \\ u(t) = -(x_1^0 - x_2^0)\,e^{-t} + \dot{\xi}(t) - \xi(t). \end{array}\right. \tag{2.39}$$

If we substitute (2.39) into (2.37), then we obtain a variational problem without differential constraint and the corresponding Euler-Lagrange equations yield:

$$\ddot{\xi}(t) - 3\,\xi(t) = (x_1^0 - x_2^0)\,e^{-t}, \quad \dot{\xi}(T) - \xi(T) = (x_1^0 - x_2^0)\,e^{-T}, \quad \xi(0) = x_2^0. \tag{2.40}$$

(2.40) corresponds to (2.36). The explicit integration of (2.40) yields:

$$\xi(t) = -\frac{1}{2}\,\frac{e^{-2\sqrt{3}\,T}\,(e^{-t} - e^{\sqrt{3}\,t}) + (2-\sqrt{3})\,(e^{-t} - e^{-\sqrt{3}t})}{e^{-2\sqrt{3}\,T} + 2 - \sqrt{3}}\,(x_1^0 - x_2^0) + \frac{e^{\sqrt{3}\,(t-2\,T)} + (2-\sqrt{3})\,e^{-\sqrt{3}\,t}}{e^{-2\sqrt{3}\,T} + 2 - \sqrt{3}}\,x_2^0.$$

Hence, if we substitute the previous expression of ξ into (2.39), then we obtain

$$\left(\begin{array}{c} x_1(t) \\ x_2(t) \end{array}\right) = P(t) \left(\begin{array}{c} x_1^0 - x_2^0 \\ x_2^0 \end{array}\right), \qquad u(t) = Q(t) \left(\begin{array}{c} x_1^0 - x_2^0 \\ x_2^0 \end{array}\right), \tag{2.41}$$

where $P = (P_{ij})_{i,j=1,2}$ and $Q = (Q_{1j})_{j=1,2}$ are defined by:

$$\left\{\begin{array}{l} P_{11} = \dfrac{e^{-2\sqrt{3}\,T}\,(e^{\sqrt{3}\,t} + e^{-t}) + (2-\sqrt{3})\,(e^{-\sqrt{3}\,t} + e^{-t})}{2\,(e^{-2\sqrt{3}\,T} + 2 - \sqrt{3})}, \\[2ex] P_{21} = \dfrac{e^{-2\sqrt{3}\,T}\,(e^{\sqrt{3}\,t} - e^{-t}) + (2-\sqrt{3})\,(e^{-\sqrt{3}\,t} - e^{-t})}{2\,(e^{-2\sqrt{3}\,T} + 2 - \sqrt{3})}, \\[2ex] P_{12} = P_{22} = P_{11} + P_{21}, \\[2ex] Q_{11} = \dfrac{(\sqrt{3}-1)\,(e^{\sqrt{3}\,(t-2\,T)} - e^{-\sqrt{3}\,t})}{2\,(e^{-2\sqrt{3}\,T} + 2 - \sqrt{3})} = \dfrac{1}{2} Q_{12}. \end{array}\right.$$

Eliminating the initial conditions $x_1^0 - x_2^0$ and x_2^0 from (2.41), we obtain the optimal controller

$$u(t) = K(t) \left(\begin{array}{c} x_1(t) \\ x_2(t) \end{array}\right),$$

where $K = (K_{11} \quad K_{12}) = Q\,P^{-1}$ is defined by:

$$K_{11} = K_{12} = \frac{Q_{11}}{P_{12}} = \frac{(\sqrt{3}-1)\,(e^{\sqrt{3}\,(t-2\,T)} - e^{-\sqrt{3}\,t})}{2\,(e^{\sqrt{3}\,(t-2\,T)} + (2-\sqrt{3})\,e^{-\sqrt{3}\,t})}.$$

Finally, if T is taken to be $+\infty$, then we only need the condition that (2.38) is stabilizable and not controllable as it is required within the behavioural approach to optimal control problems.

2.3 Characteristic variety and dimensions

> "Le savant n'étudie pas la nature parce que cela est utile ; il l'étudie parce qu'il y prend plaisir et il y prend plaisir parce qu'elle est belle. Si la nature n'était pas belle, elle ne vaudrait pas la peine d'être connue, la vie ne vaudrait pas la peine d'être vécue."
>
> Henri Poincaré, *Science et Méthodes*, Philosophia Scientiæ, Cahier Spécial 3, 1998-1999, Editions KIMÉ, p. 22.

In this section, we introduce a few classical results of algebraic analysis on the dimension of the characteristic variety of a left D-module M and on the dimension of the left D-modules $\mathrm{ext}^i_D(\mathrm{ext}^i_D(M, D), D)$'s ([10, 11, 13, 45, 66]). These results will be used in the next section to develop the purity filtration of a finitely presented left D-module $M = D^{1\times p}/(D^{1\times q}\, R)$, which will allow us to generalize the results obtained in the previous section on the Monge parametrization of the linear PD system $\ker_{\mathcal{F}}(R.)$.

In what follows, we shall assume that A is either a field k, $k[x_1, \ldots, x_n]$, $k(x_1, \ldots, x_n)$ or $k[\![x_1, \ldots, x_n]\!]$, where k is a field of characteristic 0, or $k\{x_1, \ldots, x_n\}$, where $k = \mathbb{R}$ or $\mathbb{C}$.

An element $P \in D = A\langle \partial_1, \ldots, \partial_n\rangle$ is uniquely defined by $P = \sum_{|\alpha|=0,\ldots,r} a_\alpha\, \partial^\alpha$, where $a_\alpha \in A$, $\alpha = (\alpha_1, \ldots, \alpha_n)^T \in \mathbb{N}^n$, $|\alpha| = \alpha_1 + \ldots + \alpha_n$ and $\partial^\alpha = \partial_1^{\alpha_1} \ldots \partial_n^{\alpha_n}$. Then, we can introduce the *order filtration* of D, namely, $D_r = \left\{\sum_{0\leq|\alpha|\leq r} a_\alpha\, \partial^\alpha \mid a_\alpha \in A\right\}$ for all $r \in \mathbb{N}$, with the convention that $D_{-1} = 0$. Then, we can check that the following filtration conditions hold:

1. $\forall\, r,\, s \in \mathbb{N},\ r \leq s \;\Rightarrow\; D_r \subseteq D_s$.
2. $D = \bigcup_{r\in\mathbb{N}} D_r$.
3. $\forall\, r,\, s \in \mathbb{N},\ D_r\, D_s \subseteq D_{r+s}$.

The ring D is then called a *filtered ring* and an element of D_r is said to have a *degree* less or equal to r. We can easily check that $D_0 = A$ and D_r is a finitely generated A-module.

If $d_1,\, d_2 \in D$, then we can define the *bracket* of d_1 and d_2 by $[d_1, d_2] = d_1\, d_2 - d_2\, d_1$. Now, if $d_1 \in D_r$ and $d_2 \in D_s$, then $d_1\, d_2$ and $d_2\, d_1$ belong to D_{r+s} since $D_r\, D_s \subseteq D_{r+s}$ and $D_s\, D_r \subseteq D_{r+s}$. Moreover, we can check that $[d_1, d_2] \in D_{r+s-1}$, i.e., $[D_r, D_s] \subseteq D_{r+s-1}$.

Let us now introduce the following A-module:

$$\mathrm{gr}(D) = \bigoplus_{r\in\mathbb{N}} D_r/D_{r-1}.$$

If $\pi_r : D_r \longrightarrow D_r/D_{r-1}$ is the canonical projection for all $r \in \mathbb{N}$, then the A-module $\mathrm{gr}(D)$ inherits a ring structure defined by:

$$\forall\, d_1 \in D_r, \quad \forall\, d_2 \in D_s, \quad \left\{\begin{array}{l} \pi_r(d_1) + \pi_s(d_2) \triangleq \pi_t(d_1 + d_2) \in D_t/D_{t-1},\ t = \max(r, s), \\ \pi_r(d_1)\, \pi_s(d_2) \triangleq \pi_{r+s}(d_1\, d_2) \in D_{r+s}/D_{r+s-1}. \end{array}\right.$$

$\mathrm{gr}(D)$ is called the *graded ring* associated with the order filtration of D. If we now introduce

$$\forall\, i = 1, \ldots, n, \quad \chi_i = \pi_1(\partial_i) \in D_1/D_0,$$

then $\pi_1([\partial_i, \partial_j]) = 0$ and $\pi_1([\partial_i, a]) = 0$ for all $a \in A$ and all $i, j = 1, \ldots, n$ since $[\partial_i, \partial_j] = 0$ and $[\partial_i, a] \in D_0$, which shows that $\mathrm{gr}(D) = A[\chi_1, \ldots, \chi_n]$ is the commutative polynomial ring in $\chi_1, \ldots, \chi_n$ with coefficients in the commutative noetherian ring A.

We can now generalize the concepts of filtered and graded rings to modules.

Definition 2.3.1 ([10, 13, 66])**.** Let M be a finitely generated left $D = A\langle \partial_1, \dots, \partial_n \rangle$-module.

1. A *filtration* of M is a sequence $\{M_q\}_{q \in \mathbb{N}}$ of A-submodules of M (with the convention that $M_{-1} = 0$) such that:
   (a) For all $q,\, r \in \mathbb{N}$, $q < r$ implies that $M_q \subseteq M_r$.
   (b) $M = \bigcup_{q \in \mathbb{N}} M_q$.
   (c) For all $q,\, r \in \mathbb{N}$, we have $D_r\, M_q \subseteq M_{q+r}$.

   The left D-module M is then called a *filtered module*
2. The associated *graded* $\mathrm{gr}(D)$-module $\mathrm{gr}(M)$ is defined by:
   (a) $\mathrm{gr}(M) = \bigoplus_{q \in \mathbb{N}} M_q / M_{q-1}$.
   (b) For every $d \in D_r$ and every $m \in M_q$, we set $\pi_r(d)\, \sigma_q(m) \triangleq \sigma_{q+r}(d\, m) \in M_{q+r}/M_{q+r-1}$, where $\sigma_q : M_q \longrightarrow M_q/M_{q-1}$ is the canonical projection for all $q \in \mathbb{N}$.
3. A filtration $\{M_q\}_{q \in \mathbb{N}}$ is called a *good filtration* if it satisfies one of the equivalent conditions:
   (a) M_q is a finitely generated A-module for all $q \in \mathbb{N}$ and there exists $p \in \mathbb{N}$ such that $D_r\, M_p = M_{p+r}$ for all $r \in \mathbb{N}$.
   (b) $\mathrm{gr}(M) = \bigoplus_{q \in \mathbb{N}} M_q/M_{q-1}$ is a finitely generated $\mathrm{gr}(D) = A[\chi_1, \dots, \chi_n]$-module.

Example 2.3.1. Let M be a finitely generated left D-module defined by a family of generators $\{y_1, \dots, y_p\}$. Then, the filtration $M_q = \sum_{i=1}^p D_q\, y_i$ is a good filtration of M since we then have $\mathrm{gr}(M) = \sum_{i=1}^p \mathrm{gr}(D)\, y_i$, which proves that $\mathrm{gr}(M)$ is a finitely generated left $\mathrm{gr}(D)$-module.

If M is a finitely generated left $D = A\langle \partial_1, \dots, \partial_n \rangle$-module, then $\mathrm{gr}(M)$ is a finitely generated module over the commutative polynomial ring $\mathrm{gr}(D) = A[\chi_1, \dots, \chi_n]$. Hence, we are back to the realm of commutative algebra. Based on techniques of algebraic geometry and commutative algebra, we can then characterize invariants of $\mathrm{gr}(M)$ (e.g., dimension, multiplicity) which are important invariants of the differential module M.

Let us recall the concept of *prime ideals* of a commutative ring.

Definition 2.3.2. A *prime ideal* of a commutative ring A is an ideal $\mathfrak{p} \subsetneq A$ which satisfies that $a\, b \in \mathfrak{p}$ implies $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$. The set of all the proper prime ideals of A is denoted by $\mathrm{spec}(A)$ and is a topological space endowed with the *Zariski topology* defined by the Zariski-closed sets $V(I) = \{\mathfrak{p} \in \mathrm{spec}(A) \mid I \subseteq \mathfrak{p}\}$, where I is an ideal of A.

Example 2.3.2. If $(a_1, \dots, a_n) \in \mathbb{C}^n$, then the finitely generated ideal $\mathfrak{m} = (x - a_1, \dots, x_n - a_n)$ of the ring $D = \mathbb{C}[x_1, \dots, x_n]$ is a *maximal ideal* of D, namely, $\mathfrak{m}$ is not contained in any proper ideal of D different from $\mathfrak{m}$. A maximal ideal $\mathfrak{m}$ is a prime ideal. Indeed, if we have $x \notin \mathfrak{m}$ and $x\, y \in \mathfrak{m}$, then, since $\mathfrak{m}$ is maximal, we get $A\, x + \mathfrak{m} = A$, and thus, there exist $a \in A$ and $b \in \mathfrak{m}$ such that $a\, x + b = 1$. Then, we have $y = a\,(x\, y) + (y\, b) \in \mathfrak{m}$, which proves that $\mathfrak{m}$ is prime. For instance, the twisted cubic is defined by the prime ideal $\mathfrak{p} = (x_2 - x_1^2, x_3 - x_1^2)$ of $\mathbb{C}[x_1, x_2, x_3]$.

We now introduce the important concept of a *characteristic variety* of a differential module.

Proposition 2.3.1 ([10, 13, 66])**.** Let M be a finitely generated left $D = A\langle \partial_1, \dots, \partial_n \rangle$-module and $G = \mathrm{gr}(M)$ the associated graded $\mathrm{gr}(D) = A[\chi_1, \dots, \chi_n]$-module for a good filtration of M. Then, the *characteristic ideal* $I(M)$ of M is the ideal of ring $\mathrm{gr}(D) = A[\chi_1, \dots, \chi_n]$ defined by:

$$I(M) = \sqrt{\mathrm{ann}(G)} \triangleq \{a \in \mathrm{gr}(D) \,|\, \exists\, n \in \mathbb{N} :\; a^n\, G = 0\}.$$

The characteristic ideal $I(M)$ does not depend on the good filtration of M. The *characteristic variety* of M is then the subset of spec(gr(D)) defined by:

$$\mathrm{char}_D(M) = V(I(M)) = \left\{\mathfrak{p} \in \mathrm{spec}(\mathrm{gr}(D)) \,|\, \sqrt{\mathrm{ann}(G)} \subseteq \mathfrak{p}\right\}.$$

According to Example 2.3.1, every finitely generated left $D = A\langle \partial_1, \ldots, \partial_n\rangle$-module M admits a good filtration and thus a characteristic variety. The *dimension* of the left D-module M can then be defined as the geometric dimension of the characteristic variety $\mathrm{char}_D(M)$ of M.

Definition 2.3.3 ([10, 13, 66])**.** Let M be a finitely generated left $D = A\langle \partial_1, \ldots, \partial_n\rangle$-module. Then, the *dimension* of M is the supremum of the lengths of the chains $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \mathfrak{p}_2 \subset \ldots \subset \mathfrak{p}_d$ of distinct proper prime ideals in the commutative ring $\mathrm{gr}(D)/I(M) = A[\chi_1, \ldots, \chi_n]/I(M)$. If $M = 0$, then we set $\dim_D(M) = -1$.

For simplicity reasons, we shall write $\dim(D)$ instead of $\dim_D(D)$.

Example 2.3.3 ([10, 13])**.** We have $\dim(k[x_1, \ldots, x_n]) = n$ and $\dim(B_n(k)) = n$. Now, if $A = k[x_1, \ldots, x_n]$, $k[[x_1, \ldots, x_n]]$, where k is a field of characteristic 0, or $k\{x_1, \ldots, x_n\}$, where $k = \mathbb{R}$ or $\mathbb{C}$, then we have $\dim(A\langle \partial_1, \ldots, \partial_n\rangle) = 2\,n$.

Example 2.3.4. Let us consider the following linear PD system:

$$\left\{ \begin{array}{l} \Phi_1 = (\partial_4 - x_3\,\partial_2 - 1)\,y = 0, \\ \Phi_2 = (\partial_3 - x_4\,\partial_1)\,y = 0. \end{array} \right. \tag{2.42}$$

We can check that (2.42) is not *formally integrable* ([82, 84]) since

$$(\partial_4 - x_3\,\partial_2 - 1)\,\Phi_2 + (x_4\,\partial_1 - \partial_3)\,\Phi_1 = (\partial_2 - \partial_1)\,y = 0$$

is a new non-trivial first order PD equation which does not appear in (2.42). Adding this new equation to (2.42), then we can check that the new linear PD system defined by

$$\left\{ \begin{array}{l} (\partial_4 - x_3\,\partial_2 - 1)\,y = 0, \\ (\partial_3 - x_4\,\partial_1)\,y = 0, \\ (\partial_2 - \partial_1)\,y = 0, \end{array} \right. \tag{2.43}$$

is formally integrable and *involutive* ([82, 84]). Therefore, using the Cartan-Kähler-Janet's theorem (see [82, 84]), we can obtain a formal power series (analytic) solution of (2.43) in a neighbourhood of $a = (a_1, a_2, a_3, a_4) \in \mathbb{R}^4$ which satisfies an appropriate set of initial conditions.

Using (2.43), the characteristic variety of the left $D = A_4(\mathbb{C})$-module $M = D/(D^{1\times 2}\,R)$ finitely presented by the matrix $R = (\partial_4 - x_3\,\partial_2 - 1 \quad \partial_3 - x_4\,\partial_1)^T$ is defined by the ideal

$$I(M) = (\chi_4 - x_3\,\chi_2,\ \chi_3 - x_4\,\chi_1,\ \chi_2 - \chi_1)$$

of the commutative polynomial ring $\mathrm{gr}(D) = \mathbb{C}[x_1, x_2, x_3, x_4, \chi_1, \chi_2, \chi_3, \chi_4]$. The characteristic variety $\mathrm{char}_D(M)$ of M is then the affine algebraic variety of $\mathbb{C}^8$ defined by the ideal $I(M)$ of $\mathrm{gr}(D)$. We can easily check that we have:

$$\mathrm{char}_D(M) = \{(x_1, x_2, x_3, x_4, \chi_1, \chi_1, x_4\,\chi_1, x_3\,\chi_1) \mid \chi_1,\ x_i \in \mathbb{C},\ i = 1, \ldots, 4\}.$$

Therefore, the Krull dimension of $\mathrm{char}_D(M)$ is 5, i.e., $\dim_D(M) = 5$. If instead of $D = A_4(\mathbb{C})$, we use the second Weyl algebra $B_4(\mathbb{C})$, then the characteristic variety of M becomes

$$\mathrm{char}_D(M) = \{(\chi_1, \chi_1, x_4\, \chi_1, x_3\, \chi_1) \mid \chi_1 \in \mathbb{C}\},$$

which proves that $\mathrm{char}_D(M)$ is a 1-dimensional family of algebraic varieties parametrized by the point (x_1, x_2, x_3, x_4), i.e., $\dim_D(M) = 1$. Finally, we point out that we must transform (2.42) into the involutive system (2.43) (i.e., a Gröbner basis) to study the characteristic variety of M.

Let us introduce the important concept of the *grade* of a finitely generated left D-module.

Definition 2.3.4 ([10, 11])**.** The *grade* of a non-zero finitely generated left D-module M is:

$$j_D(M) = \min\,\{i \geq 0 \mid \mathrm{ext}^i_D(M, D) \neq 0\}.$$

If $M \neq 0$, then using Proposition 1.2.8, $\mathrm{ext}^{i+1}_D(M, D) = 0$ for all $i \geq \mathrm{gld}(D)$, which yields:

$$0 \leq j_D(M) \leq \mathrm{gld}(D). \tag{2.44}$$

Theorem 2.3.1 ([10, 13])**.** *Let M be a finitely generated left $D = A\langle \partial_1, \ldots, \partial_n\rangle$-module. Then:*

$$j_D(M) = \dim(D) - \dim_D(M). \tag{2.45}$$

A similar result holds for finitely generated right D-modules.

Remark 2.3.1. A ring D satisfying $j_D(M) = \dim(D) - \dim_D(M)$ for all finitely generated left D-modules M and a dimension function $\dim_D(\,\cdot\,)$ is called a *Cohen-Macaulay ring*. Hence, the previous rings of PD operators are Cohen-Macaulay. Moreover, they are also *Auslander regular rings*, namely, noetherian rings with a finite global dimension which satisfy the *Auslander condition*, namely, for every $i \in \mathbb{N}$, every finitely generated left (resp., right) D-module M and every left (resp., right) D-module $N \subseteq \mathrm{ext}^i_D(M, D)$, then $j_D(N) \geq i$ ([10, 11, 13]).

If $M = D^{1\times p}/(D^{1\times q}\, R)$ is a left D-module finitely presented by a full row rank matrix R, then Theorem 2.3.1 can be used to check the module properties of M. If $N = D^q/(R\, D^p) \cong \mathrm{ext}^1_D(M, D)$ is the Auslander transpose right D-module of M, then a right module analogue of Theorem 1.1.1 implies $\hom_D(N, D) \cong \ker_D(.R) = 0$. Then, $j_D(N) \geq 1$ and Theorem 2.3.1 yields $\dim_D(M) \leq \dim(D) - 1$. The computation of $\dim_D(M)$ then gives $j_D(M)$, i.e., the smallest $i \geq 1$ such that $\mathrm{ext}^i_D(N, D) \neq 0$. Using Theorem 1.3.1, we obtain the following interesting result.

Corollary 2.3.1 ([89])**.** *Let $M = D^{1\times p}/(D^{1\times q}\, R)$ be a left D-module finitely presented by a full row rank matrix R, i.e., $\ker_D(.R) = 0$, and $N = D^q/(R\, D^p)$ its Auslander transpose. Then:*

1. *$t(M) \neq 0$ iff $j_D(N) = 1$, i.e., iff $\dim_D(N) = \dim(D) - 1$.*
2. *M is torsion-free iff $j_D(N) \geq 2$, i.e., iff $\dim_D(N) \leq \dim(D) - 2$.*
3. *M is reflexive iff $j_D(N) \geq 3$ i.e., iff $\dim_D(N) \leq \dim(D) - 3$.*
4. *M is projective (stably free) iff $N = 0$, i.e., iff $\dim_D(N) = -1$.*

4 of Corollary 2.3.1 was already proved in Corollary 1.3.3. Corollary 2.3.1 shows that we only need to compute $\dim_D(N)$ to check whether or not a left D-module M finitely presented by a full row rank matrix R admits torsion elements or is torsion-free, reflexive or projective. Hence, if M is finitely presented by a full row rank matrix R, then we only need to determine the dimension of the left D-module $\widetilde{N} = D^{1\times q}/(D^{1\times p}\, \widetilde{R})$ by means of a Gröbner basis computation to check the module properties of the left D-module $M = D^{1\times p}/(D^{1\times q}\, R)$.

Example 2.3.5. If we consider again the $D = \mathbb{Q}[\partial_1, \partial_2, \partial_3]$-module $M = D^{1\times 3}/(D\,R)$ finitely presented by the divergence operator $R = (\partial_1 \quad \partial_2 \quad \partial_3)$ in $\mathbb{R}^3$, then the Auslander transpose $N = D/(R\,D^3) = D/(D^{1\times 3}\,R^T)$ of M is finitely presented by the gradient operator. Since $\mathrm{char}_D(M) = \{(0,0,0)\}$, then $\dim_D(N) = 0$ and $j_D(N) = 3 - 0 = 3$. Therefore, we get $\mathrm{ext}^i_D(N, D) = 0$ for $i = 0, 1, 2$ and $\mathrm{ext}^3_D(N, D) \neq 0$. Using Theorem 1.3.1, we find again that M is a reflexive but not a projective D-module.

In the theory of linear PD systems, the following definitions are generally used.

Definition 2.3.5. Let M be a finitely generated left $D = A\langle \partial_1, \dots, \partial_n\rangle$-module.

1. M is said to be *determined* if $\mathrm{ext}^0_D(M, D) = 0$ and $\mathrm{ext}^1_D(M, D) \neq 0$.
2. M is said to be *overdetermined* if $\mathrm{ext}^i_D(M, D) = 0$ for $i = 0,\, 1$.
3. M is said to be *underdetermined* if $\mathrm{ext}^0_D(M, D) \neq 0$.

These definitions can be easily explained by means of Theorem 2.3.1: if M is determined, then $j_D(M) = 1$, and thus $\dim_D(M) = \dim(D) - 1$. Moreover, if M is overdetermined, then $j_D(M) \geq 2$, which yields $\dim_D(M) \leq \dim(D) - 2$. Finally, if M is underdetermined, then $j_D(M) = 0$, and thus $\dim_D(M) = \dim(D)$.

If $M \neq 0$, then (2.44) and (2.45) yield $\dim_D(M) \geq \dim(D) - \mathrm{gld}(D)$.

Example 2.3.6. Using Examples 1.2.13 and 2.3.3, if M is a non-zero left $D = A\langle \partial_1, \dots, \partial_n\rangle$, then $\dim_D(M) \geq n$ when $A = k[x_1, \dots, x_n]$, $k[\![x_1, \dots, x_n]\!]$, where k is a field of characteristic 0, or $k\{x_1, \dots, x_n\}$, where $k = \mathbb{R}$ or $\mathbb{C}$. Moreover, $\dim_D(M) \geq 0$ whenever $A = k$ or $k(x_1, \dots, x_n)$, where k is a field of characteristic 0.

Definition 2.3.6 ([10, 13, 66])**.** Let $A = k[x_1, \dots, x_n]$, $k[\![x_1, \dots, x_n]\!]$, where k is a field of characteristic 0, or $k\{x_1, \dots, x_n\}$, where $k = \mathbb{R}$ or $\mathbb{C}$, and M a non-zero finitely generated left $D = A\langle \partial_1, \dots, \partial_n\rangle$-module. If $\dim_D(M) = n$ then M is called a *holonomic* left D-module.

Example 2.3.7. The time-varying OD equation defined by $t\,\dot{y} - y = 0$ defines the holonomic left $D = A_1(\mathbb{C})$-module $M = D/D\,(t\,\partial - 1)$. Indeed, the characteristic variety $\mathrm{char}_D(M)$ of M is defined by the characteristic ideal $I(M) = (t\,\chi)$ of the commutative polynomial ring $\mathrm{gr}(D) = \mathbb{C}[t, \chi]$, which implies that $\mathrm{char}_D(M) = \{(t,\, 0) \,|\, t \in \mathbb{C}\} \cup \{(0,\, \chi) \,|\, \chi \in \mathbb{C}\}$ is a 1-dimensional affine algebraic variety of $\mathbb{C}^2$, i.e., $\dim_D(M) = 1$.

Example 2.3.8. If $D = A\langle \partial\rangle$, where $A = k[t]$ or $k[\![t]\!]$ and k is a field of characteristic 0, or $k\{t\}$ and $k = \mathbb{R}$ or $\mathbb{C}$, then one can prove that a left (resp., right) D-module M is holonomic iff M is a torsion left (resp., right) D-module. For more details, see [10, 11, 13, 45, 66].

Proposition 2.3.2 ([10])**.** *Any holonomic left $D = A\langle \partial_1, \dots, \partial_n\rangle$-module M is cyclic, i.e., M can be generated by one element as a left D-module. More precisely, if $\{y_j\}_{j=1,\dots,p}$ is a set of generators of the holonomic left D-module M, then there exist $d_2, \dots, d_p \in D$ such that M is generated by $z = y_1 + d_2\,y_2 + \dots + d_p\,y_p$. Similar results hold for holonomic right D-modules.*

Let us state two difficult but important results of algebraic analysis.

Proposition 2.3.3 ([10, 11, 13])**.** *Let M be a finitely generated left $D = A\langle \partial_1, \dots, \partial_n\rangle$-module.*

1. $\dim_D(\mathrm{ext}^i_D(M, D)) \leq \dim(D) - i$.
2. $\dim_D(\mathrm{ext}^{j_D(M)}_D(M, D)) = \dim(D) - j_D(M)$.

Theorem 2.3.2 ([10, 11, 13]). *Let M be a finitely generated left $D = A\langle \partial_1, \ldots, \partial_n\rangle$-module.*

1. $\mathrm{ext}_D^j(\mathrm{ext}_D^i(M, D), D) = 0$ *for* $j < i$.
2. *If* $\mathrm{ext}_D^i(\mathrm{ext}_D^i(M, D), D)$ *is non-zero, then* $\dim_D(\mathrm{ext}_D^i(\mathrm{ext}_D^i(M, D), D)) = \dim(D) - i$.
3. $j_D(\mathrm{ext}_D^{j_D(M)}(M, D)) = j_D(M)$.

In particular, 3 of Theorem 2.3.2 asserts that the first non-zero $\mathrm{ext}_D^i(M, D)$'s of a left D-module M, i.e., $\mathrm{ext}_D^{j_D(M)}(M, D)$, satisfies the following conditions:

$$\begin{cases} \mathrm{ext}_D^j(\mathrm{ext}_D^{j_D(M)}(M, D), D) = 0, \quad j = 0, \ldots, j_D(M) - 1, \\ \mathrm{ext}_D^{j_D(M)}(\mathrm{ext}_D^{j_D(M)}(M, D), D) \neq 0. \end{cases}$$

Let us introduce the concept of a *pure module* which will play an important role in Section 2.4.

Definition 2.3.7. A finitely generated left D-module M is said to be *pure* or $j_D(M)$*-pure* if $j_D(N) = j_D(M)$ for all non-zero left D-submodules N of M.

Remark 2.3.2. If M is a pure left D-module, then the cyclic left D-module $D\,m \cong D/\mathrm{ann}_D(M)$ generated by $m \in M \setminus \{0\}$ satisfies $j_D(D\,m) = j_D(M)$. Moreover, if N is a left D-submodule of a $j_D(M)$-pure left D-module M, then N is also a $j_D(M)$-pure left D-module since every left D-submodule of N is a left D-submodule of M and $j_D(N) = j_D(M)$. Finally, if M is a $j_D(M)$-pure left D-module, then using (2.45), every left D-submodule of M has dimension $\dim(D) - j_D(M)$.

Theorem 2.3.3 ([10, 11]). *If M is a non-zero finitely generated left D-module, then we have:*

1. *The left D-module* $\mathrm{ext}_D^i(\mathrm{ext}_D^i(M, D), D)$ *is pure with* $j_D(\mathrm{ext}_D^i(\mathrm{ext}_D^i(M, D), D)) = i$.
2. *M is pure iff* $\mathrm{ext}_D^i(\mathrm{ext}_D^i(M, D), D) = 0$ *for* $i \neq j_D(M)$.
3. *M is pure iff M is a left D-submodule of* $\mathrm{ext}_D^{j_D(M)}(\mathrm{ext}_D^{j_D(M)}(M, D), D)$.

Example 2.3.9. Using 3 of Theorem 2.3.3, M is 0-pure iff M is a left D-submodule of the left D-module $\hom_D(\hom_D(M, D), D)$. Using 3 of Theorem 1.3.1, we obtain that M is 0-pure iff M is a torsion-free left D-module. In particular, the left D-module $M/t(M)$ is either zero or a 0-pure left D-module.

Example 2.3.10. If the left D-module $M = D^{1\times p}/(D^{1\times p}\,R)$ is finitely presented by a full row rank square matrix $R \in D^{p\times p}$ and $R \notin \mathrm{GL}_p(D)$, i.e., $M \neq 0$, then M is a torsion left D-module, i.e., $M = t(M)$. Since $N = D^p/(R\,D^p) \cong \mathrm{ext}_D^1(M, D)$, then using 1 of Theorem 1.3.1, $M = t(M) \cong \mathrm{ext}_D^1(\mathrm{ext}_D^1(M, D), D) \neq 0$. By Theorems 2.3.1 and 2.3.2, we get

$$\dim_D(M) = \dim_D(\mathrm{ext}_D^1(\mathrm{ext}_D^1(M, D), D)) = \dim(D) - 1,$$

and M is a 1-pure left D-module. This result was first conjectured by Janet in 1921 ("Etant donné un système linéaire comprenant *autant* d'équations que de fonctions inconnues ; si ces équations sont supposées *indépendantes*, peut-on affirmer que la solution, ou bien est *entièrement déterminée*, ou bien dépend de *fonctions arbitraires de* $n-1$ *variables* ?") and proved by Johnson in 1978 ([42]). For more details, see [42, 89, 95]. See also [95] for a generalization of this result.

2.4 Purity filtration of differential modules

"Les mathématiciens "appliqués" considèrent parfois leurs collègues "purs" comme des artistes élaborant des constructions théoriques sans doute jolies pour ceux qui les comprennent, mais totalement inutiles. Et même chez les mathématiciens dits "purs" cette dichotomie se perpétue. Les analystes sont persuadés que l'intégrale de Lebesgue, c'est du concret, et laissent le maniement des diagrammes aux fanatiques de l'algèbre homologique. D'ailleurs Siegel disait en parlant de Grothendieck que ce n'est pas en répétant "Om Om" que l'on démontrera des théorèmes sérieux (jeu de mots entre le "Om" tantrique et le "Hom" des algébristes)." [1]

P. Schapira, Défense du conceptuel, *Le Monde*, 26/04/96.

Based on the concept of *purity filtration* of the left D-module $M = D^{1\times p}/(D^{1\times q}\, R)$ ([10, 11]), the purpose of this section is to generalize Theorem 2.2.1. We show that every linear PD system in n independent variables is equivalent to a linear PD system defined by an upper block-triangular matrix P of PD operators: each diagonal block of P is respectively formed by the elements of the left D-module M of $\dim(D) - j$, for $j = 0, \ldots, n$. The linear PD system $R\,\eta = 0$ can then be integrated in cascade by successively solving (inhomogeneous) linear i-dimensional PD linear systems to get a Monge parametrization of its solution space $\ker_{\mathcal{F}}(R.)$.

The existence of the purity filtration of the left D-module M is proved by means of *spectral sequences*, i.e., by means of powerful but rather involved homological algebra techniques (see, e.g., [10, 11, 85]). The spectral sequences computing the purity filtration of differential modules have recently been implemented in the GAP4 package `homalg` by Barakat ([5]), which is an important "tour de force" for symbolic computation. However, in this section, we shall show how the purity filtration of the left D-module M can be explicitly characterized and computed by simply generalizing the idea developed in Section 1.3 (particularly the characterization of $t(M)$ in terms $\mathrm{ext}^1_D(N, D)$ (see 1 of Theorem 1.3.1)) ([97, 98]). The corresponding results are implemented in the PurityFiltration package ([97]). Finally, the techniques developed here can be used to compute the closed-form solutions (if they exist) of linear PD systems which cannot be solved by means of the classical computer algebra systems such as Maple ([97]).

In this section, we shall detail the main results concerning the purity filtration since they illustrate the different techniques and results developed in the previous sections and in Chapter 1.

Let D be a noetherian domain and M a left D-module defined by the following beginning of a finite free resolution:

$$0 \longleftarrow M \xleftarrow{\pi} D^{1\times p_0} \xleftarrow{.R_1} D^{1\times p_1} \xleftarrow{.R_2} D^{1\times p_2} \xleftarrow{.R_3} D^{1\times p_3}.$$

Then, the defects of exactness of the following complex of right D-modules

$$0 \longrightarrow D^{p_0} \xrightarrow{R_1.} D^{p_1} \xrightarrow{R_2.} D^{p_2} \xrightarrow{R_3.} D^{p_3} \tag{2.46}$$

1. "Applied" mathematicians often regard their "pure" colleagues as artists (cut-rate dancers) spinning theoretical constructs which are no doubt pleasing to those who understand them but are totally useless. And among these so called "pure" mathematicians much the same dichotomy reappears. Analysts are sure that the (Lebesgue) integral is concrete, and leave diagram-chasing to fanatics of (homological) algebra. Think of Siegel (a very great mathematician) saying of Grothendieck (an even greater mathematician, in my opinion) that one can't prove serious theorems by repeating "Om, Om." (A pun between the tantric "Om" and the algebraist's "Hom."), P. Schapira, Defense of the Conceptual, *Mathematical Intelligencer*, 19 (1997), 7-8.

are defined by:

$$\begin{cases} \mathrm{ext}^2_D(M, D) \cong \ker_D(R_3.)/(R_2\, D^{p_1}), \\ \mathrm{ext}^1_D(M, D) \cong \ker_D(R_2.)/(R_1\, D^{p_0}.), \\ \mathrm{ext}^0_D(M, D) \cong \ker_D(R_1.). \end{cases} \tag{2.47}$$

To characterize the $\mathrm{ext}^i_D(M, D)$'s for all $0 \leq i \leq 2$, we need to study $\ker_D(R_i.)$. For $1 \leq k \leq 3$, considering the beginning of a finite free resolution of $\ker_D(R_k.)$, we obtain the following long exact sequence of right D-modules

$$D^{p_{(-1)k}} \xrightarrow{R_{0k}.} D^{p_{0k}} \xrightarrow{R_{1k}.} D^{p_{1k}} \xrightarrow{R_{2k}.} \dots \xrightarrow{R_{(k-1)k}.} D^{p_{(k-1)k}} \xrightarrow{R_{kk}.} D^{p_{kk}} \xrightarrow{\kappa_{kk}} N_{kk} \longrightarrow 0, \tag{2.48}$$

with, for a fixed k from 1 to 3, the notations $R_{kk} = R_k$, $p_{kk} = p_k$, $p_{(k-1)k} = p_{k-1}$ and:

$$N_{kk} = \mathrm{coker}_D(R_{kk}.) = D^{p_{kk}}/(R_{kk}\, D^{p_{(k-1)k}}).$$

The choice of these notations is natural if we consider the 3 long exact sequences (2.48) for all $k = 1, 2, 3$ on the same page, where (2.48) is written at the level k, i.e.:

$$\begin{array}{ccccccccccccc} D^{p_{-13}} & \xrightarrow{R_{03}.} & D^{p_{03}} & \xrightarrow{R_{13}.} & D^{p_{13}} & \xrightarrow{R_{23}.} & D^{p_{23}} & \xrightarrow{R_{33}.} & D^{p_{33}} & \xrightarrow{\kappa_{33}} & N_{33} & \longrightarrow 0, \\ D^{p_{-12}} & \xrightarrow{R_{02}.} & D^{p_{02}} & \xrightarrow{R_{12}.} & D^{p_{12}} & \xrightarrow{R_{22}.} & D^{p_{22}} & \xrightarrow{\kappa_{22}} & N_{22} & \longrightarrow & 0, & \\ D^{p_{-11}} & \xrightarrow{R_{01}.} & D^{p_{01}} & \xrightarrow{R_{11}.} & D^{p_{11}} & \xrightarrow{\kappa_{11}} & N_{11} & \longrightarrow & 0. & & & \end{array}$$

Then, the free right D-module $D^{p_{jk}}$ is at position (j, k) and R_{jk} arrives at $D^{p_{jk}}$ with $j \leq k$, which is a good mnemonic device.

Since (2.46) is a complex, we get $R_{kk}\, R_{(k-1)(k-1)} = R_k\, R_{k-1} = 0$ for all $k = 2, 3$, and thus:

$$R_{(k-1)(k-1)}\, D^{p_{(k-2)(k-1)}} \subseteq \ker_D(R_{kk}.) = R_{(k-1)k}\, D^{p_{(k-2)k}}.$$

Therefore, for $k = 1, 2, 3$, there exists a matrix $F_{(k-2)k} \in D^{p_{(k-2)k} \times p_{(k-2)(k-1)}}$ such that:

$$R_{(k-1)(k-1)} = R_{(k-1)k}\, F_{(k-2)k}. \tag{2.49}$$

Then, using (2.49), we get $R_{(k-1)k}\, F_{(k-2)k}\, R_{(k-2)(k-1)} = R_{(k-1)(k-1)}\, R_{(k-2)(k-1)} = 0$, i.e.,

$$\forall\, k = 2, 3, \quad F_{(k-2)k}\, R_{(k-2)(k-1)}\, D^{p_{(k-3)(k-1)}} \subseteq \ker_D(R_{(k-1)k}.) = R_{(k-2)k} D^{p_{(k-3)k}},$$

and thus, there exists a matrix $F_{(k-3)k} \in D^{p_{(k-3)k} \times p_{(k-3)(k-1)}}$ such that:

$$F_{(k-2)k}\, R_{(k-2)(k-1)} = R_{(k-2)k}\, F_{(k-3)k}. \tag{2.50}$$

Similarly, for $k = 3$, there exists $F_{-13} \in D^{p_{-13} \times p_{-12}}$ such that:

$$F_{03}\, R_{02} = R_{03}\, F_{-13}.$$

Therefore, we obtain the following commutative diagram of right D-modules

$$\begin{array}{ccccccccccccc} D^{p_{-13}} & \xrightarrow{R_{03}.} & D^{p_{03}} & \xrightarrow{R_{13}.} & D^{p_{13}} & \xrightarrow{R_{23}.} & D^{p_{23}} & \xrightarrow{R_{33}.} & D^{p_{33}} & \xrightarrow{\kappa_{33}} & N_{33} & \longrightarrow & 0 \\ \uparrow F_{-13}. & & \uparrow F_{03}. & & \uparrow F_{13}. & & \| & & & & & & \\ D^{p_{-12}} & \xrightarrow{R_{02}.} & D^{p_{02}} & \xrightarrow{R_{12}.} & D^{p_{12}} & \xrightarrow{R_{22}.} & D^{p_{22}} & \xrightarrow{\kappa_{22}} & N_{22} & \longrightarrow & 0 & & \\ \uparrow F_{-12}. & & \uparrow F_{02}. & & \| & & & & & & & & \\ D^{p_{-11}} & \xrightarrow{R_{01}.} & D^{p_{01}} & \xrightarrow{R_{11}.} & D^{p_{11}} & \xrightarrow{\kappa_{11}} & N_{11} & \longrightarrow & 0 & & & & \\ \uparrow & & \| & & & & & & & & & & \\ 0 & \longrightarrow & D^{p_{00}} & \xrightarrow{\kappa_{00}} & N_{00} & \longrightarrow & 0, & & & & & & \end{array} \tag{2.51}$$

whose horizontal sequences are exact and where:

$$R_{00} = 0, \quad N_{00} = D^{p_{00}}/0 \cong D^{1\times p_{00}}, \quad p_{00} = p_{01}, \quad p_{12} = p_{11}, \quad p_{23} = p_{22}. \tag{2.52}$$

If we denote by N_{jk} the right D-module defined by

$$N_{jk} = \operatorname{coker}_D(R_{jk}.) = D^{p_{jk}}/(R_{jk}\, D^{p_{(j-1)k}}),$$

then, using (2.51), we obtain the following commutative diagram

$$\begin{array}{ccccccccc} D^{p_{-13}} & \xrightarrow{R_{03}.} & D^{p_{03}} & \xrightarrow{R_{13}.} & D^{p_{13}} & \xrightarrow{\kappa_{13}} & N_{13} & \longrightarrow & 0 \\ \uparrow F_{-13}. & & \uparrow F_{03}. & & \uparrow F_{13}. & & & & \\ D^{p_{-12}} & \xrightarrow{R_{02}.} & D^{p_{02}} & \xrightarrow{R_{12}.} & D^{p_{12}} & \xrightarrow{\kappa_{12}} & N_{12} & \longrightarrow & 0 \\ \uparrow F_{-12}. & & \uparrow F_{02}. & & \| & & & & \\ D^{p_{-11}} & \xrightarrow{R_{01}.} & D^{p_{01}} & \xrightarrow{R_{11}.} & D^{p_{11}} & \xrightarrow{\kappa_{11}} & N_{11} & \longrightarrow & 0, \end{array} \tag{2.53}$$

whose horizontal sequences are exact. Moreover, we have the following short exact sequences:

$$\begin{array}{l} 0 \longrightarrow N_{13} \longrightarrow D^{p_{23}} \longrightarrow N_{23} \longrightarrow 0, \\ 0 \longrightarrow N_{23} \longrightarrow D^{p_{33}} \longrightarrow N_{33} \longrightarrow 0, \\ 0 \longrightarrow N_{12} \longrightarrow D^{p_{22}} \longrightarrow N_{22} \longrightarrow 0, \\ 0 \longrightarrow N_{01} \longrightarrow D^{p_{11}} \longrightarrow N_{11} \longrightarrow 0. \end{array} \tag{2.54}$$

Now, using (2.47), we obtain the following characterization of right D-modules $\operatorname{ext}^i_D(M, D)$'s:

$$\left\{ \begin{array}{l} \operatorname{ext}^2_D(M, D) \cong \ker_D(R_{33}.)/\operatorname{im}_D(R_{22}.) = (R_{23}\, D^{p_{13}})/(R_{22}\, D^{p_{12}}), \\ \operatorname{ext}^1_D(M, D) \cong \ker_D(R_{22}.)/\operatorname{im}_D(R_{11}.) = (R_{12}\, D^{p_{02}})/(R_{11}\, D^{p_{01}}), \\ \operatorname{ext}^0_D(M, D) \cong \ker_D(R_{11}.)/\operatorname{im}_D(R_{00}.) = R_{01}\, D^{p_{-11}}. \end{array} \right. \tag{2.55}$$

Then, using (2.52), (2.55) yields the following three short exact sequences of right D-modules:

$$\begin{array}{c} 0 \longrightarrow \operatorname{ext}^2_D(M, D) \longrightarrow N_{22} = D^{p_{23}}/(R_{22}\, D^{p_{12}}) \longrightarrow N_{23} = D^{p_{23}}/(R_{23}\, D^{p_{13}}) \longrightarrow 0, \\ 0 \longrightarrow \operatorname{ext}^1_D(M, D) \longrightarrow N_{11} = D^{p_{12}}/(R_{11}\, D^{p_{01}}) \longrightarrow N_{12} = D^{p_{12}}/(R_{12}\, D^{p_{02}}) \longrightarrow 0, \\ 0 \longrightarrow \operatorname{ext}^0_D(M, D) \longrightarrow N_{00} = D^{p_{00}} \longrightarrow N_{01} = D^{p_{01}}/(R_{01}\, D^{p_{01}}) \longrightarrow 0. \end{array} \tag{2.56}$$

Applying the contravariant exact functor $\hom_D(\cdot\,, D)$ to the three short exact sequences of (2.56) and using Theorem 1.2.1, we obtain the following long exact sequences of left D-modules:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \operatorname{ext}^0_D(N_{23}, D) & \longrightarrow & \operatorname{ext}^0_D(N_{22}, D) & \longrightarrow & \operatorname{ext}^0_D(\operatorname{ext}^2_D(M, D), D) \\ & \xrightarrow{\delta^1} & \operatorname{ext}^1_D(N_{23}, D) & \longrightarrow & \operatorname{ext}^1_D(N_{22}, D) & \longrightarrow & \operatorname{ext}^1_D(\operatorname{ext}^2_D(M, D), D) \\ & \xrightarrow{\delta^2} & \operatorname{ext}^2_D(N_{23}, D) & \longrightarrow & \operatorname{ext}^2_D(N_{22}, D) & \longrightarrow & \operatorname{ext}^2_D(\operatorname{ext}^2_D(M, D), D) \\ & \xrightarrow{\delta^3} & \operatorname{ext}^3_D(N_{23}, D) & \longrightarrow & \operatorname{ext}^3_D(N_{22}, D) & \longrightarrow & \ldots \end{array}$$

$$\begin{array}{ccccccc} 0 & \longrightarrow & \operatorname{ext}^0_D(N_{12}, D) & \longrightarrow & \operatorname{ext}^0_D(N_{11}, D) & \longrightarrow & \operatorname{ext}^0_D(\operatorname{ext}^1_D(M, D), D) \\ & \xrightarrow{\sigma^1} & \operatorname{ext}^1_D(N_{12}, D) & \longrightarrow & \operatorname{ext}^1_D(N_{11}, D) & \longrightarrow & \operatorname{ext}^1_D(\operatorname{ext}^1_D(M, D), D) \\ & \xrightarrow{\sigma^2} & \operatorname{ext}^2_D(N_{12}, D) & \longrightarrow & \operatorname{ext}^2_D(N_{11}, D) & \longrightarrow & \ldots \end{array}$$

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{ext}^0_D(N_{01}, D) & \longrightarrow & \mathrm{ext}^0_D(N_{00}, D) & \longrightarrow & \mathrm{ext}^0_D(\mathrm{ext}^0_D(M, D), D) \\ & \xrightarrow{\tau^1} & \mathrm{ext}^1_D(N_{01}, D) & \longrightarrow & \mathrm{ext}^1_D(N_{00}, D). & & \end{array}$$

If D is an Auslander regular ring (see Remark 2.3.1), then we have $\mathrm{ext}^i_D(\mathrm{ext}^j_D(M, D), D) = 0$ for all $0 \leq i < j$. In particular, we have:

$$\mathrm{ext}^0_D(\mathrm{ext}^1_D(M, D), D) = 0, \quad \mathrm{ext}^0_D(\mathrm{ext}^2_D(M, D), D) = 0, \quad \mathrm{ext}^1_D(\mathrm{ext}^2_D(M, D), D) = 0.$$

Moreover, $\mathrm{ext}^1_D(N_{00}, D)$ is reduced to 0 since $N_{00} = D^{p_{00}}$ is a free, and thus a projective right D-module (see Proposition 1.2.2). Therefore, the above three long exact sequences yield the following exact sequences of left D-modules:

$$\begin{array}{lllllll} 0 \longrightarrow & \mathrm{ext}^2_D(N_{23}, D) & \longrightarrow & \mathrm{ext}^2_D(N_{22}, D) & \longrightarrow & \mathrm{ext}^2_D(\mathrm{ext}^2_D(M, D), D), & \\ 0 \longrightarrow & \mathrm{ext}^1_D(N_{12}, D) & \longrightarrow & \mathrm{ext}^1_D(N_{11}, D) & \longrightarrow & \mathrm{ext}^1_D(\mathrm{ext}^1_D(M, D), D), & \\ 0 \longrightarrow & \mathrm{ext}^0_D(N_{01}, D) & \longrightarrow & \mathrm{ext}^0_D(N_{00}, D) & \longrightarrow & \mathrm{ext}^0_D(\mathrm{ext}^0_D(M, D), D) & \longrightarrow \mathrm{ext}^1_D(N_{01}, D) \longrightarrow 0. \end{array} \tag{2.57}$$

Applying Proposition 1.2.3 to the short exact sequences of (2.54), we obtain:

$$\begin{cases} \mathrm{ext}^3_D(N_{33}, D) \cong \mathrm{ext}^2_D(N_{23}, D) \cong \mathrm{ext}^1_D(N_{13}, D), \\ \mathrm{ext}^2_D(N_{22}, D) \cong \mathrm{ext}^1_D(N_{12}, D), \\ \mathrm{ext}^2_D(N_{11}, D) \cong \mathrm{ext}^1_D(N_{01}, D). \end{cases}$$

Since $N_{11} = D^{p_{11}}/(R_{11}\, D^{p_{01}})$ is the Auslander transpose of $M = D^{1\times p_{01}}/(D^{1\times p_{11}}\, R_{11})$, 1 of Theorem 1.3.1 yields $t(M) \cong \mathrm{ext}^1_D(N_{11}, D)$. Moreover, a right D-module analogue of Theorem 1.1.1 gives $\mathrm{ext}^0_D(N_{01}, D) \cong \ker_D(.R_{01})$ and (1.42) implies $M/t(M) = D^{1\times p_{00}}/\ker_D(.R_{01})$.

Therefore, (2.57) yields the following two exact sequences of left D-modules:

$$\begin{array}{lllllllll} 0 \longrightarrow & \mathrm{ext}^3_D(N_{33}, D) & \xrightarrow{\gamma_{32}} & \mathrm{ext}^2_D(N_{22}, D) & \xrightarrow{\gamma_{22}} & \mathrm{ext}^2_D(\mathrm{ext}^2_D(M, D), D) & \longrightarrow & \mathrm{coker}\,\gamma_{22} & \longrightarrow 0, \\ 0 \longrightarrow & \mathrm{ext}^2_D(N_{22}, D) & \xrightarrow{\gamma_{21}} & t(M) & \xrightarrow{\gamma_{11}} & \mathrm{ext}^1_D(\mathrm{ext}^1_D(M, D), D) & \longrightarrow & \mathrm{coker}\,\gamma_{11} & \longrightarrow 0, \\ 0 \longrightarrow & \mathrm{ext}^0_D(N_{01}, D) & \xrightarrow{\gamma_{10}} & D^{1\times p_{00}} & \xrightarrow{\gamma_{00}} & \mathrm{ext}^0_D(\mathrm{ext}^0_D(M, D), D) & \longrightarrow & \mathrm{ext}^2_D(N_{11}, D) & \longrightarrow 0. \end{array}$$

Combining the above long exact sequences with (1.26), i.e.,

$$0 \longrightarrow t(M) \longrightarrow M \xrightarrow{\varepsilon} \mathrm{ext}^0_D(\mathrm{ext}^0_D(M, D), D) \longrightarrow \mathrm{ext}^2_D(N_{11}, D) \longrightarrow 0,$$

(see 3 of Theorem 1.3.1), and using $\mathrm{coker}\,\varepsilon = M/t(M)$, we obtain the following important exact diagram of left D-modules

$$\begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & & \downarrow & & & & \\ 0 \longrightarrow & \mathrm{ext}^3_D(N_{33}, D) & \xrightarrow{\gamma_{32}} & & \mathrm{ext}^2_D(N_{22}, D) & \longrightarrow & \mathrm{coker}\,\gamma_{32} & \longrightarrow & 0 \\ & & & & \downarrow \gamma_{21} & & & & \\ & 0 & \longrightarrow & & t(M) & \xrightarrow{i} & M & \xrightarrow{\rho} & M/t(M) \longrightarrow 0, \\ & & & & \downarrow & & & & \\ & & & & \mathrm{coker}\,\gamma_{21} & & & & \\ & & & & \downarrow & & & & \\ & & & & 0 & & & & \end{array} \tag{2.58}$$

where:

$$\begin{cases} \operatorname{coker}\gamma_{32} \cong \operatorname{im}\gamma_{22} \subseteq \operatorname{ext}_D^2(\operatorname{ext}_D^2(M,D),D), \\ \operatorname{coker}\gamma_{21} \cong \operatorname{im}\gamma_{11} \subseteq \operatorname{ext}_D^1(\operatorname{ext}_D^1(M,D),D), \\ \operatorname{coker} i = M/t(M) \cong \operatorname{coker}\gamma_{10} \cong \operatorname{im}\gamma_{00} \subseteq \operatorname{ext}_D^0(\operatorname{ext}_D^0(M,D),D). \end{cases} \tag{2.59}$$

Thus, using Remark 2.3.2, $\operatorname{coker}\gamma_{32}$ is a 2-pure left D-module, $\operatorname{coker}\gamma_{21}$ is a 1-pure left D-module and $M/t(M)$ is a 0-pure left D-module (see Example 2.3.9). Moreover, using 1 of Proposition 2.3.3 and 2 of Theorem 2.3.2, we obtain:

$$\begin{cases} \dim_D(\operatorname{ext}_D^3(N_{33},D)) \leq \dim(D)-3, \\ \dim_D(\operatorname{coker}\gamma_{32}) = \dim(D)-2, \\ \dim_D(\operatorname{coker}\gamma_{21}) = \dim(D)-1, \\ \dim_D(M/t(M)) = \dim(D). \end{cases} \tag{2.60}$$

If the matrix R_3 has full row rank, i.e., $\ker_D(.R_3)=0$, then $N_{33}\cong \operatorname{ext}_D^3(M,D)$, and thus $\operatorname{ext}_D^3(N_{33},D)\cong \operatorname{ext}_D^3(\operatorname{ext}_D^3(M,D),D)$ is a 3-pure left D-module and:

$$\dim_D(\operatorname{ext}_D^3(N_{33},D)) = \dim(D)-3. \tag{2.61}$$

Then, we obtain the filtration $\{M_i\}_{i=-1,\ldots,3}$ of the left D-module M defined by:

$$M_{-1}=0 \subseteq M_0 = (\gamma_{21}\circ\gamma_{32})(\operatorname{ext}_D^3(N_{33},D)) \subseteq M_1 = \gamma_{21}(\operatorname{ext}_D^2(N_{22},D)) \subseteq M_2 = t(M) \subseteq M_3 = M. \tag{2.62}$$

We note that $M_0/M_{-1}\cong \operatorname{ext}_D^3(\operatorname{ext}_D^3(M,D),D)$ is a 3-pure left D-module, $M_1/M_0\cong \operatorname{coker}\gamma_{32}$ is a 2-pure left D-module, $M_2/M_1\cong\operatorname{coker}\gamma_{21}$ is a 1-pure left D-module and $M_3/M_2\cong M/t(M)$ is a 0-pure left D-module, i.e., the successive quotients of the elements of $\{M_i\}_{i=-1,\ldots,3}$ are all pure left D-modules. This filtration $\{M_i\}_{i=-1,\ldots,3}$ is called a *purity filtration* of M ([11]).

The purpose of the rest of the section is to apply Theorem 2.1.3 on Baer's extensions to the short exact sequences of (2.58) to find a presentation matrix of the left D-module M defined by a block-diagonal matrix P, where the block-diagonal matrices of P finitely present the (pure) left D-modules $M/t(M)$, $\operatorname{coker}\gamma_{21}$, $\operatorname{coker}\gamma_{32}$ and $\operatorname{ext}_D^3(N_{33},D)$.

Let us now precisely describe the left D-homomorphisms γ_{32} and γ_{21} and the left D-modules $\operatorname{coker}\gamma_{32}$ and $\operatorname{coker}\gamma_{21}$. Applying the contravariant left exact functor $\hom_D(\cdot\,,D)$ to the commutative exact diagram (2.53), we obtain the following commutative diagram:

$$\begin{array}{ccccc} D^{1\times p_{-13}} & \xleftarrow{.R_{03}} & D^{1\times p_{03}} & \xleftarrow{.R_{13}} & D^{1\times p_{13}} \\ \downarrow .F_{-13} & & \downarrow .F_{03} & & \downarrow .F_{13} \\ D^{1\times p_{-12}} & \xleftarrow{.R_{02}} & D^{1\times p_{02}} & \xleftarrow{.R_{12}} & D^{1\times p_{12}} \\ \downarrow .F_{-12} & & \downarrow .F_{02} & & \| \\ D^{1\times p_{-11}} & \xleftarrow{.R_{01}} & D^{1\times p_{01}} & \xleftarrow{.R_{11}} & D^{1\times p_{11}}. \end{array} \tag{2.63}$$

The defect of exactness of the first (resp., second, third) horizontal complex is $\operatorname{ext}_D^1(N_{13},D)$ (resp., $\operatorname{ext}_D^1(N_{12},D)$, $\operatorname{ext}_D^1(N_{11},D)$). Let us introduce the following canonical projections:

$$\begin{aligned} &\rho_3 : \ker_D(.R_{03}) \longrightarrow \ker_D(.R_{03})/(D^{1\times p_{13}}\,R_{13}) \cong \operatorname{ext}_D^1(N_{13},D)\cong \operatorname{ext}_D^3(N_{33},D), \\ &\rho_2 : \ker_D(.R_{02}) \longrightarrow \ker_D(.R_{02})/(D^{1\times p_{12}}\,R_{12}) \cong \operatorname{ext}_D^1(N_{12},D)\cong \operatorname{ext}_D^2(N_{22},D), \\ &\rho_1 : \ker_D(.R_{01}) \longrightarrow \ker_D(.R_{01})/(D^{1\times p_{11}}\,R_{11}) \cong \operatorname{ext}_D^1(N_{11},D)\cong t(M). \end{aligned}$$

The commutative diagram (2.63) induces the following two left D-homomorphisms:

$$\begin{array}{rcl} \alpha_{32} : \ker_D(.R_{03})/(D^{1\times p_{13}}\, R_{13}) & \longrightarrow & \ker_D(.R_{02})/(D^{1\times p_{12}}\, R_{12}) \\ \rho_3(\lambda) & \longmapsto & \rho_2(\lambda\, F_{03}), \end{array} \tag{2.64}$$

$$\begin{array}{rcl} \alpha_{21} : \ker_D(.R_{02})/(D^{1\times p_{12}}\, R_{12}) & \longrightarrow & \ker_D(.R_{01})/(D^{1\times p_{11}}\, R_{11}) \\ \rho_2(\mu) & \longmapsto & \rho_1(\mu\, F_{02}). \end{array} \tag{2.65}$$

Chases in the commutative diagram (2.63) show that ρ_3 and ρ_2 are well-defined (see, e.g., [109]).

Let us now find a finite presentation of the left D-modules $\mathrm{ext}^3_D(N_{33}, D)$, $\mathrm{ext}^2_D(N_{22}, D)$ and $\mathrm{ext}^1_D(N_{11}, D)$. Let $R'_{1k} \in D^{p_{0k}\times p'_{1k}}$ be a matrix such that $\ker_D(.R_{0k}) = D^{1\times p'_{1k}}\, R'_{1k}$ for $k = 1, 2, 3$. Moreover, since $D^{1\times p_{1k}}\, R_{1k} \subseteq D^{1\times p'_{1k}}\, R'_{1k}$, there exists a matrix $R''_{1k} \in D^{p_{1k}\times p'_{1k}}$ such that:

$$R_{1k} = R''_{1k}\, R'_{1k}. \tag{2.66}$$

If $R'_{2k} \in D^{p'_{1k}\times p'_{2k}}$ is such that $\ker_D(.R'_{1k}) = D^{1\times p'_{2k}}\, R'_{2k}$, then using Proposition 1.3.1, we obtain

$$\begin{array}{rcl} \chi_k : L_k \triangleq D^{1\times p'_{1k}}/(D^{1\times p_{1k}}\, R''_{1k} + D^{1\times p'_{2k}}\, R'_{2k}) & \longrightarrow & (D^{1\times p'_{1k}}\, R'_{1k})/(D^{1\times p_{1k}}\, R_{1k}) \cong \mathrm{ext}^1_D(N_{1k}, D), \\ \rho'_k(\lambda) & \longmapsto & \rho_k(\lambda\, R'_{1k}), \end{array} \tag{2.67}$$

where $\rho'_k : D^{1\times p'_{1k}} \longrightarrow L_k$ is the canonical projection onto the left D-module L_k.

Since $R'_{1k}\, F_{0k}\, R_{0(k-1)} = R'_{1k}\, R_{0k}\, F_{-1k} = 0$, then

$$D^{1\times p'_{1k}}\, (R'_{1k}\, F_{0k}) \subseteq \ker_D(.R_{0(k-1)}) = D^{1\times p'_{1(k-1)}}\, R'_{1(k-1)},$$

and thus there exists a matrix $F'_{1k} \in D^{p'_{1k}\times p'_{1(k-1)}}$ such that:

$$\forall\, k = 2, 3, \quad R'_{1k}\, F_{0k} = F'_{1k}\, R'_{1(k-1)}. \tag{2.68}$$

Similarly, we can prove that there exists $F'_{2k} \in D^{p'_{2k}\times p'_{2(k-1)}}$ such that:

$$\forall\, k = 2, 3, \quad R'_{2k}\, F'_{1k} = F'_{2k}\, R'_{2(k-1)}. \tag{2.69}$$

Therefore, we obtain the following commutative exact diagram of left D-modules:

$$\begin{array}{ccccccc} D^{1\times p_{-13}} & \xleftarrow{.R_{03}} & D^{1\times p_{03}} & \xleftarrow{.R'_{13}} & D^{1\times p'_{13}} & \xleftarrow{.R'_{23}} & D^{1\times p'_{23}} \\ \downarrow .F_{-13} & & \downarrow .F_{03} & & \downarrow .F'_{13} & & \downarrow .F'_{23} \\ D^{1\times p_{-12}} & \xleftarrow{.R_{02}} & D^{1\times p_{02}} & \xleftarrow{.R'_{12}} & D^{1\times p'_{12}} & \xleftarrow{.R'_{22}} & D^{1\times p'_{22}} \\ \downarrow .F_{-12} & & \downarrow .F_{02} & & \downarrow .F'_{12} & & \downarrow .F'_{22} \\ D^{1\times p_{-11}} & \xleftarrow{.R_{01}} & D^{1\times p_{01}} & \xleftarrow{.R'_{11}} & D^{1\times p'_{11}} & \xleftarrow{.R'_{21}} & D^{1\times p'_{21}}. \end{array} \tag{2.70}$$

Remark 2.4.1. If $R_{0k} = 0$, i.e., $\ker_D(R_{1k}.) = 0$, then applying the functor $\hom_D(\,\cdot\,, D)$ to the short exact sequence $0 \longrightarrow D^{p_{0k}} \xrightarrow{R_{1k}.} D^{p_{1k}} \xrightarrow{\kappa_{1k}} N_{1k} \longrightarrow 0$, we obtain the following complex:

$$0 \longleftarrow D^{1\times p_{0k}} \xleftarrow{.R_{1k}} D^{1\times p_{1k}}.$$

Hence, we get $\ker_D(.R_{0k}) = D^{1\times p_{0k}}$, i.e., $R'_{1k} = I_{p_{0k}}$, $p'_{1k} = p_{0k}$ and $R'_{2k} = 0$.

Let us now deduce two identities which will be useful in what follows. Combining (2.49) for $k=2$ with (2.66) for $k=1$ and $k=2$ and with (2.68) for $k=2$, we obtain

$$R''_{11}\,R'_{11}=R_{11}=R_{12}\,F_{02}=R''_{12}\,R'_{12}\,F_{02}=R''_{12}\,F'_{12}\,R'_{11},$$

and thus $(R''_{11}-R''_{12}\,F'_{12})\,R'_{11}=0$, i.e., $D^{1\times p_{11}}\,(R''_{11}-R''_{12}\,F'_{12})\subseteq \ker_D(.R'_{11})=D^{1\times p'_{21}}\,R'_{21}$, which proves the existence of a matrix $X_{12}\in D^{p_{11}\times p'_{21}}$ such that:

$$R''_{11}=R''_{12}\,F'_{12}+X_{12}\,R'_{21}. \tag{2.71}$$

Combining (2.50) for $k=3$ with (2.66) for $k=2$ and $k=3$ and with (2.68) for $k=3$, we obtain

$$F_{13}\,(R''_{12}\,R'_{12})=F_{13}\,R_{12}=R_{13}\,F_{03}=(R''_{13}\,R'_{13})\,F_{03}=R''_{13}\,F'_{13}\,R'_{12},$$

and thus $(F_{13}\,R''_{12}-R''_{13}\,F'_{13})\,R'_{12}=0$, i.e., $D^{1\times p_{13}}\,(F_{13}\,R''_{12}-R''_{13}\,F'_{13})\subseteq \ker_D(.R'_{12})=D^{1\times p'_{22}}\,R'_{22}$, which proves the existence of a matrix $X_{22}\in D^{p_{13}\times p'_{22}}$ such that:

$$F_{13}\,R''_{12}-R''_{13}\,F'_{13}=X_{22}\,R'_{22}. \tag{2.72}$$

Let us recall that:

$$\left\{\begin{array}{l} L_1=D^{1\times p'_{11}}/(D^{1\times p_{11}}\,R''_{11}+D^{1\times p'_{21}}\,R'_{21})\cong \mathrm{ext}^1_D(N_{11},D)\cong t(M),\\ L_2=D^{1\times p'_{12}}/(D^{1\times p_{12}}\,R''_{12}+D^{1\times p'_{22}}\,R'_{22})\cong \mathrm{ext}^2_D(N_{22},D),\\ L_3=D^{1\times p'_{13}}/(D^{1\times p_{13}}\,R''_{13}+D^{1\times p'_{23}}\,R'_{23})\cong \mathrm{ext}^3_D(N_{33},D).\end{array}\right. \tag{2.73}$$

Then, we can define the left D-homomorphism $\overline{\alpha}_{32}=\chi_2^{-1}\circ\alpha_{32}\circ\chi_3 : L_3\longrightarrow L_2$, where the χ_i's are defined by (2.67) and α_{32} is defined by (2.64). Using (2.68) for $k=3$, we have

$$\overline{\alpha}_{32}(\rho'_3(\lambda))=(\chi_2^{-1}\circ\alpha_{32})(\rho_3(\lambda\,R'_{13}))=\chi_2^{-1}(\rho_2(\lambda\,R'_{13}\,F_{03}))=\chi_2^{-1}(\rho_2(\lambda\,F'_{13}\,R'_{12}))=\rho'_2(\lambda\,F'_{13}), \tag{2.74}$$

for all $\lambda\in D^{1\times p'_{13}}$. Moreover, using (2.72) and (2.69) for $k=3$, we get

$$\left(\begin{array}{c} R''_{13}\\ R'_{23}\end{array}\right)F'_{13}=\left(\begin{array}{c} F_{13}\,R''_{12}-X_{22}\,R'_{22}\\ F'_{23}\,R'_{22}\end{array}\right)=\left(\begin{array}{cc} F_{13} & -X_{22}\\ 0 & F'_{23}\end{array}\right)\left(\begin{array}{c} R''_{12}\\ R'_{22}\end{array}\right),$$

which yields the following commutative exact diagram:

$$\begin{array}{ccccccc} D^{1\times(p_{13}+p'_{23})} & \xrightarrow{.(R''^T_{13}\quad R'^T_{23})^T} & D^{1\times p'_{13}} & \xrightarrow{\rho'_3} & L_3 & \longrightarrow & 0\\ \downarrow .\left(\begin{array}{cc} F_{13} & -X_{22}\\ 0 & F'_{23}\end{array}\right) & & \downarrow .F'_{13} & & \downarrow \overline{\alpha}_{32} & & \\ D^{1\times(p_{12}+p'_{22})} & \xrightarrow{.(R''^T_{12}\quad R'^T_{22})^T} & D^{1\times p'_{12}} & \xrightarrow{\rho'_2} & L_2 & \longrightarrow & 0.\end{array}$$

Up to isomorphism, the short exact sequence

$$0\longrightarrow \mathrm{ext}^3_D(N_{33},D)\xrightarrow{\gamma_{32}}\mathrm{ext}^2_D(N_{22},D)\longrightarrow \mathrm{coker}\,\gamma_{32}\longrightarrow 0$$

becomes the following short exact sequence:

$$0\longrightarrow L_3\xrightarrow{\overline{\alpha}_{32}} L_2\xrightarrow{\theta_2}\mathrm{coker}\,\overline{\alpha}_{32}\longrightarrow 0. \tag{2.75}$$

Using 3 of Proposition 3.4.1, the left D-module $\mathrm{coker}\,\overline{\alpha}_{32}$ is defined by:

$$\mathrm{coker}\,\overline{\alpha}_{32}=D^{1\times p'_{12}}/(D^{1\times p'_{13}}\,F'_{13}+D^{1\times p_{12}}\,R''_{12}+D^{1\times p'_{22}}\,R'_{22}).$$

Then, we can easily check that the following commutative exact diagram holds

$$
\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \downarrow & & & & \\
 & & & & D^{1\times p_{12}}\,R''_{12}+D^{1\times p'_{22}}\,R'_{22} & & & & \\
 & & & & \downarrow & & & & \\
 & & D^{1\times (p'_{13}+p_{12}+p'_{22})} & \xrightarrow{.(F'^{T}_{13} \quad R''^{T}_{12} \quad R'^{T}_{22})^{T}} & D^{1\times p'_{12}} & \xrightarrow{\sigma_2} & \operatorname{coker}\overline{\alpha}_{32} & \longrightarrow & 0\\
 & & \downarrow \psi_2 & & \downarrow \rho'_2 & & \| & & \\
0 & \longrightarrow & L_3 & \xrightarrow{\overline{\alpha}_{32}} & L_2 & \xrightarrow{\theta_2} & \operatorname{coker}\overline{\alpha}_{32} & \longrightarrow & 0,\\
 & & & & \downarrow & & & & \\
 & & & & 0 & & & &
\end{array}
$$

where $\psi_2 : D^{1\times (p'_{13}+p_{12}+p'_{22})} \longrightarrow L_3$ is the left D-homomorphism defined by:

$$\psi_2(e_i)=\left\{\begin{array}{ll} \rho'_3(e_i) & i=1,\ldots,p'_{13},\\ 0, & i=p'_{13}+1,\ldots,p'_{13}+p_{12}+p'_{22}.\end{array}\right.$$

Applying Theorem 2.1.3 to the short exact sequence (2.75) with the matrix

$$A=\left(\begin{array}{c} I_{p'_{13}}\\ 0\\ 0\end{array}\right)\in D^{(p'_{13}+p_{12}+p'_{22})\times p'_{13}},$$

(see Corollary 2.1.1), we obtain the following characterization of the left D-module L_2 in terms of the presentations of the left D-modules $L_3 \cong \operatorname{ext}^3_D(N_{33}, D)$ and $\operatorname{coker}\overline{\alpha}_{32}$.

Proposition 2.4.1 ([97, 98]). *Let D be an Auslander regular ring (e.g., $D = A\langle \partial_1, \ldots, \partial_n\rangle$, where A is either a field k, $k[x_1, \ldots, x_n]$, $k(x_1, \ldots, x_n)$ or $k[[x_1, \ldots, x_n]]$, where k is a field of characteristic 0, or $k\{x_1, \ldots, x_n\}$, where $k = \mathbb{R}$ or $\mathbb{C}$). With the previous notations, let us consider the following two matrices*

$$Q_2=\left(\begin{array}{c} R''_{12}\\ R'_{22}\end{array}\right)\in D^{(p_{12}+p'_{22})\times p'_{12}},\quad P_2=\left(\begin{array}{cc} F'_{13} & -I_{p'_{13}}\\ R''_{12} & 0\\ R'_{22} & 0\\ 0 & R''_{13}\\ 0 & R'_{23}\end{array}\right)\in D^{(p'_{13}+p_{12}+p'_{22}+p_{13}+p'_{23})\times (p'_{12}+p'_{13})},$$

and the following two finitely presented left D-modules:

$$\left\{\begin{array}{l} L_2 = D^{1\times p'_{12}}/(D^{1\times p_{12}}\,R''_{12}+D^{1\times p'_{22}}\,R'_{22}),\\ E_2 = D^{1\times (p'_{12}+p'_{13})}/(D^{1\times (p'_{13}+p_{12}+p'_{22}+p_{13}+p'_{23})}\,P_2).\end{array}\right.$$

If $\varrho_2 : D^{1\times (p'_{12}+p'_{13})} \longrightarrow E_2$ is the canonical projection, then we have $E_2 \cong L_2$, where the left D-isomorphism is defined by:

$$\begin{array}{rcl} \phi_2 : L_2 & \longrightarrow & E_2\\ \rho'_2(\mu) & \longmapsto & \varrho_2(\mu\,(I_{p'_{12}} \quad 0)),\end{array}\qquad \begin{array}{rcl} \phi_2^{-1} : E_2 & \longrightarrow & L_2\\ \varrho_2(\nu) & \longmapsto & \rho'_2(\nu\,(I^{T}_{p'_{12}} \quad F'^{T}_{13})^{T}).\end{array} \tag{2.76}$$

Now, if $\mathcal{F}$ is a left D-module, then applying the functor $\hom_D(\,\cdot\,, \mathcal{F})$ to the isomorphism $E_2 \cong L_2$ and using Theorem 1.1.1, we obtain $\ker_{\mathcal{F}}(Q_2.) \cong \ker_{\mathcal{F}}(P_2.)$. More precisely, using (2.76), we obtain the following corollary of Proposition 2.4.1.

Corollary 2.4.1 ([97, 98]). *If $\mathcal{F}$ is a left D-module, then we have $\ker_{\mathcal{F}}(Q_2.) \cong \ker_{\mathcal{F}}(P_2.)$, i.e., the following system equivalence holds*

$$\left\{ \begin{array}{l} R''_{12}\, \upsilon = 0, \\ R'_{22}\, \upsilon = 0, \end{array} \right. \quad \Leftrightarrow \quad \left\{ \begin{array}{l} F'_{13}\, \tau_2 - \tau_3 = 0, \\ R''_{12}\, \tau_2 = 0, \\ R'_{22}\, \tau_2 = 0, \\ R''_{13}\, \tau_3 = 0, \\ R'_{23}\, \tau_3 = 0, \end{array} \right.$$

under the following invertible transformations:

$$\begin{array}{lcl} \delta : \ker_{\mathcal{F}}(P_2.) & \longrightarrow & \ker_{\mathcal{F}}(Q_2.) \\ \left(\begin{array}{c} \tau_2 \\ \tau_3 \end{array} \right) & \longmapsto & \upsilon = \tau_2, \end{array} \qquad \begin{array}{lcl} \delta^{-1} : \ker_{\mathcal{F}}(Q_2.) & \longrightarrow & \ker_{\mathcal{F}}(P_2.) \\ \upsilon & \longmapsto & \left(\begin{array}{c} \tau_2 \\ \tau_3 \end{array} \right) = \left(\begin{array}{c} I_{p'_{12}} \\ F'_{13} \end{array} \right) \upsilon. \end{array} \tag{2.77}$$

Now, we can introduce the left D-homomorphism $\overline{\alpha}_{21} = \chi_1^{-1} \circ \alpha_{21} \circ \chi_2 : L_2 \longrightarrow L_1$, where the χ_i's are defined by (2.67) and α_{21} is defined by (2.65). Then, using (2.68) for $k = 2$, we get

$$\overline{\alpha}_{21}(\rho'_2(\mu)) = (\chi_1^{-1} \circ \alpha_{21})(\rho_2(\mu\, R'_{12})) = \chi_1^{-1}(\rho_1(\mu\, R'_{12}\, F_{02})) = \chi_1^{-1}(\rho_1(\mu\, F'_{12}\, R'_{11})) = \rho'_1(\mu\, F'_{12}), \tag{2.78}$$

for all $\mu \in D^{1 \times p'_{12}}$. Moreover, using (2.71) and (2.69) for $k = 2$, we have

$$\left(\begin{array}{c} R''_{12} \\ R'_{22} \end{array} \right) F'_{12} = \left(\begin{array}{c} R''_{11} - X_{12}\, R'_{21} \\ F'_{22}\, R'_{21} \end{array} \right) = \left(\begin{array}{cc} I_{p_{11}} & -X_{12} \\ 0 & F'_{22} \end{array} \right) \left(\begin{array}{c} R''_{11} \\ R'_{21} \end{array} \right),$$

which yields the following commutative exact diagram:

$$\begin{array}{ccccccc} D^{1 \times (p_{12} + p'_{22})} & \xrightarrow{.(R''^T_{12} \quad R'^T_{22})^T} & D^{1 \times p'_{12}} & \xrightarrow{\rho'_2} & L_2 & \longrightarrow & 0 \\ \downarrow .\left(\begin{array}{cc} I_{p_{11}} & -X_{12} \\ 0 & F'_{22} \end{array} \right) & & \downarrow .F'_{12} & & \downarrow \overline{\alpha}_{21} & & \\ D^{1 \times (p_{11} + p'_{21})} & \xrightarrow{.(R''^T_{11} \quad R'^T_{21})^T} & D^{1 \times p'_{11}} & \xrightarrow{\rho'_1} & L_1 & \longrightarrow & 0. \end{array}$$

Up to isomorphism, the short exact sequence

$$0 \longrightarrow \operatorname{ext}^2_D(N_{22}, D) \xrightarrow{\gamma_{21}} t(M) \longrightarrow \operatorname{coker} \gamma_{21} \longrightarrow 0,$$

becomes the following short exact sequence

$$0 \longrightarrow L_2 \xrightarrow{\overline{\alpha}_{21}} L_1 \xrightarrow{\theta_1} \operatorname{coker} \overline{\alpha}_{21} \longrightarrow 0, \tag{2.79}$$

where, using 3 of Proposition 3.4.1, the left D-module $\operatorname{coker} \overline{\alpha}_{21}$ is defined by:

$$\operatorname{coker} \overline{\alpha}_{21} = D^{1 \times p'_{11}} / (D^{1 \times p'_{12}}\, F'_{12} + D^{1 \times p_{11}}\, R''_{11} + D^{1 \times p'_{21}}\, R'_{21}).$$

Using the left D-isomorphism $\phi_2^{-1} : E_2 \longrightarrow L_2$ defined by (2.76), the short exact sequence (2.79) yields the following short exact sequence

$$0 \longrightarrow E_2 \xrightarrow{\overline{\alpha}_{21} \circ \phi_2^{-1}} L_1 \xrightarrow{\theta_1} \operatorname{coker} \overline{\alpha}_{21} \longrightarrow 0,$$

where the left D-homomorphism $\overline{\alpha}_{21} \circ \phi_2^{-1} : E_2 \longrightarrow L_1$ is defined by:

$$\forall\, \nu \in D^{1 \times (p'_{12}+p'_{13})}, \quad (\overline{\alpha}_{21} \circ \phi_2^{-1})(\varrho_2(\nu)) = \overline{\alpha}_{21}\left(\rho'_2\left(\nu \left(\begin{array}{c} I_{p'_{12}} \\ F'_{13} \end{array}\right)\right)\right) = \rho'_1\left(\nu \left(\begin{array}{c} F'_{12} \\ F'_{13}\, F'_{12} \end{array}\right)\right).$$

Now, we can check that the following commutative exact diagram holds

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & & D^{1\times p_{11}}\, R''_{11} + D^{1\times p'_{21}}\, R'_{21} \arrow[d] & & \\
 & D^{1\times(p'_{12}+p_{11}+p'_{21})} \arrow[r, ".(F'^T_{12} \quad R''^T_{11} \quad R'^T_{21})^T"] \arrow[d, "\psi_1"] & D^{1\times p'_{11}} \arrow[r, "\sigma_1"] \arrow[d, "\rho'_1"] & \operatorname{coker}\overline{\alpha}_{21} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & E_2 \arrow[r, "\overline{\alpha}_{21}\circ\phi_2^{-1}"] & L_1 \arrow[r, "\theta_1"] \arrow[d] & \operatorname{coker}\overline{\alpha}_{21} \arrow[r] & 0, \\
 & & 0 & &
\end{tikzcd}$$

where $\psi_1 : D^{1\times(p'_{12}+p_{11}+p'_{21})} \longrightarrow E_2$ is the left D-homomorphism defined by

$$\psi_1(f_j) = \begin{cases} \varrho_2(f_j\, F), & j = 1, \ldots, p'_{12}, \\ 0, & j = p'_{12}+1, \ldots, p'_{12}+p_{11}+p'_{21}, \end{cases}$$

where $\{f_j\}_{j=1,\ldots,p'_{12}+p_{11}+p'_{21}}$ is the standard basis of $D^{1\times(p'_{12}+p_{11}+p'_{21})}$ and:

$$F = \left(\begin{array}{cc} I_{p'_{12}} & 0 \\ 0 & 0 \\ 0 & 0 \end{array}\right) \in D^{(p'_{12}+p_{11}+p'_{21})\times(p'_{12}+p'_{13})}.$$

If we apply Theorem 2.1.3 to the short exact sequence

$$0 \longrightarrow E_2 \xrightarrow{\overline{\alpha}_{21}\circ\phi_2^{-1}} L_1 \xrightarrow{\theta_1} \operatorname{coker}\overline{\alpha}_{21} \longrightarrow 0$$

with the matrix $A = F$ (see Corollary 2.1.1), then we obtain the following proposition.

Proposition 2.4.2 ([97, 98])**.** *With the hypotheses of Proposition 2.4.1 and the previous notations, let us consider the following two matrices*

$$P_1 = \left(\begin{array}{ccc} F'_{12} & -I_{p'_{12}} & 0 \\ R''_{11} & 0 & 0 \\ R'_{21} & 0 & 0 \\ 0 & F'_{13} & -I_{p'_{13}} \\ 0 & R''_{12} & 0 \\ 0 & R'_{22} & 0 \\ 0 & 0 & R''_{13} \\ 0 & 0 & R'_{23} \end{array}\right) \in D^{(p'_{12}+p_{11}+p'_{21}+p'_{13}+p_{12}+p'_{22}+p_{13}+p'_{23})\times(p'_{11}+p'_{12}+p'_{13})},$$

$$Q_1 = \left(\begin{array}{c} R''_{11} \\ R'_{21} \end{array}\right) \in D^{(p_{11}+p'_{21})\times p'_{11}},$$

and the following two finitely presented left D-modules:

$$\left\{ \begin{array}{l} L_1 = D^{1 \times p'_{11}} / (D^{1 \times (p_{11} + p'_{21})} \, Q_1), \\ E_1 = D^{1 \times (p'_{11} + p'_{12} + p'_{13})} / (D^{1 \times (p'_{12} + p_{11} + p'_{21} + p'_{13} + p_{12} + p'_{22} + p_{13} + p'_{23})} \, P_1). \end{array} \right.$$

If $\varrho_1 : D^{1 \times (p'_{11} + p'_{12} + p'_{13})} \longrightarrow E_1$ is the canonical projection, then we have $E_1 \cong L_1$, where the left D-isomorphism is defined by:

$$\begin{array}{rcl} \phi_1 : L_1 & \longrightarrow & E_1 \\ \rho'_1(\nu) & \longmapsto & \varrho_1(\nu \, (I_{p'_{11}} \quad 0 \quad 0)), \end{array} \qquad \begin{array}{rcl} \phi_1^{-1} : E_1 & \longrightarrow & L_1 \\ \varrho_1(\lambda) & \longmapsto & \rho'_1\left(\lambda \begin{pmatrix} I_{p'_{11}} \\ F'_{12} \\ F'_{13} \, F'_{12} \end{pmatrix}\right). \end{array} \tag{2.80}$$

Finally, we have $L_1 \cong t(M)$, with the following left D-isomorphisms:

$$\begin{array}{rcl} \vartheta : L_1 & \longrightarrow & t(M) \\ \rho'_1(\nu) & \longmapsto & \pi(\nu \, R'_{11}), \end{array} \qquad \begin{array}{rcl} \vartheta^{-1} : t(M) & \longrightarrow & L_1 \\ \pi(\nu \, R'_{11}) & \longmapsto & \rho'_1(\nu). \end{array}$$

If $\mathcal{F}$ is a left D-module, then applying the functor $\hom_D(\,\cdot\,, \mathcal{F})$ to the isomorphism $E_1 \cong L_1$ and using Theorem 1.1.1, we obtain $\ker_{\mathcal{F}}(Q_1.) \cong \ker_{\mathcal{F}}(P_1.)$. More precisely, using (2.80), we get the following corollary.

Corollary 2.4.2 ([97, 98])**.** *If $\mathcal{F}$ is a left D-module, then we have $\ker_{\mathcal{F}}(Q_1.) \cong \ker_{\mathcal{F}}(P_1.)$, i.e., the following system equivalence holds*

$$\left\{ \begin{array}{l} R''_{11} \, \theta = 0, \\ R'_{21} \, \theta = 0, \end{array} \right. \quad \Leftrightarrow \quad \left\{ \begin{array}{l} F'_{12} \, \tau_1 - \tau_2 = 0, \\ R''_{11} \, \tau_1 = 0, \\ R'_{21} \, \tau_1 = 0, \\ F'_{13} \, \tau_2 - \tau_3 = 0, \\ R''_{12} \, \tau_2 = 0, \\ R'_{22} \, \tau_2 = 0, \\ R''_{13} \, \tau_3 = 0, \\ R'_{23} \, \tau_3 = 0, \end{array} \right.$$

under the following invertible transformations:

$$\begin{array}{rcl} \varpi : \ker_{\mathcal{F}}(P_1.) & \longrightarrow & \ker_{\mathcal{F}}(Q_1.) \\ \begin{pmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{pmatrix} & \longmapsto & \theta = \tau_1, \end{array} \qquad \begin{array}{rcl} \varpi^{-1} : \ker_{\mathcal{F}}(Q_1.) & \longrightarrow & \ker_{\mathcal{F}}(P_1.) \\ \theta & \longmapsto & \begin{pmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{pmatrix} = \begin{pmatrix} I_{p'_{12}} \\ F'_{12} \\ F'_{13} \, F'_{12} \end{pmatrix} \theta. \end{array} \tag{2.81}$$

Using Proposition 2.4.2, let $\vartheta \circ \phi_1^{-1} : E_1 \longrightarrow t(M)$ be the left D-isomorphism defined by:

$$(\vartheta \circ \phi_1^{-1})(\varrho_1(\lambda)) = \pi\left(\lambda \begin{pmatrix} R'_{11} \\ F'_{12} \, R'_{11} \\ F'_{13} \, F'_{12} \, R'_{11} \end{pmatrix}\right).$$

Then, the short exact sequence $0 \longrightarrow t(M) \xrightarrow{i} M \xrightarrow{\rho} M/t(M) \longrightarrow 0$ yields the following one:

$$0 \longrightarrow E_1 \xrightarrow{i\circ\vartheta\circ\phi_1^{-1}} M \xrightarrow{\rho} M/t(M) \longrightarrow 0. \tag{2.82}$$

Now, we can easily check that the following commutative exact diagram holds

$$\begin{array}{ccccccc} & & D^{1\times p'_{11}} & \xrightarrow{.R'_{11}} & D^{1\times p_{01}} & \xrightarrow{\pi'} & M/t(M) & \longrightarrow 0 \\ & & \downarrow \psi & & \downarrow \pi & & \| & \\ 0 \longrightarrow & & E_1 & \xrightarrow{i\circ\vartheta\circ\phi_1^{-1}} & M & \xrightarrow{\rho} & M/t(M) & \longrightarrow 0, \end{array}$$

where the left D-homomorphism $\psi : D^{1\times p'_{11}} \longrightarrow E_1$ is defined by $\psi(g_k) = \varrho_1(g_k\,(I_{p'_{11}} \quad 0 \quad 0))$, and $\{g_k\}_{k=1,\dots,p'_{11}}$ is the standard basis of $D^{1\times p'_{11}}$. Then, we can apply Theorem 2.1.3 to the short exact sequence (2.82) with $A = (I_{p'_{11}} \quad 0 \quad 0) \in D^{p'_{11}\times(p'_{11}+p'_{12}+p'_{13})}$ (see Corollary 2.1.1) and we obtain the following main theorem.

Theorem 2.4.1 ([97, 98]). *With the hypotheses of Proposition 2.4.1 and the previous notations, let us consider the following matrix*

$$P = \begin{pmatrix} R'_{11} & -I_{p'_{11}} & 0 & 0 \\ 0 & F'_{12} & -I_{p'_{12}} & 0 \\ 0 & R''_{11} & 0 & 0 \\ 0 & R'_{21} & 0 & 0 \\ 0 & 0 & F'_{13} & -I_{p'_{13}} \\ 0 & 0 & R''_{12} & 0 \\ 0 & 0 & R'_{22} & 0 \\ 0 & 0 & 0 & R''_{13} \\ 0 & 0 & 0 & R'_{23} \end{pmatrix} \in D^{(p'_{11}+p'_{12}+p_{11}+p'_{21}+p'_{13}+p_{12}+p'_{22}+p_{13}+p'_{23})\times(p_{01}+p'_{11}+p'_{12}+p'_{13})},$$

and the following two finitely presented left D-modules:

$$\begin{cases} M = D^{1\times p_{01}}/(D^{1\times p_{11}}\,R_{11}), \\ E = D^{1\times(p_{01}+p'_{11}+p'_{12}+p'_{13})}/(D^{1\times(p'_{11}+p'_{12}+p_{11}+p'_{21}+p'_{13}+p_{12}+p'_{22}+p_{13}+p'_{23})}\,P). \end{cases}$$

If $\varrho : D^{1\times(p_{01}+p'_{11}+p'_{12}+p'_{13})} \longrightarrow E$ is the canonical projection, then we have $E \cong M$, where the left D-isomorphism is defined by:

$$\begin{array}{rcl} \phi : M & \longrightarrow & E \\ \pi(\lambda) & \longmapsto & \varrho(\lambda\,(I_{p_{01}} \quad 0 \quad 0 \quad 0)), \end{array} \qquad \begin{array}{rcl} \phi : E & \longrightarrow & M \\ \varrho(\epsilon) & \longmapsto & \pi\left(\epsilon\left(\left(\begin{pmatrix} I_{p_{01}} \\ R'_{11} \\ F'_{12}\,R'_{11} \\ F'_{13}\,F'_{12}\,R'_{11} \end{pmatrix}\right)\right)\right). \end{array} \tag{2.83}$$

The different matrices used in Theorem 2.4.1 can be arranged into the diagram defined in Figure 2.1. Each diagram of Figure 2.1 commutes except for the two diagrams defined by violet matrices ("faces in the depth direction") (see (2.71) and (2.71)). The horizontal sequences are either complexes (marked in red) or are exact sequences (marked in blue and in green). The

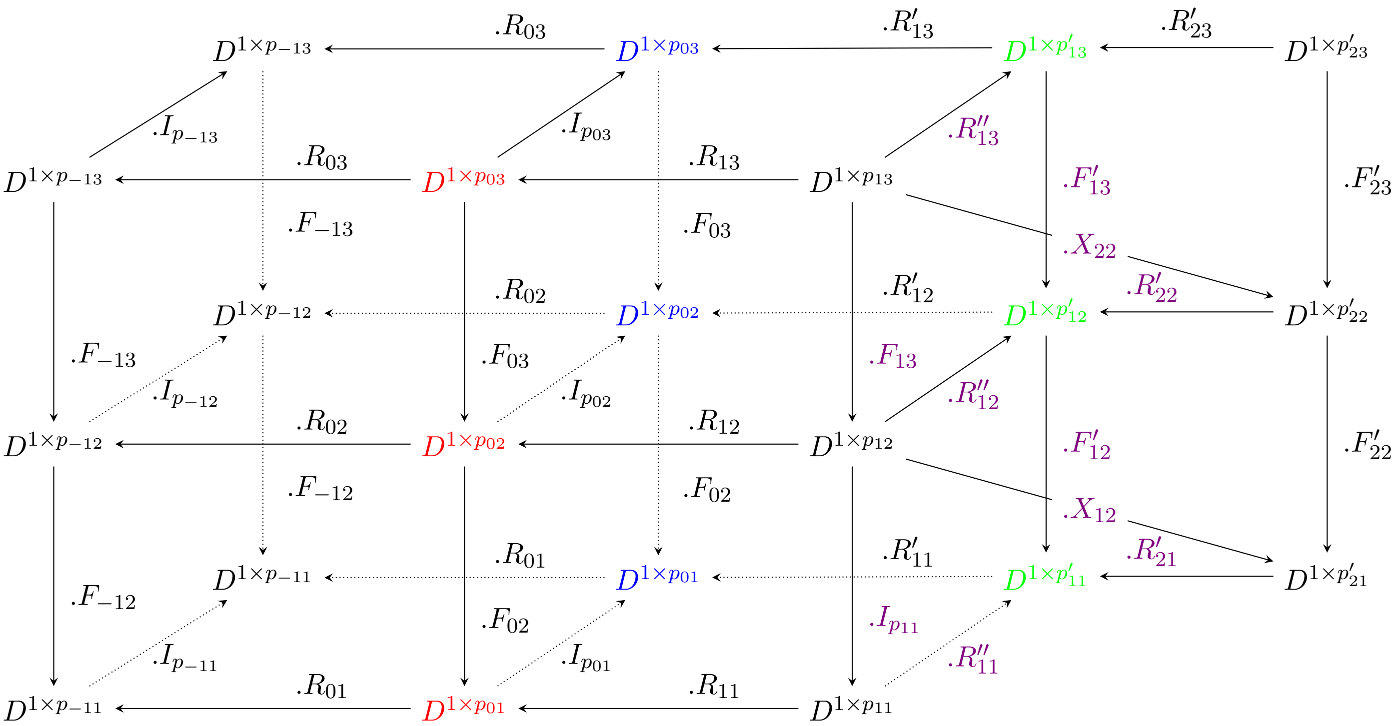

Figure 2.1: Purity filtration

vertical sequences are not complexes. The defect of exactness $\mathrm{ext}^i_D(N_{ii}, D)$ of the i^{th} horizontal complex at the red position $D^{1\times p_{0i}}$ is isomorphic to the cokernel L_i of the left D-homomorphism defined by the two maps $.R''_{1i} : D^{1\times p_{1i}} \longrightarrow D^{1\times p'_{1i}}$ and $.R'_{2i} : D^{1\times p'_{2i}} \longrightarrow D^{1\times p'_{1i}}$ arriving at the green position $D^{1\times p'_{1i}}$ on the same level, i.e., $L_i = D^{1\times p'_{1i}}/(D^{1\times(p_{1i}+p'_{2i})}\,(R''^T_{1i} \quad R'^T_{2i})^T)$. Moreover, the homomorphism $\alpha_{i(i-1)} : \mathrm{ext}^i_D(N_{ii}, D) \longrightarrow \mathrm{ext}^{i-1}_D(N_{(i-1)(i-1)}, D)$ (see (2.64) and (2.65)), defined by means of the corresponding connecting vertical arrow, defines the left D-homomorphism $\overline{\alpha}_{i(i-1)} : L_i \longrightarrow L_{i-1}$ (see (2.74) and (2.78)), defined by means of the vertical arrow connecting the corresponding two successive green positions.

Using the regular patterns of the matrix P and (2.83), we can easily generalize Theorem 2.4.1, Corollary 2.4.4 and Remark 2.4.4 when $\ker_D(.R_3) \neq 0$, i.e., for a finitely presented left D-module $M = D^{1\times p_{01}}/(D^{1\times p_{11}}\,R_{11})$ defined by a longer finite free resolution of the form:

$$0 \longleftarrow M \xleftarrow{\pi} D^{1\times p_0} \xleftarrow{.R_1} D^{1\times p_1} \xleftarrow{.R_2} D^{1\times p_2} \xleftarrow{.R_3} D^{1\times p_3} \xleftarrow{.R_4} \ldots \xleftarrow{.R_m} D^{1\times p_m}.$$

If $\ker_D(.R_m) = 0$, then the corresponding generalization defines the purity filtration of M.

We note that using (2.71), the row of P containing the matrix R''_{11} can be removed. Hence, we obtain the following straightforward corollary of Theorem 2.4.1.

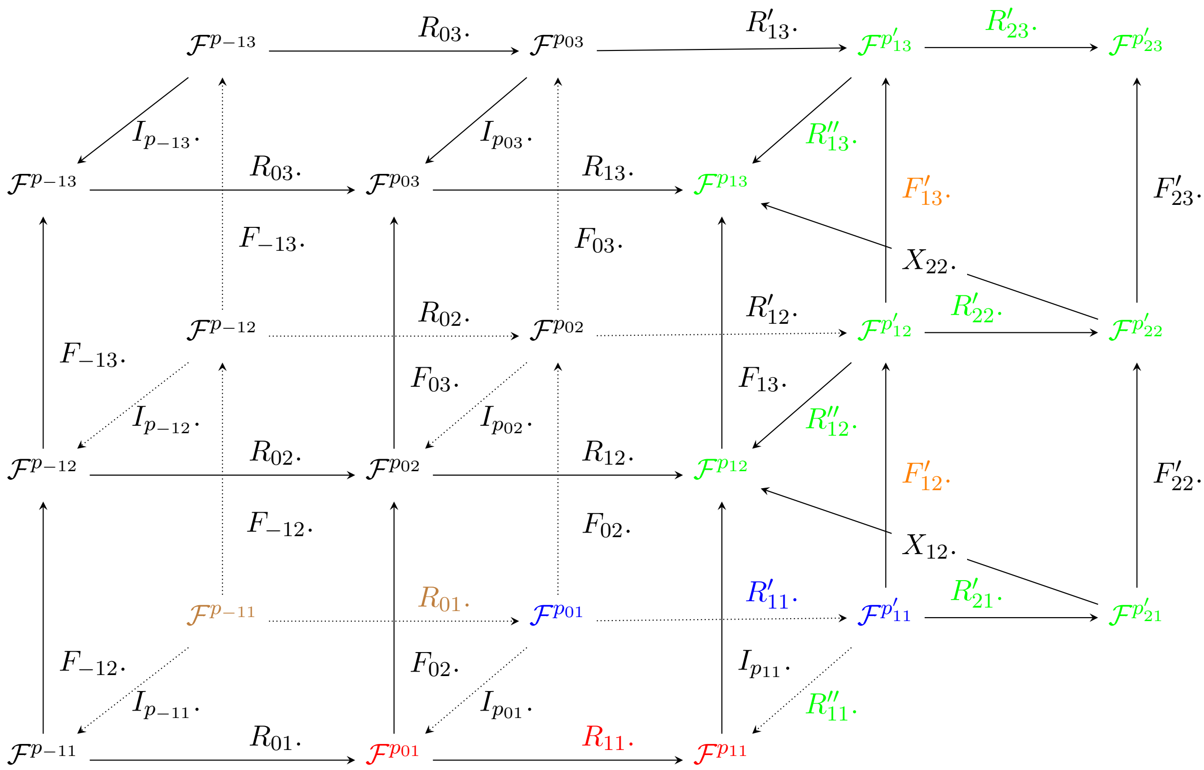

Figure 2.2: System equivalence based on the purity filtration

Corollary 2.4.3. *With the hypotheses of Proposition 2.4.1 and the previous notations, if*

$$Q = \begin{pmatrix} R'_{11} & -I_{p'_{11}} & 0 & 0 \\ 0 & F'_{12} & -I_{p'_{12}} & 0 \\ 0 & R'_{21} & 0 & 0 \\ 0 & 0 & F'_{13} & -I_{p'_{13}} \\ 0 & 0 & R''_{12} & 0 \\ 0 & 0 & R'_{22} & 0 \\ 0 & 0 & 0 & R''_{13} \\ 0 & 0 & 0 & R'_{23} \end{pmatrix} \in D^{(p'_{11}+p'_{12}+p'_{21}+p'_{13}+p_{12}+p'_{22}+p_{13}+p'_{23})\times(p_{01}+p'_{11}+p'_{12}+p'_{13})},$$

then we have

$$M = D^{1\times p_{01}}/(D^{1\times p_{11}}\, R_{11}) \cong E = D^{1\times(p_{01}+p'_{11}+p'_{12}+p'_{13})}/(D^{1\times(p'_{11}+p'_{12}+p'_{21}+p'_{13}+p_{12}+p'_{22}+p_{13}+p'_{23})}\, Q),$$

where the isomorphism is defined by (2.83).

If $\mathcal{F}$ is a left D-module, then applying the functor $\hom_D(\,\cdot\,, \mathcal{F})$ to $M \cong E$ and using Theorem 1.1.1, we obtain $\ker_{\mathcal{F}}(R_{11}.) \cong \ker_{\mathcal{F}}(P.) = \ker_{\mathcal{F}}(Q.)$. Similarly, applying the functor $\hom_D(\,\cdot\,, \mathcal{F})$ to the diagram defined in Figure 2.1, we obtain the diagram of abelian groups defined in Figure 2.2. More precisely, using (2.83), we get the following corollary.

Corollary 2.4.4 ([97, 98]). *If $\mathcal{F}$ is a left D-module, then we have*

$$\ker_{\mathcal{F}}(R_{11}.) \cong \ker_{\mathcal{F}}(P.) = \ker_{\mathcal{F}}(Q.),$$

i.e., the following system equivalence holds

$$R_{11}\,\eta = 0 \quad \Leftrightarrow \quad \left\{ \begin{array}{l} R'_{11}\,\zeta - \tau_1 = 0, \\ F'_{12}\,\tau_1 - \tau_2 = 0, \\ R''_{11}\,\tau_1 = 0, \\ R'_{21}\,\tau_1 = 0, \\ F'_{13}\,\tau_2 - \tau_3 = 0, \\ R''_{12}\,\tau_2 = 0, \\ R'_{22}\,\tau_2 = 0, \\ R''_{13}\,\tau_3 = 0, \\ R'_{23}\,\tau_3 = 0, \end{array} \right. \quad \Leftrightarrow \quad \left\{ \begin{array}{l} R'_{11}\,\zeta - \tau_1 = 0, \\ F'_{12}\,\tau_1 - \tau_2 = 0, \\ R'_{21}\,\tau_1 = 0, \\ F'_{13}\,\tau_2 - \tau_3 = 0, \\ R''_{12}\,\tau_2 = 0, \\ R'_{22}\,\tau_2 = 0, \\ R''_{13}\,\tau_3 = 0, \\ R'_{23}\,\tau_3 = 0, \end{array} \right. \tag{2.84}$$

under the following invertible transformations:

$$\begin{array}{rcl} \gamma : \ker_{\mathcal{F}}(Q.) & \longrightarrow & \ker_{\mathcal{F}}(R_{11}.) \\ \begin{pmatrix} \zeta \\ \tau_1 \\ \tau_2 \\ \tau_3 \end{pmatrix} & \longmapsto & \eta = \zeta, \end{array} \qquad \begin{array}{rcl} \gamma^{-1} : \ker_{\mathcal{F}}(R_{11}.) & \longrightarrow & \ker_{\mathcal{F}}(Q.) \\ \eta & \longmapsto & \begin{pmatrix} \zeta \\ \tau_1 \\ \tau_2 \\ \tau_3 \end{pmatrix} = \begin{pmatrix} I_{p_{01}} \\ R'_{11} \\ F'_{12}\,R'_{11} \\ F'_{13}\,F'_{12}\,R'_{11} \end{pmatrix} \eta. \end{array} \tag{2.85}$$

Remark 2.4.2. If we set

$$S_0 = R'_{11}, \quad S_1 = \begin{pmatrix} F'_{12} \\ R''_{11} \\ R'_{21} \end{pmatrix}, \quad S'_1 = \begin{pmatrix} F'_{12} \\ R'_{21} \end{pmatrix}, \quad S_2 = \begin{pmatrix} F'_{13} \\ R''_{12} \\ R'_{22} \end{pmatrix}, \quad S_3 = \begin{pmatrix} R''_{13} \\ R'_{23} \end{pmatrix},$$

then using (2.60), we get:

1. $\ker_{\mathcal{F}}(S_3.) \cong \hom_D(L_3, \mathcal{F}) \cong \hom_D(\mathrm{ext}^3_D(N_{33}, D), \mathcal{F})$ is either 0 or has dimension less or equal to $\dim(D) - 3$,
2. $\ker_{\mathcal{F}}(S_2.) \cong \hom_D(\mathrm{coker}\,\overline{\alpha}_{32}, \mathcal{F}) \cong \hom_D(\mathrm{coker}\,\gamma_{32}, \mathcal{F})$ has dimension $\dim(D) - 2$ when it is non-trivial,
3. $\ker_{\mathcal{F}}(S_1.) = \ker_{\mathcal{F}}(S'_1.) \cong \hom_D(\mathrm{coker}\,\overline{\alpha}_{21}, \mathcal{F}) \cong \hom_D(\mathrm{coker}\,\gamma_{21}, \mathcal{F})$ is either 0 or has dimension $\dim(D) - 1$,
4. $\ker_{\mathcal{F}}(S_0.) \cong \hom_D(M/t(M), \mathcal{F})$ has dimension $\dim(D)$ when it is non-trivial.

If R_3 has full row rank, i.e., $\ker_D(.R_3) = 0$, then $N_{33} \cong \mathrm{ext}^3_D(N_{33}, D)$ and thus $\mathrm{ext}^3_D(N_{33}, D) \cong \mathrm{ext}^3_D(\mathrm{ext}^3_D(M, D), D)$, and $\ker_{\mathcal{F}}(S_3.)$ is either 0 or has $\dim(D) - 3$ (see (2.61)).

The linear system $\ker_{\mathcal{F}}(R_{11}.)$ can be obtained by first integrating the linear system $\ker_{\mathcal{F}}(Q.)$, i.e., by integrating in cascade the linear system $\ker_{\mathcal{F}}(S_3.)$ of dimension less or equal to $\dim(D) - 3$, then the inhomogeneous linear systems of dimension respectively $\dim(D) - 2$, $\dim(D) - 1$ and $\dim(D)$. If $\mathcal{F}$ is an injective left D-module, then $\ker_{\mathcal{F}}(R'_{11}.) = R_{01}\,\mathcal{F}^{p_{-11}}$.

Even if the size of the matrix Q (resp., P) is larger than the one of R_{11}, Q (resp., P) is more suitable for a fine study of the module properties of the left D-module $M \cong E$ than R_{11}, for the study of the structural properties of the linear system $\ker_{\mathcal{F}}(R_{11}.) \cong \ker_{\mathcal{F}}(Q.) = \ker_{\mathcal{F}}(P.)$ as well as for computing closed-form solutions of $\ker_{\mathcal{F}}(R_{11}.)$ (if they exist). We refer the reader to [97] for examples of linear PD systems $\ker_{\mathcal{F}}(R_{11}.)$ which cannot be integrated by means of computer algebra systems such as Maple contrary to their equivalent forms $\ker_{\mathcal{F}}(Q.)$.

Let us illustrate Theorem 2.4.1 with an example coming from [86].

Example 2.4.1. Let us consider the $D = \mathbb{Q}[\partial_1, \partial_2, \partial_3]$-module $M = D^{1\times 4}/(D^{1\times 6}\, R)$ finitely presented by the following matrix:

$$R = \begin{pmatrix} 0 & -2\,\partial_1 & \partial_3 - 2\,\partial_2 - \partial_1 & -1 \\ 0 & \partial_3 - 2\,\partial_1 & 2\,\partial_2 - 3\,\partial_1 & 1 \\ \partial_3 & -6\,\partial_1 & -2\,\partial_2 - 5\,\partial_1 & -1 \\ 0 & \partial_2 - \partial_1 & \partial_2 - \partial_1 & 0 \\ \partial_2 & -\partial_1 & -\partial_2 - \partial_1 & 0 \\ \partial_1 & -\partial_1 & -2\,\partial_1 & 0 \end{pmatrix}.$$

Using Algorithm 1.2.1, the D-module M admits the following finite free resolution:

$$0 \longleftarrow M \xleftarrow{\pi} D^{1\times 4} \xleftarrow{.R} D^{1\times 6} \xleftarrow{.R_2} D^{1\times 4} \xleftarrow{.R_3} D \longleftarrow 0,$$

$$R_2 = \begin{pmatrix} 2\,\partial_2 & \partial_2 & -\partial_2 & -\partial_3 & \partial_3 & 0 \\ 2\,\partial_1 & \partial_2 & -2\,\partial_1 + \partial_2 & -\partial_3 & 8\,\partial_1 - \partial_3 & -8\,\partial_2 + 2\,\partial_3 \\ 0 & \partial_1 - \partial_2 & \partial_1 - \partial_2 & \partial_3 & -8\,\partial_1 + \partial_3 & 8\,\partial_2 - \partial_3 \\ 0 & 0 & 0 & \partial_1 & -\partial_1 & \partial_2 \end{pmatrix},$$

$$R_3 = (\partial_1 \quad \partial_2 \quad -\partial_2 \quad \partial_3).$$

Using the notations $R_{11} = R$, $R_{22} = R_2$ and $R_{33} = R_3$, the commutative diagram (2.51) becomes the following commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[r] & D \arrow[r, "R_{13}."] & D^4 \arrow[r, "R_{23}."] & D^4 \arrow[r, "R_{33}."] & D \arrow[r, "\kappa_{33}"] & N_{33} \arrow[r] & 0 \\
 & 0 \arrow[r] & D^3 \arrow[r, "R_{12}."] \arrow[u, "F_{03}."'] & D^6 \arrow[r, "R_{22}."] \arrow[u, "F_{13}."'] & D^4 \arrow[r, "\kappa_{22}"] \arrow[u, equal] & N_{22} \arrow[r] & 0 & \\
0 \arrow[r] & D \arrow[r, "R_{01}."] & D^4 \arrow[r, "R_{11}."] \arrow[u, "F_{02}."'] & D^6 \arrow[r, "\kappa_{11}"] \arrow[u, equal] & N_{11} \arrow[r] & 0 & & \\
 & 0 \arrow[r] & D^4 \arrow[r, "\kappa_{00}"] \arrow[u, equal] & N_{00} \arrow[r] & 0, & & &
\end{tikzcd}$$

whose horizontal sequences are exact and with the following notations:

$$R_{01} = \begin{pmatrix} 1 \\ -1 \\ 1 \\ \partial_1 - 2\,\partial_2 + \partial_3 \end{pmatrix}, \quad R_{12} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 4\,\partial_1 - \partial_3 & 0 \\ 1 & 4\,\partial_1 - \partial_3 & \partial_3 \\ 0 & \partial_1 - \partial_2 & 0 \\ 0 & \partial_1 - \partial_2 & 0 \\ 0 & 0 & \partial_1 \end{pmatrix}, \quad R_{23} = \begin{pmatrix} -\partial_3 & \partial_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & \partial_1 & -1 & \partial_3 \\ \partial_1 & 0 & 0 & \partial_2 \end{pmatrix},$$

$$R_{13} = \begin{pmatrix} -\partial_2 \\ -\partial_3 \\ 0 \\ \partial_1 \end{pmatrix}, \quad F_{02} = \begin{pmatrix} 0 & -2\,\partial_1 & -\partial_1 - 2\,\partial_2 + \partial_3 & -1 \\ 0 & -1 & -1 & 0 \\ 1 & -1 & -2 & 0 \end{pmatrix},$$

$$F_{13} = \begin{pmatrix} 0 & 0 & 0 & 1 & -1 & 0 \\ 2 & 1 & -1 & 0 & 0 & 0 \\ 2\,\partial_1 & \partial_2 & -2\,\partial_1 + \partial_2 & -\partial_3 & 8\,\partial_1 - \partial_3 & -8\,\partial_2 + 2\,\partial_3 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad F_{03} = (0 \quad 0 \quad 1),$$

$R_{03} = 0$ and $R_{02} = 0$. Using Remark 2.4.1 with $p_{03} = 1$ and $p_{02} = 3$, we obtain $R'_{13} = 1$, $R'_{12} = I_3$, $R'_{23} = 0$ and $R'_{13} = 0$. The commutative diagram (2.70) becomes the following one

$$\begin{array}{ccccccc} 0 & \longleftarrow & D & \xleftarrow{.R'_{13}} & D & \longleftarrow & 0 \\ & & \downarrow .F_{03} & & \downarrow .F'_{13} & & \\ 0 & \longleftarrow & D^{1\times 3} & \xleftarrow{.R'_{12}} & D^{1\times 3} & \longleftarrow & 0 \\ & & \downarrow .F_{02} & & \downarrow .F'_{12} & & \\ D & \xleftarrow{.R_{01}} & D^{1\times 4} & \xleftarrow{.R'_{11}} & D^{1\times 3} & \longleftarrow & 0, \end{array}$$

with the following notations:

$$R'_{11} = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & \partial_1 - 2\,\partial_2 + \partial_3 & -1 \end{pmatrix}, \quad F'_{13} = F_{03}, \quad F'_{12} = \begin{pmatrix} 0 & -2\,\partial_1 & 1 \\ 0 & -1 & 0 \\ 1 & -1 & 0 \end{pmatrix}.$$

Moreover, using (2.66), we have $R''_{13} = R_{13}$, $R''_{12} = R_{12}$ and:

$$R''_{11} = \begin{pmatrix} 0 & -2\,\partial_1 & 1 \\ 0 & -2\,\partial_1 + \partial_3 & -1 \\ \partial_3 & -6\,\partial_1 & 1 \\ 0 & -\partial_1 + \partial_2 & 0 \\ \partial_2 & -\partial_1 & 0 \\ \partial_1 & -\partial_1 & 0 \end{pmatrix}.$$

Since $\ker_D(.R_3) = 0$, $N_{33} \cong \mathrm{ext}^3_D(M, D)$ and thus $\mathrm{ext}^3_D(N_{33}, D) \cong \mathrm{ext}^3_D(\mathrm{ext}^3_D(M, D), D)$, which shows that the filtration $\{M_i\}_{i=-1,\ldots,3}$ of the left D-module M defined by (2.62) is a purity filtration of M.

Using (2.73), if $N_{11} = D^6/(R_{11}\, D^4)$, $N_{22} = D^4/(R_{22}\, D^6)$ and $N_{33} = D/(R_{33}\, D^4)$, then we obtain the finitely left D-modules:

$$\begin{cases} L_1 = D^{1\times 3}/(D^{1\times 6}\, R''_{11}) \cong \mathrm{ext}^1_D(N_{11}, D) \cong t(M), \\ L_2 = D^{1\times 3}/(D^{1\times 6}\, R_{12}) \cong \mathrm{ext}^2_D(N_{22}, D), \\ L_3 = D/(D^{1\times 4}\, R_{13}) \cong \mathrm{ext}^3_D(N_{33}, D). \end{cases}$$

Theorem 2.4.1 yields $M \cong E = D^{1\times 11}/(D^{1\times 17}\, Q)$, where the matrix Q is defined by:

$$Q = \left(\begin{array}{ccccccccccc}
1 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \partial_1 - 2\,\partial_2 + \partial_3 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -2\,\partial_1 & 1 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 4\,\partial_1 - \partial_3 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 4\,\partial_1 - \partial_3 & \partial_3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \partial_1 - \partial_2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \partial_1 - \partial_2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \partial_1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\partial_2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\partial_3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \partial_1
\end{array}\right).$$

If $\mathcal{F} = C^\infty(\mathbb{R}^3)$, then let us explicitly compute $\ker_{\mathcal{F}}(Q.)$. We first integrate the last diagonal block of Q, i.e., the 0-dimensional linear system $\ker_{\mathcal{F}}(R_{13}.)$:

$$\left\{\begin{array}{l} -\partial_2\, \tau_3 = 0, \\ -\partial_3\, \tau_3 = 0, \\ \partial_1\, \tau_3 = 0, \end{array}\right. \quad \Leftrightarrow \quad \tau_3 = c_1 \in \mathbb{R}.$$

Then, we integrate the inhomogeneous linear system in $\tau_2 = (\tau_{21} \quad \tau_{22} \quad \tau_{23})^T$ and τ_3 formed by the third triangular block of Q, namely:

$$\left\{\begin{array}{l} \tau_{23} - \tau_3 = 0, \\ \tau_{21} = 0, \\ -\tau_{21} + (4\,\partial_1 - \partial_3)\,\tau_{22} = 0, \\ \tau_{21} + (4\,\partial_1 - \partial_3)\,\tau_{22} + \partial_3\,\tau_{23} = 0, \\ (\partial_1 - \partial_2)\,\tau_{22} = 0, \end{array}\right. \quad \Leftrightarrow \quad \left\{\begin{array}{l} \tau_{23} = \tau_3 = c_1, \\ \tau_{21} = 0, \\ (4\,\partial_1 - \partial_3)\,\tau_{22} = 0, \\ (\partial_1 - \partial_2)\,\tau_{22} = 0, \end{array}\right.$$

We obtain $\tau_{21} = 0$, $\tau_{22} = f_1(x_3 + \frac{1}{4}\,(x_1 + x_2))$, where f_1 is an arbitrary smooth function, and $\tau_{23} = c_1$, where c_1 is an arbitrary constant. Then, we have to integrate the inhomogeneous linear system in $\tau_1 = (\tau_{11} \quad \tau_{12} \quad \tau_{13})^T$ and τ_2 formed by the second triangular block of Q, namely:

$$\left\{\begin{array}{l} -2\,\partial_1\,\tau_{12} + \tau_{13} - \tau_{21} = 0, \\ -\tau_{12} - \tau_{22} = 0, \\ \tau_{11} - \tau_{12} - \tau_{23} = 0, \end{array}\right. \quad \Leftrightarrow \quad \left\{\begin{array}{l} \tau_{12} = -\tau_{22} = -f_1(x_3 + \frac{1}{4}\,(x_1 + x_2)), \\ \tau_{11} = -\tau_{22} + \tau_{23} = -f_1(x_3 + \frac{1}{4}\,(x_1 + x_2)) + c_1, \\ \tau_{13} = -2\,\partial_1\,\tau_{22} + \tau_{21} = -\frac{1}{2}\,\dot{f}_1(x_3 + \frac{1}{4}\,(x_1 + x_2)). \end{array}\right.$$

The entries of τ_1 are 1-dimensional and not 2-dimensional. This result can be explained by the fact that the matrix S_1' defined in Remark 2.4.2 admits a left inverse, and thus $\ker_{\mathcal{F}}(S_1'.) \cong$

$\hom_D(\operatorname{coker}\overline{\alpha}_{21}, \mathcal{F}) \cong \hom_D(\operatorname{coker}\gamma_{21}, \mathcal{F}) = 0$. Finally, we integrate the inhomogeneous linear system in $\zeta = (\zeta_1 \ \dots \ \zeta_4)^T$ and τ_1 formed by the first triangular block of P, namely:

$$\left\{\begin{array}{l} \zeta_1 - \zeta_3 - \tau_{11} = 0, \\ \zeta_2 + \zeta_3 - \tau_{12} = 0, \\ (\partial_1 - 2\,\partial_2 + \partial_3)\,\zeta_3 - \zeta_4 - \tau_{13} = 0, \end{array}\right. \Leftrightarrow \left\{\begin{array}{l} \zeta_1 - \zeta_2 = -f_1(x_3 + \frac{1}{4}\,(x_1 + x_2)) + c_1, \\ \zeta_2 + \zeta_3 = -f_1(x_3 + \frac{1}{4}\,(x_1 + x_2)), \\ (\partial_1 - 2\,\partial_2 + \partial_3)\,\zeta_3 - \zeta_4 = -\frac{1}{2}\,\dot{f}_1(x_3 + \frac{1}{4}\,(x_1 + x_2)). \end{array}\right. \tag{2.86}$$

The D-module $M/t(M) = D^{1\times 4}/(D^{1\times 3}\,R'_{11})$ is parametrized by R_{01}, i.e., $M/t(M) \cong D^{1\times 4}\,R_{01}$. Since $\mathcal{F}$ is an injective D-module (see Example 1.4.2), the linear system $\ker_{\mathcal{F}}(R'_{11}.)$ is parametrized by R_{01}, i.e., $\ker_{\mathcal{F}}(R'_{11}.) = R_{01}\,\mathcal{F}$. Since the matrix R'_{11} admits the right inverse

$$X = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

2 of Corollary 1.3.3 shows that $M/t(M)$ is a stably free D-module, and thus $M/t(M)$ is a free D-module of rank 1 by the Quillen-Suslin theorem (see 2 of Theorem 1.1.2). Hence, Corollary 2.2.2 proves that the general $\mathcal{F}$-solution of (2.86) is defined by $\zeta = R_{01}\,\xi + X\,\tau_1$, i.e.:

$$\forall\,\xi \in C^\infty(\mathbb{R}^3), \quad \forall\,f_1 \in C^\infty(\mathbb{R}), \quad \forall\,c_1 \in \mathbb{R}, \quad \left\{\begin{array}{l} \zeta_1 = \xi - f_1(x_3 + \frac{1}{4}\,(x_1 + x_2)) + c_1, \\ \zeta_2 = -\xi - f_1(x_3 + \frac{1}{4}\,(x_1 + x_2)), \\ \zeta_3 = \xi, \\ \zeta_4 = (\partial_1 - 2\,\partial_2 + \partial_3)\,\xi + \frac{1}{2}\,\dot{f}_1(x_3 + \frac{1}{4}\,(x_1 + x_2)). \end{array}\right.$$

Finally, using the D-isomorphism γ defined by (2.85), we obtain

$$\left\{\begin{array}{l} -2\,\partial_1\,\eta_2 + \partial_3\,\eta_3 - 2\,\partial_2\,\eta_3 - \partial_1\,\eta_3 - \eta_4 = 0, \\ \partial_3\,\eta_2 - 2\,\partial_1\,\eta_2 + 2\,\partial_2\,\eta_3 - 3\,\partial_1\,\eta_3 + \eta_4 = 0, \\ \partial_3\,\eta_1 - 6\,\partial_1\,\eta_2 - 2\,\partial_2\,\eta_3 - 5\,\partial_1\,\eta_3 - \eta_4 = 0, \\ \partial_2\,\eta_2 - \partial_1\,\eta_2 + \partial_2\,\eta_3 - \partial_1\,\eta_3 = 0, \\ \partial_2\,\eta_1 - \partial_1\,\eta_2 - \partial_2\,\eta_3 - \partial_1\,\eta_3 = 0, \\ \partial_1\,\eta_1 - \partial_1\,\eta_2 - 2\,\partial_1\,\eta_3 = 0, \end{array}\right. \Leftrightarrow \left\{\begin{array}{l} \eta_1 = \xi - f_1(x_3 + \frac{1}{4}\,(x_1 + x_2)) + c_1, \\ \eta_2 = -\xi - f_1(x_3 + \frac{1}{4}\,(x_1 + x_2)), \\ \eta_3 = \xi, \\ \eta_4 = (\partial_1 - 2\,\partial_2 + \partial_3)\,\xi + \frac{1}{2}\,\dot{f}_1(x_3 + \frac{1}{4}\,(x_1 + x_2)), \end{array}\right.$$

where ξ (resp., f_1, c_1) is an arbitrary function of $C^\infty(\mathbb{R}^3)$ (resp., $C^\infty(\mathbb{R})$, constant).

For more results, details and examples on Baer's extensions and purity filtrations, see [100]. See also the PurityFiltration package ([97]) for an implementation of these results.

Chapter 3

Factorization, reduction and decomposition problems

Nowadays, mathematics focuses on the concept of *categories* (see [15, 65, 109]) which simultaneously study objects and homomorphisms between objects. In Chapter 1, we studied the objects of the category ${}_D\mathrm{Mod}^f$ formed by finitely generated left D-modules and left D-homomorphisms between finitely generated left D-modules, where D is a noetherian domain or a noncommutative polynomial ring for which Buchberger's algorithm terminates for any admissible term order. In this chapter, we study the left D-homomorphisms between two finitely generated left D-modules, i.e., between two finitely presented left D-modules since D is a left noetherian domain.

We shall explain that the computation of homomorphisms has many interesting applications in mathematical systems theory. In particular, the elements of the endomorphism ring $\mathrm{end}_D(M) = \hom_D(M, M)$ of a finitely presented left D-module $M = D^{1\times p}/(D^{1\times q}\, R)$ naturally define the internal symmetries of the linear system $\ker_{\mathcal{F}}(R.)$, where $\mathcal{F}$ is a left D-module, namely, linear transformations which send elements of $\ker_{\mathcal{F}}(R.)$ to elements of $\ker_{\mathcal{F}}(R.)$. The subgroup $\mathrm{aut}_D(M)$ of $\mathrm{end}_D(M)$ formed by the *automorphisms* of M (namely, the bijective left D-homomorphisms of M) defines Galois-like transformations of $\ker_{\mathcal{F}}(R.)$. A first application of the computation of homomorphisms is the computation of quadratic conservation laws of linear PD systems coming from mathematical physics. They can be obtained in a purely algorithmic way without any knowledge of physics. Other applications of the computation of $\mathrm{end}_D(M)$ are the so-called *factorization*, *reduction* and *decomposition problems* largely studied in the symbolic computation literature. These problems aim at factoring a matrix of functional operators (e.g., PD operators, OD time-delay operators, difference operators) or at finding an equivalence matrix having a block-triangular or block-diagonal structure. We study those problems by generalizing the *eigenring* approach developed for linear OD systems by Singer and others ([7, 94, 113]) to more general linear functional (determined/underdetermined/overdetermined) systems.

3.1 Homomorphisms between two finitely presented modules

As explained in Chapter 1, if $M = D^{1\times p}/(D^{1\times q}\, R)$ (resp., $M' = D^{1\times p'}/(D^{1\times q'}\, R')$) is a left D-module finitely presented by $R \in D^{q\times p}$ (resp., $R' \in D^{q'\times p'}$) and if $\{e_j\}_{j=1,\ldots,p}$ (resp., $\{e'_k\}_{k=1,\ldots,p'}$) is the standard basis of $D^{1\times p}$ (resp., $D^{1\times p'}$), then $\{\pi(e_j)\}_{j=1,\ldots,p}$ (resp., $\{\pi'(e'_k)\}_{k=1,\ldots,p'}$) is a family of generators of M (resp., M'). Now, $f \in \hom_D(M, M')$ sends the generators of M to some elements of M', i.e., we have $f(\pi(e_j)) = \sum_{k=1}^{p'} P_{jk}\, \pi'(e'_k)$ for $j = 1, \ldots, p$, where the P_{jk}'s are elements of D which must satisfy the relations coming from $f(0)=0$, i.e., f must send the left D-linear relations $\sum_{j=1}^{p} R_{ij}\,\pi(e_j)=0$ for $i=1,\ldots,q$ between the generators $\pi(e_j)$'s of M to 0. Hence, for $i=1,\ldots,q$, by left D-linearity, we have:

$$f\left(\sum_{j=1}^{p} R_{ij}\,\pi(e_j)\right)=\sum_{j=1}^{p} R_{ij}\,f(\pi(e_j))=\sum_{j=1}^{p} R_{ij}\left(\sum_{k=1}^{p'} P_{jk}\,\pi'(e'_k)\right)=\pi'\left(\sum_{k=1}^{p'}\left(\sum_{j=1}^{p} R_{ij}\,P_{jk}\right)\,e'_k\right)=0,$$

and thus, $(\sum_{j=1}^{p} R_{ij}\,P_{j1},\ldots,\sum_{j=1}^{p} R_{ij}\,P_{jp'})\in D^{1\times q'}\,R'$, i.e., there exists $Q_i\in D^{1\times q'}$ such that $(\sum_{j=1}^{p} R_{ij}\,P_{j1},\ldots,\sum_{j=1}^{p} R_{ij}\,P_{jp'})=Q_i\,R'$. If $Q=(Q_1^T\ldots Q_q^T)^T\in D^{q\times q'}$, then we obtain:

$$R\,P=Q\,R'.$$

We can check that the P_{jk}'s are not uniquely defined by $f\in\hom_D(M,M')$. Indeed, if we have $f(\pi(e_j))=\sum_{k=1}^{p'}\overline{P}_{jk}\,\pi'(e'_k)$, where the $\overline{P}_{jk}$'s are elements of D, then we have

$$\forall\, j=1,\ldots,p,\quad \pi'\left(\sum_{k=1}^{p'}(\overline{P}_{jk}-P_{jk})\,e'_k\right)=\sum_{k=1}^{p'}(\overline{P}_{jk}-P_{jk})\,\pi'(e'_k)=0,$$

and thus, the row vector $\overline{P}_{j\bullet}-P_{j\bullet}=(\overline{P}_{j1}-P_{j1},\ldots,\overline{P}_{jp'}-P_{jp'})$ belongs to $D^{1\times q'}\,R'$, i.e., there exists $Z_j\in D^{1\times q'}$ satisfying $\overline{P}_{j\bullet}-P_{j\bullet}=Z_j\,R'$. Hence, we obtain $\overline{P}-P=Z\,R'$, where $Z=(Z_1^T\ldots Z_p^T)^T\in D^{p\times q'}$. Finally, if $R'_2\in D^{r'\times q'}$ is a matrix satisfying $\ker_D(.R')=D^{1\times r'}\,R'_2$ and $Z'\in D^{q\times r'}$ is any arbitrary matrix, then we have

$$R\,\overline{P}=R\,P+R\,Z\,R'=Q\,R'+R\,Z\,R'=(Q+R\,Z)\,R'=(Q+R\,Z+Z'\,R'_2)\,R',$$

which proves that we have $R\,\overline{P}=\overline{Q}\,R'$ where $\overline{Q}=Q+R\,Z+Z'\,R'_2\in D^{q\times q'}$.

Proposition 3.1.1 ([19]). *Let $R\in D^{q\times p}$ and $R'\in D^{q'\times p'}$ be two matrices, $M=D^{1\times p}/(D^{1\times q}\,R)$ and $M'=D^{1\times p'}/(D^{1\times q'}\,R')$ two finitely presented left D-modules and the canonical projections $\pi: D^{1\times p}\longrightarrow M$ and $\pi': D^{1\times p'}\longrightarrow M'$. Then, $f\in\hom_D(M,M')$ is defined by*

$$\forall\, m=\pi(\lambda),\ \lambda\in D^{1\times p}:\ f(m)=\pi'(\lambda\,P), \tag{3.1}$$

where $P\in D^{p\times p'}$ is such that $D^{1\times q}\,(R\,P)\subseteq D^{1\times q'}\,R'$, i.e., such that the following identity holds

$$R\,P=Q\,R', \tag{3.2}$$

for a certain matrix $Q\in D^{q\times q'}$. Then, we have the following commutative exact diagram:

$$\begin{array}{ccccccc} D^{1\times q} & \xrightarrow{.R} & D^{1\times p} & \xrightarrow{\pi} & M & \longrightarrow 0 \\ \downarrow .Q & & \downarrow .P & & \downarrow f & \\ D^{1\times q'} & \xrightarrow{.R'} & D^{1\times p'} & \xrightarrow{\pi'} & M' & \longrightarrow 0. \end{array} \tag{3.3}$$

Conversely, a pair of matrices (P,Q) satisfying (3.2) defines $f\in\hom_D(M,M')$ by (3.1), i.e.:

$$\hom_D(M,M')\cong\{P\in D^{p\times p'}\mid \exists\, Q\in D^{q\times q'}:\ R\,P=Q\,R'\}/(D^{p\times q'}\,R') \tag{3.4}$$

The matrices P and Q are defined up to a homotopy equivalence: the matrices defined by

$$\left\{\begin{array}{l}\overline{P}=P+Z\,R',\\ \overline{Q}=Q+R\,Z+Z'\,R'_2,\end{array}\right. \tag{3.5}$$

where $Z\in D^{p\times q'}$ and $Z'\in D^{q\times r'}$ are arbitrary matrices and the matrix $R'_2\in D^{r'\times q'}$ is such that $\ker_D(.R')=D^{1\times r'}\,R'_2$, satisfy the relation $R\,\overline{P}=\overline{Q}\,R'$ and define the left D-homomorphism f.

Remark 3.1.1. Applying the contravariant functor $\hom_D(\cdot\,, M')$ to the finite presentation $D^{1\times q} \xrightarrow{.R} D^{1\times p} \xrightarrow{\pi} M \longrightarrow 0$ of M, we obtain the following exact sequence of abelian groups:

$$M'^{q} \xleftarrow{R.} M'^{p} \longleftarrow \ker_{M'}(R.) \longleftarrow 0.$$

Theorem 1.1.1 shows that $\hom_D(M, M') \cong \ker_{M'}(R.) = \{\eta \in M'^{p} \mid R\,\eta = 0\}$. More precisely, if $\eta = (\pi'(\mu_1) \ \ldots \ \pi'(\mu_p))^T \in \ker_{M'}(R.)$ and $P = (\mu_1^T \ \ldots \ \mu_p^T)^T \in D^{p\times p'}$, then, using (1.2), $\chi(\eta) = \phi_\eta \in \hom_D(M, M')$ is defined by

$$\phi_\eta(\pi(\lambda)) = \lambda\,\eta = \sum_{j=1}^{p} \lambda_j\,\pi'(\mu_j) = \pi'\left(\sum_{j=1}^{p} \lambda_j\,\mu_j\right) = \pi'(\lambda\,P),$$

where the $\mu_j \in D^{1\times p'}$ for $j = 1, \ldots, p$ satisfy $R\,\eta = 0$, i.e.,

$$\forall\, i = 1, \ldots, q, \quad \sum_{j=1}^{p} R_{ij}\,\pi'(\mu_j) = \pi'\left(\sum_{j=1}^{p} R_{ij}\,\mu_j\right) = 0,$$

which implies the existence of $\nu_i \in D^{1\times q'}$ for $i = 1, \ldots, q$ such that $\sum_{j=1}^{p} R_{ij}\,\mu_j = \nu_i\,R'$, i.e., such that (3.2) holds where $Q = (\nu_1^T \ \ldots \ \nu_q^T)^T \in D^{q\times q'}$, which also leads to Proposition 3.1.1.

Let us now explain one of the main interests of characterizing $\hom_D(M, M')$.

Applying the contravariant left exact functor $\hom_D(\cdot\,, \mathcal{F})$ to the commutative exact diagram (3.3) and using Theorem 1.1.1, i.e., the $\mathbb{Z}$-isomorphism $\ker_{\mathcal{F}}(R.) \cong \hom_D(M, \mathcal{F})$ (resp., $\ker_{\mathcal{F}}(R'.) \cong \hom_D(M', \mathcal{F})$), we get the following commutative exact diagram of abelian groups

$$\begin{array}{ccccccc}
\mathcal{F}^{q} & \xleftarrow{R.} & \mathcal{F}^{p} & \xleftarrow{i} & \ker_{\mathcal{F}}(R.) & \longleftarrow & 0 \\
\uparrow Q. & & \uparrow P. & & \uparrow f^\star & & \\
\mathcal{F}^{q'} & \xleftarrow{R'.} & \mathcal{F}^{p'} & \xleftarrow{i'} & \ker_{\mathcal{F}}(R'.) & \longleftarrow & 0,
\end{array}$$

where $f^\star : \ker_{\mathcal{F}}(R'.) \longrightarrow \ker_{\mathcal{F}}(R.)$ is defined by $f^\star(\zeta) = P\,\zeta$ for all $\zeta \in \ker_{\mathcal{F}}(R'.)$. Indeed, $R\,P = Q\,R'$ and $R'\,\zeta = 0$ yield $R\,(P\,\zeta) = Q'\,(R'\,\zeta) = 0$, i.e., $\eta = P\,\zeta \in \ker_{\mathcal{F}}(R.)$.

Corollary 3.1.1 ([19]). *Let $\mathcal{F}$ be a left D-module, $R \in D^{q\times p}$, $R' \in D^{q'\times p'}$ and the linear systems $\ker_{\mathcal{F}}(R.) = \{\eta \in \mathcal{F}^{p} \mid R\,\eta = 0\}$ and $\ker_{\mathcal{F}}(R'.) = \{\eta' \in \mathcal{F}^{p'} \mid R'\,\eta' = 0\}$. Then, an element $f \in \hom_D(M, M')$ defined by matrices $P \in D^{p\times p'}$ and $Q \in D^{q\times q'}$ satisfying (3.2) induces the following abelian group homomorphism:*

$$\begin{array}{rcl}
f^\star : \ker_{\mathcal{F}}(R'.) & \longrightarrow & \ker_{\mathcal{F}}(R.) \\
\eta' & \longmapsto & \eta = P\,\eta'.
\end{array}$$

Corollary 3.1.1 shows that an element of $\hom_D(M, M')$ defines a transformation which sends the elements of $\ker_{\mathcal{F}}(R'.) \cong \hom_D(M', \mathcal{F})$ to those of $\ker_{\mathcal{F}}(R.) \cong \hom_D(M, \mathcal{F})$. If $M' = M$, then the elements of the D-endomorphism ring $\mathrm{end}_D(M) = \hom_D(M, M)$ of M define internal transformations of $\ker_{\mathcal{F}}(R.)$. We note that the ring $\mathrm{end}_D(M)$ contains the subgroup $\mathrm{aut}_D(M)$ formed by the left D-*automorphisms* of M, namely, the bijective endomorphisms of M. The elements of $\mathrm{aut}_D(M)$ define *Galois-like transformations* of the linear system $\ker_{\mathcal{F}}(R.)$.

Proposition 3.1.1 and Corollary 3.1.1 allow us to find again the theory of *eigenrings* ([7, 113]).

Example 3.1.1. Let $D = A\langle \partial \rangle$ be the ring of OD operators with coefficients in a differential ring A, $E,\, F \in A^{p\times p}$, $R = \partial\, I_p - E \in D^{p\times p}$, $R' = \partial\, I_p - F \in D^{p\times p}$, $M = D^{1\times p}/(D^{1\times p}\, R)$ and $M' = D^{1\times p}/(D^{1\times p}\, R')$. Let π (resp., π') be the canonical projection of $D^{1\times p}$ onto M (resp., M') and $\{e_j\}_{j=1,\dots,p}$ the standard basis of the free left D-module $D^{1\times p}$. As explained in Section 1.1, $\{y_j = \pi(e_j)\}_{j=1,\dots,p}$ (resp., $\{z_j = \pi'(e_j)\}_{j=1,\dots,p}$) defines a family of generators of M (resp., M') and the y_j's (resp., z_j's) satisfy the following left D-linear relations:

$$\forall\, i = 1,\dots,p, \quad \partial\, y_i = \sum_{j=1}^{p} E_{ij}\, y_j, \quad \left(\text{resp.}, \partial\, z_i = \sum_{j=1}^{p} F_{ij}\, z_j\right). \tag{3.6}$$

Let us now consider a non-trivial $f \in \hom_D(M, M')$. Then, f sends the generators y_j's of M to left D-linear combinations of the generators z_j's of M', i.e., there exists a matrix $P \in D^{p\times p}$ such that $f(y_i) = \sum_{j=1}^{p} P_{ij}\, z_j$ for $i = 1,\dots,p$. Using (3.6), every left D-linear combination of the z_j's can be rewritten in the form of an A-linear combination of the z_j's, i.e., we can suppose without loss of generality that all the entries P_{ij} of P belong to A, i.e., $P \in A^{p\times p}$. By Proposition 3.1.1, there exists a matrix $Q \in D^{p\times p}$ such that (3.2), and thus:

$$(3.2) \;\Leftrightarrow\; (\partial\, I_p - E)\, P = Q\, (\partial\, I_p - F) \;\Leftrightarrow\; P\, \partial + \dot{P} - E\, P = Q\, \partial - Q\, F. \tag{3.7}$$

Since the degrees of $P\, \partial$ and $Q\, \partial$ are respectively 1 and $r+1$, where r is the maximum of the degrees of the entries of Q, then we must have $r = 0$, i.e., $Q \in A^{p\times p}$, a fact yielding

$$(3.7) \;\Leftrightarrow\; (P - Q)\, \partial + (\dot{P} - E\, P + Q\, F) = 0 \;\Leftrightarrow\; \left\{ \begin{array}{l} Q = P, \\ \dot{P} = E\, P - P\, F. \end{array} \right. \tag{3.8}$$

Any $f \in \hom_D(M, M')$ can then be defined by $f(\pi(\lambda)) = \pi'(\lambda\, P)$, where $P \in A^{p\times p}$ satisfies $\dot{P} = E\, P - P\, F$. If $\mathcal{F}$ is a left D-module, $\zeta \in \ker_{\mathcal{F}}(R'.)$, i.e., $\partial\, \zeta - F\, \zeta = 0$, and $\eta = P\, \zeta$, then:

$$R\, \eta = \partial\, (P\, \zeta) - E\, (P\, \zeta) = P\, \partial\, \zeta + \dot{P}\, \zeta - (E\, P)\, \zeta = P\, (\partial\, \zeta - F\, \zeta) = 0 \;\Rightarrow\; \eta \in \ker_{\mathcal{F}}(R.).$$

If $P \in \mathrm{GL}_p(A)$, then the second equation of (3.8) yields $F = P^{-1}\, E\, P - P^{-1}\, \dot{P}$. In particular, if P is a constant matrix, i.e., $\dot{P} = 0$, then we find again the transformation $F = P^{-1}\, E\, P$ classically used in the integration of first order linear OD systems with constant coefficients.

If $F = E$, then the second equation of (3.8) defines the *eigenring* of the linear OD system $\partial\, \eta = E\, \eta$, namely, $\mathcal{E} = \{P \in A^{p\times p} \mid \dot{P} = E\, P - P\, E\}$, introduced by Singer in [113]. Using the properties of the trace $\mathrm{tr}(P_1 + P_2) = \mathrm{tr}(P_2 + P_1)$ and $\mathrm{tr}(P_1\, P_2) = \mathrm{tr}(P_2\, P_1)$ for all $P \in \mathcal{E}$, we get

$$\begin{aligned} \forall\, k \in \mathbb{N}, \quad \frac{d\, \mathrm{tr}(P^k)}{dt} &= \mathrm{tr}\left(\frac{d\, P^k}{dt}\right) = \mathrm{tr}\left(\frac{d\, (P \dots P)}{dt}\right) \\ &= \mathrm{tr}(\dot{P}\, P^{k-1} + P\, \dot{P}\, P^{k-2} + P^2\, \dot{P}\, P^{k-3} + \dots + P^{k-1}\, \dot{P}) = k\, \mathrm{tr}(\dot{P}\, P^{k-1}) \\ &= k\, \mathrm{tr}((E\, P - P\, E)\, P^{k-1}) = k\, \mathrm{tr}(E\, P^k - P\, E\, P^{k-1}) \\ &= k\, \mathrm{tr}(E\, P^k - E\, P^k) = 0, \end{aligned}$$

i.e., the $\mathrm{tr}(P^k)$'s are first integrals. Since the coefficients a_i's of the characteristic polynomial of P are symmetric functions of the eigenvalues of P and they can be expressed in terms of the $\mathrm{tr}(P^k)$'s (Newton's formulas), they are also first integrals. Therefore, the eigenvalues of P are first integrals because they are algebraic functions of the a_i's, i.e., $P \in \mathcal{E}$ is isospectral. Following the ideas of [7, 94, 113], we can then compute a Jordan normal form of $P \in \mathcal{E}$ and use the corresponding change of bases to transform the linear OD system $\partial\,\eta = E\,\eta$ into $\partial\,\zeta = \overline{E}\,\zeta$, where $\overline{E} \in A^{p\times p}$ is either a block-triangular or a block-diagonal matrix.

Let us illustrate the results with the following explicit example over $A = \mathbb{Q}[t]$:

$$\dot{\eta} = E\,\eta, \quad E = \begin{pmatrix} t\,(2\,t+1) & -2\,t^3 - 2\,t^2 + 1 \\ 2\,t & -t\,(2\,t+1) \end{pmatrix} \in A^{2\times 2}. \tag{3.9}$$

Using algorithms which compute polynomial solutions of linear OD systems ([1, 7]), we get:

$$\mathcal{E} = \left\{ P = \begin{pmatrix} a_1 - a_2\,(t+1) & a_2\,t\,(t+1) \\ -a_2 & a_2\,t + a_1 \end{pmatrix} \mid a_1,\, a_2 \in \mathbb{Q} \right\}.$$

If $P \in \mathcal{E}$, then $\det(P - \lambda\,I_2) = (\lambda - a_1)\,(\lambda - a_1 + a_2)$ and the Jordan normal form of P is:

$$J = U^{-1}\,P\,U = \begin{pmatrix} a_1 & 0 \\ 0 & a_1 - a_2 \end{pmatrix}, \quad U = \begin{pmatrix} -t & t+1 \\ -1 & 1 \end{pmatrix}, \quad U^{-1} = \begin{pmatrix} 1 & -(t+1) \\ 1 & -t \end{pmatrix}.$$

If $\zeta = U^{-1}\,\eta = (\eta_1 - (t+1)\,\eta_2 \quad \eta_1 - t\,\eta_2)^T$, then the linear OD system $\dot{\eta} = E\,\eta$ is equivalent to:

$$\dot{\zeta} = U^{-1}\,(E\,U - \dot{U})\,\zeta = \begin{pmatrix} -t & 0 \\ 0 & t \end{pmatrix} \zeta \quad \Leftrightarrow \quad \forall\, C_1,\, C_2 \in \mathbb{R}, \quad \begin{cases} \zeta_1 = C_1\,e^{-t^2/2}, \\ \zeta_2 = C_2\,e^{t^2/2}. \end{cases}$$

Finally, using the invertible transformation $\eta = U\,\zeta$, we obtain the general solution of (3.9):

$$\forall\, C_1,\, C_2 \in \mathbb{R}, \quad \begin{cases} \eta_1 = -C_1\,t\,e^{-t^2/2} + C_2\,(t+1)\,e^{t^2/2}, \\ \eta_2 = -C_1\,e^{-t^2/2} + C_2\,e^{t^2/2}. \end{cases}$$

Example 3.1.1 can be generalized to the so-called *integrable algebraic connections* ([94]).

Let $D = B_n(k)$ be the second Weyl algebra, where k is a field, and $E_i \in k(x_1, \ldots, x_n)^{p\times p}$ for $i = 1, \ldots, n$. Then, an *algebraic connection* is a linear PD system of the form:

$$\begin{cases} \partial_1\,y - E_1\,y = 0, \\ \quad\vdots \\ \partial_n\,y - E_n\,y = 0. \end{cases} \tag{3.10}$$

Let $\nabla_i = \partial_i\,I_p - E_i \in D^{p\times p}$ for $i = 1, \ldots, n$. Then, the algebraic connection (3.10) is said to be *integrable* if the following integrability conditions are satisfied:

$$[\nabla_i, \nabla_j] \triangleq \nabla_i\,\nabla_j - \nabla_j\,\nabla_i = \frac{\partial E_i}{\partial x_j} - \frac{\partial E_j}{\partial x_i} + E_i\,E_j - E_j\,E_i = 0, \quad 1 \leq i < j \leq n. \tag{3.11}$$

The next proposition characterizes the ring of endomorphisms of an integrable connection.

Proposition 3.1.2 ([19]). *Let $D = B_n(k)$ be the second Weyl algebra over a field k, n matrices $E_1, \ldots, E_n \in k(x_1, \ldots, x_n)^{p\times p}$ satisfying (3.11), $R = ((\partial_1\,I_p - E_1)^T \cdots (\partial_n\,I_p - E_n)^T)^T \in D^{n\,p\times p}$, and the left D-module $M = D^{1\times p}/(D^{1\times n\,p}\,R)$. Then, $f \in \mathrm{end}_D(M)$ is defined by the matrices $P \in k(x_1, \ldots, x_n)^{p\times p}$ and $Q \in k(x_1, \ldots, x_n)^{n\,p\times n\,p}$ satisfying the following relations*

$$\begin{cases} \dfrac{\partial P}{\partial x_i} + P\,E_i - E_i\,P = 0, \quad i = 1, \ldots, n, \\ Q = \mathrm{diag}(P, \ldots, P), \end{cases} \tag{3.12}$$

where $\mathrm{diag}(P, \ldots, P)$ *denotes the diagonal matrix formed by n matrices P on the diagonal.*

Example 3.1.2. The strain tensor $\epsilon = (\epsilon_{ij})_{i,j=1,2}$ is defined by the *Killing operator*, i.e., the Lie derivative of the euclidean metric of $\mathbb{R}^2$ defined by $\omega_{ij} = 1$ for $i = j$ and 0 otherwise, namely

$$\left\{ \begin{array}{l} \epsilon_{11} = \partial_1 \xi_1, \\ \epsilon_{12} = \epsilon_{21} = \frac{1}{2} (\partial_2 \xi_1 + \partial_1 \xi_2), \\ \epsilon_{22} = \partial_2 \xi_2, \end{array} \right. \tag{3.13}$$

where, using the euclidean metric of $\mathbb{R}^2$, $\xi_i = \xi^i$, $i = 1, 2$, and $\xi = (\xi^1, \xi^2)$ is a displacement.

Let us consider (3.13) with $\epsilon = 0$, i.e., the system corresponding to the Lie algebra of the Lie group of rigid motions in $\mathbb{R}^2$ ([83, 84]). (3.13) can be written as the integrable connection:

$$\forall\, i = 1, 2, \quad \nabla_i = \partial_i I_3 - E_i, \quad E_1 = \left(\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array} \right), \quad E_2 = \left(\begin{array}{ccc} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right), \quad y = \left(\begin{array}{c} \xi_1 \\ \xi_2 \\ \partial_1 \xi_2 \end{array} \right).$$

Let $D = \mathbb{R}[\partial_1, \partial_2]$, $R = (\nabla_1^T \quad \nabla_2^T)^T$ and $M = D^{1\times 3}/(D^{1\times 6} R)$. According to Proposition 3.1.2, $f \in \mathrm{end}_D(M)$ can be defined by $P \in \mathbb{R}^{3\times 3}$ satisfying:

$$\left\{ \begin{array}{l} P E_1 - E_1 P = 0, \\ P E_2 - E_2 P = 0, \end{array} \right. \quad \Leftrightarrow \quad P = \left(\begin{array}{ccc} \alpha & 0 & \gamma \\ 0 & \alpha & \beta \\ 0 & 0 & \alpha \end{array} \right), \quad \forall\, \alpha, \beta, \gamma \in \mathbb{R}. \tag{3.14}$$

We can easily check that the general solution of $\nabla_i \eta(x_1, x_2) = 0$ for $i = 1, 2$ is defined by:

$$\forall\, a, b, c \in \mathbb{R}, \quad \eta_1(x_1, x_2) = -a\, x_2 + b, \quad \eta_2(x_1, x_2) = a\, x_1 + c, \quad \eta_3(x_1, x_2) = a.$$

Finally, if P is defined by (3.14), then according to Corollary 3.1.1,

$$\zeta = P\, \eta = \left(\begin{array}{c} -(\alpha\, a)\, x_2 + (\alpha\, b + \gamma\, a) \\ (\alpha\, a)\, x_1 + (\alpha\, c + \beta\, a) \\ \alpha\, a \end{array} \right)$$

is another solution of the integrable algebraic connection $\nabla_i \eta(x_1, x_2) = 0$ for $i = 1, 2$.

3.2 Computation of left D-homomorphisms

We now turn to the problem of solving the general equation $R\, P = Q\, R'$. The situation is different if we consider a commutative or a noncommutative ring D. Indeed, if D is a commutative ring, then $\hom_D(M, M')$ is a D-module whereas $\hom_D(M, M')$ is usually an abelian group if D is a noncommutative ring (see Section 1.1). If D is a noetherian commutative ring, then M'^k is a noetherian D-module for all $k \in \mathbb{N}$, and thus so is the D-module $\ker_{M'}(R.) \cong \hom_D(M, M')$ (see, e.g., [54, 109]). Thus, $\hom_D(M, M')$ is a finitely generated D-module, and thus a finitely presented D-module since D is a noetherian ring (see Section 1.1). Hence, $\hom_D(M, M')$ can be defined by a finite number of generators and of D-linear relations, i.e., by a finite presentation.

If D is a noetherian commutative ring, then let us explain how to find a finite presentation of the D-module $\hom_D(M, M')$. Let $R \in D^{q\times p}$, $R' \in D^{q'\times p'}$, $P \in D^{p\times p'}$ and $Q \in D^{q\times q'}$ be four matrices satisfying (3.2). Since D is a commutative ring, then using Lemma 2.1.2, we obtain

$$\left\{ \begin{array}{l} \mathrm{row}(R\, P) = \mathrm{row}(R\, P\, I_{p'}) = \mathrm{row}(P)\, (R^T \otimes I_{p'}), \\ \mathrm{row}(Q\, R') = \mathrm{row}(I_q\, Q\, R') = \mathrm{row}(Q)\, (I_q \otimes R'), \end{array} \right.$$

$$(3.2) \quad \Leftrightarrow \quad (\mathrm{row}(P) \quad -\mathrm{row}(Q))\, L = 0, \quad L = \begin{pmatrix} R^T \otimes I_{p'} \\ I_q \otimes R' \end{pmatrix} \in D^{(p\,p'+q\,q')\times q\,p'}.$$

Let $L_2 \in D^{s\times(p\,p'+q\,q')}$ be such that $\ker_D(.L) = D^{1\times s}\, L_2$. Augmenting the rows of L_2, we find a set of matrices $\{P_i\}_{i=1,\ldots,s}$ and $\{Q_i\}_{i=1,\ldots,s}$, where $P_i \in D^{p\times p'}$ and $Q_i \in D^{q\times q'}$, satisfying the relation $R\,P_i = Q_i\,R'$ for $i = 1,\ldots,s$. Moreover, every solution $P \in D^{p\times p'}$ and $Q \in D^{q\times q'}$ of (3.2) has the form

$$\begin{cases} P = \sum_{i=1}^{s} \alpha_i\, P_i, \\ Q = \sum_{i=1}^{s} \alpha_i\, Q_i, \end{cases}$$

where $\alpha_i \in D$ for $i = 1,\ldots,s$, i.e., $\{P_i\}_{i=1,\ldots,s}$ is a set of generators of the following D-module:

$$E = \{P \in D^{p\times p'} \mid \exists\, Q \in D^{q\times q'} :\ R\,P = Q\,R'\}.$$

Therefore, the set $\{\overline{P}_i\}_{i=1,\ldots,s}$ of the residue classes of the matrices P_i's in the D-module $E/(D^{p\times q'}\,R')$ generates $E/(D^{p\times q'}\,R')$, i.e., generates $\hom_D(M,M') \cong E/(D^{p\times q'}\,R')$ up to isomorphism (see (3.4)). In particular, if $\overline{P}_i = P_i + Z_i\,R'$ for certain matrices $Z_i \in D^{p\times q'}$ and $i = 1,\ldots,s$, then we can introduce the matrices $\overline{Q}_i = Q_i + R\,Z_i$ for $i = 1,\ldots,s$, and $\overline{P_i}$ and $\overline{Q_i}$ satisfy $R\,\overline{P}_i = \overline{Q_i}\,R'$ for $i = 1,\ldots,s$, i.e., they induce $f_i \in \hom_D(M,M')$ defined by:

$$\forall\, \lambda \in D^{1\times p}, \quad f_i(\pi(\lambda)) = \pi'(\lambda\,\overline{P}_i), \quad i = 1,\ldots,s.$$

Then, $\{f_i\}_{i=1,\ldots,s}$ is a family of generators of $\hom_D(M,M')$. A D-linear relation $\sum_{j=1}^{s} d_j\, f_j = 0$ between the f_i's is equivalent to the existence of $Z \in D^{p\times q'}$ satisfying $\sum_{j=1}^{s} d_j\,\overline{P}_j = Z\,R'$, i.e.:

$$\sum_{j=1}^{s} d_j\,\mathrm{row}(\overline{P}_j) - \mathrm{row}(Z)\,(I_p \otimes R') = 0 \quad \Leftrightarrow \quad (d_1\ \ldots\ d_s\ \ -\mathrm{row}(Z)) \begin{pmatrix} \mathrm{row}(\overline{P}_1) \\ \vdots \\ \mathrm{row}(\overline{P}_s) \\ I_p \otimes R' \end{pmatrix} = 0.$$

If we introduce the matrices $U = \left(\mathrm{row}(\overline{P}_1)^T\ \ldots\ \mathrm{row}(\overline{P}_s)^T\right)^T \in D^{s\times p\,p'}$, $V = I_p \otimes R' \in D^{p\,q'\times p\,p'}$ and $W = (U^T \quad V^T)^T \in D^{(s+p\,q')\times p\,p'}$, then there exist $X \in D^{l\times s}$ and $Y \in D^{l\times p\,q'}$ satisfying $\ker_D(.W) = D^{1\times l}\,(X \quad -Y)$. If $Y_{i,j}$ denotes the $i\times j$ entry of the matrix Y and

$$Z_i = \begin{pmatrix} Y_{i,1} & \ldots & Y_{i,q'} \\ Y_{i,(q'+1)} & \ldots & Y_{i,2\,q'} \\ \vdots & & \vdots \\ Y_{i,(p-1)\,q'+1} & \ldots & Y_{i,p\,q'} \end{pmatrix} \in D^{p\times q'}, \quad i = 1,\ldots,l,$$

then $\sum_{j=1}^{s} X_{ij}\,\overline{P}_j = Z_i\,R'$, and thus the f_i's satisfy the following D-linear relations:

$$\sum_{j=1}^{s} X_{ij}\, f_j = 0, \quad i = 1,\ldots,l. \tag{3.15}$$

Hence, $\hom_D(M,M') \cong D^{1\times s}/(D^{1\times l}\,X)$, i.e., $\hom_D(M,M')$ is finitely presented by $X \in D^{l\times s}$.

Let us now study the particular case $M' = M$, i.e., using (3.4):

$$\mathrm{end}_D(M) \cong \{P \in D^{p\times p} \mid \exists\, Q \in D^{q\times q} :\ R\,P = Q\,R\}/(D^{p\times q}\,R).$$

We note that $A \triangleq \{P \in D^{p\times p} \mid \exists\, Q \in D^{q\times q} :\ R\,P = Q\,R\}$ is a ring. Indeed, $0 \in A$, $I_p \in A$ and if $P_1,\, P_2 \in A$, i.e., $R\,P_1 = Q_1\,R$ and $R\,P_2 = Q_2\,R$ for certain matrices $Q_1,\, Q_2 \in D^{q\times q}$, then:

$$\left\{\begin{array}{l} R\,(P_1+P_2) = (Q_1+Q_2)\,R, \\ R\,(P_1\,P_2) = (Q_1\,R)\,P_2 = Q_1\,(R\,P_2) = (Q_1\,Q_2)\,R, \end{array}\right. \quad \Rightarrow \quad \left\{\begin{array}{l} P_1+P_2 \in A, \\ P_1\,P_2 \in A. \end{array}\right.$$

The other properties of a ring can easily be checked. Ring A is generally a noncommutative ring since $P_1\,P_2$ is generally different from $P_2\,P_1$. Moreover, $I \triangleq D^{p\times q}\,R$ is a two-sided ideal of A. Indeed, if $P_1,\, P_2 \in A$ and $Z_1\,R,\, Z_2\,R \in I$, where $Z_i \in D^{p\times q}$ for $i = 1,\, 2$, then:

$$\left\{\begin{array}{l} P_1\,(Z_1\,R) + P_2\,(Z_2\,R) = (P_1\,Z_1 + P_2\,Z_2)\,R, \\ (Z_1\,R)\,P_1 + (Z_2\,R)\,P_2 = Z_1\,Q_1\,R + Z_2\,Q_2\,R = (Z_1\,Q_1 + Z_2\,Q_2)\,R. \end{array}\right.$$

Therefore, $B \triangleq A/I$ is a ring. If $\kappa : A \longrightarrow B$ is the canonical projection onto B, then the product of B is defined by $\kappa(P_1)\,\kappa(P_2) \triangleq \kappa(P_1\,P_2)$ for all $P_1,\, P_2 \in A$.

The ring structure of $\mathrm{end}_D(M)$ can be characterized by the expressing of the compositions $f_i \circ f_j$ in the family of generators $\{f_k\}_{k=1,\dots,s}$ for $i, j = 1, \dots, s$, i.e.:

$$\forall\, i, j = 1, \dots, s, \quad f_i \circ f_j = \sum_{k=1}^{s} \gamma_{ijk}\, f_k, \quad \gamma_{ijk} \in D. \tag{3.16}$$

The γ_{ijk}'s look like the *structure constants* appearing in the theory of finite-dimensional algebras. Hence, if $F = (f_1 \ \dots \ f_s)^T$, then the matrix Γ formed by the γ_{ijk} satisfies $F \otimes F = \Gamma\, F$. Γ is called a *multiplication table* in group theory. Finally, if $D\langle f_1, \dots f_s\rangle$ is the free associative D-algebra generated by the f_i's and if

$$I = \left\langle \sum_{j=1}^{s} X_{ij}\, f_j,\ i = 1, \dots, l,\ f_i \circ f_j - \sum_{k=1}^{s} \gamma_{ijk}\, f_k,\ i, j = 1, \dots, s \right\rangle$$

is the two-sided ideal of D generated by the polynomials corresponding to the identities (3.15) and (3.16), then the noncommutative ring $\mathrm{end}_D(M)$ is defined by

$$\mathrm{end}_D(M) = D\langle f_1, \dots f_s\rangle / I, \tag{3.17}$$

which shows that $\mathrm{end}_D(M)$ can be defined as the quotient of a free associative algebra by a two-sided ideal generated by linear and quadratic relations ([20]).

We sum up the previous results in the following algorithm.

Algorithm 3.2.1.
- **Input:** Two matrices $R \in D^{q\times p}$ and $R' \in D^{q'\times p'}$ defined over a commutative polynomial ring D over a computable field k.
- **Output:** A finite family of generators $\{f_1, \dots, f_s\}$ of the D-module $\hom_D(M, M')$, where $M = D^{1\times p}/(D^{1\times q}\,R)$ (resp., $M' = D^{1\times p'}/(D^{1\times q'}\,R')$) and a set of D-linear relations of the f_i's defining the D-module structure of $\hom_D(M, M')$.

1. Compute the matrix $L = \begin{pmatrix} R^T \otimes I_{p'} \\ I_q \otimes R' \end{pmatrix} \in D^{(p\,p'+q\,q')\times q\,p'}$.
2. Using Algorithm 1.2.1, compute a matrix $L_2 \in D^{s\times(p\,p'+q\,q')}$ satisfying $\ker_D(.L) = D^{1\times s}\,L_2$.

3. For $i = 1, \ldots, s$, construct the matrices $P_i \in D^{p \times p'}$ and $Q_i \in D^{q \times q'}$ defined by

$$\begin{cases} P_i(j,k) = L(i, (j-1)\,p' + k), & j = 1, \ldots, p, \quad k = 1, \ldots, p', \\ Q_i(l,m) = -L(i, p\,p' + (l-1)\,q' + m), & l = 1, \ldots, q, \quad m = 1, \ldots, q', \end{cases}$$

   where $L(i,j)$ is the $i \times j$ entry of the matrix L. We then have:

$$R\,P_i = Q_i\,R', \quad i = 1, \ldots, s.$$

4. Compute a Gröbner basis G of the rows of R' for a total degree order.
5. For $i = 1, \ldots, s$, reduce the rows of P_i with respect to G by computing their normal forms with respect to G. We obtain the matrices $\overline{P}_i$ which satisfy $\overline{P}_i = P_i + Z_i\,R'$, for certain matrices $Z_i \in D^{p \times q'}$ which can be obtained by means of factorizations.
6. For $i = 1, \ldots, s$, define the following matrices $\overline{Q}_i = Q_i + R\,Z_i$. The pair $(\overline{P_i}, \overline{Q_i})$ then satisfies the relation $R\,\overline{P}_i = \overline{Q_i}\,R'$ and the D-module $\hom_D(M, M')$ is finitely generated by $\{f_i\}_{i=1,\ldots,s}$, where $f_i \in \hom_D(M, M')$ is defined by $f_i(\pi(\lambda)) = \pi'(\lambda\,\overline{P}_i)$, for all $\lambda \in D^{1\times p}$, and $\pi : D^{1\times p} \longrightarrow M$ (resp., $\pi' : D^{1\times p'} \longrightarrow M'$) is the projection onto M (resp., M').
7. Form the three matrices $U = (\mathrm{row}(\overline{P}_1)^T \; \ldots \; \mathrm{row}(\overline{P}_s)^T)^T \in D^{s \times p\,p'}$, $V = I_p \otimes R' \in D^{p\,q' \times p\,p'}$ and $W = (U^T \quad V^T) \in D^{(s+p\,q') \times p\,p'}$.
8. Using Algorithm 1.2.1, compute a matrix $(X \quad -Y)$, where $X \in D^{l \times s}$ and $Y \in D^{l \times p\,q'}$, such that $\ker_D(.W) = D^{1\times l}\,(X \quad -Y)$. Then, the family of generators $\{f_i\}_{i=1,\ldots,s}$ of the D-module $\hom_D(M, M')$ satisfies the D-linear relations $X\,F = 0$, where $F = (f_1 \; \ldots \; f_s)^T$, i.e., $\hom_D(M, M') \cong D^{1\times s}/(D^{1\times l}\,X)$.
9. If $R' = R$, then, for $i, j = 1, \ldots, s$, compute the normal form of $\mathrm{row}(\overline{P}_i\,\overline{P}_j)$ with respect to a Gröbner basis of the D-module $D^{1\times(s+p\,q)}\,W$. With these formal forms, form the matrix $(\Gamma_1 \quad \Gamma_2) \in D^{s^2 \times (s^2 + p\,q)}$, where $\Gamma_1 \in D^{s^2 \times s}$ and $\Gamma_2 \in D^{s^2 \times p\,q}$. Then, the matrix Γ_1 defines the multiplication table of family of generators $\{f_i\}_{i=1,\ldots,s}$ of the D-module $\mathrm{end}_D(M)$.

Example 3.2.1. Let us consider a commutative ring D, $R \in D^q$ a column vector with entries in D, $I = D^{1\times q}\,R$ the ideal of D generated by the entries R_i of R and $M = D/I$ the D-module finitely presented by the matrix R. Then, a D-endomorphism f of M is defined by $f(\pi(\lambda)) = \pi(\lambda\,P)$, where $\pi : D \longrightarrow M$ is the canonical projection onto M, $\lambda \in D$ and $P \in D$ is such that there exists $Q \in D^{q\times q}$ satisfying $R\,P = Q\,R$. Since D is a commutative ring, we can choose any $P \in D$ and $Q = P\,I_q$, a fact showing that we can take $P = 1$ and $f = \mathrm{id}_M$ generates the endomorphism ring $\mathrm{end}_D(M)$. The relations satisfied by id_M are obtained by computing $\ker_D(.W)$, where $W = (1 \quad R^T)^T$: if $\lambda = (\lambda_1 \quad \lambda_2) \in \ker_D(.W)$, where $\lambda_1 \in D$ and $\lambda_2 \in D^{1\times q}$, i.e., $\lambda_1 + \lambda_2\,R = 0$, then $\lambda_1 = -\lambda_2\,R$, i.e., $\lambda = -\lambda_2\,(R \quad -1)$, a fact showing that we can take $X = R$ and $Y = 1$. Hence, we get $R\,\mathrm{id}_M = 0$ and $\mathrm{end}_D(M) \cong M = D/I$ as a D-module. Finally, $\mathrm{id}_M \circ \mathrm{id}_M = \mathrm{id}_M$ defines a trivial ring structure on $\mathrm{end}_D(M)$ and:

$$\mathrm{end}_D(M) \cong D\langle \mathrm{id}_M \rangle / \langle R_1\,\mathrm{id}_M, \ldots, R_q\,\mathrm{id}_M, \mathrm{id}_M \circ \mathrm{id}_M - \mathrm{id}_M \rangle \;\cong\; D/I = M.$$

We note that we could have directly obtained $\mathrm{end}_D(M) \cong M = D/I$ by applying the left contravariant functor $\hom_D(\,\cdot\,, D/I)$ to the finite presentation $D^{1\times q} \xrightarrow{.R} D \xrightarrow{\pi} D/I \longrightarrow 0$ of the D-module D/I to get the following exact sequence of D-modules

$$(D/I)^q \xleftarrow{R.} D/I \longleftarrow \mathrm{end}_D(D/I) \longleftarrow 0,$$

i.e., $\ker_{D/I}(R.) \cong \mathrm{end}_D(D/I)$. Using the fact that all the R_i's belong to I, we then get

$$\forall\, d \in D, \quad R\,\pi(d) = \begin{pmatrix} R_1 \\ \vdots \\ R_q \end{pmatrix} \pi(d) = \begin{pmatrix} \pi(R_1\, d) \\ \vdots \\ \pi(R_q\, d) \end{pmatrix} = \begin{pmatrix} \pi(d\, R_1) \\ \vdots \\ \pi(d\, R_q) \end{pmatrix} = 0,$$

which finally shows that $\mathrm{end}_D(D/I) \cong \ker_{D/I}(R.) = D/I$.

Example 3.2.2. Let us consider again the model of the motion of a fluid in a one-dimensional tank studied in Example 2.2.5. Let $D = \mathbb{Q}(\alpha)[\partial, \delta]$ be the commutative polynomial ring of OD time-delay operators with rational constant coefficients (i.e., $\partial\, y(t) = \dot{y}(t)$, $\delta\, y(t) = y(t-h)$),

$$R = \begin{pmatrix} \delta^2 & 1 & -2\,\partial\,\delta \\ 1 & \delta^2 & -2\,\partial\,\delta \end{pmatrix} \in D^{2\times 3}. \tag{3.18}$$

the presentation matrix of (2.32) and the D-module $M = D^{1\times 3}/(D^{1\times 2}\, R)$ finitely presented by R. Applying Algorithm 3.2.1 to R, we obtain that $\mathrm{end}_D(M)$ is generated by the D-endomorphisms f_{e_1}, f_{e_2}, f_{e_3} and f_{e_4} defined by $f_\alpha(\pi(\lambda)) = \pi(\lambda\, P_\alpha)$ for all $\lambda \in D^{1\times 3}$, where

$$P_\alpha = \begin{pmatrix} \alpha_1 & \alpha_2 & 2\,\alpha_3\,\partial\,\delta \\ \alpha_2 + 2\,\alpha_4\,\partial & \alpha_1 - 2\,\alpha_4\,\partial & 2\,\alpha_3\,\partial\,\delta \\ \alpha_4\,\delta & -\alpha_4\,\delta & \alpha_1 + \alpha_2 + \alpha_3\,(\delta^2+1) \end{pmatrix}, \quad Q_\alpha = \begin{pmatrix} \alpha_1 - 2\,\alpha_4\,\partial & \alpha_2 + 2\,\alpha_4\,\partial \\ \alpha_2 & \alpha_1 \end{pmatrix},$$

$\alpha = (\alpha_1 \;\; \alpha_2 \;\; \alpha_3 \;\; \alpha_4) \in D^{1\times 4}$ and $\{e_i\}_{i=1,\ldots,4}$ is the standard basis of $D^{1\times 4}$. To simplify the notations, we denote by $f_i = f_{e_i}$. We can check that the generators $\{f_i\}_{i=1,\ldots,4}$ of the D-module $\mathrm{end}_D(M)$ satisfy the following D-linear relations:

$$(\delta^2 - 1)\, f_4 = 0, \quad \delta^2\, f_1 + f_2 - f_3 = 0, \quad f_1 + \delta^2\, f_2 - f_3 = 0. \tag{3.19}$$

A complete description of the noncommutative ring $\mathrm{end}_D(M)$ is given by the knowledge of the expressions of the compositions $f_i \circ f_j$ in the family of generators $\{f_k\}_{k=1,\ldots,4}$ for $i, j = 1, \ldots, 4$:

$$\left\{ \begin{array}{l} f_1 \circ f_i = f_i \circ f_1 = f_i, \quad i = 1, \ldots, 4, \\ f_2 \circ f_2 = f_1, \\ f_2 \circ f_3 = f_3 \circ f_2 = f_3, \\ f_2 \circ f_4 = 2\,\partial\, f_1 - 2\,\partial\, f_2 + f_4, \\ f_4 \circ f_2 = -f_4, \end{array} \right. \quad \left\{ \begin{array}{l} f_3 \circ f_3 = (\delta^2+1)\, f_3, \\ f_3 \circ f_4 = 2\,\partial\, f_1 - 2\,\partial\, f_2 + 2\, f_4, \\ f_4 \circ f_3 = 0, \\ f_4 \circ f_4 = -2\,\partial\, f_4. \end{array} \right. \tag{3.20}$$

Denoting by $f_c \circ f_r$ the composition of an element f_c in the first column by an element f_r in the first row, we can write (3.20) in the form of the following multiplication table:

$f_c \circ f_r$	f_1	f_2	f_3	f_4
f_1	f_1	f_2	f_3	f_4
f_2	f_2	f_1	f_3	$2\,\partial\, f_1 - 2\,\partial\, f_2 + f_4$
f_3	f_3	f_3	$(\delta^2+1)\, f_3$	$2\,\partial\, f_1 - 2\,\partial\, f_2 + 2\, f_4$
f_4	f_4	$-f_4$	0	$-2\,\partial\, f_4$

We finally obtain $\mathrm{end}_D(M) = D\langle f_1, f_2, f_3, f_4\rangle / I$, where

$$I = \langle (\delta^2 - 1)\, f_4, \delta^2\, f_1 + f_2 - f_3, f_1 + \delta^2\, f_2 - f_3, f_1 \circ f_1 - f_1, \ldots, f_4 \circ f_4 + 2\,\partial\, f_4 \rangle$$

is the two-sided ideal of the free D-algebra $D\langle f_1, f_2, f_3, f_4\rangle$ generated by the polynomials defined by the identities (3.19) and (3.20).

If D is a noncommutative polynomial k-algebra, where k is a field, then $\hom_D(M, M')$ has generally no D-module structure but is a k-vector space. Thus, we cannot repeat what we have done for commutative rings. Let us explain what we can be done if $D = A_n(k)$ or $B_n(k)$ and k is a field. For r, s, $t \in \mathbb{N}$, let us introduce the finite-dimensional k-vector spaces:

$$\begin{cases} k[x_1, \ldots, x_m]_s = \{a \in k[x_1, \ldots, x_m] \mid \deg a \leq s\}, \\ k(x_1, \ldots, x_m)_{s,t} = \{a/b \in k(x_1, \ldots, x_m) \mid 0 \neq b, a \in k[x_1, \ldots, x_n],\ \deg a \leq s,\ \deg b \leq t\}, \\ E^r_s = \{P = \sum_{|\mu|=0,\ldots,r} a_\mu\, \partial^\mu \mid a_\mu \in k[x_1, \ldots, x_m]_s^{p\times p'}\}, \\ E^r_{s,t} = \{P = \sum_{|\mu|=0,\ldots,r} a_\mu\, \partial^\mu \mid a_\mu \in k(x_1, \ldots, x_m)_{s,t}^{p\times p'}\}. \end{cases}$$

We note that $E^r_{s,0} = E^r_s$ and $E^r_0 = \{P = \sum_{|\mu|=0,\ldots,r} a_\mu\, \partial^\mu \mid a_\mu \in k\}$. Even if $\hom_D(M, M')$ is generally an infinite-dimensional k-vector space, we can compute the finite-dimensional k-vector space $\{P \in E^r_{s,t} \mid R\, P \in D^{q\times p'}\, R'\}$ by solving the algebraic systems of equations in the coefficients of an ansatz $P \in E^r_{s,t}$ obtained by reducing to zero the normal forms of the rows of the matrix $R\, P$ with respect to a Gröbner basis of the left D-module $D^{1\times q'}\, R'$. More precisely, we have the following algorithm which computes the elements of $\hom_D(M, M')$ defined by means of a matrix P with a fixed total order in the operators ∂_i and a fixed degree (resp., degrees) in x_i for the polynomial (resp., for the numerators and denominators of the rational) coefficients.

Algorithm 3.2.2.
- **Input:** A noncommutative polynomial ring D for which Buchberger's algorithm terminates for any admissible term order, $R \in D^{q\times p}$ and $R' \in D^{q'\times p'}$ and three non-negative integers α, β and γ.
- **Output:** A finite family $\{f_i\}_{i\in I}$ of elements of $\hom_D(M, M')$, where $M = D^{1\times p}/(D^{1\times q}\, R)$ and $M' = D^{1\times p'}/(D^{1\times q'}\, R')$, defined by matrices $P_i \in E^\alpha_{\beta,\gamma}$, i.e., satisfying $R\, P_i \in D^{q\times p}\, R'$ and $f_i(\pi(\lambda)) = \pi'(\lambda\, P_i)$, where $\pi : D^{1\times p} \longrightarrow M$ (resp., $\pi' : D^{1\times p'} \longrightarrow M'$) is the canonical projection onto M (resp., M') and $\lambda \in D^{1\times p}$.

1. Take an ansatz $L = \sum_{|\mu|=0,\ldots,\alpha} a_\mu\, \partial^\mu \in E^\alpha_{\beta\gamma}$.
2. Compute the product $R\, L$ and denote the result by F.
3. Compute a Gröbner basis G of the left D-module $D^{1\times p'}\, R'$ for a total degree order.
4. Compute the normal forms of the rows of F with respect to G.
5. Solve the system for the coefficients a_μ so that all the normal forms vanish.
6. Substitute the solutions into the matrix L. Denote the set of solutions by $\{L_i\}_{i\in I}$.
7. For $i \in I$, form the matrix P_i obtained by computing the normal forms of the rows of L_i with respect to G.

Remark 3.2.1. We note that algebraic systems obtained in the case $E^\alpha_\beta = E^\alpha_{\beta,0}$ are linear, and thus, their solutions belong to the field k, whereas the solutions of systems of algebraic equations corresponding to $E^\alpha_{\alpha,\gamma}$ for $\gamma \geq 1$ belong to the algebraic closure $\overline{k}$ of k.

Example 3.2.3. Let us consider the *Euler-Tricomi equation* ([23]) appearing in transonic flow:

$$\partial_1^2 \, u(x_1, x_2) - x_1 \, \partial_2^2 \, u(x_1, x_2) = 0.$$

Let $D = A_2(\mathbb{Q})$ be the first Weyl algebra, $R = (\partial_1^2 - x_1 \, \partial_2^2) \in D$ and $M = D/(D\,R)$. We can easily check that $\mathrm{end}_D(M)$ is an infinite-dimensional $\mathbb{Q}$-vector space. Let us denote by $\mathrm{end}_D(M)_s^r$ the $\mathbb{Q}$-vector space formed by the elements of $\mathrm{end}_D(M)$ defined by PD operators P whose total orders (resp., degrees) in the ∂_i's (resp., x_j's) are less or equal to r (resp., s).

Below, we list of a few examples of $\mathrm{end}_D(M)_s^r$, where the a_i's belong to $\mathbb{Q}$:
- $\mathrm{end}_D(M)_0^0$ is defined by $P = Q = a_1$.
- $\mathrm{end}_D(M)_1^1$ is defined by $P = a_1 + a_2 \, \partial_2 + \frac{3}{2} \, a_3 \, x_2 \, \partial_2 + a_3 \, x_1 \, \partial_1$ and $Q = P + 2 \, a_3$.
- $\mathrm{end}_D(M)_0^2$ is defined by $P = Q = a_1 + a_2 \, \partial_2 + a_3 \, \partial_2^2$.
- $\mathrm{end}_D(M)_1^2$ is defined by:

$$\begin{cases} P = a_1 + a_2 \, \partial_2 + \frac{3}{2} \, a_3 \, x_2 \, \partial_2 + a_3 \, x_1 \, \partial_1 + a_4 \, \partial_2^2 + \frac{3}{2} \, a_5 \, x_2 \, \partial_2^2 + a_5 \, x_1 \, \partial_1 \, \partial_2, \\ Q = P + 2 \, a_3 + 2 \, a_5 \, \partial_2. \end{cases}$$

Example 3.2.4. Let us consider the first Weyl algebra $D = A_2(\mathbb{Q})$ and the finitely presented left D-module $M = D^{1\times 2}/(D^{1\times 2}\,R)$ defined by the following matrix of PD operators:

$$R = \begin{pmatrix} \partial_1 & -x_1 \partial_2 \\ \partial_2 & x_1 \, \partial_1 \end{pmatrix} \in D^{2\times 2}.$$

The left D-module M is associated with the so-called *conjugate Beltrami equations*. The endomorphism ring $\mathrm{end}_D(M)$ is an infinite-dimensional $\mathbb{Q}$-vector space and, using the notations defined in Example 3.2.3, we obtain the following examples of $\mathrm{end}_D(M)_s^r$:
- $\mathrm{end}_D(M)_1^0$ is defined by $P = Q = a_1 \, I_2$, where $a_1 \in \mathbb{Q}$.
- $\mathrm{end}_D(M)_0^1$ is defined by:

$$P = Q = \begin{pmatrix} a_1 + a_2 \, \partial_2 & 0 \\ 0 & a_1 + a_2 \, \partial_2 \end{pmatrix}, \quad a_1, a_2 \in \mathbb{Q}.$$

- $\mathrm{end}_D(M)_1^1$ is defined by:

$$\begin{aligned} P &= \begin{pmatrix} a_3 \, (x_2 \, \partial_2 + x_1 \, \partial_1 - 1) + a_2 \, \partial_2 + a_1 & 0 \\ -a_3 \, \partial_2 & a_3 \, x_2 \, \partial_2 + a_2 \, \partial_2 + a_1 \end{pmatrix}, \\ Q &= \begin{pmatrix} a_3 \, (x_2 \, \partial_2 + x_1 \, \partial_1) + a_2 \, \partial_2 + a_1 & a_3 \, x_1 \, \partial_2 \\ 0 & a_2 \, \partial_2 + a_3 \, x_2 \, \partial_2 + a_1 \end{pmatrix}, \quad a_1, a_2, a_3 \in \mathbb{Q}. \end{aligned}$$

3.3 Conservations laws of linear PD systems

Let $D = A\langle \partial_1, \ldots, \partial_n \rangle$ be a ring of PD operators with coefficients in a differential ring A and $R \in D^{q\times p}$. One can prove that the formal adjoint $\widetilde{R} \in D^{p\times q}$ of R satisfies the following identity

$$(\lambda, R\,\eta) = (\widetilde{R}\,\lambda, \eta) + \sum_{i=1}^{n} \partial_i \, \Phi_i(\lambda, \eta), \tag{3.21}$$

where $(\cdot\,,\,\cdot)$ denotes the standard inner product of $\mathbb{R}^q$ and the Φ_i's are bilinear forms in the variables η_i's and λ_j's (see, e.g., [66, 85]). If $\mathcal{F}$ is a left D-module (e.g., $\mathcal{F} = A$) and $\eta \in \ker_{\mathcal{F}}(R.)$, then (3.21) yields $(\widetilde{R}\,\lambda, \eta) + \sum_{i=1}^{n} \partial_i \, \Phi_i(\lambda, \eta) = 0$. Now, if we choose $\lambda \in \ker_{\mathcal{F}}(\widetilde{R}.)$, then the vector $\vec{\Phi} = (\Phi_1(\lambda, \eta), \ldots, \Phi_n(\lambda, \eta))^T$ satisfies

$$\sum_{i=1}^{n} \partial_i \, \Phi_i(\lambda, \eta) = 0,$$

i.e., Φ is a *conservation law* of the linear PD system $\ker_{\mathcal{F}}(R.)$ ([51, 52]).

If $n = 1$, then $\Phi = \Phi_1$ is a *first integral* of the linear OD system $\ker_{\mathcal{F}}(R.)$ (see, e.g., [50, 88]). Moreover, if R has full row rank and A is either k, $k[t]$, $k(t)$, $k[\![t]\!]$ or $k\{t\}$, where $k = \mathbb{R}$ or $\mathbb{C}$, then Corollary 2.3.1 shows that $M = D^{1\times p}/(D^{1\times q}\, D)$ is torsion-free, i.e., stably free (see Example 1.2.13 and Corollary 1.3.3), iff $N = D^q/(R\, D^p) = 0$, i.e., iff $\widetilde{N} = D^{1\times q}/(D^{1\times p}\, \widetilde{R}) = 0$, which yields $\ker_{\mathcal{F}}(\widetilde{R}.) \cong \hom_D(\widetilde{N}, \mathcal{F}) = 0$. Hence, if $\mathcal{F}$ is a cogenerator left D-module (see Remark 1.4.2) and M admits a torsion element, i.e., $\widetilde{N} \neq 0$, then $\ker_{\mathcal{F}}(\widetilde{R}.) \cong \hom_D(\widetilde{N}, \mathcal{F}) \neq 0$, and thus $\ker_{\mathcal{F}}(R.)$ admits a first integral.

Example 3.3.1. Let us consider the following linear OD control system:

$$\begin{cases} \dot{x}_1 = x_2 + u, \\ \dot{x}_2 = x_1 - u. \end{cases}$$

Let $D = \mathbb{Q}[\partial]$ be the commutative polynomial ring of OD operators, $M = D^{1\times 3}/(D^{1\times 2}\, R)$ and $\widetilde{N} = D^{1\times 2}/(D^{1\times 3}\, \widetilde{R})$ the D-modules respectively presented by the following matrices:

$$R = \begin{pmatrix} \partial & -1 & -1 \\ -1 & \partial & 1 \end{pmatrix}, \quad \widetilde{R} = \theta(R) = \begin{pmatrix} -\partial & -1 \\ -1 & -\partial \\ -1 & 1 \end{pmatrix}.$$

We can check that $z = x_1 + x_2$ satisfies $\partial\, z = 0$, i.e., is a torsion element of M. Thus, if $\mathcal{F} = C^\infty(\mathbb{R}_+)$, then the linear OD system $\ker_{\mathcal{F}}(R.)$ admits a first integral. Integrating the linear OD system $\ker_{\mathcal{F}}(\widetilde{R}.)$, we obtain:

$$\forall\, C \in \mathbb{R}, \quad \begin{cases} \lambda_1 = C\, e^{-t}, \\ \lambda_2 = C\, e^{-t}. \end{cases}$$

Using the identity $\lambda^T\,(R\,\eta) = \eta^T\,(\widetilde{R}\,\lambda) + \partial\,(\lambda_1\, x_1 + \lambda_2\, x_2)$, where $\eta = (x_1 \quad x_2 \quad u)^T \in \ker_{\mathcal{F}}(R.)$, the first integrals of $\ker_{\mathcal{F}}(R.)$ are defined by $\Phi = C\, e^{-t}\,(x_1 + x_2)$, i.e., $\dot{\Phi} = 0$.

Example 3.3.2. Let us consider again the first set of Maxwell equations defined by (1.45). In Example 1.3.6, we proved that the corresponding differential module was torsion-free, and thus parametrizable (see Example 1.4.4). If $\vec{B}$ and $\vec{E}$ satisfy (1.45), and $\vec{C}$ and $\vec{G}$ satisfy (1.49), using (1.48), we obtain that (1.45) admits the following conservation law:

$$\frac{\partial}{\partial t}\left(\vec{C}\,.\,\vec{B}\right) + \vec{\nabla}\,.\,\left(G\,\vec{B} - \vec{C} \wedge \vec{E}\right) = 0.$$

Now, if we substitute the quadri-potential $(\vec{A}, V)$ by $(\vec{C}, G)$ in Example 1.3.6, we obtain that the smooth solutions of (1.49) are parametrized by

$$\begin{cases} -\dfrac{\partial \vec{C}}{\partial t} - \vec{\nabla}\, G = \vec{0}, \\ \vec{\nabla} \wedge \vec{C} = \vec{0}, \end{cases} \quad \Leftrightarrow \quad \begin{cases} \vec{C} = -\vec{\nabla}\,\xi, \\ G = \dfrac{\partial \xi}{\partial t}, \end{cases} \quad \xi \in \mathcal{F} = C^\infty(\mathbb{R}^4),$$

a fact proving that (1.45) admits the following family of conservation laws:

$$\forall\, \xi \in \mathcal{F}, \quad \frac{\partial}{\partial t}\left(-\vec{\nabla}\,\xi\,.\,\vec{B}\right) + \vec{\nabla}\,.\,\left(\frac{\partial \xi}{\partial t}\,\vec{B} + \vec{\nabla}\,\xi \wedge \vec{E}\right) = 0.$$

The differential module defined by the first set of Maxwell equations is torsion-free (see Example 1.3.6). Hence, contrary to the OD case (see above), a PD linear system can admit conversation laws even if its underlying differential module is torsion-free.

The above computation of conservation laws of the linear PD system $\ker_{\mathcal{F}}(R.)$ requires the knowledge of a solution of the adjoint system $\ker_{\mathcal{F}}(\widetilde{R}.)$. The computation of a particular solution of $\ker_{\mathcal{F}}(\widetilde{R}.)$ is generally a difficult issue. If $M = D^{1\times p}/(D^{1\times q}\,R)$ and $\widetilde{N} = D^{1\times q}/(D^{1\times p}\,\widetilde{R})$, then $f \in \hom_D(\widetilde{N}, M)$ is defined by $P \in D^{q\times p}$ and $Q \in D^{p\times q}$ satisfying $\widetilde{R}\,P = Q\,R$ and Corollary 3.1.1 shows that f induces the $\mathbb{Z}$-homomorphism $f^\star : \ker_{\mathcal{F}}(R.) \longrightarrow \ker_{\mathcal{F}}(\widetilde{R}.)$ defined by $f^\star(\eta) = P\,\eta$. We can consider $\lambda = P\,\eta$, which yields a quadratic conservation law of $\ker_{\mathcal{F}}(R.)$.

Theorem 3.3.1 ([19])**.** *Let $D = A\langle\partial_1, \ldots, \partial_n\rangle$ be a ring of PD operators with coefficients in a differential ring A, $R \in D^{q\times p}$, $\mathcal{F}$ a left D-module (e.g., $\mathcal{F} = A$) and the linear PD system $\ker_{\mathcal{F}}(R.)$. Moreover, let $\widetilde{R} \in D^{q\times p}$ be the formal adjoint of R and let us introduce the left D-modules $M = D^{1\times p}/(D^{1\times q}\,R)$ and $\widetilde{N} = D^{1\times q}/(D^{1\times p}\,\widetilde{R})$. Then, $f \in \hom_D(\widetilde{N}, M)$, defined by $P \in D^{q\times p}$ and $Q \in D^{p\times q}$ satisfying $\widetilde{R}\,P = Q\,R$, induces the quadratic conservation law*

$$\Phi = (\Phi_1(P\,\eta, \eta)\ \ldots\ \Phi_n(P\,\eta, \eta))^T$$

of $\ker_{\mathcal{F}}(R.)$, i.e., $\sum_{i=1}^n \partial_i\,\Phi_i = 0$, where the Φ_i's are the bilinear forms defined by (3.21).

We point out that no integration of the formal adjoint linear PD system is needed to compute the quadratic conversation laws of the system. Only Gröbner basis techniques is needed.

Example 3.3.3. Let us consider the *Maxwell equations* in the vacuum ([51, 83, 84])

$$\left\{\begin{array}{l} \dfrac{\partial \vec{B}}{\partial t} + \vec{\nabla} \wedge \vec{E} = \vec{0}, \\ \dfrac{1}{\mu_0}\,\vec{\nabla} \wedge \vec{B} - \epsilon_0\,\dfrac{\partial \vec{E}}{\partial t} = \vec{0}, \end{array}\right. \tag{3.22}$$

where $\vec{B}$ (resp., $\vec{E}$) is the *magnetic* (resp., *electric*) *field*, μ_0 (resp., ϵ_0) the *magnetic* (resp., *electric*) *constant*. Let $D = \mathbb{Q}(\mu_0, \epsilon_0)[\partial_t, \partial_1, \partial_2, \partial_3]$ be the polynomial ring of PD operators,

$$R = \begin{pmatrix} \partial_t & 0 & 0 & 0 & -\partial_3 & \partial_2 \\ 0 & \partial_t & 0 & \partial_3 & 0 & -\partial_1 \\ 0 & 0 & \partial_t & -\partial_2 & \partial_1 & 0 \\ 0 & -\partial_3/\mu_0 & \partial_2/\mu_0 & -\epsilon_0\,\partial_t & 0 & 0 \\ \partial_3/\mu_0 & 0 & -\partial_1/\mu_0 & 0 & -\epsilon_0\,\partial_t & 0 \\ -\partial_2/\mu_0 & \partial_1/\mu_0 & 0 & 0 & 0 & -\epsilon_0\,\partial_t \end{pmatrix} \in D^{6\times 6}$$

the presentation matrix of (3.22) and $M = D^{1\times 6}/(D^{1\times 6}\,R)$. Then, the formal adjoint $\widetilde{R}$ of R is:

$$\widetilde{R} = \begin{pmatrix} -\partial_t & 0 & 0 & 0 & -\partial_3/\mu_0 & \partial_2/\mu_0 \\ 0 & -\partial_t & 0 & \partial_3/\mu_0 & 0 & -\partial_1/\mu_0 \\ 0 & 0 & -\partial_t & -\partial_2/\mu_0 & \partial_1/\mu_0 & 0 \\ 0 & -\partial_3 & \partial_2 & \epsilon_0\,\partial_t & 0 & 0 \\ \partial_3 & 0 & -\partial_1 & 0 & \epsilon_0\,\partial_t & 0 \\ -\partial_2 & \partial_1 & 0 & 0 & 0 & \epsilon_0\,\partial_t \end{pmatrix} \in D^{6\times 6}.$$

If we denote by $\eta = (B_1 \; B_2 \; B_3 \; E_1 \; E_2 \; E_3)^T$ and $\lambda = (C_1 \; C_2 \; C_3 \; F_1 \; F_2 \; F_3)^T$, then we have:

$$(\lambda, R\,\eta) = (\eta, \widetilde{R}\,\lambda) + \partial_t \left(\sum_{i=1}^{3} C_i\, B_i - \epsilon_0 \sum_{i=1}^{3} F_i\, E_i \right) + \vec{\nabla}. \begin{pmatrix} C_3\, E_2 - C_2\, E_3 + (F_3\, B_2 - F_2\, B_3)/\mu_0 \\ C_1\, E_3 - C_3\, E_1 + (F_1\, B_3 - F_3\, B_1)/\mu_0 \\ C_2\, E_1 - C_1\, E_2 + (F_2\, B_1 - F_1\, B_2)/\mu_0 \end{pmatrix}. \tag{3.23}$$

Denoting by $\widetilde{N} = D^{1\times 6}/(D^{1\times 6}\,\widetilde{R})$ the adjoint D-module of M, an element $f \in \hom_D(\widetilde{N}, M)$ can be defined by the following two matrices:

$$P = \begin{pmatrix} 1/\mu_0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/\mu_0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/\mu_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix} = -Q.$$

We can easily check that f is an isomorphism, i.e., $\widetilde{N} \cong M$. Hence, if η is a solution of the system $R\,\eta = 0$, then $\lambda = P\,\eta$, i.e., $C_i = B_i/\mu_0$, $F_i = -E_i$ for $i = 1, 2, 3$, is a solution of $\widetilde{R}\,\lambda = 0$. Using (3.23), we obtain the following conservation law of (3.22):

$$\partial_t \left(\frac{1}{\mu_0} \parallel \vec{B} \parallel^2 + \epsilon_0 \parallel \vec{E} \parallel^2 \right) + \vec{\nabla}\,.\left(\frac{1}{\mu_0} (\vec{E} \wedge \vec{B}) \right) = 0.$$

$\omega = \frac{1}{\mu_0} \parallel \vec{B} \parallel^2 + \epsilon_0 \parallel \vec{E} \parallel^2$ is the *electromagnetic energy* and $\Pi = (\vec{E} \wedge \vec{B})/\mu_0$ the *Poynting vector*. Other conservation laws can be obtained by considering different elements of $\mathrm{end}_D(M)$.

Example 3.3.4. The movement of an incompressible fluid rotating with a small velocity around the axis lying along the x_3 axis can be defined by

$$\begin{cases} \rho_0 \dfrac{\partial u_1}{\partial t} - 2\,\rho_0\,\Omega_0\,u_2 + \dfrac{\partial p}{\partial x_1} = 0, \\ \rho_0 \dfrac{\partial u_2}{\partial t} + 2\,\rho_0\,\Omega_0\,u_1 + \dfrac{\partial p}{\partial x_2} = 0, \\ \rho_0 \dfrac{\partial u_3}{\partial t} + \dfrac{\partial p}{\partial x_3} = 0, \\ \dfrac{\partial u_1}{\partial x_1} + \dfrac{\partial u_2}{\partial x_2} + \dfrac{\partial u_3}{\partial x_3} = 0, \end{cases} \tag{3.24}$$

where $\vec{u} = (u_1 \quad u_2 \quad u_3)^T$ is the local rate of velocity, p the pressure, ρ_0 the constant fluid density and Ω_0 the constant angle speed ([52]). Let $D = \mathbb{Q}(\rho_0, \Omega_0)[\partial_t, \partial_1, \partial_2, \partial_3 t]$ be the commutative polynomial ring of PD operators,

$$R = \begin{pmatrix} \rho_0\,\partial_t & -2\,\rho_0\,\Omega_0 & 0 & \partial_1 \\ 2\,\rho_0\,\Omega_0 & \rho_0\,\partial_t & 0 & \partial_2 \\ 0 & 0 & \rho_0\,\partial_t & \partial_3 \\ \partial_1 & \partial_2 & \partial_3 & 0 \end{pmatrix} \in D^{4\times 4}$$

the presentation matrix of (3.24) and the D-module $M = D^{1\times 4}/(D^{1\times 4}\,R)$ associated with (3.24).

If we denote by $\eta = (u_1 \quad u_2 \quad u_2 \quad p)^T$, then we have the following identity

$$(\lambda, R\,\eta) = (\eta, \widetilde{R}\,\lambda) + (\partial_t \quad \partial_1 \quad \partial_2 \quad \partial_3) \left(\begin{array}{c} \rho_0\,(\lambda_1\,u_1 + \lambda_2\,u_2 + \lambda_3\,u_3) \\ \lambda_1\,p + \lambda_4\,u_1 \\ \lambda_2\,p + \lambda_4\,u_2 \\ \lambda_3\,p + \lambda_4\,u_3 \end{array} \right), \tag{3.25}$$

where $\widetilde{R} = -R$ is the formal adjoint of R. Hence, we get $\widetilde{N} = D^{1\times 4}/(D^{1\times 4}\,\widetilde{R}) = M$ and $\hom_D(\widetilde{N}, M) = \mathrm{end}_D(M)$. Hence, if $(u_1 \quad u_2 \quad u_2 \quad p)^T$ is a solution of (3.24), then $\lambda_1 = u_1$, $\lambda_2 = u_2$, $\lambda_3 = u_3$ and $\lambda_4 = p$ is a solution of $\widetilde{R}\,\lambda = 0$. Taking $\lambda = \eta$, i.e., $\mathrm{id}_M \in \mathrm{end}_D(M)$, and using (3.25), we obtain $\partial_t\,(\rho_0\,(u_1^2 + u_2^2 + u_3^2)) + \partial_1\,(2\,p\,u_1) + \partial_2\,(2\,p\,u_2) + \partial_3\,(2\,p\,u_3) = 0$, i.e., (3.24) admits the following quadratic conservation of law:

$$\partial_t \left(\frac{\rho_0}{2} \parallel \vec{u} \parallel^2 \right) + \vec{\nabla}\,.\,(p\,\vec{u}) = 0.$$

Other conservation laws can be obtained by considering different elements of $\mathrm{end}_D(M)$.

More examples of quadratic conservation laws of physical systems can be found in [100].

3.4 System equivalences

If $f \in \hom_D(M, M')$, then we have the following left D-modules:

$$\left\{ \begin{array}{l} \ker f = \{m \in M \mid f(m) = 0\}, \\ \mathrm{im}\, f = \{m' \in M' \mid \exists\, m \in M : \; m' = f(m)\}, \end{array} \right. \qquad \left\{ \begin{array}{l} \mathrm{coim}\, f = M/\ker f \\ \mathrm{coker}\, f = M'/\mathrm{im}\, f. \end{array} \right.$$

Let us explicitly characterize the kernel, image, coimage and cokernel of $f \in \hom_D(M, M')$, where M and M' are two finitely presented left D-modules.

Proposition 3.4.1 ([19]). *Let $M = D^{1\times p}/(D^{1\times q}\,R)$ (resp., $M' = D^{1\times p'}/(D^{1\times q'}\,R')$) be a left D-module finitely presented by $R \in D^{q\times p}$ (resp., $R' \in D^{q'\times p'}$). Let $f \in \hom_D(M, M')$ be defined by the matrices $P \in D^{p\times p'}$ and $Q \in D^{q\times q'}$ satisfying the relation $R\,P = Q\,R'$. Then, we have:*

1. $\ker f = (D^{1\times r}\,S)/(D^{1\times q}\,R)$, *where $S \in D^{r\times p}$ is a matrix defined by:*

$$\ker_D \left(.\left(\begin{array}{c} P \\ R' \end{array} \right) \right) = D^{1\times r}\,(S \quad -T), \quad T \in D^{r\times q'}. \tag{3.26}$$

2. $\mathrm{coim}\, f = D^{1\times p}/(D^{1\times r}\,S) \cong \mathrm{im}\, f = \left(D^{1\times (p+q')} \left(\begin{array}{c} P \\ R' \end{array} \right) \right) /(D^{1\times q'}\,R')$,

3. $\mathrm{coker}\, f = D^{1\times p'} / \left(D^{1\times (p+q')} \left(\begin{array}{c} P \\ R' \end{array} \right) \right)$.

   The left D-module $\mathrm{coker}\, f$ *admits the following beginning of a finite free resolution:*

$$D^{1\times r} \xrightarrow{.(S \quad -T)} D^{1\times (p+q')} \xrightarrow{.\left(\begin{array}{c} P \\ R' \end{array} \right)} D^{1\times p'} \xrightarrow{\epsilon} \mathrm{coker}\, f \longrightarrow 0. \tag{3.27}$$

4. *We have the following commutative exact diagram of left D-modules*

$$\begin{tikzcd} & & 0 \arrow[d] & \\ D^{1\times r} \arrow[r, ".S"] \arrow[d, ".T"] & D^{1\times p} \arrow[r, "\kappa"] \arrow[d, ".P"] & \operatorname{coim} f \arrow[r] \arrow[d, "f^\sharp"] & 0 \\ D^{1\times q'} \arrow[r, ".R'"] & D^{1\times p'} \arrow[r, "\pi'"] & M' \arrow[r] \arrow[d, "\sigma"] & 0, \\ & & \operatorname{coker} f \arrow[d] & \\ & & 0 & \end{tikzcd} \tag{3.28}$$

where $f^\sharp : \operatorname{coim} f \longrightarrow M'$ is defined by $f^\sharp(\kappa(\lambda)) = \pi'(\lambda\, P)$ for all $\lambda \in D^{1\times p}$.

Corollary 3.4.1 ([19]). *With the notations of Proposition 3.4.1, $f \in \hom_D(M, M')$ is:*

1. *The zero homomorphism, i.e., $f = 0$, iff one of the following equivalent conditions holds:*
   (a) *There exists a matrix $Z \in D^{p\times q'}$ such that $P = Z\, R'$. Then, there exists $Z' \in D^{q\times q_2'}$ such that $Q = R\, Z + Z'\, R_2'$, where $R_2' \in D^{q_2'\times q'}$ is such that $\ker_D(.R') = D^{1\times q_2'}\, R_2'$.*
   (b) *The matrix S admits a left inverse, i.e., there exits $X \in D^{p\times r}$ such that $X\, S = I_p$.*
2. *Injective, i.e., $\ker f = 0$, iff one of the following equivalent conditions holds:*
   (a) *There exists a matrix $F \in D^{r\times q}$ such that $S = F\, R$. Then, if $\rho : M \longrightarrow \operatorname{coim} f = M/\ker f$ is the canonical projection onto $\operatorname{coim} f$, then we have the following commutative exact diagram of left D-modules:*

$$\begin{tikzcd} & 0 & 0 & \\ D^{1\times q} \arrow[r, ".R"] & D^{1\times p} \arrow[r, "\pi"] \arrow[u] & M \arrow[r] \arrow[u] & 0 \\ D^{1\times r} \arrow[r, ".S"] \arrow[u, ".F"] & D^{1\times p} \arrow[r, "\kappa"] \arrow[u, equal] & \operatorname{coim} f \arrow[r] \arrow[u, "\rho^{-1}"] & 0. \\ & 0 \arrow[u] & 0 \arrow[u] & \end{tikzcd}$$

   (b) *The matrix $(L^T \quad S_2^T)^T$ admits a left inverse, where $L \in D^{q\times r}$ is such that $R = L\, S$.*
3. *Surjective, i.e., $\operatorname{im} f = M'$, iff $(P^T \quad R'^T)^T$ admits a left inverse.*
   Then, the long exact sequence (3.27) splits. In particular, there exist $(X \quad Y) \in D^{p'\times(p+q')}$ and $(U^T \quad V^T)^T \in D^{(p+q')\times r}$, where $X \in D^{p'\times p}$, $Y \in D^{p'\times q'}$, $U \in D^{p\times r}$ and $V \in D^{q'\times r}$, such that the following identities hold:

$$\begin{cases} X\, P + Y\, R' = I_{p'}, \\ P\, X + U\, S = I_p, \\ P\, Y - U\, T = 0, \\ R'\, X + V\, S = 0, \\ R'\, Y - V\, T = I_{q'}. \end{cases} \tag{3.29}$$

Moreover, we have the following commutative exact diagram of left D-modules:

$$\begin{array}{ccccccc} & & & & 0 & & \\ & & & & \uparrow & & \\ D^{1\times r} & \xrightarrow{.S} & D^{1\times p} & \xrightarrow{\kappa} & \operatorname{coim} f & \longrightarrow & 0 \\ \uparrow .-V & & \uparrow .X & & \uparrow f^{\sharp -1} & & \\ D^{1\times q'} & \xrightarrow{.R'} & D^{1\times p'} & \xrightarrow{\pi'} & M' & \longrightarrow & 0. \\ & & & & \uparrow & & \\ & & & & 0 & & \end{array}$$

4. *An isomorphism, i.e., $M \cong M'$, if the matrices $(L^T \quad S_2^T)^T$ and $(P^T \quad R'^T)^T$ admit left inverses. The inverse f^{-1} of f is then defined by*

$$\forall\, \lambda' \in D^{1\times p'}, \quad f^{-1}(\pi'(\lambda')) = \pi(\lambda'\, X),$$

where $X \in D^{p'\times p}$ is defined in 3 and we have the following commutative exact diagram

$$\begin{array}{ccccccc} D^{1\times q} & \xrightarrow{.R} & D^{1\times p} & \xrightarrow{\pi} & M & \longrightarrow & 0 \\ \uparrow .-V\,F & & \uparrow .X & & \uparrow f^{-1} & & \\ D^{1\times q'} & \xrightarrow{.R'} & D^{1\times p'} & \xrightarrow{\pi'} & M' & \longrightarrow & 0, \end{array} \tag{3.30}$$

where $F \in D^{r\times q}$ is such that $S = F\, R$.

Example 3.4.1. Let us consider two PD systems used in the theory of elasticity: the Lie derivative of the euclidean metric of $\mathbb{R}^2$ defined in Example 3.1.2 and its *Spencer operator*:

$$\left\{\begin{array}{l} \partial_1\,\xi_1 = 0, \\ \frac{1}{2}\,(\partial_2\,\xi_1 + \partial_1\,\xi_2) = 0, \\ \partial_2\,\xi_2 = 0, \end{array}\right. \qquad \left\{\begin{array}{l} \partial_1\,\zeta_1 = 0, \\ \partial_2\,\zeta_1 - \zeta_2 = 0, \\ \partial_1\,\zeta_2 = 0, \\ \partial_1\,\zeta_3 + \zeta_2 = 0, \\ \partial_2\,\zeta_3 = 0, \\ \partial_2\,\zeta_2 = 0. \end{array}\right.$$

For more details, see [82, 84] and Example 3.1.2. Let $D = \mathbb{Q}[\partial_1, \partial_2]$ be the commutative polynomial ring of PD operators with rational constant coefficients,

$$R = \begin{pmatrix} \partial_1 & 0 \\ \frac{1}{2}\,\partial_2 & \frac{1}{2}\,\partial_1 \\ 0 & \partial_2 \end{pmatrix} \in D^{3\times 2}, \quad R' = \begin{pmatrix} \partial_1 & \partial_2 & 0 & 0 & 0 & 0 \\ 0 & -1 & \partial_1 & 1 & 0 & \partial_2 \\ 0 & 0 & 0 & \partial_1 & \partial_2 & 0 \end{pmatrix}^T \in D^{6\times 3}, \tag{3.31}$$

and the finitely presented D-modules $M = D^{1\times 2}/(D^{1\times 3}\, R)$ and $M' = D^{1\times 3}/(D^{1\times 6}\, R')$. We can check that the following matrices

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad Q = \frac{1}{2}\begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \end{pmatrix}, \tag{3.32}$$

satisfy the relation $R\,P = Q\,R'$, i.e., define $f \in \hom_D(M, M')$ by $f(\xi_1) = \zeta_1$ and $f(\xi_2) = \zeta_3$. With the notations of Proposition 3.4.1, we obtain that f is injective since the matrices

$$S = \begin{pmatrix} \partial_2 & \partial_1 & \partial_2^2 & 0 \\ \partial_1 & 0 & 0 & \partial_2 \end{pmatrix}^T, \quad F = \begin{pmatrix} 0 & 2 & 0 \\ 1 & 0 & 0 \\ 0 & 2\,\partial_2 & -\partial_1 \\ 0 & 0 & 1 \end{pmatrix},$$

satisfy the relation $S = F\,R$. Moreover, f is surjective since the matrix $(P^T \quad R'^T)^T$ admits the left inverse $(X \quad Y)$ defined by:

$$X = \begin{pmatrix} 1 & 0 \\ 0 & -\partial_1 \\ 0 & 1 \end{pmatrix} \in D^{3\times 2}, \quad Y = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \in D^{3\times 6}. \tag{3.33}$$

These results prove that f is a D-isomorphism, $M \cong M'$ and f^{-1} is defined by:

$$f^{-1}(\zeta_1) = \xi_1, \quad f^{-1}(\zeta_2) = -\partial_1\,\xi_2 = \partial_2\,\xi_1, \quad f^{-1}(\zeta_3) = \xi_2.$$

Example 3.4.2. In Example 1.6.11, without giving a proof, we stated that (1.112) defined by

$$\begin{pmatrix} \dfrac{\nu\left(\partial_y^2+\partial_z^2\right)}{1+\nu} & \dfrac{\nu\,\partial_y^2-\partial_z^2}{1+\nu} & \dfrac{-\partial_y^2+\nu\,\partial_z^2}{1+\nu} & 2\,\partial_y\,\partial_z & 0 & 0 \\ \dfrac{\nu\,\partial_x^2-\partial_z^2}{1+\nu} & \dfrac{\nu\,(\partial_x^2+\partial_z^2)}{1+\nu} & \dfrac{-\partial_x^2+\nu\,\partial_z^2}{1+\nu} & 0 & 2\,\partial_x\,\partial_z & 0 \\ \dfrac{\nu\,\partial_x^2-\partial_y^2}{1+\nu} & \dfrac{-\partial_x^2+\nu\,\partial_y^2}{1+\nu} & \dfrac{\nu\left(\partial_x^2+\partial_y^2\right)}{1+\nu} & 0 & 0 & 2\,\partial_x\,\partial_y \\ \dfrac{\partial_y\,\partial_z}{1+\nu} & -\dfrac{\partial_y\,\partial_z\,\nu}{1+\nu} & -\dfrac{\nu\,\partial_y\,\partial_z}{1+\nu} & \partial_x^2 & -\partial_x\,\partial_y & -\partial_x\,\partial_z \\ -\dfrac{\nu\,\partial_x\,\partial_z}{1+\nu} & \dfrac{\partial_x\,\partial_z}{1+\nu} & -\dfrac{\nu\,\partial_x\,\partial_z}{1+\nu} & -\partial_x\,\partial_y & \partial_y^2 & -\partial_y\,\partial_z \\ -\dfrac{\nu\,\partial_x\,\partial_y}{1+\nu} & -\dfrac{\nu\,\partial_x\,\partial_y}{1+\nu} & \dfrac{\partial_x\,\partial_y}{1+\nu} & -\partial_x\,\partial_z & -\partial_y\,\partial_z & \partial_z^2 \\ \partial_x & 0 & 0 & 0 & \partial_z & \partial_y \\ 0 & \partial_y & 0 & \partial_z & 0 & \partial_x \\ 0 & 0 & \partial_z & \partial_y & \partial_x & 0 \end{pmatrix} \begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{zx} \\ \tau_{xy} \end{pmatrix} = 0, \tag{3.34}$$

was equivalent to (1.113) defined by

$$\begin{pmatrix} \Delta+\frac{\partial_x^2}{1+\nu} & \frac{\partial_x^2}{1+\nu} & \frac{\partial_x^2}{1+\nu} & 0 & 0 & 0 \\ \frac{\partial_y^2}{1+\nu} & \Delta+\frac{\partial_y^2}{1+\nu} & \frac{\partial_y^2}{1+\nu} & 0 & 0 & 0 \\ \frac{\partial_z^2}{1+\nu} & \frac{\partial_z^2}{1+\nu} & \Delta+\frac{\partial_z^2}{1+\nu} & 0 & 0 & 0 \\ \frac{\partial_y\,\partial_z}{1+\nu} & \frac{\partial_y\,\partial_z}{1+\nu} & \frac{\partial_y\,\partial_z}{1+\nu} & \Delta & 0 & 0 \\ \frac{\partial_x\,\partial_z}{1+\nu} & \frac{\partial_x\,\partial_z}{1+\nu} & \frac{\partial_x\,\partial_z}{1+\nu} & 0 & \Delta & 0 \\ \frac{\partial_x\,\partial_y}{1+\nu} & \frac{\partial_x\,\partial_y}{1+\nu} & \frac{\partial_x\,\partial_y}{1+\nu} & 0 & 0 & \Delta \\ \partial_x & 0 & 0 & 0 & \partial_z & \partial_y \\ 0 & \partial_y & 0 & \partial_z & 0 & \partial_x \\ 0 & 0 & \partial_z & \partial_y & \partial_x & 0 \end{pmatrix} \begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{zx} \\ \tau_{xy} \end{pmatrix} = 0, \tag{3.35}$$

where $\Delta = \partial_x^2 + \partial_y^2 + \partial_z^2$ is the Laplacian operator in $\mathbb{R}^3$. Using Corollary 3.4.1, let us prove this result. Let $D = \mathbb{Q}(\nu)[\partial_x, \partial_y, \partial_z]$ be the commutative polynomial ring of PD operators with coefficients in $\mathbb{Q}(\nu)$ and $R \in D^{9\times 6}$ (resp., $R' \in D^{9\times 6}$) the presentation matrix of (3.34) (resp., (3.35)). Using the OreMorphisms package ([20]), we can prove that $R = V\,R'$, where V is the unimodular matrix defined by:

$$V = \begin{pmatrix} \frac{1+\nu}{2+\nu} & -\frac{1}{2+\nu} & -\frac{1}{2+\nu} & 0 & 0 & 0 & -\partial_x & \partial_y & \partial_z \\ -\frac{1}{2+\nu} & \frac{1+\nu}{2+\nu} & -\frac{1}{2+\nu} & 0 & 0 & 0 & \partial_x & -\partial_y & \partial_z \\ -\frac{1}{2+\nu} & -\frac{1}{2+\nu} & \frac{1}{2+\nu} & 0 & 0 & 0 & \partial_x & \partial_y & -\partial_z \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -\partial_z & -\partial_y \\ 0 & 0 & 0 & 0 & 1 & 0 & -\partial_z & 0 & -\partial_x \\ 0 & 0 & 0 & 0 & 0 & 1 & -\partial_y & -\partial_x & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \in \mathrm{GL}_9(D).$$

We have the following consequences of Corollary 3.4.1.

Corollary 3.4.2 ([100]). *Let $\mathcal{F}$ be a left D-module, $R \in D^{q\times p}$, $R' \in D^{q'\times p'}$, $M = D^{1\times p}/(D^{1\times q}\,R)$, $M' = D^{1\times p'}/(D^{1\times q'}\,R')$ and $f \in \hom_D(M, M')$ defined by two $P \in D^{p\times p'}$ and $Q \in D^{q\times q'}$ satisfying $R\,P = Q\,R'$. Then, we have:*

1. *If $\operatorname{coker} f = 0$, then the following $\mathbb{Z}$-homomorphism is injective:*

$$\begin{array}{rcl} f^\star : \ker_{\mathcal{F}}(R'.) & \longrightarrow & \ker_{\mathcal{F}}(R.) \\ \zeta & \longmapsto & P\,\zeta. \end{array}$$

2. *If* $\ker f = 0$ *and* $\mathrm{ext}^1_D(\mathrm{coker}\, f, \mathcal{F}) = 0$ *(i.e.,* $\mathcal{F}$ *is an injective left* D*-module), then the* $\mathbb{Z}$*-homomorphism* $f^\star$ *is surjective. Moreover, if* $\hom_D(\mathrm{coker}\, f, \mathcal{F}) = 0$*, then* $f^\star$ *is bijective.*
3. *If* f *is a left* D*-isomorphism, then so is* $f^\star$ *and* $f^{\star -1}$ *is defined by*

$$\begin{array}{rcl} f^{\star -1} : \ker_{\mathcal{F}}(R.) & \longrightarrow & \ker_{\mathcal{F}}(R'.) \\ \eta & \longmapsto & X\,\eta, \end{array}$$

where the matrix $(X \quad Y)$ *is a left inverse of* $(P^T \quad R'^T)^T$ *with* $X \in D^{p' \times p}$ *and* $Y \in D^{p' \times q'}$ *and we have the following commutative exact diagram of abelian groups:*

$$\begin{array}{ccccccc} \mathcal{F}^q & \xleftarrow{R.} & \mathcal{F}^p & \longleftarrow & \ker_{\mathcal{F}}(R.) & \longleftarrow & 0 \\ \downarrow -V\,F. & & \downarrow X. & & \downarrow f^{\star -1} & & \\ \mathcal{F}^{q'} & \xleftarrow{R'.} & \mathcal{F}^{p'} & \longleftarrow & \ker_{\mathcal{F}}(R'.) & \longleftarrow & 0. \end{array}$$

The next result is due to Fitting. But, we give here an explicit formulation.

Theorem 3.4.1 ([22]). *Let* $M = D^{1\times p}/(D^{1\times q}\,R)$ *and* $M' = D^{1\times p'}/(D^{1\times q'}\,R')$ *be two left* D*-modules finitely presented respectively by* $R \in D^{q\times p}$ *and* $R' \in D^{q'\times p'}$ *and* $\phi : M \longrightarrow M'$ *a left* D*-isomorphism. Moreover, let* $R_2 \in D^{r\times q}$ *(resp.,* $R_2' \in D^{r'\times q'}$*) be a matrix such that* $\ker_D(.R) = D^{1\times r}\,R_2$ *(resp.,* $\ker_D(.R') = D^{1\times r'}\,R_2'$*). Then, there exist* $P \in D^{p\times p'}$*,* $P' \in D^{p'\times p}$*,* $Q \in D^{q\times q'}$*,* $Q' \in D^{q'\times q}$*,* $Z \in D^{p\times q}$*,* $Z' \in D^{p'\times q'}$*,* $Z_2 \in D^{q\times r}$ *and* $Z_2' \in D^{q'\times r'}$ *such that:*

$$\left\{\begin{array}{l} R\,P = Q\,R', \\ R'\,P' = Q'\,R, \end{array}\right. \quad \left\{\begin{array}{l} P\,P' + Z\,R = I_p, \\ P'\,P + Z'\,R' = I_{p'}, \end{array}\right. \quad \left\{\begin{array}{l} Q\,Q' + R\,Z + Z_2\,R_2 = I_q, \\ Q'\,Q + R'\,Z' + Z_2'\,R_2' = I_{q'}. \end{array}\right.$$

1. *The following two matrices*

$$X = \begin{pmatrix} I_p & P \\ -P' & I_{p'} - P'\,P \end{pmatrix}, \quad Y = \begin{pmatrix} I_q & 0 & R & Q \\ 0 & I_{p'} & -P' & Z' \\ -Z & P & 0 & P\,Z' - Z\,Q \\ -Q' & -R' & 0 & Z_2'\,R_2' \end{pmatrix},$$

are unimodular, i.e., $X \in \mathrm{GL}_{p+p'}(D)$ *and* $Y \in \mathrm{GL}_{q+p'+p+q'}(D)$*, and:*

$$X^{-1} = \begin{pmatrix} I_p - P\,P' & -P \\ P' & I_{p'} \end{pmatrix}, \quad Y^{-1} = \begin{pmatrix} Z_2\,R_2 & 0 & -R & -Q \\ P'\,Z - Z'\,Q' & 0 & P' & -Z' \\ Z & -P & I_p & 0 \\ Q' & R' & 0 & I_{q'} \end{pmatrix}.$$

2. *The following commutative diagram of left* D*-modules holds*

$$\begin{array}{ccccccc} 0 & & 0 & & 0 & & \\ \downarrow & & \downarrow & & \downarrow & & \\ D^{1\times(q+p'+p+q')} & \xrightarrow{.L} & D^{1\times(p+p')} & \xrightarrow{\pi\oplus 0} & M & \longrightarrow & 0 \\ \downarrow .Y & & \downarrow .X & & \downarrow \phi & & \\ D^{1\times(q+p'+p+q')} & \xrightarrow{.L'} & D^{1\times(p+p')} & \xrightarrow{0\oplus\pi'} & M' & \longrightarrow & 0, \\ \downarrow & & \downarrow & & \downarrow & & \\ 0 & & 0 & & 0 & & \end{array} \tag{3.36}$$

where $\pi \oplus 0$ and $0 \oplus \pi'$ are respectively defined by

$$\begin{array}{ccc} D^{1\times(p+p')} & \xrightarrow{\pi \oplus 0} & M \\ (\lambda \quad \lambda') & \longmapsto & \pi(\lambda), \end{array} \qquad \begin{array}{ccc} D^{1\times(p'+p)} & \xrightarrow{0 \oplus \pi'} & M' \\ (\lambda \quad \lambda') & \longmapsto & \pi'(\lambda'), \end{array}$$

and with the following notations:

$$L = \begin{pmatrix} R & 0 \\ 0 & I_{p'} \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \in D^{(q+p'+p+q')\times(p+p')}, \quad L' = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ I_p & 0 \\ 0 & R' \end{pmatrix} \in D^{(q+p'+p+q')\times(p+p')}.$$

Hence, we have $L\,X = Y\,L'$, i.e., $L' = Y^{-1}\,L\,X$ or equivalently $L = Y\,L'\,X^{-1}$.

Example 3.4.3. Let us consider again Example 3.4.1. With the notations of Theorem 3.4.1, the matrices $Y \in \mathrm{GL}_{14}(D)$ and $X \in \mathrm{GL}_5(D)$ are defined by

$$Y = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & \partial_1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & \frac{1}{2}\,\partial_2 & \frac{1}{2}\,\partial_1 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & \partial_2 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & \partial_1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & -\partial_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -2 & 0 & -\partial_2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\partial_2 & 2\,\partial_1 & 0 & 0 & -\partial_1 & 0 & 0 & 0 & -\partial_2 & \partial_1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & -\partial_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & -\partial_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \partial_1 & 0 & -\partial_2 & 0 & 0 & 0 & 0 & 0 & 0 & -\partial_2 & \partial_1 & 1
\end{pmatrix},$$

$$Y^{-1} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -\partial_1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2}\,\partial_2 & -\frac{1}{2}\,\partial_1 & 0 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\partial_2 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\partial_1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & \partial_1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & \partial_2 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \partial_2 & -2\,\partial_1 & 0 & 0 & \partial_1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \partial_1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & \partial_2 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -\partial_1 & 0 & \partial_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$X = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & \partial_1 & 0 & 1 & \partial_1 \\ 0 & -1 & 0 & 0 & 0 \end{pmatrix}, \quad X^{-1} = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & -\partial_1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix}.$$

Then, the matrices $L = (\operatorname{diag}(R, I_3)^T \quad 0^T)^T \in D^{14\times 5}$ and $L' = (0^T \quad \operatorname{diag}(I_2, R')^T) \in D^{14\times 5}$ are equivalent, namely, we have:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \partial_1 & 0 & 0 \\ 0 & 0 & \partial_2 & -1 & 0 \\ 0 & 0 & 0 & \partial_1 & 0 \\ 0 & 0 & 0 & 1 & \partial_1 \\ 0 & 0 & 0 & 0 & \partial_2 \\ 0 & 0 & 0 & \partial_2 & 0 \end{pmatrix} = Y^{-1} \begin{pmatrix} \partial_1 & 0 & 0 & 0 & 0 \\ \frac{1}{2}\,\partial_2 & \frac{1}{2}\,\partial_1 & 0 & 0 & 0 \\ 0 & \partial_2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} X.$$

Finally, let us show how to use Theorem 3.4.1 to prove the result stated in Remark 1.3.1 on the Auslander transposes. Let $M = D^{1\times p}/(D^{1\times q}\,R)$ and $M' = D^{1\times p'}/(D^{1\times q'}\,R')$ be two left D-modules finitely presented respectively by $R \in D^{q\times p}$ and $R' \in D^{q'\times p'}$ and $\phi : M \longrightarrow M'$ a left D-isomorphism. Moreover, let $N = D^q/(R\, D^p)$ (resp., $N' = D^{q'}/(R'\, D^{p'})$) be the Auslander transpose right D-module of M (resp., M') and $\kappa : D^q \longrightarrow N$ (resp., $\kappa' : D^{q'} \longrightarrow N'$) the canonical projection onto N (resp., N'). With the notations of Theorem 3.4.1, we get:

$$\begin{aligned}
\operatorname{coker}_D(L.) &= D^{(q+p'+p+q')}/(L\, D^{(p+p')}) \cong D^q/(R\, D^p) \oplus D^{(p'+p+q')}/(D^{p'}) \cong N \oplus D^{(p+q')},\\
\operatorname{coker}_D(L'.) &= D^{(q+p'+p+q')}/(L'\, D^{(p+p')}) \cong D^{(q+p'+p)}/(D^p) \oplus D^{q'}/(R'\, D^{p'}) \cong D^{(q+p')} \oplus N'.
\end{aligned}$$

Now, applying the contravariant left exact functor $\hom_D(\cdot, D)$ to the commutative exact diagram (3.36), we obtain the following one:

$$\begin{tikzcd}
 & & 0 & 0 & 0 & \\
0 & N \oplus D^{(p+q')} \arrow[l] & D^{(q+p'+p+q')} \arrow[l, "\kappa \oplus \mathrm{id}_{(p+q')}"'] \arrow[u] & D^{(p+p')} \arrow[l, "L."'] \arrow[u] & \hom_D(M, D) \arrow[l] \arrow[u] & 0 \arrow[l] \\
0 & D^{(q+p')} \oplus N' \arrow[l] & D^{(q+p'+p+q')} \arrow[l, "\mathrm{id}_{(q+p')} \oplus \kappa'"'] \arrow[u, "Y."] & D^{(p+p')} \arrow[l, "L'."'] \arrow[u, "X."] \arrow[d] & \hom_D(M', D) \arrow[l] \arrow[u, "\phi^\star"] & 0. \arrow[l] \\
 & & 0 \arrow[u] & 0 & 0 \arrow[u] &
\end{tikzcd} \quad (3.37)$$

Since $Y \in \mathrm{GL}_{(q+p'+p+q')}(D)$, (3.37) induces the following right D-isomorphism

$$\begin{aligned}
\gamma : D^{(q+p')} \oplus N' &\longrightarrow N \oplus D^{(p+q')}\\
(\mathrm{id}_{q+p'} \oplus \kappa')(\lambda') &\longmapsto (\kappa \oplus \mathrm{id}_{p+q'})(Y\, \lambda'),
\end{aligned} \quad (3.38)$$

which proves that $N \oplus D^{(p+q')} \cong N' \oplus D^{(q+p')}$. We have just explicitly proved a result first due to Auslander (see, e.g., [2]) which plays an important role in Chapter 1 (see Remark 1.3.1).

Theorem 3.4.2 ([2, 22, 91]). *Let us consider two finite presentations of a left D-module M:*

$$D^{1\times q} \xrightarrow{.R} D^{1\times p} \xrightarrow{\pi} M \longrightarrow 0, \quad D^{1\times q'} \xrightarrow{.R'} D^{1\times p'} \xrightarrow{\pi'} M \longrightarrow 0.$$

If we denote by $N = D^p/(R\, D^q)$ and $N' = D^{q'}/(R'\, D^{p'})$ the Auslander transposes, then we have the right D-isomorphism γ defined by (3.38), i.e., $N \oplus D^{(p+q')} \cong N' \oplus D^{(q+p')}$, which proves that N and N' are two projectively equivalent right D-modules.

Example 3.4.4. Let us consider again Example 3.4.1. Using Theorem 3.4.2, the Auslander transposes $N = D^3/(R\, D^2) = D^{1\times 3}/(D^{1\times 2}\, R^T)$ of the D-module $M = D^{1\times 2}/(D^{1\times 3}\, R)$ and $N' = D^6/(R'\, D^3) = D^{1\times 6}/(D^{1\times 3}\, R'^T)$ of the D-module $M' = D^{1\times 3}/(D^{1\times 6}\, R')$ satisfy:

$$N \oplus D^8 \cong N' \oplus D^6.$$

In particular, the above D-isomorphism is defined by (3.38), where the matrix $Y \in \mathrm{GL}_{14}(D)$ is defined in Example 3.4.3. The D-module N corresponds to the following linear PD system

$$R_1^T \begin{pmatrix} \sigma^{11} \\ 2\,\sigma^{12} \\ \sigma^{22} \end{pmatrix} = 0 \quad \Leftrightarrow \quad \begin{cases} \partial_1\, \sigma^{11} + \partial_2\, \sigma^{12} = 0, \\ \partial_1\, \sigma^{12} + \partial_2\, \sigma^{22} = 0, \end{cases} \quad (3.39)$$

where $(\sigma^{11}, \sigma^{12}, \sigma^{22})$ is the *symmetric stress tensor* ([53]). Moreover, the D-module N' corresponds to the following linear PD system

$$R_1'^T \begin{pmatrix} \sigma^{11} \\ \sigma^{12} \\ \mu^1 \\ \sigma^{21} \\ \sigma^{22} \\ \mu^2 \end{pmatrix} = 0 \quad \Leftrightarrow \quad \begin{cases} \partial_1 \sigma^{11} + \partial_2 \sigma^{12} = 0, \\ \partial_1 \mu^1 + \partial_2 \mu^2 + \sigma^{21} - \sigma^{12} = 0, \\ \partial_1 \sigma^{21} + \partial_2 \sigma^{22} = 0, \end{cases} \tag{3.40}$$

where $(\sigma^{11}, \sigma^{12}, \sigma^{21}, \sigma^{22})$ is a possibly non-symmetric stress tensor and (μ^1, μ^2) a *couple-stress* ([53]). In particular, if the couple-stress vanishes, then (3.40) becomes (3.39). (3.39) corresponds to the equilibrium of the stress tensor (i.e., without couple-stress and *density of forces*) and (3.40) corresponds to the equilibrium of the stress and couple-stress tensors (i.e., without density of forces and *volume density of couple*) ([53]). This last system was discovered by E. and F. Cosserat in 1909 and it is nowadays used in the study of liquid crystals, rocks and granular media. See [83, 84] for a general variational formulation of Cosserat's equations based on the Spencer operator and *Lie pseudogroups* ([83, 84]) and extensions of Cosserat's ideas in mathematical physics (e.g., electromagnetism, general relativity).

3.5 Factorization problem

The next theorem gives a sufficient condition for the existence of a factorization of R.

Theorem 3.5.1 ([19]). *Let $M = D^{1\times p}/(D^{1\times q}\, R)$ and $M' = D^{1\times p'}/(D^{1\times q'}\, R')$ be two finitely presented left D-modules and $f \in \hom_D(M, M')$. Every element $f \in \hom_D(M, M')$ defines a factorization of the matrix $R \in D^{q\times p}$ of the form*

$$R = L\, S, \tag{3.41}$$

where $L \in D^{q\times r}$ and $S \in D^{r\times p}$ are such that $\operatorname{coim} f = D^{1\times p}/(D^{1\times r}\, S)$.

The following commutative exact diagram of left D-modules holds

$$\begin{array}{ccccccc} & & & & 0 & & \\ & & & & \downarrow & & \\ & & 0 & & \ker f & & \\ & & \downarrow & & \downarrow i & & \\ D^{1\times q} & \xrightarrow{.R} & D^{1\times p} & \xrightarrow{\pi} & M & \longrightarrow 0 & \\ \downarrow .L & & \| & & \downarrow \rho & & \\ D^{1\times r} & \xrightarrow{.S} & D^{1\times p} & \xrightarrow{\kappa} & \operatorname{coim} f & \longrightarrow 0, & \\ & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & \end{array} \tag{3.42}$$

where $\rho : M \longrightarrow \operatorname{coim} f$ is the canonical projection onto $\operatorname{coim} f = M/\ker\, f$ and ρ is defined by $\rho(\pi(\lambda)) = \kappa(\lambda)$ for all $\lambda \in D^{1\times p}$. In particular, if f is not injective, i.e., $\ker f \neq 0$, then the factorization $R = L\, S$ is non-trivial.

If $\mathcal{F}$ is a left D-module and $R = L\, S$ is a factorization, then $\ker_{\mathcal{F}}(S.) \subseteq \ker_{\mathcal{F}}(R.)$, i.e., every $\mathcal{F}$-solution of the linear system $S\, \eta = 0$ is a $\mathcal{F}$-solution of the linear system $R\, \eta = 0$.

Corollary 3.5.1 ([19]). *With the notations of Proposition 3.4.1, if* $L \in D^{q \times r}$ *(resp.,* $S_2 \in D^{r_2 \times r}$*) is a matrix such that* $R = L\,S$ *(resp.,* $\ker_D(.S) = D^{1 \times r_2}\,S_2$*), then we have:*

$$\ker f \cong D^{1\times r} / \left(D^{1 \times (q+r_2)} \begin{pmatrix} L \\ S_2 \end{pmatrix} \right).$$

Moreover, if $U = (L^T \quad S_2^T)^T \in D^{(q+r_2)\times r}$ *and* $\mathcal{F}$ *is a left* D*-module, then the following short exact sequence of abelian groups holds*

$$0 \longrightarrow \ker_{\mathcal{F}}(S.) \xrightarrow{\iota} \ker_{\mathcal{F}}(R.) \xrightarrow{\varpi} \ker_{\mathcal{F}}(U.), \tag{3.43}$$

where the $\mathbb{Z}$*-homomorphisms* ι *and* ϖ *are respectively defined by:*

$$\begin{array}{rcl} \iota : \ker_{\mathcal{F}}(S.) & \longrightarrow & \ker_{\mathcal{F}}(R.) \\ \zeta & \longmapsto & \zeta, \end{array} \qquad \begin{array}{rcl} \varpi : \ker_{\mathcal{F}}(R.) & \longrightarrow & \ker_{\mathcal{F}}(U.) \\ \eta & \longmapsto & S\,\eta. \end{array}$$

Finally, if $\mathcal{F}$ *is an injective left* D*-module, then* ϖ *is a surjective* $\mathbb{Z}$*-homomorphism and:*

$$\ker_{\mathcal{F}}(R.) / \ker_{\mathcal{F}}(S.) \cong \ker_{\mathcal{F}}(U.).$$

Example 3.5.1. Let us consider the acoustic wave for a compressible perfect gas

$$\left\{ \begin{array}{l} \rho_0\,\vec{\nabla}\,.\,\vec{v}(x,t) + \dfrac{1}{c^2}\,\dfrac{\partial p(x,t)}{\partial t} = 0, \\ \rho_0\,\dfrac{\partial \vec{v}(x,t)}{\partial t} + \vec{\nabla}\,p(x,t) = 0, \end{array} \right. \tag{3.44}$$

where $x = (x_1,\, x_2,\, x_3)$, $\vec{v} = (v_1 \quad v_2 \quad v_3)^T$ (resp., p) is the perturbations of the speed (resp., pressure), ρ_0 the average density of gas and c the speed of sound ([52]). Let $D = \mathbb{Q}(\rho_0, c)[\partial_t, \partial_1, \partial_2, \partial_3]$ be the commutative polynomial ring of PD operators with coefficients in $\mathbb{Q}(\rho_0, c)$,

$$R = \begin{pmatrix} \rho_0\,\partial_1 & \rho_0\,\partial_2 & \rho_0\,\partial_3 & \dfrac{\partial_t}{c^2} \\ \rho_0\,\partial_t & 0 & 0 & \partial_1 \\ 0 & \rho_0\,\partial_t & 0 & \partial_2 \\ 0 & 0 & \rho_0\,\partial_t & \partial_3 \end{pmatrix} \in D^{4\times 4},$$

and the finitely generated D-module $M = D^{1\times 4}/(D^{1\times 4}\,R)$ associated with (3.44). Computing the set of generators of the D-module $\mathrm{end}_D(M)$ and their D-linear relations by means of Algorithm 3.2.1, we obtain that a D-endomorphism f of M is defined by the following matrices:

$$P = \begin{pmatrix} 0 & \partial_3 & -\partial_2 & 0 \\ -\partial_3 & 0 & \partial_1 & 0 \\ \partial_2 & -\partial_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \partial_3 & -\partial_2 \\ 0 & -\partial_3 & 0 & \partial_1 \\ 0 & \partial_2 & -\partial_1 & 0 \end{pmatrix}.$$

Using Algorithm 1.2.1, we can compute $\ker_D(.(P^T \quad R^T)^T)$ and we obtain a presentation matrix S of $\mathrm{coim}\, f$ and the factorization $R = L\,S$ defined by:

$$S = \begin{pmatrix} \partial_1 & \partial_2 & \partial_3 & 0 \\ \rho_0\,\partial_t & 0 & 0 & 0 \\ 0 & \rho_0\,\partial_t & 0 & 0 \\ 0 & 0 & \rho_0\,\partial_t & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad L = \begin{pmatrix} \rho_0 & 0 & 0 & 0 & \dfrac{\partial_t}{c^2} \\ 0 & 1 & 0 & 0 & \partial_1 \\ 0 & 0 & 1 & 0 & \partial_2 \\ 0 & 0 & 0 & 1 & \partial_3 \end{pmatrix}.$$

We can check that $\ker f = (D^{1\times 5}\, S)/(D^{1\times 4}\, R) \neq 0$, which shows that $R = L\, S$ is a non-trivial factorization of R and $\operatorname{coim} f = D^{1\times 4}/(D^{1\times 5}\, S)$ is a non-trivial D-submodule of M. If we consider $\mathcal{F} = C^\infty(\Omega)$, where Ω is an open convex subset of $\mathbb{R}^4$ (e.g., $\Omega = \mathbb{R}_+ \times \mathbb{R}^3$), then all $\mathcal{F}$-solutions of $S\,\eta = 0$ have the form:

$$\left\{\begin{array}{l} \vec{v}(x,t) = \vec{v}(x), \\ \vec{\nabla}\,.\,\vec{v}(x) = 0, \\ p(x,t) = 0, \end{array}\right. \quad \Leftrightarrow \quad \left\{\begin{array}{l} \vec{v}(x,t) = \vec{\nabla} \wedge \vec{\psi}(x), \\ p(x,t) = 0, \end{array}\right. \quad \vec{\psi} = (\psi_1 \quad \psi_2 \quad \psi_3)^T \in C^\infty(\Omega \cap \mathbb{R}^3).$$

Finally, we can check that this solution of $S\,\eta = 0$ is a particular solution of (3.44).

Let us introduce the concept of a *generic solution* of the linear system $\ker_{\mathcal{F}}(R.)$.

Definition 3.5.1. Let $\mathcal{F}$ be a left D-module, $M = D^{1\times p}/(D^{1\times q}\, R)$ a finitely presented left D-module and $\pi : D^{1\times p} \longrightarrow M$ the canonical projection. Then, $\eta \in \ker_{\mathcal{F}}(R.)$ is called a *generic solution* if the left D-homomorphism $\phi_\eta : M \longrightarrow \mathcal{F}$ defined by $\phi_\eta(\pi(\lambda)) = \lambda\, \eta$ is injective.

Equivalently, $\eta \in \ker_{\mathcal{F}}(R.)$ is generic if the left D-homomorphism $\phi_\eta : M \longrightarrow \mathcal{F}$ defined by $\phi(y_j) = \eta_j$ for all $j = 1, \ldots p$, is injective, where $\{y_j = \pi(f_j)\}_{j=1,\ldots,p}$ is the set of generators of M defined in Section 1.1 and $\{f_j\}_{j=1,\ldots,p}$ is the standard basis of $D^{1\times p}$. In particular, we have

$$\forall\, d_j \in D, \quad j = 1, \ldots, p, \quad \phi_\eta\left(\sum_{j=1}^{p} d_j\, y_j\right) = \sum_{i=1}^{p} d_j\, \eta_j = 0 \quad \Rightarrow \quad \sum_{j=1}^{p} d_j\, y_j = \pi\left(\sum_{j=1}^{p} d_j\, f_j\right) = 0,$$

and thus $(d_1 \; \ldots \; d_p) \in D^{1\times q}\, R$. This is equivalent to saying that the solution η does not satisfy other equations than those defined by the left D-module $D^{1\times q}\, R$.

Example 3.5.2. Let $M = D^{1\times p}/(D^{1\times q}\, R)$ be a non-trivial finitely presented left D-module and $\{y_j\}_{j=1,\ldots,p}$ a family of generators of M, where $\pi : D^{1\times p} \longrightarrow M$ is the canonical projection onto M and $\{f_j\}_{j=1,\ldots,p}$ the standard basis of $D^{1\times p}$. As explained at the beginning of Section 1.1, $y = (y_1 \; \ldots \; y_p) \in M^p$ satisfies $R\, y = 0$ and y corresponds to $\phi_y = \mathrm{id}_M \in \mathrm{end}_D(M)$ by the isomorphism $\chi : \ker_M(R.) \longrightarrow \mathrm{end}_D(M)$ defined in Theorem 1.1.1, which shows that y is a generic solution of the linear system $\ker_M(R.) \cong \mathrm{end}_D(M)$.

Example 3.5.3. Let us consider the commutative polynomial ring $D = \mathbb{Q}[\partial]$ of OD operators, the matrix $R = (\partial^2 \quad -\partial) \in D^{1\times 2}$, the D-module $M = D^{1\times 2}/(D\, R)$ and the D-module $\mathcal{F} = \mathcal{D}(\mathbb{R})$ of compactly supported smooth functions on $\mathbb{R}$. If $\eta = (\eta_1 \quad \eta_2)^T \in \ker_{\mathcal{F}}(R.)$, i.e., $\partial^2\, \eta_1 - \partial\, \eta_2 = 0$, then $\partial\,(\partial\, \eta_1 - \eta_2) = 0$, i.e., $\partial\, \eta_1 - \eta_2$ must be a constant of $\mathcal{F}$. Since the only constant of $\mathcal{F}$ is 0, we get $\partial\, \eta_1 - \eta_2 = 0$, which proves that every $\eta \in \ker_{\mathcal{F}}(R.)$ satisfies the new equation $\partial\, \eta_1 - \eta_2 = 0$, i.e., $\ker_{\mathcal{F}}(R.) = \ker_{\mathcal{F}}((\partial \quad -1).) \cong \mathcal{F}$ and shows that no solution of $\ker_{\mathcal{F}}(R.)$ is generic since $(\partial \quad -1) \notin D\, R$.

Let us study the converse of Theorem 3.5.1.

Corollary 3.5.2 ([100])**.** *If $R \in D^{q\times p}$, then the following assertions are equivalent:*

1. *There exist $L \in D^{q\times r}$ and $S \in D^{r\times q}$ such that $D^{1\times q}\, R \subsetneq D^{1\times r}\, S$ and $R = L\, S$.*
2. *There exist a finitely presented left D-module $\mathcal{F}$ and $f \in \hom_D(M, \mathcal{F})$ such that:*

$$\ker f \neq 0.$$

3. *There exists a finitely presented left D-module $\mathcal{F}$ such that the linear system $\ker_{\mathcal{F}}(R.)$ admits a non-generic solution in the sense of Definition 3.5.1.*

Example 3.5.4. In this example, we show that an operator $R \in D$ can admit a non-trivially factorization $R = L\,S$ even if $\mathrm{end}_D(M)$ is trivial (see [7, 94, 113]). Let us consider the OD operator $R = \partial^2 + t\,\partial \in D = B_1(\mathbb{Q})$. Without loss of generality, any element of $\mathrm{end}_D(M)$ can be defined by $P = a\,\partial + b$, where $a, b \in \mathbb{Q}(t)$, which satisfies $R\,P = Q\,R$ for a certain $Q \in D$. But, we first have:

$$R\,P = (\partial^2 + t\,\partial)\,(a\,\partial + b) = a\,\partial^3 + (2\,\dot{a} + t\,a + b)\,\partial^2 + (\ddot{a} + t\,(\dot{a} + b) + 2\,\dot{b})\,\partial + \ddot{b} + t\,\dot{b}.$$

Hence, Q has the form $Q = a\,\partial + c$, where $c \in \mathbb{Q}(t)$, which yields

$$Q\,R = (a\,\partial + c)\,(\partial^2 + t\,\partial) = a\,\partial^3 + (t\,a + c)\,\partial^2 + (a + t\,c)\,\partial,$$

and thus $R\,P = Q\,R$ is equivalent to the following linear OD system:

$$\begin{cases} 2\,\dot{a} + b - c = 0, \\ \ddot{a} + t\,(\dot{a} + b - c) + 2\,\dot{b} - a = 0, \\ \ddot{b} + t\,\dot{b} = 0. \end{cases}$$

If we denote by $d = \dot{b}$, then the last equation gives $\dot{d} + t\,d = 0$, i.e., $d = C_1\,e^{-t^2/2}$, and thus $b = C_1 \int_0^t e^{-s^2/2}\,ds + C_2$, where C_1 and C_2 are two arbitrary constants of $\mathbb{Q}$. Since $b \in \mathbb{Q}(t)$, then $C_1 = 0$, i.e., $b = C_2$. The above system then becomes:

$$\begin{cases} \ddot{a} - t\,\dot{a} - a = \frac{d}{dt}\,(\dot{a} - t\,a) = 0, \\ b = C_2, \\ c = 2\,\dot{a} + C_2. \end{cases}$$

The integration of the first equation gives $\dot{a} - t\,a = C_3$ and thus $a = (C_4 + C_3 \int_0^t e^{-s^2/2}\,ds)\,e^{t^2/2}$, where C_3 and C_4 are two arbitrary constants of $\mathbb{Q}$. Since, $a \in \mathbb{Q}(t)$, we must have $C_3 = C_4 = 0$, i.e., $a = 0$ and $b = c = C_2$. Hence, we obtain $P = Q = C_2$, i.e., any element of $\mathrm{end}_D(M)$ has the form of $f = C_2\,\mathrm{id}_M$, where C_2 is an arbitrary constant of $\mathbb{Q}$, and thus $\ker f = 0$. Efficient algorithms for computing rational solutions of linear OD systems, which do not need an explicitly computation of the whole linear OD system, can be found in [1, 6] and the references therein.

Corollary 3.5.2 asserts that R admits a non-trivial factorization iff there exists a finitely presented left D-module $\mathcal{F}$ and $f \in \hom_D(M, \mathcal{F})$ such that $\ker f \neq 0$. If we consider the finitely presented left D-module $\mathcal{F} = D/(D\,\partial) \cong \mathbb{Q}(t)$, then the OD equation $\ddot{\eta} + t\,\dot{\eta} = 0$ admits the non-generic solution $\eta = C \in \mathbb{Q}$ since $\dot{\eta} = 0$, which shows that $f \in \hom_D(M, \mathcal{F})$ defined by $f(\pi(\lambda)) = \kappa(C\,\lambda)$ for all $\lambda \in D$, where $\kappa : D \longrightarrow \mathcal{F}$ is the canonical projection onto $\mathcal{F}$, admits the kernel $\ker f = (D\,\partial)/(D\,R) \neq 0$, which yields the non-trivial factorization $R = L\,S$, where:

$$L = \partial + t, \quad S = \partial.$$

Let us now introduce the concept of a *simple module*.

Definition 3.5.2. A non-zero left D-module M is called *simple* if M has only 0 and M as left D-submodules.

Example 3.5.5. The holonomic left $D = A_2(\mathbb{Q})$-module $M = D/(D\,\partial_1 + D\,\partial_2) \cong k[x_1, x_2]$ is simple. Indeed, if L is a left D-submodule of M and $z = d\,y$ is an element of L, where $d \in D$, $y = \pi(1)$ is the generator of M and $\pi : D \longrightarrow M$ the canonical projection onto M, then we can assume without loss of generality that $d \in k[x_1, x_2]$ since y satisfies the following equations:

$$\left\{ \begin{array}{l} \partial_1\, y = 0, \\ \partial_2\, y = 0. \end{array} \right. \tag{3.45}$$

Differentiating z with respect to x_1 and x_2 a certain number of times and using (3.45), we obtain $y = d'\, z$ for a certain $d' \in D$, i.e., $y \in L$, which proves $L = M$ and M is a simple left D-module.

Using Theorem 3.5.1, we obtain that the existence of a non-trivial factorization of R of the form $R = L\,S$, i.e., $D^{1\times q}\, R \subsetneq D^{1\times r}\, S$, implies that $\ker f \neq 0$, which shows that M is not a simple left D-module. Hence, if M is a simple left D-module, then any non-zero left D-endomorphism of M is injective. Moreover, since $\operatorname{im} f$ is a non-zero left D-submodule of M and M is simple, we get $\operatorname{im} f = M$, which shows that any non-trivial $f \in \operatorname{end}_D(M)$ is an automorphism, i.e., $f \in \operatorname{aut}_D(M)$. This last result is the classical *Schur's lemma* stating that the endomorphism ring $\operatorname{end}_D(M)$ of a simple left D-module M is a division ring (see, e.g., [71]).

3.6 Reduction problem

Let us now state the second main result of this chapter on the reduction problem.

Theorem 3.6.1 ([19]). *Let $R \in D^{q\times p}$, $M = D^{1\times p}/(D^{1\times q}\, R)$ and $f \in \operatorname{end}_D(M)$ be defined by two matrices $P \in D^{p\times p}$ and $Q \in D^{q\times q}$ such that $R\,P = Q\,R$. If the left D-modules*

$$\ker_D(.P), \quad \operatorname{coim}_D(.P), \quad \ker_D(.Q), \quad \operatorname{coim}_D(.Q),$$

are free of rank m, $p-m$, l, $q-l$, then there exist four matrices $U_1 \in D^{m\times p}$, $U_2 \in D^{(p-m)\times p}$, $V_1 \in D^{l\times q}$ and $V_2 \in D^{(q-l)\times q}$ such that

$$U = (U_1^T \quad U_2^T)^T \in \operatorname{GL}_p(D), \quad V = (V_1^T \quad V_2^T)^T \in \operatorname{GL}_q(D), \tag{3.46}$$

and

$$\overline{R} = V\,R\,U^{-1} = \begin{pmatrix} V_1\,R\,W_1 & 0 \\ V_2\,R\,W_1 & V_2\,R\,W_2 \end{pmatrix} \in D^{q\times p},$$

where $U^{-1} = (W_1 \quad W_2) \in D^{p\times p}$, $W_1 \in D^{p\times m}$ and $W_2 \in D^{p\times(p-m)}$.

In particular, the full row rank matrix U_1 (resp., U_2, V_1 and V_2) defines a basis of the free left D-module $\ker_D(.P)$ (resp., $\operatorname{coim}_D(.P)$, $\ker_D(.Q)$ and $\operatorname{coim}_D(.Q)$), namely, we have

$$\left\{ \begin{array}{l} \ker_D(.P) = D^{1\times m}\, U_1, \\ \operatorname{coim}_D(.P) = \kappa(D^{1\times(p-m)}\, U_2), \\ \ker_D(.Q) = D^{1\times l}\, V_1, \\ \operatorname{coim}_D(.Q) = \rho(D^{1\times(q-l)}\, V_2), \end{array} \right.$$

where $\kappa : D^{1\times p} \longrightarrow \operatorname{coim}_D(.P)$ (resp., $\rho : D^{1\times q} \longrightarrow \operatorname{coim}_D(.Q)$) is the canonical projection onto $\operatorname{coim}_D(.P)$ (resp., $\operatorname{coim}_D(.Q)$) and satisfy (3.46). In particular, we have the following two split *exact sequences*

$$0 \longrightarrow D^{1\times m} \underset{\xleftarrow{.W_1}}{\xrightarrow{.U_1}} D^{1\times p} \underset{\xleftarrow{.U_2}}{\xrightarrow{.W_2}} D^{1\times (p-m)} \longrightarrow 0,$$

$$0 \longrightarrow D^{1\times l} \underset{\xleftarrow{.Z_1}}{\xrightarrow{.V_1}} D^{1\times q} \underset{\xleftarrow{.V_2}}{\xrightarrow{.Z_2}} D^{1\times (q-l)} \longrightarrow 0,$$

where $U^{-1} = (W_1 \quad W_2)$ *and* $V^{-1} = (Z_1 \quad Z_2)$.

Example 3.6.1. Let us consider the following four complex matrices:

$$\gamma^1 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \; \gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \; \gamma^3 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, \; \gamma^4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

The *Dirac equation* for a massless particle has the form

$$\sum_{j=1}^{4} \gamma^j \, \frac{\partial \psi(x)}{\partial x_j} = 0, \tag{3.47}$$

where $\psi = (\psi_1 \quad \psi_2 \quad \psi_3 \quad \psi_4)^T$ ([23]). Let $D = \mathbb{Q}(i)[\partial_1, \partial_2, \partial_3, \partial_4]$ be the commutative polynomial ring of PD operators ($\partial_4 = -i\,\partial_t$),

$$R = \begin{pmatrix} \partial_4 & 0 & -i\,\partial_3 & -(i\,\partial_1 + \partial_2) \\ 0 & \partial_4 & -i\,\partial_1 + \partial_2 & i\,\partial_3 \\ i\,\partial_3 & i\,\partial_1 + \partial_2 & -\partial_4 & 0 \\ i\,\partial_1 - \partial_2 & -i\,\partial_3 & 0 & -\partial_4 \end{pmatrix} \in D^{4\times 4}$$

the presentation matrix of (3.47) and the finitely presented D-module $M = D^{1\times 4}/(D^{1\times 4}\, R)$.

Using Algorithm 3.2.1, we obtain that a D-endomorphism f of M is defined by:

$$P = \frac{1}{2} \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}, \quad Q = \frac{1}{2} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}.$$

Since the entries of P and Q belong to $\mathbb{Q}$, using linear linear techniques, we can easily compute bases of the free $\mathbb{Q}$-modules $\ker_{\mathbb{Q}}(.P)$, $\operatorname{coim}_{\mathbb{Q}}(.P)$, $\ker_{\mathbb{Q}}(.Q)$ and $\operatorname{coim}_{\mathbb{Q}}(.Q)$, i.e., bases of the free D-modules $\ker_D(.P)$, $\operatorname{coim}_D(.P)$, $\ker_D(.Q)$ and $\operatorname{coim}_D(.Q)$:

$$\left\{ \begin{array}{l} U_1 = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}, \\ U_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \end{array} \right. \qquad \left\{ \begin{array}{l} V_1 = \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}, \\ V_2 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \end{array} \right.$$

Forming the unimodular matrices $U = (U_1^T \quad U_2^T)^T \in \mathrm{GL}_4(D)$ and $V = (V_1^T \quad V_2^T)^T \in \mathrm{GL}_4(D)$, we then obtain that the matrix R is equivalent to the following block-triangular one:

$$\overline{R} = V\, R\, U^{-1} = \begin{pmatrix} -\partial_4 + i\,\partial_3 & i\,\partial_1 + \partial_2 & 0 & 0 \\ i\,\partial_1 - \partial_2 & -\partial_4 - i\,\partial_3 & 0 & 0 \\ -i\,\partial_3 & -i\,\partial_1 - \partial_2 & \partial_4 + i\,\partial_3 & i\,\partial_1 + \partial_2 \\ -i\,\partial_1 + \partial_2 & i\,\partial_3 & i\,\partial_1 - \partial_2 & \partial_4 - i\,\partial_3 \end{pmatrix}.$$

Example 3.6.2. Let us consider the linear PD system defined by

$$\sigma\, \partial_t\, \vec{A} + \frac{1}{\mu}\, \vec{\nabla} \wedge \vec{\nabla}\, \vec{A} - \sigma\, \vec{\nabla}\, V = 0, \tag{3.48}$$

where $(\vec{A}, V)$ is the *electromagnetic quadri-potential*, σ the *electric conductivity* and μ the *magnetic permeability*. This system corresponds to the equations satisfied by $(\vec{A}, V)$ when it is assumed that the term $\partial_t\, \vec{D}$ can be neglected in the Maxwell equations, i.e., the electric displacement $\vec{D}$ is constant in time. For more details, see [28]. It seems that Maxwell was led to introduce the term $\partial_t\, \vec{D}$ in his famous equations for pure mathematical reasons ([28]).

Let $D = \mathbb{Q}[\partial_t, \partial_1, \partial_2, \partial_3]$ be the commutative polynomial ring of PD operators,

$$R = \frac{1}{\mu} \begin{pmatrix} \sigma\,\mu\,\partial_t - (\partial_2^2 + \partial_3^2) & \partial_1\,\partial_2 & \partial_1\,\partial_3 & -\sigma\,\mu\,\partial_1 \\ \partial_1\,\partial_2 & \sigma\,\mu\,\partial_t - (\partial_1^2 + \partial_3^2) & \partial_2\,\partial_3 & -\sigma\,\mu\,\partial_2 \\ \partial_1\,\partial_3 & \partial_2\,\partial_3 & \sigma\,\mu\,\partial_t - (\partial_1^2 + \partial_2^2) & -\sigma\,\mu\,\partial_3 \end{pmatrix}$$

the presentation matrix of (3.48) and the finitely presented D-module $M = D^{1\times 4}/(D^{1\times 3}\, R)$.

The matrices P and Q defined by

$$P = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \sigma\,\mu\,\partial_t & 0 & -\sigma\,\mu\,\partial_2 \\ 0 & 0 & \sigma\,\mu\,\partial_t & -\sigma\,\mu\,\partial_3 \\ 0 & \partial_t\,\partial_2 & \partial_t\,\partial_3 & -(\partial_2^2 + \partial_3^2) \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & 0 & 0 \\ -\partial_1\,\partial_2 & \sigma\,\mu\,\partial_t - \partial_2^2 & -\partial_2\,\partial_3 \\ -\partial_1\,\partial_3 & -\partial_2\,\partial_3 & \sigma\,\mu\,\partial_t - \partial_3^2 \end{pmatrix},$$

satisfy the relation $R\,P = Q\,R$, and thus, define a D-endomorphism f of M. Using Theorem 1.3.1 and Quillen-Suslin theorem (see 2 of Theorem 1.1.2), we can check that $\ker_D(.P)$, $\mathrm{coim}_D(.P)$, $\ker_D(.Q)$ and $\mathrm{coim}_D(.Q)$ are free D-modules of rank 2, 2, 1 and 2. Computing bases of these free D-modules by means of a constructive version of the Quillen-Suslin theorem implemented in the QuillenSuslin package (see Section 1.5), we obtain the following matrices:

$$\left\{ \begin{array}{l} U_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \partial_2 & \partial_3 & -\sigma\,\mu \end{pmatrix}, \\ U_2 = \dfrac{1}{\sigma\,\mu} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \end{array} \right. \qquad \left\{ \begin{array}{l} V_1 = (1 \quad 0 \quad 0), \\ V_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \end{array} \right.$$

Defining $U = (U_1^T \quad U_2^T)^T \in \mathrm{GL}_4(D)$ and $V = (V_1^T \quad V_2^T)^T \in \mathrm{GL}_3(D)$, we get that $\overline{R} = V\,R\,U^{-1}$ is the following block-triangular matrix:

$$\overline{R} = \frac{1}{\mu} \begin{pmatrix} \sigma\,\mu\,\partial_t - (\partial_2^2 + \partial_3^2) & \partial_1 & 0 & 0 \\ \partial_1\,\partial_2 & \partial_2 & \sigma\,\mu\,(\sigma\,\mu\,\partial_t - (\partial_1^2 + \partial_2^2 + \partial_3^2)) & 0 \\ \partial_1\,\partial_3 & \partial_3 & 0 & \sigma\,\mu\,(\sigma\,\mu\,\partial_t - (\partial_1^2 + \partial_2^2 + \partial_3^2)) \end{pmatrix}.$$

3.7 Decomposition of finitely presented left D-modules

Let us introduce a few more definitions which will play important roles in this section.

Definition 3.7.1. 1. An element a of a ring A satisfying $a^2 = a$ is called an *idempotent*.

2. A non-zero left D-module M is said to be *decomposable* if it can be written as a direct sum of two proper left D-submodules M_1 and M_2 of M, i.e., $M = M_1 \oplus M_2$. A left D-module M which is not decomposable, i.e., which is not the direct sum of two proper left D-submodules, is *indecomposable*.

In linear algebra, projectors, i.e., idempotent endomorphisms, play an important role for decomposing vector spaces into direct sums. Idempotents of the endomorphism ring $\mathrm{end}_D(M)$ of a finitely presented left D-module M will play the same role. Hence, we first need to characterize idempotents of $\mathrm{end}_D(M)$.

Lemma 3.7.1 ([19]). *Let $R \in D^{q\times p}$, $M = D^{1\times p}/(D^{1\times q}\, R)$ and $f \in \mathrm{end}_D(M)$ be defined by two matrices $P \in D^{p\times p}$ and $Q \in D^{q\times q}$ satisfying $R\,P = Q\,R$. Then, f is an idempotent of the ring $\mathrm{end}_D(M)$, namely $f^2 = f$, iff there exists a matrix $Z \in D^{p\times q}$ such that:*

$$P^2 = P + Z\,R. \tag{3.49}$$

Moreover, if we denote by $R_2 \in D^{q_2\times q}$ a matrix satisfying $\ker_D(.R) = D^{1\times q_2}\, R_2$, then there exists a matrix $Z' \in D^{q\times q_2}$ such that $Q^2 = Q + R\,Z + Z'\,R_2$. In particular, if R has full row rank, i.e., $\ker_D(.R) = 0$, then we have $Q^2 = Q + R\,Z$.

Let us explain how to compute idempotents of the ring $\mathrm{end}_D(M)$.

Algorithm 3.7.1.
- **Input:** A matrix $R \in D^{q\times p}$ and the output of Algorithm 3.2.2 for $R' = R$ and for fixed positive integers α, β and γ.
- **Output:** A finite family $\{f_j\}_{j\in J}$ of idempotents of the endomorphism ring $\mathrm{end}_D(M)$ of $M = D^{1\times p}/(D^{1\times q}\, R)$ defined by matrices $P_j \in E^{\alpha}_{\beta,\gamma}$, i.e., $P_j^2 = P_j + Z_j\,R$ for certain matrices $Z_j \in D^{p\times q}$, $R\,P_j \in D^{q\times p}\,R$ and $f_j(\pi(\lambda)) = \pi(\lambda\,P_j)$ for all $\lambda \in D^{1\times p}$, where $\pi : D^{1\times p} \longrightarrow M$ is the canonical projection onto M.

1. Consider a generic element $L = \sum_{i\in I} c_i\,L_i$ of the output of Algorithm 3.2.2 for fixed α, β and γ, where c_i are new independent variables for all $i \in I$.
2. Compute $L^2 - L$ and denote the result by F.
3. Compute a Gröbner basis G of the left D-module $D^{1\times q}\, R$.
4. Compute the normal forms of the rows of F with respect to G.
5. Solve the system in the coefficients c_i's so that all the previous normal forms vanish.
6. Substitute the solutions into the matrix L and denote the set of solutions by $\{L_j\}_{j\in J}$.
7. For $j \in J$, form the matrix P_j obtained by computing the normal forms of the rows of L_j with respect to G.

Example 3.7.1. Let us consider $D = A_1(\mathbb{Q})$, $R = (\partial^2 \quad -t\,\partial - 1)$ and $M = D^{1\times 2}/(D\,R)$. Searching for idempotents of $\mathrm{end}_D(M)$ defined by matrices P and Q of total order 1 and total degree 2, Algorithm 3.7.1 gives $P_1 = Q_1 = 0$, $P_2 = Q_2 = I_2$ and

$$\left\{\begin{array}{l} P_3 = \begin{pmatrix} -(t+a)\,\partial + 1 & t^2 + a\,t \\ 0 & 1 \end{pmatrix}, \\ Q_3 = -((t+a)\,\partial + 1), \end{array}\right. \quad \left\{\begin{array}{l} P_4 = \begin{pmatrix} (t-a)\,\partial & -t^2 + a\,t \\ 0 & 0 \end{pmatrix}, \\ Q_4 = (t-a)\,\partial + 2, \end{array}\right. \tag{3.50}$$

where a is an arbitrary constant of $\mathbb{Q}$. We can check that $P_i^2 = P_i + Z_i\,R$ for $i = 3,\ 4$, where:

$$Z_3 = ((t+a)^2 \quad 0)^T, \quad Z_4 = ((t-a)^2 \quad 0)^T.$$

Lemma 3.7.2 ([19]). *Let $R \in D^{q \times p}$, $M = D^{1 \times p}/(D^{1 \times q}\, R)$ and $f \in \mathrm{end}_D(M)$ be an idempotent. Then, we have the following left D-isomorphism:*

$$M \cong \ker f \oplus \mathrm{coim}\, f.$$

More precisely, the following split exact sequence of left D-modules holds

$$0 \longrightarrow \ker f \;\underset{\xleftarrow{\mathrm{id}_M - f}}{\xrightarrow{i}}\; M \;\underset{\xleftarrow{f^\sharp}}{\xrightarrow{\rho}}\; \mathrm{coim}\, f \longrightarrow 0,$$

where $f^\sharp : \mathrm{coim}\, f \longrightarrow M$ is defined by $f^\sharp(\rho(m)) = f(m)$ for all $m \in M$.

According to Lemma 3.7.2, we obtain that the existence of a non-trivial idempotent f of $\mathrm{end}_D(M)$ yields $M \cong \ker f \oplus \mathrm{coim}\, f$, i.e., M is a decomposable left D-module. Conversely, if there exist two left D-modules M_1 and M_2 such that M is isomorphic to $M_1 \oplus M_2$ and if $\phi : M \longrightarrow M_1 \oplus M_2$ is an isomorphism and $p_1 : M_1 \oplus M_2 \longrightarrow M_1 \oplus 0$ is the canonical projection (i.e., $p_1^2 = p_1$), then $p = \phi^{-1} \circ p_1 \circ \phi$ is an idempotent of $\mathrm{end}_D(M)$.

We obtain the following well-known corollary of Lemma 3.7.2.

Corollary 3.7.1 ([71, 54]). *M is decomposable iff $\mathrm{end}_D(M)$ admits a non-trivial idempotent.*

Example 3.7.2. In Example 3.2.1, we proved that the endomorphism ring of D/I, where D was a commutative ring and I an ideal of D, satisfied $\mathrm{end}_D(D/I) \cong D/I$. Hence, the D-module D/I is decomposable iff the commutative ring D/I admits non-trivial idempotents. For instance, if we consider the commutative polynomial ring $D = \mathbb{Q}[\partial_t, \partial_x]$ of PD operators with rational constant coefficients and $I = (\partial_t - \partial_x, \partial_t - \partial_x^2)$ the ideal of D formed by the transport and the heat operators, then $\partial_t^2 - \partial_t = (\partial_t + \partial_x)\,(\partial_t - \partial_x) - (\partial_t - \partial_x^2) \in I$, a fact showing that the residue class $\pi(\partial_t)$ of ∂_t in D/I is a non-trivial idempotent of D/I, i.e., $\pi(\partial_t)^2 = \pi(\partial_t)$. Hence, the D-module D/I is decomposable. Now, if I is a prime ideal of D, then D/I is an integral domain, a fact showing that $\mathrm{end}_D(D/I) \cong D/I$ only admits the trivial idempotents 0 and $\mathrm{id}_{D/I}$. Then, Corollary 3.7.1 proves that D/I is indecomposable. For instance, if we consider $D = \mathbb{Q}[\partial_t, \partial_x]$ and the principal ideal of D generated by the heat operator $I = (\partial_t - \partial_x^2)$, then $D/I \cong \mathbb{Q}[\partial_x]$ is an integral domain, which proves that the D-module D/I is indecomposable.

The next proposition gives another characterization of an idempotent of the ring $\mathrm{end}_D(M)$.

Proposition 3.7.1 ([19]). *Let $R \in D^{q \times p}$, $M = D^{1 \times p}/(D^{1 \times q}\, R)$ and $f \in \mathrm{end}_D(M)$ be defined by two matrices $P \in D^{p \times p}$ and $Q \in D^{q \times q}$ such that $R\,P = Q\,R$. Then, f is an idempotent of $\mathrm{end}_D(M)$ iff there exists $X \in D^{p \times r}$ such that*

$$P = I_p - X\,S, \tag{3.51}$$

where $S \in D^{r \times p}$ is the matrix defined in 1 of Proposition 3.4.1, i.e., $\mathrm{coim}\, f = D^{1 \times p}/(D^{1 \times r}\, S)$. Then, there exist two matrices $X \in D^{p \times r}$ and $X_2 \in D^{r \times r_2}$ such that the following identity holds

$$S\,X + X_2\,S_2 = I_r - T\,L, \tag{3.52}$$

where $S_2 \in D^{r_2 \times r}$ (resp., $T \in D^{r \times q}$) is such that $\ker_D(.S) = D^{1 \times r_2}\, S_2$ (resp., (3.26) holds).

Remark 3.7.1. If S has full row rank, i.e., $\ker_D(.S) = 0$, then (3.52) becomes:

$$S\,X + T\,L = I_r. \tag{3.53}$$

Then, the factorization $R = L\,S$ satisfies (3.53), which is nothing else than the generalization for matrices and noncommutative rings of the classical decomposition of a commutative polynomial into coprime factors. Indeed, if $R \in D = k[x_1, \ldots, x_n]$, where k is a field, then (3.53) becomes $X\,S + T\,L = 1$ (Bézout identity), i.e., the ideal of D generated by S and L is equal to D, and shows that $R = L\,S$ is a factorization of the polynomial R into the coprime factors L and S.

The knowledge of idempotents of $\mathrm{end}_D(M)$ allows us to decompose the system $R\,y = 0$ into two uncoupled systems $T_1\,y_1 = 0$ and $T_2\,y_2 = 0$, where T_1 and T_2 are two matrices with entries in D. Consequently, as it is shown in the next theorem, the integration of the system $R\,y = 0$ is then equivalent to the integration of the two independent systems $T_1\,y_1 = 0$ and $T_2\,y_2 = 0$.

Theorem 3.7.1. *Let $R \in D^{q\times p}$, $M = D^{1\times p}/(D^{1\times q}\,R)$, $f \in \mathrm{end}_D(M)$ be a non-trivial idempotent and $\mathcal{F}$ a left D-module. Moreover, let $S \in D^{r\times p}$, $L \in D^{q\times r}$, $X \in D^{p\times r}$ and $S_2 \in D^{r_2\times r}$ be four matrices such that:*

$$\begin{cases} \mathrm{coim}\, f = D^{1\times p}/(D^{1\times r}\,S), \\ R = L\,S, \\ I_p - P = X\,S, \\ \ker_D(.S) = D^{1\times r_2}\,S_2. \end{cases}$$

Then, every element of the form $\eta = \zeta + X\,\tau$, where $\zeta \in \ker_{\mathcal{F}}(S.)$ and $\tau \in \mathcal{F}^r$ satisfies

$$\begin{cases} L\,\tau = 0, \\ S_2\,\tau = 0, \end{cases} \tag{3.54}$$

belongs to $\ker_{\mathcal{F}}(R.)$. Conversely, every element $\eta \in \ker_{\mathcal{F}}(R.)$ has the form $\eta = \zeta + X\,\tau$ for a certain $\zeta \in \ker_{\mathcal{F}}(S.)$ and a certain $\tau \in \ker_{\mathcal{F}}((L^T \quad S_2^T)^T.)$. In other words, we have:

$$\ker_{\mathcal{F}}(R.) = \ker_{\mathcal{F}}(S.) \oplus X\ \ker_{\mathcal{F}}((L^T \quad S_2^T)^T.).$$

Example 3.7.3. Let us consider the commutative polynomial ring $D = \mathbb{Q}[\partial_t, \partial_x]$ of PD operators with rational constant coefficients and $I = (\partial_t - \partial_x, \partial_t - \partial_x^2)$ the ideal of D formed by the transport and the heat operators. In Example 3.7.2, we proved that $\pi(\partial_x)$ defined a non-trivial idempotent of D/I, where $\pi : D \longrightarrow D/I$ is the canonical projection onto D/I. Hence, the D-endomorphism $f \in \mathrm{end}_D(D/I) \cong D/I$ defined by $f(\pi(1)) = \partial_t$ is an idempotent. Using the notations of Theorem 3.7.1, we have $R = (\partial_t - \partial_x \quad \partial_t - \partial_x^2)^T$, $P = \partial_t$, $Q = \partial_t\,I_2$,

$$S = \begin{pmatrix} \partial_x - 1 \\ \partial_t - 1 \\ 0 \end{pmatrix}, \quad L = \begin{pmatrix} -1 & 1 & 1 \\ -\partial_x - 1 & 1 & 0 \end{pmatrix}, \quad S_2 = \begin{pmatrix} \partial_t - 1 & -\partial_x + 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and $X = (-1 \quad 0 \quad 0)$. Considering the injective D-module $\mathcal{F} = C^\infty(\mathbb{R}^2)$, we can easily check that we have $\ker_{\mathcal{F}}(S.) = \{\zeta = c_1\,e^{x+t} \mid c_1 \in \mathbb{R}\}$. Finally, (3.54) is defined by

$$\begin{cases} -\tau_1 + \tau_2 + \tau_3 = 0, \\ -\partial_x\,\tau_1 - \tau_1 + \tau_2 = 0, \\ \partial_t\,\tau_1 - \tau_1 - \partial_x\,\tau_2 + \tau_2 = 0, \ \tau_3 = 0, \end{cases} \Leftrightarrow \begin{cases} \partial_x\,\tau_1 = 0, \\ \partial_t\,\tau_1 = 0, \\ \tau_2 = \tau_1, \\ \tau_3 = 0, \end{cases} \Leftrightarrow \begin{cases} \tau_1 = c_2, \\ \tau_2 = c_2, \\ \tau_3 = 0, \end{cases} \quad c_2 \in \mathbb{R},$$

which proves that $\ker_{\mathcal{F}}(R.) = \{\eta = c_1\, e^{x+t} - c_2 \mid c_1,\, c_2 \in \mathbb{R}\} = \{\eta = c_1\, e^{x+t} + c_3 \mid c_1,\, c_3 \in \mathbb{R}\}$.

Similarly, if we consider the ideal $J = (\partial_t^2 - \partial_x^2, \partial_t - \partial_x^2)$ defined by the wave and the heat operators, then $\pi(\partial_t)$ is an idempotent of the ring D/J and, using the notations of Theorem 3.7.1, we get $R = (\partial_t^2 - \partial_x^2 \quad \partial_t - \partial_x^2)^T$, $P = \partial_t$, $Q = \partial_t\, I_2$,

$$S = \begin{pmatrix} \partial_t - 1 \\ \partial_x^2 - 1 \\ 0 \end{pmatrix}, \quad L = \begin{pmatrix} \partial_t + 1 & -1 & 0 \\ 1 & -1 & 0 \end{pmatrix}, \quad S_2 = \begin{pmatrix} \partial_x^2 - 1 & -\partial_t + 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and $X = (-1 \quad 0 \quad 0)$. We can easily check that $\ker_{\mathcal{F}}(S.) = \{\zeta = c_1\, e^{t-x} + c_2\, e^{t+x} \mid c_1,\, c_2 \in \mathbb{R}\}$ and $\ker_{\mathcal{F}}((L^T \quad S_2^T)^T.) = \{\tau = (c_3\, x + c_4 \quad c_3\, x + c_4 \quad 0)^T \mid c_3,\, c_4 \in \mathbb{R}\}$, which finally proves that $\ker_{\mathcal{F}}(R.) = \{\eta = c_1\, e^{t-x} + c_2\, e^{t+x} - c_3\, x - c_4 \mid c_i \in \mathbb{R},\, i = 1, \ldots, 4\}$.

Finally, let us explain another way to obtain Theorem 3.7.1.

If $R = L\, S$, then Corollary 3.5.1 (see (3.43)) shows that $\ker_{\mathcal{F}}(S.) \subseteq \ker_{\mathcal{F}}(R.)$ for all left D-modules $\mathcal{F}$. If we introduce the new unknown $\tau = S\, \eta$, then we have $S_2\, \tau = 0$, where the matrix $S_2 \in D^{r_2 \times r}$ is such that $\ker_D(.S) = D^{1 \times r_2}\, S_2$ (see Corollary 3.5.1). Moreover, the linear system $R\, \eta = L(S\, \eta) = 0$, where $\eta \in \mathcal{F}^p$, can be integrated in cascade as follows:

$$\begin{cases} S\, \eta - \tau = 0, \\ L\, \tau = 0, \\ S_2\, \tau = 0. \end{cases}$$

This remark can easily be understood using Theorem 2.1.3 on Baer's extensions developed in Section 2.1. As explained in Theorem 3.5.1, we have the short exact sequence

$$0 \longrightarrow \ker f \xrightarrow{i} M \xrightarrow{\rho} \operatorname{coim} f \longrightarrow 0,$$

where $M = D^{1\times p}/(D^{1\times q}\, R)$, $\ker f = (D^{1\times r}\, S)/(D^{1\times q}\, R) \cong P \triangleq D^{1\times r}/(D^{1\times q}\, L + D^{1\times r_2}\, S_2)$ (see Corollary 3.5.1) and $\operatorname{coim} f = D^{1\times p}/(D^{1\times r}\, S)$. Therefore, the above short exact sequence yields the following one $0 \longrightarrow P \xrightarrow{j} M \xrightarrow{\rho} \operatorname{coim} f \longrightarrow 0$, i.e., yields an extension of P by $\operatorname{coim} f$.

Proposition 3.7.2. *Using the notations of Corollary 3.5.1, if $\mathcal{F}$ is a left D-module,*

$$A = I_r + U_1\, L + U_2\, S_2 + S\, V \in D^{r\times r},$$

where $U_1 \in D^{r\times q}$, $U_2 \in D^{r\times r_2}$ and $V \in D^{p\times r}$ are three arbitrary matrices (e.g., $U_1 = 0$, $U_2 = 0$, $V = 0$ which yields $A = I_r$) and

$$Q = \begin{pmatrix} S & -A \\ 0 & L \\ 0 & S_2 \end{pmatrix} \in D^{(r+q+r_2)\times(p+r)},$$

then the following equivalence of linear systems holds

$$R\, \eta = 0 \quad \Leftrightarrow \quad \begin{cases} S\, \zeta - A\, \tau = 0, \\ L\, \tau = 0, \\ S_2\, \tau = 0, \end{cases}$$

under the following invertible transformations:

$$\begin{array}{rcl} \phi : \ker_{\mathcal{F}}(R.) & \longrightarrow & \ker_{\mathcal{F}}(Q.) \\ \eta & \longmapsto & \left\{ \begin{array}{l} \zeta = \eta + V\,S\,\eta, \\ \tau = S\,\eta, \end{array} \right. \end{array} \qquad \begin{array}{rcl} \phi^{-1} : \ker_{\mathcal{F}}(Q.) & \longrightarrow & \ker_{\mathcal{F}}(R.) \\ \left(\begin{array}{c} \zeta \\ \tau \end{array} \right) & \longmapsto & \eta = \zeta - V\,\tau. \end{array}$$

Moreover, if there exist three matrices $U_1 \in D^{r \times q}$, $U_2 \in D^{r \times r_2}$ *and* $V \in D^{p \times r}$ *such that*

$$I_r + U_1\,L + U_2\,S_2 + S\,V = 0,$$

then $M \cong \ker f \oplus \operatorname{coim} f$ *and the linear system* $R\,\eta = 0$ *is equivalent to* $\eta = \zeta + V\,\tau$*, where:*

$$S\,\zeta = 0, \quad \left\{ \begin{array}{l} L\,\tau = 0, \\ S_2\,\tau = 0. \end{array} \right.$$

In other words, we have $\ker_{\mathcal{F}}(R.) = \ker_{\mathcal{F}}(S.) \oplus V\, \ker_{\mathcal{F}}((L^T \quad S_2^T)^T.)$.

3.8 Decomposition problem

Let us start with two simple lemmas.

Lemma 3.8.1 ([19]). *Let* $R \in D^{q \times p}$ *be a full row rank matrix, i.e.,* $\ker_D(.R) = 0$*, and* $P \in D^{p \times p}$, $Q \in D^{q \times q}$ *two matrices satisfying* $R\,P = Q\,R$*. If* P *is an idempotent of* $D^{p \times p}$*, i.e.,* $P^2 = P$*, then so is* Q*, i.e.,* $Q^2 = Q$*.*

Lemma 3.8.2 ([19]). *Let* $R \in D^{q \times p}$ *be a full row rank matrix and* $M = D^{1 \times p}/(D^{1 \times q}\,R)$*. Let* $f \in \operatorname{end}_D(M)$ *be an idempotent defined by two matrices* $P \in D^{p \times p}$ *and* $Q \in D^{q \times q}$ *satisfying the relations* $R\,P = Q\,R$, $P^2 = P + Z\,R$ *and* $Q^2 = Q + R\,Z$*. If there exists a solution* $\Lambda \in D^{p \times q}$ *of the following* algebraic Riccati equation

$$\Lambda\,R\,\Lambda + (P - I_p)\,\Lambda + \Lambda\,Q + Z = 0, \tag{3.55}$$

then the matrices defined by

$$\left\{ \begin{array}{l} \overline{P} = P + \Lambda\,R, \\ \overline{Q} = Q + R\,\Lambda, \end{array} \right. \tag{3.56}$$

satisfy the following relations:

$$R\,\overline{P} = \overline{Q}\,R, \quad \overline{P}^2 = \overline{P}, \quad \overline{Q}^2 = \overline{Q}.$$

Example 3.8.1. Let us consider again Example 3.7.1 where we proved that the matrices P_3 and P_4 defined by (3.50) were such that $P_i^2 = P_i + Z_i\,R$, for $i = 3,\,4$, where the matrices Z_1 and Z_2 are defined in Example 3.7.1. Searching for solutions of (3.55) of order 1 and degree 1, we obtain the solutions $\Lambda_3 = (a\,t \quad a\,\partial - 1)^T$ and $\Lambda_4 = (a\,t \quad a\,\partial + 1)^T$. Then, the matrices (3.56) defined by

$$\left\{ \begin{array}{l} \overline{P}_3 = \left(\begin{array}{cc} a\,t\,\partial^2 - (t + a)\,\partial + 1 & t^2\,(1 - a\,\partial) \\ (a\,\partial - 1)\,\partial^2 & -a\,t\,\partial^2 + (t - 2\,a)\,\partial + 2 \end{array} \right), \\ \overline{Q}_3 = 0, \end{array} \right.$$

$$\left\{ \begin{array}{l} \overline{P}_4 = \left(\begin{array}{cc} a\,t\,\partial^2 + (t - a)\,\partial & -t^2\,(1 + a\,\partial) \\ (a\,\partial + 1)\,\partial^2 & -a\,t\,\partial^2 - (t + 2\,a)\,\partial - 1 \end{array} \right), \\ \overline{Q}_4 = 1, \end{array} \right.$$

satisfy the relations $R_i\,\overline{P}_i = \overline{Q}_i\,R$, $\overline{P}_i^2 = \overline{P}_i$ and $\overline{Q}_i^2 = \overline{Q}_i$ for $i = 3,\,4$.

Remark 3.8.1. If $\overline{P}^2 = \overline{P}$, then Proposition 1.3.2 shows that $O = D^{1\times p}/(D^{1\times p}\,\overline{P})$ is a projective left D-module. Therefore, the short exact sequence $0 \longrightarrow D^{1\times p}\,\overline{P} \longrightarrow D^{1\times p} \longrightarrow O \longrightarrow 0$ splits by Proposition 1.2.5, i.e., $D^{1\times p} \cong D^{1\times p}\,\Pi \oplus O$, which proves that $D^{1\times p}\,\overline{P}$ is a projective left D-module. Moreover, we have $\ker_D(.\overline{P}) = \mathrm{im}_D(.(I_p - \overline{P}))$, which shows that $\ker_D(.\overline{P})$ is also a projective left D-module since the matrix $I_p - \overline{P}$ is an idempotent.

The next theorem shows that the matrix R is equivalent to a block-diagonal matrix if the ring $\mathrm{end}_D(M)$ admits an idempotent f which can be defined by two idempotent matrices P and Q such that their kernels and images are free left D-modules.

Theorem 3.8.1 ([19]). *Let $R \in D^{q\times p}$, $M = D^{1\times p}/(D^{1\times q}\,R)$ and $f \in \mathrm{end}_D(M)$ be an idempotent, i.e., $f^2 = f$, defined by two idempotents matrices $P \in D^{p\times p}$ and $Q \in D^{q\times q}$ satisfying the relations $R\,P = Q\,R$, $P^2 = P$ and $Q^2 = Q$. If the left D-modules*

$$\ker_D(.P), \quad \mathrm{im}_D(.P) = \ker_D(.(I_p - P)), \quad \ker_D(.Q), \quad \mathrm{im}_D(.Q) = \ker_D(.(I_q - Q)),$$

are free of rank m, $p - m = \mathrm{tr}(P)$, l, $q - l = \mathrm{tr}(Q)$, then there exist four matrices $U_1 \in D^{m\times p}$, $U_2 \in D^{(p-m)\times p}$, $V_1 \in D^{l\times q}$ and $V_2 \in D^{(q-l)\times q}$ such that

1. $U = (U_1^T \quad U_2^T)^T \in \mathrm{GL}_p(D)$,
2. $V = (V_1^T \quad V_2^T)^T \in \mathrm{GL}_q(D)$,
3. $\overline{R} = V\,R\,U^{-1} = \begin{pmatrix} V_1\,R\,W_1 & 0 \\ 0 & V_2\,R\,W_2 \end{pmatrix} \in D^{q\times p}$,

where $U^{-1} = (W_1 \quad W_2)$, $W_1 \in D^{p\times m}$ and $W_2 \in D^{p\times(p-m)}$.

In particular, the full row rank matrix U_1 (resp., U_2, V_1, V_2) defines a basis of the free left D-module $\ker_D(.P)$, (resp., $\mathrm{im}_D(.P)$, $\ker_D(.Q)$, $\mathrm{im}_D(.Q)$) of rank m (resp., $p-m$, l, $q-l$), i.e.:

$$\begin{cases} \ker_D(.P) = D^{1\times m}\,U_1, \\ \mathrm{im}_D(.P) = D^{1\times(p-m)}\,U_2, \\ \ker_D(.Q) = D^{1\times l}\,V_1, \\ \mathrm{im}_D(.Q) = D^{1\times(q-l)}\,V_2. \end{cases} \tag{3.57}$$

Finally, we have $\ker f \cong D^{1\times m}/(D^{1\times l}\,(V_1\,R\,W_1))$ and $\mathrm{im}\, f \cong D^{1\times(p-m)}/(D^{1\times(q-l)}\,(V_2\,R\,W_2))$, i.e., up to isomorphism, the first (resp., second) diagonal block of $\overline{R}$ corresponds to $\ker f$ (resp., $\mathrm{im}\, f$) and $M \cong \ker f \oplus \mathrm{im}\, f$.

Let us illustrate Theorem 3.8.1.

Example 3.8.2. Let us consider again the Dirac equation for a massless particle studied in Example 3.6.1. We can check that the matrices P and Q defined in Example 3.6.1 are idempotents of $D^{4\times 4}$, i.e., $P^2 = P$ and $Q^2 = Q$. Since the entries of P and Q belong to $\mathbb{Q}$, the D-modules $\ker_D(.P)$, $\mathrm{im}_D(.P)$, $\ker_D(.Q)$ and $\mathrm{im}_D(.Q)$ are free. Hence, by Theorem 3.8.1, the presentation matrix R of the Dirac equation defined in Example 3.6.1 is equivalent to a block-diagonal matrix. In order to compute this equivalent form, we have to compute a basis of the free D-modules $\mathrm{im}_D(.P)$ and $\mathrm{im}_D(.Q)$ instead of a basis of the free D-modules $\mathrm{coim}_D(.P)$ and $\mathrm{coim}_D(.Q)$ computed in Example 3.6.1 for the reduction problem. Using linear algebra techniques, we obtain $\mathrm{im}_D(.P) = D^{1\times 2}\,U_2'$ and $\mathrm{im}_D(.Q) = D^{1\times 2}\,V_2'$, where:

$$U_2' = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}, \quad V_2' = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}.$$

Hence, if we define by $U' = (U_1^T \quad U_2'^T)^T \in \mathrm{GL}_4(D)$ and $V' = (V_1^T \quad V_2'^T)^T \in \mathrm{GL}_4(D)$, where the matrices U_1 and V_1 are defined in Example 3.6.1, then we obtain:

$$\overline{\overline{R}} = V'\, R\, U'^{-1} = \begin{pmatrix} -\partial_4 + i\,\partial_3 & \partial_2 + i\,\partial_1 & 0 & 0 \\ -\partial_2 + i\,\partial_1 & -\partial_4 - i\,\partial_3 & 0 & 0 \\ 0 & 0 & \partial_4 + i\,\partial_3 & \partial_2 + i\,\partial_1 \\ 0 & 0 & -\partial_2 + i\,\partial_1 & \partial_4 - i\,\partial_3 \end{pmatrix}.$$

Finally, let us study whether or not the block-diagonal submatrices of $\overline{\overline{R}}$ can also be decomposed. Let $S \in D^{2\times 2}$ be the first block-diagonal submatrix of $\overline{\overline{R}}$ and $N = D^{1\times 2}/(D^{1\times 2}\, S)$. Using Algorithm 3.2.1, the D-modules $\mathrm{end}_D(N)$ is generated by $\{g_i\}_{i=1,2,3}$, where $g_i(\kappa(\mu)) = \kappa(\mu\, X_i)$ for all $\mu \in D^{1\times 2}$, $\kappa : D^{1\times 2} \longrightarrow N$ is the canonical projection onto N and:

$$X_1 = I_2, \quad X_2 = \begin{pmatrix} 0 & -\partial_2 - i\,\partial_1 \\ 0 & -\partial_4 + i\,\partial_3 \end{pmatrix}, \quad X_3 = \begin{pmatrix} 0 & -\partial_4 - i\,\partial_3 \\ 0 & \partial_2 - i\,\partial_1 \end{pmatrix}.$$

Moreover, the generators g_i's satisfy the following D-linear relations:

$$\left\{ \begin{array}{l} (\partial_4 - i\,\partial_3)\, g_1 + g_2 = 0, \\ (\partial_2 - i\,\partial_1)\, g_1 - g_3 = 0, \\ -(\partial_4 + i\,\partial_3)\, g_2 + (\partial_2 + i\,\partial_1)\, g_3 = 0, \\ (\partial_2 - i\,\partial_1)\, g_2 + (\partial_4 - i\,\partial_3)\, g_3 = 0. \end{array} \right.$$

The first two equations of the above system yield $g_2 = -(\partial_4 - i\,\partial_3)\, g_1$ and $g_3 = (\partial_2 - i\,\partial_1)\, g_1$, which shows that $\mathrm{end}_D(N)$ is a cyclic D-module generated by $g_1 = \mathrm{id}_N$. Hence, using Example 1.2.2, we get $\mathrm{end}_D(N) = D\, g_1 \cong D/(\mathrm{ann}_D(g_1))$, where $\mathrm{ann}_D(g_1) = \Delta = \partial_1^2 + \partial_2^2 + \partial_3^2 + \partial_4^2$. Since Δ is an irreducible polynomial, $D/(\mathrm{ann}_D(g_1))$ is an integral domain which shows that it does not admit idempotents and proves that N cannot be decomposed and S is not equivalent to a block-diagonal matrix. The same result holds for the second block-diagonal of the matrix $\overline{\overline{R}}$.

Example 3.8.3. Let us consider again Example 2.2.4, namely, the model of a tank containing a fluid and subjected to a one-dimensional horizontal move studied in [79]:

$$\left\{ \begin{array}{l} \dot{y}_1(t) - \dot{y}_2(t - 2\,h) + \alpha\, \ddot{y}_3(t - h) = 0, \\ \dot{y}_1(t - 2\,h) - \dot{y}_2(t) + \alpha\, \ddot{y}_3(t - h) = 0. \end{array} \right.$$

Let $D = \mathbb{Q}(\alpha)[\partial, \delta]$ be the commutative polynomial ring of OD time-delay operators with rational constant coefficients (i.e., $\partial\, y(t) = \dot{y}(t)$, $\delta\, y(t) = y(t - h)$),

$$R = \begin{pmatrix} \partial & -\partial\, \delta^2 & \alpha\, \partial^2\, \delta \\ \partial\, \delta^2 & -\partial & \alpha\, \partial^2\, \delta \end{pmatrix} \in D^{2\times 3},$$

the presentation matrix of (2.27) and the D-module $M = D^{1\times 3}/(D^{1\times 2}\, R)$ finitely presented by R. Using Algorithm 3.7.1, we obtain that the matrices defined by

$$P = \frac{1}{2} \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad Q = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},$$

satisfy the relations $R\, P = Q\, R$, $P^2 = P$ and $Q^2 = Q$, i.e., define an idempotent $f \in \mathrm{end}_D(M)$.

Since the entries of P and Q belong to $\mathbb{Q}$, $\ker_D(.P)$, $\mathrm{im}_D(.P)$, $\ker_D(.Q)$, $\mathrm{im}_D(.Q)$ are free D-modules. Computing basis of these $\mathbb{Q}$-vector spaces, we get:

$$U = \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{GL}_3(D), \quad V = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \in \mathrm{GL}_2(D).$$

Therefore, we obtain that the matrix R is equivalent to the following block-diagonal matrix:

$$\overline{R} = V\, R\, U^{-1} = \begin{pmatrix} \partial\,(1-\delta)\,(1+\delta) & 0 & 0 \\ 0 & \partial\,(\delta^2+1) & 2\,\alpha\,\partial^2\,\delta \end{pmatrix}.$$

Hence, we obtain $M \cong M_1 \oplus M_2$, where:

$$M_1 = D/(D\,(\partial\,(\delta^2-1))), \quad M_2 = D^{1\times 2}/(D\,(\partial\,(\delta^2+1) \quad 2\,\alpha\,\partial^2\,\delta)).$$

Let us now consider the D-module $\mathcal{F} = C^\infty(\mathbb{R})$ and the linear system $\ker_{\mathcal{F}}(R.)$. Let us characterize $\ker_{\mathcal{F}}(\overline{R}.)$, and thus, $\ker_{\mathcal{F}}(R.)$. If we denote by C_1 and C_2 two arbitrary real constants and ψ a $2\,h$-periodic of $\mathcal{F}$, then we can check that we have:

$$\overline{R} \begin{pmatrix} z_1(t) \\ z_2(t) \\ z_3(t) \end{pmatrix} = 0 \quad \Leftrightarrow \quad \left\{ \begin{array}{l} z_1(t) = \psi(t) + C_1\,t, \\ z_2(t) = -2\,\alpha\,\dot{\xi}(t-h) + C_2, \\ z_3(t) = \xi(t-2\,h) + \xi(t), \end{array} \right. \quad \forall\, \xi \in \mathcal{F}.$$

Finally, using the invertible transformation defined by the matrix U, we obtain:

$$\begin{pmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{pmatrix} = U^{-1} \begin{pmatrix} z_1(t) \\ z_2(t) \\ z_3(t) \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\,(\psi(t) + C_1\,t + C_2) - \alpha\,\dot{\xi}(t-h) \\ \frac{1}{2}\,(\psi(t) + C_1\,t - C_2) + \alpha\,\dot{\xi}(t-h) \\ \xi(t-2\,h) + \xi(t) \end{pmatrix}.$$

We find again the parametrization of $\ker_{\mathcal{F}}(R.)$ obtained in Example 2.2.4 and [79].

The choice of another idempotent of $\mathrm{end}_D(M)$ defined by the two idempotent matrices

$$P' = \begin{pmatrix} 0 & 0 & 0 \\ -\delta^2 & 1 & -\alpha\,\delta\,\partial \\ 0 & 0 & 0 \end{pmatrix}, \quad Q' = \begin{pmatrix} 0 & \delta^2 \\ 0 & 1 \end{pmatrix},$$

gives another decomposition of M. Indeed, the matrices $X \in \mathrm{GL}_3(D)$ and $Y \in \mathrm{GL}_2(D)$ obtained by stacking bases of free D-modules $\ker_D(.P')$ and $\mathrm{im}_D(.P')$ (resp., $\ker_D(.Q')$ and $\mathrm{im}_D(.Q')$),

$$X = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ \delta^2 & -1 & \alpha\,\delta\,\partial \end{pmatrix}, \quad Y = \begin{pmatrix} -1 & \delta^2 \\ 0 & 1 \end{pmatrix},$$

are such that $\overline{\overline{R}} = Y\,R\,X^{-1}$ is the following block-diagonal matrix:

$$\overline{\overline{R}} = \begin{pmatrix} \partial\,(\delta^2-1)\,(\delta^2+1) & \alpha\,\partial^2\,\delta\,(\delta^2-1) & 0 \\ 0 & 0 & \partial \end{pmatrix}.$$

Hence, we obtain $M \cong M_3 \oplus M_4$, where:

$$M_3 = D^{1\times 2}/(D\,(\partial\,(\delta^2-1)\,(\delta^2+1) \quad \alpha\,\partial^2\,\delta\,(\delta^2-1))), \quad M_4 = D/(D\,\partial).$$

Since M_1 and M_4 are torsion D-modules and $M_2/t(M_2) \neq 0$ and $M_3/t(M_3) \neq 0$, we obtain that $M_1 \not\cong M_3$ and $M_2 \not\cong M_4$. Moreover, we have $M_1 \not\cong M_4$ since $\hom_D(M_4, M_1)$ is generated by the injective but not surjective D-homomorphism $\phi(\pi_1(\lambda)) = \pi_4(\lambda\,(\delta^2-1))$ for all $\lambda \in D$, where $\pi_1 : D \longrightarrow M_1$ (resp., $\pi_4 : D \longrightarrow M_4$) is the canonical projection onto M_1 (resp., M_4). Moreover, we have $t(M_2) \cong M_4$ and $t(M_3) \cong M_1$, a fact implying that $M_2 \not\cong M_3$. Hence, the D-module M admits the two decompositions formed by pairwise non-isomorphic D-modules:

$$M \cong M_1 \oplus M_2 \cong M_3 \oplus M_4.$$

The converse of Theorem 3.8.1 is also true as it is explained in the next corollary.

Corollary 3.8.1 ([100]). *A matrix $R \in D^{q\times p}$ is equivalent to a block-diagonal matrix $\overline{R} \in D^{q\times p}$, i.e., there exist two matrices $U \in \mathrm{GL}_p(D)$ and $V \in \mathrm{GL}_q(D)$ such that*

$$\overline{R} = V\,R\,U^{-1} = \begin{pmatrix} \overline{R}_{11} & 0 \\ 0 & \overline{R}_{22} \end{pmatrix}, \quad \overline{R}_{11} \in D^{l\times m}, \quad \overline{R}_{22} \in D^{(q-l)\times(p-m)}, \tag{3.58}$$

iff there exist two idempotent matrices $P \in D^{p\times p}$ and $Q \in D^{q\times q}$, i.e., $P^2 = P$, $Q^2 = Q$, such that $R\,P = Q\,R$ and $\ker_D(.P)$, $\mathrm{im}_D(.P)$, $\ker_D(.Q)$ and $\mathrm{im}_D(.Q)$ are free left D-modules of rank respectively m, $p-m$, l and $q-l$.

According to Remark 3.8.1, the kernel and the image of an idempotent matrix are projective modules. Theorem 3.8.1 shows that the matrix R is equivalent to a block-diagonal matrix if the kernels and the images of certain idempotent matrices are free. Hence, using Theorems 1.1.2 and 1.5.4, we obtain the following result.

Theorem 3.8.2 ([19]). *Let $R \in D^{q\times p}$, $M = D^{1\times p}/(D^{1\times q}\,R)$ and $f \in \mathrm{end}_D(M)$ be an idempotent defined by two matrices $P \in D^{p\times p}$ and $Q \in D^{q\times q}$ satisfying $R\,P = Q\,R$, $P^2 = P$ and $Q^2 = Q$.*

Assume further that one of the following conditions holds:

1. *$D = A\langle\partial\rangle$ is a ring of OD operators with coefficients in a differential field A such as k, $k(t)$ and $k[\![t]\!][t^{-1}]$, where k is a field of characteristic 0, or $k\{t\}[t^{-1}]$, where $k = \mathbb{R}$ or $\mathbb{C}$,*
2. *$D = k[x_1, \ldots, x_n]$ is a commutative polynomial ring over a field k,*
3. *$D = A_n(k)$, $B_n(k)$, $k[\![t]\!][\partial]$, where k is a field of characteristic 0, or $k\{t\}[\partial]$, where $k = \mathbb{R}$ or $\mathbb{C}$, and:*
$$\begin{cases} \mathrm{rank}_D(\ker_D(.P)) \geq 2, \\ \mathrm{rank}_D(\mathrm{im}_D\,(.P)) \geq 2, \end{cases} \quad \begin{cases} \mathrm{rank}_D(\ker_D(.Q)) \geq 2, \\ \mathrm{rank}_D(\mathrm{im}_D\,(.Q)) \geq 2. \end{cases}$$

Then, there exist $U \in \mathrm{GL}_p(D)$ and $V \in \mathrm{GL}_q(D)$ such that

$$\overline{R} = V\,R\,U^{-1} = \begin{pmatrix} \overline{R}_{11} & 0 \\ 0 & \overline{R}_{22} \end{pmatrix} \in D^{q\times p},$$

where $\overline{R}_{11} \in D^{l\times m}$, $\overline{R}_{22} \in D^{(q-l)\times(p-m)}$ and:

$$m = \mathrm{rank}_D(\ker_D(.P)) = p - \mathrm{tr}(P), \quad l = \mathrm{rank}_D(\ker_D(.Q)) = q - \mathrm{tr}(Q).$$

Example 3.8.4. Let us consider again Example 2.2.6, namely, the model of a flexible rod with a torque studied in [74]:

$$\begin{cases} \dot{y}_1(t) - \dot{y}_2(t-1) - u(t) = 0, \\ 2\,\dot{y}_1(t-1) - \dot{y}_2(t) - \dot{y}_2(t-2) = 0. \end{cases} \tag{3.59}$$

Let us consider the commutative polynomial algebra $D = \mathbb{Q}[\partial, \delta]$ of OD time-delay operators (i.e., $\partial\, y(t) = \dot{y}(t)$, $\delta\, y(t) = y(t-h)$, where $h \in \mathbb{R}_+$), the corresponding presentation matrix

$$R = \begin{pmatrix} \partial & -\partial\,\delta & -1 \\ 2\,\partial\,\delta & -\partial\,(1+\delta^2) & 0 \end{pmatrix} \in D^{2\times 3},$$

and the D-module $M = D^{1\times 3}/(D^{1\times 2}\, R)$. Using Algorithm 3.7.1, we obtain that the matrices

$$P = \begin{pmatrix} 1+\delta^2 & -\frac{1}{2}\,\delta^2\,(1+\delta) & 0 \\ 2\,\delta & -\delta^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad Q = \begin{pmatrix} 1 & -\frac{1}{2}\,\delta \\ 0 & 0 \end{pmatrix},$$

are idempotents, i.e., $P^2 = P$ and $Q^2 = Q$, and define an idempotent element f of $\mathrm{end}_D(M)$. Using the implementation of the Quillen-Suslin theorem in QuillenSuslin, we obtain:

$$U = \begin{pmatrix} -2\,\delta & \delta^2+1 & 0 \\ 2\,\partial\,(1-\delta^2) & \partial\,\delta\,(\delta^2-1) & -2 \\ -1 & \dfrac{1}{2}\,\delta & 0 \end{pmatrix} \in \mathrm{GL}_3(D), \quad V = \begin{pmatrix} 0 & -1 \\ 2 & -\delta \end{pmatrix} \in \mathrm{GL}_2(D).$$

Then, the matrix R is equivalent to the following block-diagonal matrix:

$$\overline{R} = V\,R\,U^{-1} = \begin{pmatrix} \partial & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Hence, we get the following D-isomorphisms

$$M \cong D^{1\times 3}/(D^{1\times 2}\,\overline{R}) = D/\,(D\,\partial) \oplus D^{1\times 2}/(D\,(1 \quad 0)) \cong D/\,(D\,\partial) \oplus D,$$

which show that $t(M) \cong D/\,(D\,\partial)$ and $M/t(M) \cong D$. We note that M is *extended* from the ring $E = \mathbb{Q}[\partial]$, namely, $M \cong D \otimes_E L$, where $L = E^{1\times 3}/(E^{1\times 2}\,\overline{R})$ (see [109]). This result shows that the first scalar diagonal block (resp., second diagonal block) of $\overline{R}$ corresponds to the autonomous elements (resp., flat subsystem of $\ker_{\mathcal{F}}(R.)$) of $\ker_{\mathcal{F}}(R.)$, where $\mathcal{F}$ is a D-module (e.g., $C^\infty(\mathbb{R})$).

Finally, all smooth solutions of $\overline{R}\,z = 0$ are defined by $z = (c \quad 0 \quad z_3)^T$, where $c \in \mathbb{R}$ and z_3 is an arbitrary smooth function. Hence, all smooth solutions of (3.59) are parametrized by

$$\begin{pmatrix} y_1(t) \\ y_2(t) \\ u(t) \end{pmatrix} = U^{-1} \begin{pmatrix} c \\ 0 \\ z_3(t) \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\,c - z_3(t-2) - z_3(t) \\ c - 2\,z_3(t-1) \\ \dot{z}_3(t-2) - \dot{z}_3(t) \end{pmatrix},$$

where c is an arbitrary constant and z_3 an arbitrary smooth function.

For more results on the factorization, reduction and decomposition problems, see [19, 20, 100].

Chapter 4

Serre's reduction

"Comme tout être vivant, pour ne pas mourir la mathématique doit se recréer sans cesse. Ainsi la mort de la recherche mathématique serait la mort de la pensée mathématique, c'est-à-dire du langage même de la science. Car expérimenter n'est pas seulement employer nos sens et nos mains, c'est aussi schématiser la petite partie de la réalité physique que nous observons, c'est mettre en relation le monde physique et le monde abstrait que nous révèlent les mathématiques. Notre civilisation n'est pas mécanique mais scientifique : il est vital qu'elle transmette l'essentiel de sa science aux jeunes générations ; la science ne peut se stocker exclusivement dans des bibliothèques ; elle n'est pas lettre morte, elle est une pensée vivante ; il faut qu'elle vive dans nos esprits ; si elle y meure, ni nos machines, ni nous-mêmes n'y survivrions. Nous avons donc tous besoin que la jeunesse développe toutes ses capacités intellectuelles en ayant bonne conscience et foi en son avenir."

Jean Leray, *Remise du prix Feltrinelli, Roma 1971 et Congrès Pan-Africain, Rabat 1976.*

4.1 Introduction

Let $R \in D^{q\times p}$ be a full row rank matrix, i.e., $\ker_D(.R) = 0$, and $M = D^{1\times p}/(D^{1\times q}\, R)$ the left D-module finitely presented by R. Then, the following short exact sequence holds:

$$0 \longrightarrow D^{1\times q} \xrightarrow{.R} D^{1\times p} \xrightarrow{\pi} M \longrightarrow 0, \tag{4.1}$$

The purpose of this section is to study the existence of extensions of $D^{1\times(q-r)}$ by M, where $0 \le r \le q-1$, which define free left D-modules E (see Definition 2.1.1). If such an extension of $D^{1\times(q-r)}$ by M exists, then applying Proposition 1.4.1 to the following short exact sequence

$$0 \longrightarrow D^{1\times(q-r)} \xrightarrow{\alpha} E \xrightarrow{\beta} M \longrightarrow 0,$$

we get $\operatorname{rank}_D(E) = \operatorname{rank}_D(D^{1\times(q-r)}) + \operatorname{rank}_D(M) = (q-r)+(p-q) = p-r$, i.e., E is a free left D-module of rank $p-r$. Thus, if $\psi : D^{1\times(q-r)} \longrightarrow E$ is a left D-isomorphism, then we obtain the commutative exact diagram

$$\begin{array}{ccccccccc}
0 \longrightarrow & D^{1\times(q-r)} & \xrightarrow{\alpha} & E & \xrightarrow{\beta} & M & \longrightarrow 0, \\
 & \| & & \downarrow \psi & & \| & \\
0 \longrightarrow & D^{1\times(q-r)} & \xrightarrow{\psi\circ\alpha} & D^{1\times(p-r)} & \xrightarrow{\beta\circ\psi^{-1}} & M & \longrightarrow 0,
\end{array} \tag{4.2}$$

which proves that a representative of the equivalence class of the extension of $D^{1\times(q-r)}$ by M defined by the left D-module E is defined by the second horizontal short exact sequence of (4.2) (see Definition 2.1.1). If we write the left D-homomorphism $\beta\circ\psi^{-1}: D^{1\times(q-r)} \longrightarrow D^{1\times(p-r)}$ in the standard bases of the free left D-modules $D^{1\times(q-r)}$ and $D^{1\times(p-r)}$, then there exists a matrix $\overline{R}\in D^{(q-r)\times(p-r)}$ such that the second short exact sequence of (4.2) becomes the following one

$$0 \longrightarrow D^{1\times(q-r)} \xrightarrow{.\overline{R}} D^{1\times(p-r)} \xrightarrow{\gamma} M \longrightarrow 0,$$

which yields $M \cong D^{1\times(p-r)}/(D^{1\times(q-r)}\,\overline{R})$, i.e., M admits a finite presentation by a matrix $\overline{R}\in D^{(q-r)\times(p-r)}$. In terms of unknowns and equations, it means that the linear system $\ker_{\mathcal{F}}(R.)$ defined by q left D-linearly independent equations in p unknowns is equivalent to the linear system $\ker_{\mathcal{F}}(\overline{R}.)$ defined by $q-r$ left D-linearly independent equations in $p-r$ unknowns. Hence, the existence of an extension of $D^{1\times(q-r)}$ by M defined by a free left D-module E is equivalent to the possibility of reducing the number of equations and unknowns of the linear system $\ker_{\mathcal{F}}(R.)$ by r. Motivated by the study of complete intersections of algebraic varieties, Serre first studied this problem in [112]. Hence, we shall call it *Serre's reduction problem*. The purpose of this section is to study this problem within a constructive viewpoint.

4.2 Generalization of Serre's theorem

According to Theorem 2.1.2, the extensions of $D^{1\times(q-r)}$ by M are classified by the right D-module $\mathrm{ext}^1_D(M, D^{1\times(q-r)})$. A classical result of homological algebra asserts that

$$\mathrm{ext}^1_D(M, D^{1\times(q-r)}) \cong \mathrm{ext}^1_D(M, D)\otimes_D D^{1\times(q-r)},$$

where $\cdot\otimes_D\cdot$ denotes the tensor product. See, e.g., [15, 65, 109]. Moreover, since R has full row rank, Remark 2.1.2 shows that $\Omega = D^{q\times(q-r)}$. Applying Theorem 2.1.3 to the left D-modules M and $N = D^{1\times(q-r)} \cong D^{1\times(q-r)}/(D\,S)$, where $S = (0\ \ldots\ 0)\in D^{1\times(q-r)}$, then any extension of $D^{1\times(q-r)}$ by M can be defined by a left D-module $E = D^{1\times(p+q-r)}/(D^{1\times(q+1)}\,Q)$, where

$$Q = \begin{pmatrix} R & -\Lambda \\ 0 & 0 \end{pmatrix} \in D^{(q+1)\times(p+q-r)},$$

and $\Lambda\in\Omega = D^{q\times(q-r)}$, i.e., by the the left D-module $E = D^{1\times(p+q-r)}/(D^{1\times q}\,P)$, where:

$$P = (R \quad -\Lambda) \in D^{q\times(p+q-r)}.$$

Since R has full row rank, so has P, and we have the following short exact sequence

$$0 \longrightarrow D^{1\times q} \xrightarrow{.P} D^{1\times(p+q-r)} \xrightarrow{\varrho} E \longrightarrow 0, \tag{4.3}$$

where $\varrho: D^{1\times(p+q-r)} \longrightarrow E$ is the canonical projection onto E, i.e., the left D-homomorphism which sends $\zeta\in D^{1\times(p+q-r)}$ to its residue class $\varrho(\zeta)$ in E.

Since both R and P have full row rank, we get:

$$\mathrm{ext}^1_D\,(M, D) \cong D^q/\,(R\,D^p)\,, \quad \mathrm{ext}^1_D\,(E, D) \cong D^q/\left(P\,D^{(p+q-r)}\right).$$

Using the following inclusions of right D-modules $R\,D^p \subseteq P\,D^{(p+q-r)} = R\,D^p + \Lambda\,D^{(q-r)} \subseteq D^q$, we get the following short exact sequence of right D-modules

$$0 \longrightarrow \left(P\,D^{(p+q-r)}\right)/(R\,D^p) \xrightarrow{j} \mathrm{ext}^1_D\,(M, D) \xrightarrow{\sigma} \mathrm{ext}^1_D\,(E, D) \longrightarrow 0, \tag{4.4}$$

where j is the canonical injection and σ the canonical projection. Hence, (4.4) shows that

$$\begin{aligned}
\operatorname{ext}^1_D(E,D)=0 \quad &\Leftrightarrow \quad \operatorname{ext}^1_D(M,D)=\left(R\,D^p+\Lambda\,D^{(q-r)}\right)/(R\,D^p)\\
&\Leftrightarrow \quad \operatorname{ext}^1_D(M,D)=\left(R\,D^p+\sum_{i=1}^{q-r}\Lambda_{\bullet i}\,D\right)/(R\,D^p),\\
&\Leftrightarrow \quad \operatorname{ext}^1_D(M,D)=\sum_{i=1}^{q-r}\tau(\Lambda_{\bullet i})\,D,
\end{aligned}$$

where $\tau: D^p \longrightarrow \operatorname{ext}^1_D(M,D)=D^p/(R\,D^q)$ is the canonical projection. Hence, $\operatorname{ext}^1_D(E,D)=0$ iff the right D-module $\operatorname{ext}^1_D(M,D)$ is generated by the family $\{\tau(\Lambda_{\bullet i})\}_{i=1,\ldots,q-r}$ of $q-r$ elements.

Let us now study the condition $\operatorname{ext}^1_D(E,D)=0$. By definition, $\operatorname{ext}^1_D(E,D)=0$ is equivalent to the existence of a matrix $S=(S_1\ \ldots\ S_q)\in D^{(p+q-r)\times q}$ satisfying $P\,S=I_q$, which, by 2 of Corollary 1.3.3, is equivalent to E is a stably free left D-module of rank $p-r$.

Theorem 4.2.1 ([14]). *Let D be a noetherian domain, $R\in D^{q\times p}$ a full row rank matrix, i.e., $\ker_D(.R)=0$, $\Lambda\in D^{q\times(q-r)}$, $P=(R\quad -\Lambda)\in D^{q\times(p+q-r)}$ and $M=D^{1\times p}/(D^{1\times q}\,R)$ (resp., $E=D^{1\times(p+q-r)}/(D^{1\times q}\,P)$) the left D-module finitely presented by R (resp., P) which defines the following extension of $D^{1\times(q-r)}$ by M:*

$$0\longrightarrow D^{1\times(q-r)}\xrightarrow{\alpha}E\xrightarrow{\beta}M\longrightarrow 0.$$

Then, the following results are equivalent:

1. *The left D-module E is stably free of rank $p-r$.*
2. *The matrix $P=(R\quad -\Lambda)\in D^{q\times(p+q-r)}$ admits a right inverse.*
3. $\operatorname{ext}^1_D(E,D)\cong D^q/\left(P\,D^{(p+q-r)}\right)=0.$
4. *The right D-module $D^q/(R\,D^p)\cong\operatorname{ext}^1_D(M,D)$ is generated by $\{\tau(\Lambda_{\bullet i})\}_{i=1,\ldots,q-r}$, where $\tau: D^q\longrightarrow D^q/(R\,D^p)$ is the canonical projection.*

Finally, the previous equivalences depend only on the residue class $\rho(\Lambda)$ of $\Lambda\in D^{q\times(q-r)}$ in

$$D^{q\times(q-r)}/\left(R\,D^{p\times(q-r)}\right)\cong\operatorname{ext}^1_D\left(M,D^{1\times(q-r)}\right)\cong\operatorname{ext}^1_D(M,D)^{1\times(q-r)},$$

i.e., they depend only on the row vector $(\tau(\Lambda_{\bullet 1})\ \ldots\ \tau(\Lambda_{\bullet(q-r)}))$.

Remark 4.2.1. Theorem 4.2.1 was first obtained by J.-P. Serre in [112] for a commutative ring D and $r=q-1$. In this case, $\operatorname{ext}^1_D(M,D)$ is the (right) D-module generated by $\tau(\Lambda)$, i.e., $\operatorname{ext}^1_D(M,D)$ is the cyclic (right) D-module generated by $\tau(\Lambda)$.

Example 4.2.1. Theorem 4.2.1 is fulfilled if $\operatorname{ext}^1_D(M,D)=0$, i.e., if M is a stably free left D-module or, equivalently, if R admits a right inverse (see Corollary 1.3.3) since we can take $\Lambda=0$. Another explanation of this result is that $\operatorname{ext}^1_D(M,D)$ is then the trivial cyclic left D-module. Equivalently, the short exact sequence (4.4) yields $\operatorname{ext}^1_D(E,D)=0$.

On simple examples over a commutative polynomial ring $D=k[x_1,\ldots,x_n]$ with coefficients in a computable field k (e.g., $k=\mathbb{Q}$ or $\mathbb{F}_p$ for a prime p), we can take a generic matrix $\Lambda\in D^{q\times(q-r)}$ with a fixed total degree in the x_i's and, using Gröbner basis techniques, check whether or not the D-module $\operatorname{ext}^1_D(E,D)\cong D^{1\times q}/\left(D^{1\times(p+q-r)}\,P^T\right)$ vanishes on certain branches of the corresponding *tree of integrability conditions* ([90]) or on certain constructible sets of the k-parameters of Λ ([59]). See [59] for a survey explaining these techniques and their implementations in SINGULAR. In particular, we can test whether or not a non-zero constant belongs to the annihilator of $\mathrm{ext}^1_D(E, D)$,

$$\mathrm{ann}_D(\mathrm{ext}^1_D(E, D)) = \{d \in D \mid \forall\, n \in \mathrm{ext}^1_D(E, D),\ d\, n = 0\},$$

i.e., whether or not $\mathrm{ann}_D(\mathrm{ext}^1_D(E, D)) = D$. Indeed, since $\mathrm{ext}^1_D(E, D)$ is a torsion right D-module by Proposition 1.2.1, $\mathrm{ext}^1_D(E, D) = 0$ iff $\mathrm{ann}_D(\mathrm{ext}^1_D(E, D)) = D$.

These techniques are interesting when the $D = k[x_1, \ldots, x_n]$-module $\mathrm{ext}^1_D(M, D) \cong D^q/(R\, D^p)$ is 0-dimensional, i.e., $\dim_D(D^q/(R\, D^p)) = 0$, or equivalently, when the ring $A = D/I$ is a finite k-vector space, where $I = \mathrm{ann}_D(\mathrm{ext}^1_D(M, D))$ (see Section 2.3). Indeed, a Gröbner basis computation of the D-module $R\, D^p$ then gives a finite set of row vectors $\{\lambda_k\}_{k=1,\ldots,s}$, where $\lambda_k \in D^q$ and $s = \dim_k(A)$, such that $\mathrm{ext}^1_D(M, D) = \bigoplus_{k=1}^s k\, \tau(\lambda_k)$. Then, we can consider a generic matrix of the form

$$\Lambda = \begin{pmatrix} \sum_{k=1}^s a_{1k}\, \lambda_k & \ldots & \sum_{k=1}^s a_{(q-r)k}\, \lambda_k \end{pmatrix} \in D^{q \times (q-r)},$$

where the a_{lk}'s are arbitrary elements of k for $l = 1, \ldots, (q-r)$ and $k = 1, \ldots, s$, and compute the possible constructible sets of the k-parameters a_{kl}'s corresponding to the vanishing of the D-module $D^q/\left(P\, D^{(p+q-r)}\right) \cong \mathrm{ext}^1_D(E, D)$.

Example 4.2.2. We consider the model of a string with an interior mass defined by

$$\begin{cases} \phi_1(t) + \psi_1(t) - \phi_2(t) - \psi_2(t) = 0, \\ \dot{\phi}_1(t) + \dot{\psi}_1(t) + \eta_1\, \phi_1(t) - \eta_1\, \psi_1(t) - \eta_2\, \phi_2(t) + \eta_2\, \psi_2(t) = 0, \\ \phi_1(t - 2\, h_1) + \psi_1(t) - u(t - h_1) = 0, \\ \phi_2(t) + \psi_2(t - 2\, h_2) - v(t - h_2) = 0, \end{cases} \tag{4.5}$$

introduced and studied in [76], where $h_1, h_2 \in \mathbb{R}_+$ are such that $\mathbb{Q}\, h_1 + \mathbb{Q}\, h_2$ is a 2-dimensional $\mathbb{Q}$-vector space, and η_1 and η_2 are two constant parameters. Let $D = \mathbb{Q}(\eta_1, \eta_2)\, [\partial, \sigma_1, \sigma_2]$ be the commutative polynomial algebra of OD incommensurable time-delay operators in ∂, σ_1 and σ_2, where $\partial\, f(t) = \dot{f}(t)$, $\sigma_1 f(t) = f(t - h_1)$ and $\sigma_2 f(t) = f(t - h_2)$. Let $M = D^{1\times 6}/(D^{1\times 4}\, R)$ be the D-module finitely presented by the following matrix:

$$R = \begin{pmatrix} 1 & 1 & -1 & -1 & 0 & 0 \\ \partial + \eta_1 & \partial - \eta_1 & -\eta_2 & \eta_2 & 0 & 0 \\ \sigma_1^2 & 1 & 0 & 0 & -\sigma_1 & 0 \\ 0 & 0 & 1 & \sigma_2^2 & 0 & -\sigma_2 \end{pmatrix} \in D^{4\times 6}.$$

Then, we have $\mathrm{ext}^1_D(M, D) \cong D^4/(R\, D^6) \cong D^{1\times 4}/(D^{1\times 6}\, R^T)$. Computing a Gröbner basis of the D-module $D^4/(R\, D^6)$, we obtain that $D^4/(R\, D^6)$ is a 1-dimensional $\mathbb{Q}(\eta_1, \eta_2)$-vector space and $\tau((0 \quad 1 \quad 0 \quad 0)^T)$ is a basis, where $\tau : D^4 \longrightarrow D^4/(R\, D^6)$ is the canonical projection. Hence, the only possible Λ's such that $P = (R \quad -\Lambda)$ admits a right inverse must belong to $V = \left\{a\, (0 \quad 1 \quad 0 \quad 0)^T \mid a \in \mathbb{Q}(\eta_1, \eta_2)\right\}$. If we consider the column vector $\Lambda = (0 \quad 1 \quad 0 \quad 0)^T$, then the matrix $P = (R \quad -\Lambda) \in D^{4\times 7}$ admits the following right inverse:

$$S = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 & -\sigma_2 & -\eta_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 & -\sigma_1 & 0 & -2\, \eta_1 \\ 0 & 0 & 1 & -1 & 0 & -\sigma_2 & -2\, \eta_2 \end{pmatrix}^T.$$

Hence, the D-module $D^4/(R\,D^6) \cong \text{ext}^1_D(M, D)$ is cyclic and is generated by $\tau(\Lambda)$.

Remark 4.2.2. If $D = k[x_1, x_2]$ is a commutative polynomial ring over a field k, $R \in D^{q\times p}$ and $M = D^{1\times p}/(D^{1\times q}\,R)$, then, using Theorem 1.3.1, one of the following exclusive cases holds: M admits a non-trivial torsion submodule $t(M)$, M is torsion-free but not projective or M is projective, i.e., free by the Quillen-Suslin (see 2 of Theorem 1.1.2). Hence, if $p > q$ and R has full row rank, then the generic situation is that M is a torsion-free D-module, which implies that $\text{ext}^1_D(M, D)$ is generically 0-dimensional by 2 of Corollary 2.3.1 since $\dim(D) = 2$. Hence, as previously explained, we can check whether or not there exists a matrix $\Lambda \in D^{q\times(q-r)}$ such that $P = (R \quad -\Lambda)$ admits a right inverse, when R is a generic full row rank matrix with $p > q$ and the columns of the matrix Λ are generic k-linear combinations of the basis of the finite-dimensional k-vector $\text{ext}^1_D(M, D)$. This situation particularly holds in the study of control linear OD time-delay systems defined by full row rank matrices with entries in the ring $D = k[\partial, \delta]$, where k is a computable field (see [16, 17, 19, 20]).

Apart from the previous 0-dimensional case, we do not know yet how to recognize the existence of $\Lambda \in D^{q\times(q-r)}$ satisfying 2 of Theorem 4.2.1. However, using an ansatz, we can give the sketch of an algorithm in the case of the second Weyl algebra $B_n(k)$. This case contains the cases of a commutative polynomial ring and the first Weyl algebra $A_n(k)$ since we have:

$$k[x_1, \ldots, x_n] \subset A_n(k) \subset B_n(k).$$

Algorithm 4.2.1.
- **Input:** Let k be an algebraically closed computable field, $D = B_n(k)$, $R \in D^{q\times p}$ a full row rank matrix and three non-negative integers α, β and γ.
- **Output:** A set (possibly empty) of $\{\Lambda_i\}_{i\in I}$ such that the matrix $(R \quad -\Lambda_i)$ admits a right inverse.

1. Consider an ansatz $\Lambda \in D^{q\times(q-r)}$ whose entries have a fixed total order α in the ∂_i's and a fixed total degree β (resp., γ) for the polynomial numerators (resp., denominators) in the x_j's of the arbitrary coefficients of the ansatz Λ.
2. Compute a Gröbner basis of the right D-module $R\,D^p$.
3. Compute the normal form $\overline{\Lambda}_{\bullet i} \in D^q$ of the i^{th} column $\Lambda_{\bullet i}$ of Λ in the right D-module $D^q/(R\,D^p) \cong \text{ext}^1_D(M, D)$ for all $i = 1, \ldots, q-r$.
4. Compute the obstructions for projectivity of $\overline{E} = D^{1\times(p+q-r)}/(D^{1\times q}\,(R \quad -\overline{\Lambda}))$ (e.g., compute a Gröbner basis of the right D-module $(R \quad -\overline{\Lambda})\,D^{(p+q-r)}$ or the π-*polynomials* of $\overline{E}$ ([16, 73]), namely, the generators of the ideal $\bigcap_{\{i\geq 1\,|\,\text{ext}^i_D(L,D)\neq 0\}} \text{ann}_D(\text{ext}^i_D(L, D))$, where $L = D^q/((R \quad -\overline{\Lambda})\,D^{(p+q-r)}) \cong \text{ext}^1_D(\overline{E}, D)$ is the Auslander transpose of $\overline{E}$).
5. Solve the systems in the arbitrary coefficients of the ansatz Λ obtained by making the obstructions vanish.
6. Return the set of solutions for Λ.

Example 4.2.3. Let us consider a general transmission line defined by

$$\begin{cases} \dfrac{\partial V}{\partial x} + L\,\dfrac{\partial I}{\partial t} + R\,I = 0, \\ C\,\dfrac{\partial V}{\partial t} + G\,V + \dfrac{\partial I}{\partial x} = 0, \end{cases} \tag{4.6}$$

where I denotes the current, V the voltage, L the self-inductance, R the resistance, C the capacitor and G the conductance. Let $D = \mathbb{Q}(L, R, C, G)[\partial_t, \partial_x]$ be the commutative polynomial ring of PD operators in ∂_t and ∂_x with coefficients in the field $\mathbb{Q}(L, R, C, G)$, the presentation matrix $J \in D^{2\times 2}$ of (4.6) defined by

$$J = \left(\begin{array}{cc} \partial_x & L\,\partial_t + R \\ C\,\partial_t + G & \partial_x \end{array}\right) \in D^{2\times 2}, \tag{4.7}$$

and the D-module $M = D^{1\times 2}/(D^{1\times 2}\,J)$. In this example, we slightly change the previous notations since we want to keep the standard notation R for a resistance. Let us consider $\Lambda = (\alpha \quad \beta)^T$, where α and β are two new variables, $A = D[\alpha, \beta]$, $P = (J \quad -\Lambda) \in A^{2\times 3}$ and the A-module $E = A^{1\times 3}/(A^{1\times 2}\,P)$ finitely presented by P. The obstructions for E to be a stably free A-module are defined by $A/(\pi_1, \pi_2)$, where the π-polynomials π_1 and π_2 are respectively:

$$\left\{\begin{array}{l} \pi_1 = (C\,\alpha^2 - L\,\beta^2)\,\partial_t + G\,\alpha^2 - R\,\beta^2, \\ \pi_2 = (C\,\alpha^2 - L\,\beta^2)\,\partial_x + (L\,G - R\,C)\,\alpha\,\beta. \end{array}\right.$$

They can be computed by OreModules. Hence, if $C\,\alpha^2 = L\,\beta^2$ and $G\,\alpha^2 - R\,\beta^2 \neq 0$ (resp., $(L\,G - R\,C)\,\alpha\,\beta \neq 0$), then π_1 (resp., π_2) is a non-zero constant. In particular, if we consider

$$\beta = C \neq 0, \quad \alpha^2 = L\,C \neq 0, \quad L\,G - R\,C \neq 0,$$

the ring $B = (\mathbb{Q}(L, R, C, G)[\alpha]/(\alpha^2 - L\,C))[\partial_t, \partial_x]$ and $\Lambda = (\alpha \quad C)^T \in B^2$, then the matrix $P = (J \quad -\Lambda) \in B^{2\times 3}$ admits the following right inverse:

$$S = \frac{1}{(G\,L - R\,C)} \left(\begin{array}{cc} -\alpha & L \\ -C & \alpha \\ -(C\,\partial_x + \alpha\,C\,\partial_t + \alpha\,G)/C & (\alpha\,\partial_x + L\,C\,\partial_t + R\,C)/C \end{array}\right).$$

Therefore, the B-module $B^2/(J\,B^3) \cong \mathrm{ext}^1_B(M, B)$ is cyclic and is generated by $\tau(\Lambda)$, where $\tau : B^2 \longrightarrow B^2/(J\,B^3)$ is the canonical projection.

Example 4.2.4. Let us consider the conjugate Beltrami equations with $\sigma = x^{-1}$:

$$\left\{\begin{array}{l} \dfrac{\partial u}{\partial x} - x\,\dfrac{\partial v}{\partial y} = 0, \\[2mm] \dfrac{\partial u}{\partial y} + x\,\dfrac{\partial v}{\partial x} = 0. \end{array}\right. \tag{4.8}$$

Let $D = A_2(\mathbb{Q}(a, b))$, $R \in D^{2\times 2}$ be the presentation matrix of (4.8) defined by

$$R = \left(\begin{array}{cc} \partial_x & -x\,\partial_y \\ \partial_y & x\,\partial_x \end{array}\right) \in D^{2\times 2}, \tag{4.9}$$

and $M = D^{1\times 2}/(D^{1\times 2}\,R)$ the left D-module finitely presented by R. If we consider the column vector $\Lambda = (a \quad b)^T$, the matrix $P = (R \quad -\Lambda) \in D^{2\times 3}$ and the left D-module $E = D^{1\times 3}/(D^{1\times 2}\,P)$, then, when both a and b are non-zero, we can check that P admits the following right inverse:

$$S = \left(\begin{array}{cc} x\,(a\,x\,\partial_x + b\,x\,\partial_y + a)/a & -x\,(a\,x\,\partial_x + b\,x\,\partial_y + a)/b \\ -(a\,x\,\partial_y - b\,x\,\partial_x - 2\,b)/a & (a\,x\,\partial_y - b\,x\,\partial_x - 2\,b)/b \\ x\,(x\,\partial_x^2 + x\,\partial_y^2 + 3\,\partial_x)/a & -(x^2\,\partial_x^2 + x^2\,\partial_y^2 + 3\,x\,\partial_x + 1)/b \end{array}\right) \in D^{3\times 2}.$$

Hence, the right D-module $D^2/(R\,D^3) \cong \mathrm{ext}^1_D(M, D)$ is cyclic and is generated by $\tau(\Lambda)$, where $\tau : D^2 \longrightarrow D^2/(R\,D^3)$ is the canonical projection.

We can now use Theorem 4.2.1 to study Serre's reduction.

Theorem 4.2.2 ([14]). *Let D be a noetherian domain, $R \in D^{q \times p}$ a full row rank matrix, $0 \leq r \leq q-1$ and $\Lambda \in D^{q \times (q-r)}$ a matrix such that there exists $U \in \mathrm{GL}_{p+q-r}(D)$ satisfying:*

$$(R \quad -\Lambda)\, U = (I_q \quad 0).$$

If we decompose the unimodular matrix U as follows

$$U = \begin{pmatrix} S_1 & Q_1 \\ S_2 & Q_2 \end{pmatrix}, \tag{4.10}$$

where $S_1 \in D^{p \times q}$, $S_2 \in D^{(q-r) \times q}$, $Q_1 \in D^{p \times (p-r)}$, $Q_2 \in D^{(q-r) \times (p-r)}$, and if we introduce the left D-module $L = D^{1 \times (p-r)}/(D^{1 \times (q-r)}\, Q_2)$ finitely presented by the full row rank matrix Q_2, i.e., defined by the following short exact sequence

$$0 \longrightarrow D^{1 \times (q-r)} \xrightarrow{.Q_2} D^{1 \times (p-r)} \xrightarrow{\kappa} L \longrightarrow 0, \tag{4.11}$$

then we have:

$$M = D^{1 \times p}/(D^{1 \times q}\, R) \;\cong\; L = D^{1 \times (p-r)}/(D^{1 \times (q-r)}\, Q_2). \tag{4.12}$$

Conversely, if M is isomorphic to a left D-module L defined by the short exact sequence (4.11), then there exist two matrices $\Lambda \in D^{q \times (q-r)}$ and $U \in \mathrm{GL}_{p+q-r}(D)$ such that:

$$(R \quad -\Lambda)\, U = (I_q \quad 0).$$

We now can give an explicit description of the isomorphism (4.12).

Corollary 4.2.1 ([14]). *With the notations of Theorem 4.2.2, the left D-isomorphism (4.12) is explicitly defined by:*

$$\begin{array}{rcl} \varphi : M = D^{1 \times p}/(D^{1 \times q}\, R) & \longrightarrow & L = D^{1 \times (p-r)}/(D^{1 \times (q-r)}\, Q_2) \\ \pi(\lambda) & \longmapsto & \kappa(\lambda\, Q_1). \end{array}$$

Moreover, its inverse $\varphi^{-1} : L \longrightarrow M$ is defined by $\varphi^{-1}(\kappa(\mu)) = \pi(\mu\, T_1)$, where:

$$U^{-1} = \begin{pmatrix} R & -\Lambda \\ T_1 & -T_2 \end{pmatrix} \in \mathrm{GL}_{p+q-r}(D), \quad T_1 \in D^{(p-r) \times p}, \quad T_2 \in D^{(p-r) \times (q-r)}. \tag{4.13}$$

These results depend only on the residue class $\rho(\Lambda)$ of $\Lambda \in D^{q \times (q-r)}$ in the right D-module:

$$\mathrm{ext}^1_D\left(M, D^{1 \times (q-r)}\right) \cong D^{q \times (q-r)}/(R\, D^{p \times (q-r)}).$$

A straightforward consequence of Corollary 4.2.1 is the following result.

Corollary 4.2.2 ([14]). *Let D be a noetherian domain, $R \in D^{q \times p}$ a full row rank matrix, $0 \leq r \leq q-1$ and $\Lambda \in D^{q \times (q-r)}$ a matrix such that there exists $U \in \mathrm{GL}_{p+q-r}(D)$ satisfying:*

$$(R \quad -\Lambda)\, U = (I_q \quad 0).$$

If $\mathcal{F}$ is a left D-module and if we introduce the following two linear systems

$$\ker_{\mathcal{F}}(R.) = \{\eta \in \mathcal{F}^p \mid R\, \eta = 0\}, \quad \ker_{\mathcal{F}}(Q_2.) = \{\zeta \in \mathcal{F}^{(p-r)} \mid Q_2\, \zeta = 0\},$$

where the matrix $Q_2 \in D^{(q-r) \times (p-r)}$ is defined by (4.10), then the following isomorphism holds:

$$\ker_{\mathcal{F}}(R.) \cong \ker_{\mathcal{F}}(Q_2.).$$

More precisely, we have $\ker_{\mathcal{F}}(R.) = Q_1 \ker_{\mathcal{F}}(Q_2.)$ and $\ker_{\mathcal{F}}(Q_2.) = T_1 \ker_{\mathcal{F}}(R.)$, where the matrix $Q_1 \in D^{p \times (p-r)}$ (resp., $T_1 \in D^{(p-r) \times p}$) is defined by (4.10) (resp., (4.13)).

Using Theorems 1.1.2 and 1.5.4, we obtain the following corollary of Theorem 4.2.2.

Corollary 4.2.3 ([14]). *Let $R \in D^{q \times p}$ be a full row rank matrix and $\Lambda \in D^{q \times (q-r)}$ a matrix such that $P = (R \quad -\Lambda) \in D^{q \times (p+q-r)}$ admits a right inverse. If D satisfies one of the following properties*

1. *D is a left principal ideal domain (e.g., the ring $A\langle \partial \rangle$ of OD operators with coefficients in a differential field A such as $A = k$, $k(t)$, $k[\![t]\!][t^{-1}]$, where k is a field),*
2. *$D = k[x_1, \ldots, x_n]$ is a commutative polynomial ring over a field k,*
3. *D is either $A_n(k)$ or $B_n(k)$, where k is a field of characteristic 0, and $p - r \geq 2$.*
4. *$D = A\langle \partial \rangle$ is the ring of OD operators with coefficients in $A = k[\![t]\!]$, where k is a field of characteristic 0, or $A = k\{t\}$, where $k = \mathbb{R}$ or $\mathbb{C}$, and $p - r \geq 2$,*

then there exists a matrix $U \in \mathrm{GL}_{p+q-r}(D)$ such that $P\,U = (I_q \quad 0)$ and Theorem 4.2.2 holds.

If D satisfies the conditions of Corollary 4.2.3, then, by 2 of Corollary 1.3.3, it is enough to search for $\Lambda \in D^{q \times (q-r)}$ such that $P = (R \quad -\Lambda) \in D^{q \times (p+q-r)}$ admits a right inverse.

Remark 4.2.3. Corollary 4.2.3 can also be understood as follows: if the noetherian domain D is a so-called *Hermite ring*, namely, if every finitely generated stably free left D-module is free, and $M = D^{1 \times p}/(D^{1 \times q}\, R)$ is the left D-module finitely presented by the full row rank matrix R, then M can be generated by $p - r$ elements iff its Auslander transpose right D-module $\mathrm{ext}^1_D(M, D) \cong D^q/(R\, D^p)$ can be generated by $q - r$ elements (see Theorem 4.2.2).

Example 4.2.5. Let us consider again Example 4.2.2 where the $D = \mathbb{Q}(\eta_1, \eta_2)\,[\partial, \sigma_1, \sigma_2]$-module $E = D^{1 \times 7}/(D^{1 \times 4}\, P)$ was proved to be a stably free, i.e., free by Quillen-Suslin theorem (see 2 of Corollary 4.2.3). Using a constructive version of the Quillen-Suslin theorem ([29]) and its implementation in the QuillenSuslin package ([29]), we obtain that

$$U = \begin{pmatrix} 0 & 0 & -1 & 0 & 1 & \sigma_1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -\sigma_1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -\sigma_2 \\ -1 & 0 & 0 & -1 & 1 & 0 & \sigma_2 \\ 0 & 0 & -\sigma_1 & 0 & \sigma_1 & \sigma_1^2 - 1 & 0 \\ -\sigma_2 & 0 & 0 & -\sigma_2 & \sigma_2 & 0 & \sigma_2^2 - 1 \\ -\eta_2 & -1 & -2\,\eta_1 & -2\,\eta_2 & \partial + \eta_1 + \eta_2 & 2\,\eta_1\,\sigma_1 & 2\,\eta_2\,\sigma_2 \end{pmatrix} \in \mathrm{GL}_7(D),$$

satisfies $(R \quad -\Lambda)\, U = (I_4 \quad 0)$, and thus we get $Q_2 = (\partial + \eta_1 + \eta_2 \quad 2\,\eta_1\,\sigma_1 \quad 2\,\eta_2\,\sigma_2)$. We then have $M = D^{1 \times 6}/(D^{1 \times 4}\, R) \cong L = D^{1 \times 3}/(D\, Q_2)$, i.e., using Corollary 4.2.2, (4.5) is equivalent to the following sole OD time-delay equation:

$$\dot{x}_1(t) + (\eta_1 + \eta_2)\, x_1(t) + 2\,\eta_1\, x_2(t - h_1) + 2\,\eta_2\, x_3(t - h_2) = 0. \tag{4.14}$$

This result was also obtained in [20] after the resolutions of algebraic Riccati equations of the form $X\, R\, X = X$ (see Lemma 3.8.2). But, Serre's reduction allows us to obtain this result in a more direct and simpler way. Finally, the study of the algebraic properties of (4.5) is now highly simplified and we can easily check that $M \cong L$ is torsion-free and σ_1 and σ_2-free (see [74]).

Example 4.2.6. Let us consider again the general transmission line (4.6) studied in Example 4.2.3. If $B = K[\partial_t, \partial_x]$ is the commutative polynomial ring of PD operators in ∂_t and

∂_x with coefficients in the field $K = \mathbb{Q}(L, R, C, G)[\alpha]/(\alpha^2 - L\,C)$ and $P = (J \quad -\Lambda) \in B^{2\times 3}$ is the matrix formed by the matrix J defined by (4.7) and $\Lambda = (\alpha \quad C)^T$, then the stably free B-module $E = B^{1\times 3}/(B^{1\times 2}\,P)$ is free by the Quillen-Suslin theorem. Computing a basis of E using a constructive version of the Quillen-Suslin theorem explained in [29] and implemented in the QuillenSuslin package ([29]), we obtain that the matrix $U = (S^T \quad Q^T)^T \in \mathrm{GL}_3(B)$, where the matrix $S \in B^{3\times 2}$ is defined in Example 4.2.3 and $Q = (Q_1^T \quad Q_2^T)^T$ is defined by

$$\left\{\begin{array}{l} Q_1 = (\alpha\,\partial_x - L\,C\,\partial_t - R\,C \quad C\,\partial_x - \alpha\,C\,\partial_t - \alpha\,G)^T, \\ Q_2 = \partial_x^2 - L\,C\,\partial_t^2 - (L\,C + R\,C)\,\partial_t - R\,G, \end{array}\right.$$

satisfies $(J \quad -\Lambda)\,U = (I_2 \quad 0)$. Hence, if $C \neq 0$, $L \neq 0$ and $L\,G - R\,C \neq 0$, then (4.6) is equivalent to the following PD equation:

$$(\partial_x^2 - L\,C\,\partial_t^2 - (L\,C + R\,C)\,\partial_t - R\,G)\,Z(t, x) = 0.$$

Example 4.2.7. Let us consider again Example 4.2.4 where the left $D = A_2(\mathbb{Q}(a, b))$-module $E = D^{1\times 3}/(D^{1\times 2}\,P)$ was proved to be stably free and $P = (R \quad -\Lambda)$ is formed by the matrix R defined by (4.9) and by $\Lambda = (a \quad b)^T$. Since the rank of E is $3 - 2 = 1$, we cannot use Stafford's theorem (see 3 of Theorem 1.1.2) to conclude that E is a free left D-module of rank 1. We need to investigate when E is a free left D-module of rank 1 for particular values of a and b. Using Algorithm 1.4.1, the stably free left D-module E admits the minimal parametrization:

$$Q = \left(\begin{array}{c} -a^2\,b + b\,a^2\,x\,\partial_x - a^3\,x\,\partial_y - a\,\left(a^2 + b^2\right)\,x^2\,\partial_x\,\partial_y - b\,\left(a^2 + b^2\right)\,x^2\,\partial_y^2 \\ a\,b^2\,\partial_x - b\,\left(2\,b^2 + 3\,a^2\right)\,\partial_y - b\,\left(a^2 + b^2\right)\,x\,\partial_x\,\partial_y + a\,(a^2 + b^2)\,x\,\partial_y^2 \\ -a^2\,\partial_y - \left(a^2 + b^2\right)\,x^2\,\partial_y\,\partial_x^2 + a\,b\,x\,\partial_x^2 - 3\,\left(a^2 + b^2\right)\,x\,\partial_x\,\partial_y + a\,b\,x\,\partial_y^2 - \left(a^2 + b^2\right)\,x^2\,\partial_y^3 \end{array}\right).$$

Hence, $E \cong D^{1\times 3}\,Q = \sum_{i=1}^3 D\,Q_{i1}$, i.e., E is isomorphic to the left ideal of D generated by the three entries of Q. Therefore, the following long exact sequence holds

$$0 \longrightarrow D^{1\times 2} \xrightarrow{.P} D^{1\times 3} \xrightarrow{.Q} D \xrightarrow{\sigma} L \longrightarrow 0,$$

where $\sigma : D \longrightarrow L$ is the canonical projection onto $L = D/(D^{1\times 3}\,Q)$. If there exists a set of values for the arbitrary parameters a and b such that the left D-module L vanishes, then the above long exact sequence shows that $D^{1\times 3}\,Q = D$, and thus $E \cong D^{1\times 3}\,Q = D$ is a free left D-module of rank 1. Computing a Gröbner basis of the left D-module $D^{1\times 3}\,Q$, we obtain that the generator $z = \sigma(1)$ of the left D-module L satisfies $d\,z = 0$, where:

$$d = -\left(a^2 + b^2\right)^2\,x^2\,\partial_y^2 + 2\,a\,b\,\left(a^2 + b^2\right)\,x\,\partial_y - a^2\,b^2 \in D.$$

Therefore, if we consider a solution of the following polynomial system

$$\left\{\begin{array}{l} (a^2 + b^2)^2 = 0, \\ a\,b\,(a^2 + b^2) = 0, \\ a^2\,b^2 = -1, \end{array}\right. \quad \Leftrightarrow \quad \left\{\begin{array}{l} a^2 + b^2 = 0, \\ a^2\,b^2 = -1, \end{array}\right. \quad \Leftrightarrow \quad \left\{\begin{array}{l} b^2 = -a^2, \\ a^4 = 1, \end{array}\right. \quad \Leftrightarrow \quad \left\{\begin{array}{l} b = \pm\,i\,a, \\ a \in \{\pm 1,\, \pm i\}, \end{array}\right.$$

such as $a = 1$ and $b = i$, then d is reduced to 1. If we consider the new ring $A = A_2(\mathbb{Q}(i))$, then the left A-module $E = A^{1\times 3}/(A^{1\times 2}\,P)$, where $\Lambda = (1 \quad i)^T$, admits the following parametrization

$$Q = \left(\begin{array}{c} x\,(i\,\partial_x - \partial_y) - i \\ -(\partial_x + i\,\partial_y) \\ i\,x\,(\partial_x^2 + \partial_y^2) - \partial_y \end{array}\right), \tag{4.15}$$

and $T = (i \quad -x \quad 0)$ is a left inverse of Q, which shows that Q is an injective parametrization of E and E is a free left A-module of rank 1. Finally, using Theorem 4.2.2 and Corollary 4.2.2, we obtain $M \cong A/(A\,(i\,x\,(\partial_x^2+\partial_y^2)-\partial_y))$ and:

$$(4.2.4) \quad \Leftrightarrow \quad (i\,x\,(\partial_x^2+\partial_y^2)-\partial_y))\,u=0 \quad \Leftrightarrow \quad (x\,(\partial_x^2+\partial_y^2)+i\,\partial_y)\,u=0.$$

Since holonomic right D-modules are cyclic (see Proposition 2.3.2), using Stafford's theorem (see 3 of Theorem 1.1.2), we obtain the following interesting result.

Corollary 4.2.4 ([21])**.** *Let $D = A\langle\partial_1,\ldots,\partial_n\rangle$, where A is either $k[x_1,\ldots,x_n]$, $k[\![x_1,\ldots,x_n]\!]$ and k is a field of characteristic 0, or $k\{x_1,\ldots,x_n\}$ and $k=\mathbb{R}$ or $\mathbb{C}$, $R \in D^{q\times p}$ be a full row rank matrix and $M = D^{1\times p}/(D^{1\times q}\,R)$. If $\mathrm{ext}_D^1(M,D) \cong D^q/(R\,D^p)$ is a holonomic right D-module, then Theorem 4.2.1 holds and we can choose a column vector $\Lambda \in D^q$ which admits a left inverse and which is such that $\tau(\Lambda)$ generates the right D-module $D^q/(R\,D^p)$, where $\tau : D^q \longrightarrow D^q/(R\,D^p)$ is the canonical projection. Finally, if $A = k[x_1,\ldots,x_n]$ and $p-q\geq 1$, then Theorem 4.2.2 and Corollaries 4.2.1 and 4.2.2 hold.*

Example 4.2.8. Let us consider the commutative polynomial ring $D=\mathbb{Q}\,[\partial_x,\partial_y]$ of PD operators and the D-module $M=D^{1\times 3}/(D^{1\times 2}\,R)$ finitely presented by R defined by:

$$R=\begin{pmatrix} \partial_x & \partial_y & 0 \\ 0 & \partial_x & \partial_y \end{pmatrix} \in D^{2\times 3}. \tag{4.16}$$

The matrix R defines the equation of the equilibrium of the *stress tensor* in $\mathbb{R}^2$ ([85]), namely:

$$\left\{\begin{array}{l} \partial_x\,\sigma^{11}+\partial_y\,\sigma^{12}=0, \\ \partial_x\,\sigma^{12}+\partial_y\,\sigma^{22}=0. \end{array}\right. \tag{4.17}$$

We can easily check that the D-module $\mathrm{ext}_D^1(M,D)\cong D^{1\times 2}/\left(D^{1\times 3}\,R^T\right)$ is a $\mathbb{Q}$-vector space of dimension 3 and a basis of $\mathrm{ext}_D^1(M,D)$ is defined by the vectors $\tau((1 \quad 0)^T)$, $\tau((0 \quad 1)^T)$ and $\tau((0 \quad \partial_x)^T)$, where $\tau : D^2 \longrightarrow D^2/(R\,D^3)$ is the canonical projection. Hence, without loss of generality, we can assume that Λ has the form $\Lambda = (a \quad b+c\,\partial_x)^T$, where a, b and c are three arbitrary constants. Considering the new ring $A=\mathbb{Q}[a,b,c]\,[\partial_x,\partial_y]$, $P=(R \quad -\Lambda)\in A^{2\times 4}$, the A-module $E=A^{1\times 4}/(A^{1\times 2}\,P)$ and the A-module $\mathrm{ext}_A^1(E,A)\cong N=A^{1\times 2}/(A^{1\times 4}\,P^T)$ and using Algorithm 1.3.1 implemented in OreModules, we can check that $t(E)\cong \mathrm{ext}_A^1(N,A)=0$ and $\mathrm{ext}_A^2(N,A)\cong A/(\partial_x,\partial_y)\neq 0$. According to Theorem 1.3.1, we obtain that the A-module E is a torsion-free but not projective whatever the values of the parameters a, b and c, which proves that (4.17) cannot be defined by a PD equation with constant coefficients, and the minimal number of generators $\mu(M)$ of the D-module M is 3.

We can now introduce the left $B=A_2(\mathbb{Q})$-module $M'=B\otimes_D M=B^{1\times 3}/(B^{1\times 2}\,R)$. Clearly, the right B-module $\mathrm{ext}_B^1(M',B)\cong B^2/(R\,B^3)$ is holonomic and thus cyclic by Proposition 2.3.2. Moreover, the element $\tau(\Lambda)$ of $\mathrm{ext}_B^1(M',B)$, where $\Lambda=(1 \quad x)^T$, generates $\mathrm{ext}_B^1(M',B)$ because the matrix $P=(R \quad -\Lambda)\in B^{2\times 4}$ admits the following right inverse:

$$T=\begin{pmatrix} -x & 1 \\ -x^2 & x \\ -x^3 & x^2 \\ -x\,(x\,\partial_y+\partial_x)-2 & \partial_x+x\,\partial_y \end{pmatrix}.$$

The left B-module $E' = B^{1\times 4}/(B^{1\times 2}\, P)$ is then stably free of rank 2, i.e., free by Stafford's theorem (see 3 of Theorem 1.1.2). Using the STAFFORD package ([103]), we obtain an injective parametrization Q of the free left B-module E' defined by

$$Q = \begin{pmatrix} \partial_y & \partial_x \\ x\,\partial_y & x\,\partial_x - 1 \\ x^2\,\partial_y - 1 & x\,\partial_x - x \\ (\partial_x + x\,\partial_y)\,\partial_y & (\partial_x + x\,\partial_y)\,\partial_x - \partial_y \end{pmatrix},$$

which yields $M' \cong B^{1\times 2}/(B\,((\partial_x + x\,\partial_y)\,\partial_y \quad (\partial_x + x\,\partial_y)\,\partial_x - \partial_y))$.

4.3 Equivalence to Serre's reduction

Corollary 4.3.1 ([14])**.** *With the notations of Theorem 4.2.2 and Corollary 4.2.1, if the matrix $\Lambda \in D^{q\times (q-r)}$ admits a left inverse $\Gamma \in D^{(q-r)\times q}$, i.e., $\Gamma\,\Lambda = I_{q-r}$, then the matrix Q_1 admits the left inverse $T_1 - T_2\,\Gamma\,R \in D^{(p-r)\times p}$ and the left D-module $\ker_D(.Q_1)$ is stably free of rank r.*

Moreover, if the left D-module $\ker_D(.Q_1)$ is free of rank r, then there exists $Q_3 \in D^{p\times r}$ such that $W \triangleq (Q_3 \quad Q_1) \in \mathrm{GL}_p(D)$. If we write $W^{-1} = (Y_3^T \quad Y_1^T)^T$, where $Y_3 \in D^{r\times p}$ and $Y_1 \in D^{(p-r)\times p}$, then the matrix $X \triangleq (R\,Q_3 \quad \Lambda)$ is unimodular, i.e., $X \in \mathrm{GL}_q(D)$ and:

$$V \triangleq X^{-1} = \begin{pmatrix} Y_3\,S_1 \\ Q_2\,Y_1\,S_1 - S_2 \end{pmatrix}.$$

The matrix R is then equivalent to the matrix $X\,\mathrm{diag}(I_r, Q_2)\,W^{-1}$ or equivalently:

$$V\,R\,W = \begin{pmatrix} I_r & 0 \\ 0 & Q_2 \end{pmatrix}.$$

Finally, the left D-module $\ker_D(.Q_1)$ is free when D satisfies 1 or 2 of Corollary 4.2.3 or if D is $A_n(k)$ or $B_n(k)$, where k is a field of characteristic 0, and $r \geq 2$ (e.g., if $q \geq 3$ in Corollary 4.2.4) or if $D = A\langle\partial\rangle$, where $A = k[\![t]\!]$ and k a field of characteristic 0, or $A = k\{t\}$ and $k = \mathbb{R}$ or $\mathbb{C}$, and $r \geq 2$.

Let us illustrate Corollary 4.3.1 with explicit examples.

Example 4.3.1. Let us consider again Examples 4.2.2 and 4.2.5. Since Λ clearly admits a left inverse, we can check that the matrix $Q_1 \in D^{6\times 3}$ defined by the first 6 rows of Q also admits a right inverse. Using a constructive version of the Quillen-Suslin theorem and its implementation in the QUILLENSUSLIN package ([29]), we can complete the matrix Q_1 to the following unimodular matrix:

$$W = (Q_3 \quad Q_1) = \begin{pmatrix} 1 & 0 & 0 & 1 & \sigma_1 & 0 \\ 0 & -1 & 0 & 0 & -\sigma_1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -\sigma_2 \\ 0 & -1 & -1 & 1 & 0 & \sigma_2 \\ 0 & 0 & 0 & \sigma_1 & \sigma_1^2 - 1 & 0 \\ 0 & -\sigma_2 & -\sigma_2 & \sigma_2 & 0 & \sigma_2^2 - 1 \end{pmatrix}^T \in \mathrm{GL}_6(D).$$

We can now check that the following matrix

$$X = (R\,Q_3 \quad \Lambda) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \partial + \eta_1 & -\partial + \eta_1 - \eta_2 & -2\,\eta_2 & 1 \\ \sigma_1^2 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \in D^{4 \times 4}$$

is unimodular, i.e., $X \in \mathrm{GL}_4(D)$, and satisfies

$$R\,W = X\,\mathrm{diag}(I_3, Q_2) \quad \Leftrightarrow \quad \mathrm{diag}(I_3, Q_2) = X^{-1}\,R\,W,$$

which finally proves that the matrix R is equivalent to $\mathrm{diag}(I_3, Q_2)$.

Example 4.3.2. Let us consider again Examples 4.2.3 and 4.2.6. We can easily check that Λ admits a left inverse. Using Corollary 4.3.1, the matrix $Q_1 \in B^2$ defined by the first 2 entries of Q admits a right inverse. Then, using a constructive version of the Quillen-Suslin theorem and its implementations in the QuillenSuslin package ([29]), we can complete Q_1 to the following unimodular matrix:

$$W = (Q_3 \quad Q_1) = \begin{pmatrix} \dfrac{\alpha}{C\,(R\,C - G\,L)} & -C\,(L\,\partial_t + R) + \alpha\,\partial_x \\ \dfrac{1}{R\,C - G\,L} & C\,(\partial_x - \alpha\,\partial_t) - \alpha\,G \end{pmatrix} \in \mathrm{GL}_2(A).$$

Moreover, we can check that the matrix

$$X = (J\,Q_3 \quad \Lambda) = \begin{pmatrix} \dfrac{\alpha\,\partial_x + C\,(L\,\partial_t + R)}{C\,(R\,C - L\,G)} & \alpha \\ \dfrac{C\,(\partial_x + \alpha\,\partial_t) + \alpha\,G}{C\,(R\,C - L\,G)} & C \end{pmatrix} \in B^{2 \times 2}$$

is unimodular, i.e., $X \in \mathrm{GL}_2(B)$, and satisfies

$$J\,W = X\,\mathrm{diag}(1, Q_2) \quad \Leftrightarrow \quad X^{-1}\,J\,W = \mathrm{diag}(1, Q_2),$$

which proves that the matrix R is equivalent to $\mathrm{diag}(1, Q_2)$.

Example 4.3.3. Let us consider again Examples 4.2.4 and 4.2.7. Since $\Lambda = (1 \quad i)^T$ admits the left inverse $\Gamma = (1 \quad 0)$, Corollary 4.3.1 shows that the matrix R defined by (4.9) is equivalent to $\mathrm{diag}(1, i\,x\,(\partial_x^2 + \partial_y^2) - \partial_y))$. If Q_1 denotes the column vector formed by the first two entries of (4.15), then $\ker_A(.Q_1) = A\,(-i\,\partial_x + \partial_y \quad x\,(\partial_x + i\,\partial_y)) \cong A$, i.e., $\ker_A(.Q_1)$ is a free left A-module of rank 1. Since $Q_3 = (i\,x \quad -1)^T$ is a right inverse of $(-i\,\partial_x + \partial_y \quad x\,(\partial_x + i\,\partial_y))$, we obtain the unimodular matrix W defined by:

$$W = \begin{pmatrix} i\,x & x\,(i\,\partial_x - \partial_y) - i \\ -1 & -\partial_x - i\,\partial_y \end{pmatrix}, \quad W^{-1} = \begin{pmatrix} -i\,\partial_x + \partial_y & x\,(\partial_x + i\,\partial_y) \\ i & -x \end{pmatrix}.$$

Moreover, using Corollary 4.3.1, we can also introduce the unimodular matrices:

$$X = (R\,Q_3 \quad \Lambda) = \begin{pmatrix} x\,(i\,\partial_x + \partial_y) + i & 1 \\ -x\,(\partial_x - i\,\partial_y) & i \end{pmatrix},$$

$$V = X^{-1} = \begin{pmatrix} -i & 1 \\ -x\,(\partial_x - i\,\partial_y) & -x\,(i\,\partial_x + \partial_y) - i \end{pmatrix}.$$

Finally, we can easily check that $V\,R\,W = \mathrm{diag}(1, i\,x\,(\partial_x^2 + \partial_y^2) - \partial_y))$.

Example 4.3.4. Let us consider again Example 4.2.8. Since $\Gamma = (1 \quad 0)$ is a left inverse of Λ and using Corollary 4.3.1, we obtain the following unimodular matrices:

$$W = \begin{pmatrix} -1 & \partial_y & \partial_x \\ -x & x\,\partial_y & x\,\partial_x - 1 \\ -x^2 & x^2\,\partial_y - 1 & x\,(x\,\partial_x - 1) \end{pmatrix}, \quad W^{-1} = \begin{pmatrix} x\,\partial_x & x\,\partial_y - \partial_x & -\partial_y \\ 0 & x & -1 \\ x & -1 & 0 \end{pmatrix},$$

$$X = \begin{pmatrix} -(\partial_x + x\,\partial_y) & 1 \\ -x\,(\partial_x + x\,\partial_y) - 1 & x \end{pmatrix}, \quad X^{-1} = \begin{pmatrix} x & -1 \\ x^2\,\partial_y + x\,\partial_x + 2 & -(\partial_x + x\,\partial_y) \end{pmatrix}.$$

Hence, the matrix R defined by (4.16) is equivalent to

$$\overline{R} = X^{-1}\,R\,W = \begin{pmatrix} 1 & 0 & 0 \\ 0 & (\partial_x + x\,\partial_y)\,\partial_y & (\partial_x + x\,\partial_y)\,\partial_x - \partial_y \end{pmatrix},$$

which proves that (4.17) is equivalent to the following PD equation with varying coefficients

$$(\partial_x + x\,\partial_y)\,\partial_y\,\tau_2 + (\partial_x + x\,\partial_y)\,\partial_x\,\tau_3 - \partial_y\,\tau_3 = 0,$$

under the following invertible transformations:

$$\begin{cases} \sigma^{11} = \partial_y\,\tau_2 + \partial_x\,\tau_3, \\ \sigma^{12} = x\,\partial_y\,\tau_2 + x\,\partial_x\,\tau_3 - \tau_3, \\ \sigma^{22} = x^2\,\partial_y\,\tau_2 - \tau_2 + x^2\,\partial_x\,\tau_3 - x\,\tau_3, \end{cases} \qquad \begin{cases} \tau_1 = x\,(\partial_x\,\sigma^{11} + \partial_y\,\sigma^{12}) - (\partial_x\,\sigma^{12} + \partial_y\,\sigma^{22}) = 0, \\ \tau_2 = x\,\sigma^{12} - \sigma^{22}, \\ \tau_3 = x\,\sigma^{11} - \sigma^{12}. \end{cases}$$

We note that we have lost the symmetry of (4.17). It would be interesting to get a more symmetric equivalent PD equation by considering another cyclic vector of $\mathrm{ext}^1_E(M', E)$.

Let us illustrate the interest of Serre's reduction with a larger example.

Example 4.3.5. Let us consider a model of a two reflector antenna studied in [47, 75] which is defined by the linear OD time-delay system $\ker_{\mathcal{F}}(R.)$, where

$$R = \begin{pmatrix} \partial & -K_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \partial + \frac{K_2}{T_e} & 0 & 0 & 0 & 0 & -\frac{K_p}{T_e}\,\delta & -\frac{K_c}{T_e}\,\delta & -\frac{K_c}{T_e}\,\delta \\ 0 & 0 & \partial & -K_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \partial + \frac{K_2}{T_e} & 0 & 0 & -\frac{K_c}{T_e}\,\delta & -\frac{K_p}{T_e}\,\delta & -\frac{K_c}{T_e}\,\delta \\ 0 & 0 & 0 & 0 & \partial & -K_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \partial + \frac{K_2}{T_e} & -\frac{K_c}{T_e}\,\delta & -\frac{K_c}{T_e}\,\delta & -\frac{K_p}{T_e}\,\delta \end{pmatrix},$$

$\partial\,y(t) = \dot{y}(t)$, $\delta\,y(t) = y(t-1)$ for all $y \in \mathcal{F} = C^\infty(\mathbb{R})$, and K_1, K_2, K_c, K_e, K_p and T_e are constant parameters. Let $D = \mathbb{Q}(K_1, K_2, K_c, K_e, T_e)\,[\partial, \delta]$ be the commutative polynomial ring of OD time-delay operators and $M = D^{1\times 9}/(D^{1\times 6}\,R)$ the D-module finitely presented by R. If

$$\Lambda = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in D^{6\times 3},$$

then the matrix $S \in D^{12\times 6}$ defined in Figure 4.1 is a right inverse of $P = (R \quad -\Lambda) \in D^{6\times 12}$. Hence, the D-module $E = D^{1\times 12}/(D^{1\times 6}\, P)$ is projective, and thus free by the Quillen-Suslin theorem. Using the QuillenSuslin package ([29]), we can compute a basis and an injective parametrization of E. We get that the matrix $Q \in D^{12\times 6}$ given in Figure 4.1 defines an injective parametrization of E, i.e., $\ker_D(.Q) = D^{1\times 6}\, P \cong D^{1\times 6}$. Using Theorem 4.2.2 and Corollary 4.2.2, we obtain that $M \cong L = D^{1\times 6}/(D^{1\times 3}\, Q_2)$, where Q_2 is the matrix defined by the last three rows of Q, and thus $\ker_{\mathcal{F}}(R.) \cong \ker_{\mathcal{F}}(Q_2.)$, i.e.:

$$\begin{cases} T_e\, \ddot{\zeta}_1(t) + K_2\, \dot{\zeta}_1(t) + (K_p + 2\, K_c)\, (K_c - K_p)\, \zeta_2(t-1) = 0, \\ T_e\, \ddot{\zeta}_3(t) + K_2\, \dot{\zeta}_3(t) + (K_p + 2\, K_c)\, (K_c - K_p)\, \zeta_4(t-1) = 0, \\ T_e\, \ddot{\zeta}_5(t) + K_2\, \dot{\zeta}_5(t) + (K_p + 2\, K_c)\, (K_c - K_p)\, \zeta_6(t-1) = 0. \end{cases}$$

We note that the equations of the above system are uncoupled, i.e.:

$$M \cong [D^{1\times 2}/(D\, ((T_e\, \partial + K_2)\, \partial \quad (K_p + 2\, K_c)\ (K_c - K_p)\ \delta)]^3. \tag{4.18}$$

The matrix Λ admits a left inverse Γ defined by:

$$\Gamma = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Hence, let us compute $V \in \mathrm{GL}_6(D)$ and $W \in \mathrm{GL}_9(D)$ such that $V\, R\, W = \mathrm{diag}(I_3, Q_2)$. The D-module $\ker_D(.Q_1)$ is a stably free and thus a free D-module of rank 3 by the Quillen-Suslin theorem. This last result can be checked again by computing the D-module $\ker_D(.Q_1)$: we have $\ker_D(.Q_1) = D^{1\times 3}\, F \cong D^{1\times 3}$, where the full row rank matrix $F \in D^{3\times 9}$ is defined by:

$$F = \begin{pmatrix} \partial & -K_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \partial & -K_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \partial & -K_1 & 0 & 0 & 0 \end{pmatrix}.$$

Computing a right inverse of F, we obtain that the matrix $Q_3 \in D^{9\times 3}$ defined by

$$Q_3 = -\frac{1}{K_1} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$S = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{K_1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{K_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{K_1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{T_e+K_2}{K_1\,T_e}\,\partial & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{T_e+K_2}{K_1\,T_e}\,\partial & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{T_e+K_2}{K_1\,T_e}\,\partial & -1 \end{pmatrix}$$

$$Q = \begin{pmatrix} K_1\,T_e & 0 & 0 & 0 & 0 & 0 \\ T_e\,\partial & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & K_1\,T_e & 0 & 0 & 0 \\ 0 & 0 & T_e\,\partial & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & K_1\,T_e & 0 \\ 0 & 0 & 0 & 0 & T_e\,\partial & 0 \\ 0 & T_e\,(K_p+K_c) & 0 & -K_c\,T_e & 0 & -K_c\,T_e \\ 0 & -K_c\,T_e & 0 & T_e\,(K_p+K_c) & 0 & -K_c\,T_e \\ 0 & -K_c\,T_e & 0 & -K_c\,T_e & 0 & T_e\,(K_p+K_c) \\ (T_e\,\partial+K_2)\,\partial & (K_p+2\,K_c)\,(K_c-K_p)\,\delta & 0 & 0 & 0 & 0 \\ 0 & 0 & (T_e\,\partial+K_2)\,\partial & (K_p+2\,K_c)\,(K_c-K_p)\,\delta & 0 & 0 \\ 0 & 0 & 0 & 0 & (T_e\,\partial+K_2)\,\partial & (2\,K_c+K_p)\,(K_c-K_p)\,\delta \end{pmatrix}$$

Figure 4.1: Matrices S and Q

is such that the matrix $W = (Q_3 \quad Q_1)$ defined by

$$W = \begin{pmatrix} 0 & 0 & 0 & K_1\,T_e & 0 & 0 & 0 & 0 & 0 \\ -K_1^{-1} & 0 & 0 & T_e\,\partial & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & K_1\,T_e & 0 & 0 & 0 \\ 0 & -K_1^{-1} & 0 & 0 & 0 & T_e\,\partial & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & K_1\,T_e & 0 \\ 0 & 0 & -K_1^{-1} & 0 & 0 & 0 & 0 & T_e\,\partial & 0 \\ 0 & 0 & 0 & 0 & T_e\,(K_p+K_c) & 0 & -K_c\,T_e & 0 & -K_c\,T_e \\ 0 & 0 & 0 & 0 & -K_c\,T_e & 0 & T_e\,(K_p+K_c) & 0 & -K_c\,T_e \\ 0 & 0 & 0 & 0 & -K_c\,T_e & 0 & -K_c\,T_e & 0 & T_e\,(K_p+K_c) \end{pmatrix}$$

is unimodular, i.e., $W \in \mathrm{GL}_9(D)$. Forming the matrix $X = (R\,Q_3 \quad \Lambda) \in D^{6\times 6}$, namely,

$$X = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -\dfrac{T_e\,\partial + K_2}{K_1\,T_e} & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -\dfrac{T_e\,\partial + K_2}{K_1\,T_e} & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -\dfrac{T_e\,\partial + K_2}{K_1\,T_e} & 0 & 0 & 1 \end{pmatrix},$$

then $X \in \mathrm{GL}_6(D)$. Its inverse is defined by

$$V = X^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ \dfrac{T_e\,\partial + K_2}{K_1\,T_e} & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \dfrac{T_e\,\partial + K_2}{K_1\,T_e} & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dfrac{T_e\,\partial + K_2}{K_1\,T_e} & 1 \end{pmatrix}$$

and the matrix $\overline{R} = V\,R\,W$ has the form $\mathrm{diag}(I_3, Q_2)$:

$$\overline{R} = V\,R\,W = \left(\begin{array}{cccccccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & (T_e\,\partial + K_2)\,\partial & (K_p + 2\,K_c)\,(K_c - K_p)\,\delta & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & (T_e\,\partial + K_2)\,\partial & (K_p + 2\,K_c)\,(K_c - K_p)\,\delta & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & & 0 & 0 & (T_e\,\partial + K_2)\,\partial & (K_p + 2\,K_c)\,(K_c - K_p)\,\delta \end{array}\right).$$

Finally, the D-module $L = D^{1\times 2}/(D\,((T_e\,\partial + K_2)\,\partial \quad (K_p + 2\,K_c)\,(K_c - K_p)\,\delta)$ is clearly torsion-free and δ-free ([73, 75]) and, using (4.18), so is $M \cong N^3$ (see also [75]).

We have the following consequence of Corollary 4.2.4, Example 2.3.8 and Theorem 1.5.4.

Corollary 4.3.2 ([21])**.** *Let $D = A\langle\partial\rangle$, where $A = k[t]$ or $k[[t]]$ and k is a field of characteristic 0, or $A = k\{t\}$ and $k = \mathbb{R}$ or $\mathbb{C}$, $R \in D^{q\times p}$ a full row rank matrix and $M = D^{1\times p}/(D^{1\times q}\,R)$ the left D-module finitely presented by R. Then, Theorem 4.2.1 holds and $\Lambda \in D^q$ can be chosen so that it admits a left inverse and $\tau(\Lambda)$ generates the right D-module $D^q/(R\,D^p) \cong \mathrm{ext}^1_D(M, D)$. Moreover, if $p - q \geq 1$, then Theorem 4.2.2 and Corollaries 4.2.1 and 4.2.2 hold. Finally, if $q \geq 3$, then Corollary 4.3.1 holds, i.e., the matrix R is equivalent to a matrix of the form $\mathrm{diag}(I_{q-1}, Q_2)$, where $Q_2 \in D^{1\times(p-q+1)}$.*

Example 4.3.6. Let $M = D^{1\times 4}/(D^{1\times 3}\,R)$ be the left $D = A_1(\mathbb{Q})$-module finitely presented by:

$$R = \begin{pmatrix} 1 & 0 & 0 & \partial \\ \partial & 1 & 1 & t \\ 0 & 0 & t\,\partial & t\,\partial^2 - t \end{pmatrix}.$$

The matrix $P = (R \quad -\Lambda)$, where $\Lambda = (0 \quad 1 \quad 1)^T$, admits the following right inverse:

$$S = \begin{pmatrix} 1 & -\partial & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & -1 \end{pmatrix}^T.$$

Therefore, the right D-module $\mathrm{ext}^1_D(M, D) \cong D^3/(R\, D^4)$ is cyclic and is generated by $\tau(\Lambda)$, and thus the left D-module $E = D^{1\times 5}/(D^{1\times 3}\, P)$ is stably free of rank 2, i.e., is free of rank 2 by Stafford's theorem (see 3 of Theorem 1.1.2). An injective parametrization of E is defined by the matrix $Q = (Q_1^T \quad Q_2^T) \in D^{5\times 2}$, where

$$Q_1 = \begin{pmatrix} \partial & 0 \\ -\partial^2 - \partial + 2\,t & t\,\partial - 1 \\ \partial & 1 \\ -1 & 0 \end{pmatrix}, \quad Q_2 = (t \quad t\,\partial),$$

i.e., we have $\ker_D(.Q) = D^{1\times 3}\, P$ and $T\, Q = I_2$, where:

$$T = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 \\ -1 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

Thus, we have $M \cong D^{1\times 2}/(D\, Q_2)$. Moreover, since Λ admits the left inverse $\Gamma = (0 \quad 0 \quad 1)$, the matrix R is equivalent to $\mathrm{diag}(I_2, Q_2)$. More precisely, we have $\ker_D(.Q_1) = D^{1\times 2}\, K$, where

$$K = \begin{pmatrix} 1 & 0 & 0 & \partial \\ (t+1)\,\partial & 1 & -t\,\partial + 1 & 2\,t \end{pmatrix},$$

and right inverse Q_3 of the matrix K, defined by

$$Q_3 = \begin{pmatrix} 1 & -\partial - 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}^T,$$

is such that $W = (Q_3 \quad Q_1) \in \mathrm{GL}_4(D)$. Finally, if we introduce the following two matrices

$$X = (R\, Q_3 \quad \Lambda) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ t\,\partial & 0 & 1 \end{pmatrix}, \quad V = X^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ t\,\partial & 1 & -1 \\ -t\,\partial & 0 & 1 \end{pmatrix},$$

then we have:

$$\overline{R} = V\, R\, W = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & t & t\,\partial \end{pmatrix}.$$

Example 4.3.7. Let us consider the following linear analytic OD system:

$$\begin{cases} \dot{x}_2(t) = 0, \\ \dot{x}_1(t) - \sin(t)\, u(t) = 0. \end{cases}$$

Corollary 4.3.2 shows that the $D = \mathbb{R}\{t\}\langle\partial\rangle$-module $M = D^{1\times 3}/(D^{1\times 2}\, R)$ finitely presented by

$$R = \begin{pmatrix} 0 & \partial & 0 \\ \partial & 0 & -\sin(t) \end{pmatrix}$$

admits a presentation defined by a row vector $Q_2 \in D^{1\times 2}$, i.e., $M \cong L = D^{1\times 2}/(D\,Q_2)$. if If we consider $\Lambda = (1 \quad 0)^T$, then the matrix $P = (R \quad -\Lambda) \in D^{2\times 4}$ is exactly the matrix R defined in Example 1.5.10. Then, Example 1.5.10 shows that the left D-module $E = D^{1\times 4}/(D^{1\times 2}\,P)$ is free of rank 2 and Q_2 is the last two entries of the last row of the matrix V defined in Example 1.5.10:

$$Q_2 = \\ (\sin(t) - \cos(t) + \cos^3(t))\,\partial - 3\,\cos^2(t)\,\sin(t) + \sin(t) + \cos(t) \;\; (\cos(t)\,\sin(t) - 1)\,\partial^2 - 2\,\sin^2(t)\,\partial).$$

Since Λ admits a left inverse, the matrix R is then equivalent to $\operatorname{diag}(1, Q_2)$. Using the notations of Example 1.5.10, we have:

$$Q_1 = \left(\begin{array}{cc} -\cos(t)\,\sin^2(t) & \cos(t)\,\sin(t)\,\partial - 1 \\ -\sin(t)\,(\cos(t)\,\sin(t) - 1) & (\cos(t)\,\sin(t) - 1)\,\partial - 1 \\ -\cos(t)\,\sin(t)\,\partial - 3\,\cos^2(t) + 1 & (\cos(t)\,\partial - 2\,\sin(t))\,\partial \end{array}\right).$$

Now, $\ker_D(.Q_1) = D\,K$, where $K = (\partial \quad 0 \quad -\sin(t))$, and the row column K admits the right inverse $Q_3 = (\cos(t)\,\sin(t) \quad 0 \quad \cos(t)\,\partial - 2\,\sin(t))^T$. Hence, we have $W = (Q_3 \quad Q_1) \in \operatorname{GL}_3(D)$. Moreover, we can easily check that:

$$X = (R\,Q_3 \quad \Lambda) = \left(\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right), \quad V = X^{-1} = X.$$

Finally, we obtain $\overline{R} = V\,R\,W = \operatorname{diag}(1, Q_2)$.

Since the rings $D = B_1(k)$, $k[\![t]\!][t^{-1}]\langle\partial\rangle$, where k is a field of characteristic 0, or $k\{t\}[t^{-1}]\langle\partial\rangle$, where $k = \mathbb{R}$ or $\mathbb{C}$, are simple principal left ideal domains (see, e.g., [10, 13]), using the concept of *Jacobson normal form*, namely, a generalization of the Smith normal form to principal left or right principal ideal domains (see, e.g., [25, 121]), one can prove that for every matrix $R \in D^{q\times p}$, there exist $V \in \operatorname{GL}_q(D)$, $W \in \operatorname{GL}_p(D)$ and $d \in D$ such that $V\,R\,W = \operatorname{diag}(1, \ldots, 1, d, 0, \ldots, 0)$, i.e., R is equivalent to the diagonal matrix $\overline{R} = \operatorname{diag}(1, \ldots, 1, d, 0, \ldots, 0)$, for a certain $d \in D$. In particular, if R has full row rank, i.e., $\ker_D(.R) = 0$, then R is equivalent to $\operatorname{diag}(1, \ldots, 1, d)$.

Now, if $D = A_1(k)$, $k[\![t]\!]\langle\partial\rangle$, where k is a field of characteristic 0, or $k\{t\}\langle\partial\rangle$, where $k = \mathbb{R}$ or $\mathbb{C}$, and $R \in D^{q\times p}$, then the Jacobson normal form of R can be computed by considering the injection of D into the simple principal left ideal domain E, where E is respectively $B_1(k)$, $k[\![t]\!][t^{-1}]\langle\partial\rangle$ and $k\{t\}[t^{-1}]\langle\partial\rangle$. Therefore, there exist $V \in \operatorname{GL}_q(E)$, $W \in \operatorname{GL}_p(E)$ and $e \in E$ such that $V\,R\,W = \operatorname{diag}(1, \ldots, 1, e, 0, \ldots, 0)$. However, artificial singularities may have been introduced in e, V and W. The main interest of Corollary 4.3.2 is to show that there exist three matrices $Q_2 \in D^{1\times(p-q+1)}$, $X \in \operatorname{GL}_q(D)$ and $Y \in \operatorname{GL}_p(D)$ such that:

$$X\,R\,Y = \left(\begin{array}{cc} I_{q-1} & 0 \\ 0 & Q_2 \end{array}\right).$$

In particular, the entries of Q_2, X, Y, X^{-1} and Y^{-1} belong to D, i.e., do not contain singularities.

For more results, details and examples on Serre's reduction, see [100].

"Ce qui fait la qualité de l'inventivité et de l'imagination du chercheur, c'est la **qualité de son attention**, à l'écoute de la voix des choses. Car les choses de l'Univers ne se lassent jamais de parler d'elles-mêmes et de se révéler, à celui qui se soucie d'entendre".

Alexandre Grothendieck, *Récoltes et Semailles, Réflexions et témoignage sur un passé de mathématicien.*

La Nature est un temple où de vivants piliers
Laissent parfois sortir de confuses paroles ;
L'homme y passe à travers des forêts de symboles
Qui l'observent avec des regards familiers.

Comme de longs échos qui de loin se confondent
Dans une ténébreuse et profonde unité,
Vaste comme la nuit et comme la clarté,
Les parfums, les couleurs et les sons se répondent...

Charles Baudelaire, Correspondances, *Les Fleurs du Mal*.

Chapter 5

Implementations

The purpose of this chapter is to shortly demonstrate the Maple packages I have been developing over the last years with my colleagues, namely, Chyzak (INRIA Rocquencourt) and Robertz (RWTH Aachen University) for OreModules ([17]), Cluzeau (ENSIL, University of Limoges) for OreMorphisms ([20]), Robertz for Stafford ([103]) and Culianez (internship) for Jacobson ([25]). The Serre package is being developed in collaboration with Cluzeau ([21]). The PurityFiltration package ([98]), that I developed on my own, will be soon available.

5.1 The OreModules package

Example 5.1.1. Let us consider the linearized model of a bipendulum studied in [85], i.e., a system composed of a bar where two pendula are fixed, one of length l_1 and one of length l_2. We first introduce the ring $A = \mathbb{Q}(l_1, l_2, g)[d]$ of OD operators in d with coefficients in $\mathbb{Q}(l_1, l_2, g)$:

```
> A:=DefineOreAlgebra(diff=[d,t],polynom=[t],comm=[g,l[1],l[2]]):
```

The presentation matrix of the corresponding system is defined by:

```
> R:=evalm([[d^2+g/l[1],0,-g/l[1]],[0 d^2+g/l[2],-g/l[2]]]);
```

$$R := \begin{bmatrix} d^2 + \dfrac{g}{l_1} & 0 & -\dfrac{g}{l_1} \\ 0 & d^2 + \dfrac{g}{l_2} & -\dfrac{g}{l_2} \end{bmatrix}$$

In terms of equations, the linearized model of the bipendulum is described by:

```
> ApplyMatrix(R,[x[1](t),x[2](t),u(t)],A)=evalm([[0]$2]);
```

$$\begin{bmatrix} (\frac{d^2}{dt^2}\, x_1(t)) + \dfrac{g\, x_1(t)}{l_1} - \dfrac{g\, u(t)}{l_1} \\ (\frac{d^2}{dt^2}\, x_2(t)) + \dfrac{g\, x_2(t)}{l_2} - \dfrac{g\, u(t)}{l_2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Using the involution θ defined by (1.20), the adjoint $\widetilde{R}$ of R is defined by R^T:

```
> R_adj:=Involution(R,A);
```

$$R\_adj := \begin{bmatrix} d^2 + \frac{g}{l_1} & 0 \\ 0 & d^2 + \frac{g}{l_2} \\ -\frac{g}{l_1} & -\frac{g}{l_2} \end{bmatrix}$$

Using Algorithm 1.3.1, the A-module $M = A^{1\times 3}/(A^{1\times 2}\, R)$ is torsion-free iff the A-module $\text{ext}^1_A(N, A)$ vanishes, where $N = A^{1\times 2}/(A^{1\times 3}\, R^T)$ is the Auslander transpose of M:

```
>  Ext:=Exti(R_adj,A,1);
```

$$Ext := \left[\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} d^2\, l_1 + g & 0 & -g \\ 0 & d^2\, l_2 + g & -g \end{bmatrix}, \begin{bmatrix} l_2\, d^2\, g + g^2 \\ g^2 + d^2\, l_1\, g \\ l_2\, l_1\, d^4 + l_2\, d^2\, g + d^2\, l_1\, g + g^2 \end{bmatrix} \right]$$

The fact that the first matrix $Ext[1]$ of Ext is the identity matrix means that M is generically torsion-free, i.e., torsion-free for at most all values of the system parameters l_1, l_2 and g. We can only conclude that it is generically the case because OreModules considers the system parameters as independent variables which do not fulfill algebraic relations. The second matrix $Ext[2]$ of Ext is the matrix R' defined in Algorithm 1.3.1. The last matrix $Ext[3]$ of Ext is to the matrix Q of Algorithm 1.3.1, i.e., the parametrization of the torsion-free A-module M.

If $\mathcal{F} = C^\infty(\mathbb{R}_+)$, then, for almost all the values of the system parameters g, l_1 and l_2, $\ker_{\mathcal{F}}(R.)$ does not admit autonomous elements (see 1 of Definition 1.6.1). Below, we shall actually determine the only configuration where $\ker_{\mathcal{F}}(R.)$ is not parametrizable. Let us write down the parametrization $Ext[3]$ of $\ker_{\mathcal{F}}(R.)$ in terms of arbitrary functions of $\mathcal{F}$:

```
>  Q:=Parametrization(R,A);
```

$$Q := \begin{bmatrix} l_2\, \frac{d^2}{dt^2}\, \xi_1(t) + g\, (g\, \xi_1(t)) \\ l_1\, \frac{d^2}{dt^2}\, \xi_1(t) + g\, (g\, \xi_1(t)) \\ l_1\, l_2\, (\frac{d^4}{dt^4}\, \xi_1(t)) + g\, l_2\, \frac{d^2}{dt^2}\, \xi_1(t) + g\, l_1\, \frac{d^2}{dt^2}\, \xi_1(t) + g^2\, \xi_1(t) \end{bmatrix}$$

We have $\ker_{\mathcal{F}}(R.) = Q\, \mathcal{F}$, i.e., $R\, (x_1 \quad x_2 \quad u)^T = 0 \Leftrightarrow (x_1 \quad x_2 \quad u)^T = Q\, \xi_1$ for a certain $\xi_1 \in \mathcal{F}$. Since M is generically torsion-free over the principal ideal domain A, it is generically free (see 1 of Theorem 1.1.2). Hence, $\ker_{\mathcal{F}}(R.)$ is generically flat (see 6 of Definition 1.6.1). A flat output of $\ker_{\mathcal{F}}(R.)$ corresponds to a left inverse of the parametrization Q of $\ker_{\mathcal{F}}(R.)$

```
>  T:=LeftInverse(Ext[3],A);
```

$$T := \begin{bmatrix} \frac{l_1}{g^2\, (l_1 - l_2)} & -\frac{l_2}{g^2\, (l_1 - l_2)} & 0 \end{bmatrix}$$

i.e., a flat output of the system $\ker_{\mathcal{F}}(R.)$ is defined by $\xi_1 = T\, (x_1 \quad x_2 \quad u)^T$, namely:

```
>  xi[1](t)=ApplyMatrix(T,[x[1](t),x[2](t),u(t)],A)[1,1];
```

$$\xi_1(t) = \frac{l_1\, x_1(t)}{g^2\, (l_1 - l_2)} - \frac{l_2\, x_2(t)}{g^2\, (l_1 - l_2)}$$

Let us compute the Brunovský normal form of $\ker_{\mathcal{F}}(R.)$, namely, a simple first order representation of $\ker_{\mathcal{F}}(R.)$.

```
>  B:=Brunovsky(R,A);
```

$$B := \begin{bmatrix} \frac{l_1}{g^2\,(l_1-l_2)} & -\frac{l_2}{g^2\,(l_1-l_2)} & 0 \\ \frac{d\,l_1}{g^2\,(l_1-l_2)} & -\frac{d\,l_2}{g^2\,(l_1-l_2)} & 0 \\ -\frac{1}{g\,(l_1-l_2)} & \frac{1}{g\,(l_1-l_2)} & 0 \\ -\frac{d}{g\,(l_1-l_2)} & \frac{d}{g\,(l_1-l_2)} & 0 \\ \frac{1}{(l_1-l_2)\,l_1} & -\frac{1}{(l_1-l_2)\,l_2} & \frac{1}{l_1\,l_2} \end{bmatrix}$$

The matrix B defines the Brunovský transformation between the system variables x_1, x_2 and u and the Brunovský variables z_i's, $i = 1, \ldots, 4$, and v:

```
> evalm([seq([z[i](t)],i=1..4),[v(t)]])=ApplyMatrix(B,[x[1](t),x[2](t),u(t)],A);
```

$$\begin{bmatrix} z_1(t) \\ z_2(t) \\ z_3(t) \\ z_4(t) \\ v(t) \end{bmatrix} = \begin{bmatrix} \frac{l_1\,x_1(t)}{g^2\,(l_1-l_2)} - \frac{l_2\,x_2(t)}{g^2\,(l_1-l_2)} \\ \frac{l_1\,(\frac{d}{dt}\,x_1(t))}{g^2\,(l_1-l_2)} - \frac{l_2\,(\frac{d}{dt}\,x_2(t))}{g^2\,(l_1-l_2)} \\ -\frac{x_1(t)}{g\,(l_1-l_2)} + \frac{x_2(t)}{g\,(l_1-l_2)} \\ -\frac{\frac{d}{dt}\,x_1(t)}{g\,(l_1-l_2)} + \frac{\frac{d}{dt}\,x_2(t)}{g\,(l_1-l_2)} \\ \frac{x_1(t)}{(l_1-l_2)\,l_1} - \frac{x_2(t)}{(l_1-l_2)\,l_2} + \frac{u(t)}{l_1\,l_2} \end{bmatrix}$$

Let us check that the new variables z_i's and v satisfy the Brunovský normal form:

```
> F:=Elimination(linalg[stackmatrix](B,R),[x[1],x[2],u],
> [seq(z[i],i=1..4),v,0,0], A):
> ApplyMatrix(F[1],[x[1](t),x[2](t),u(t)],A)=ApplyMatrix(F[2],
> [seq(z[i](t),i=1..4),v(t)],A);
```

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ u(t) \\ x_2(t) \\ x_1(t) \end{bmatrix} = \begin{bmatrix} -(\frac{d}{dt}\,z_4(t)) + v(t) \\ -(\frac{d}{dt}\,z_3(t)) + z_4(t) \\ -(\frac{d}{dt}\,z_2(t)) + z_3(t) \\ -(\frac{d}{dt}\,z_1(t)) + z_2(t) \\ g^2\,z_1(t) + (g\,l_2 + g\,l_1)\,z_3(t) + l_1\,l_2\,v(t) \\ g^2\,z_1(t) + g\,l_1\,z_3(t) \\ g^2\,z_1(t) + g\,l_2\,z_3(t) \end{bmatrix}$$

The first four equations define the Brunovský normal form of $\ker_{\mathcal{F}}(R.)$. The last three equations express u, x_1 and x_2 in terms of the z_i's and v.

We note that the above flat output of $\ker_{\mathcal{F}}(R.)$ is only defined for $l_1 - l_2 \neq 0$. Then, the non-generic situation $l_1 = l_2$ corresponds to the only case where $\ker_{\mathcal{F}}(R.)$ may admit non-trivial autonomous elements. We now turn to the case where the lengths of the pendula are equal:

```
> U:=subs(l[2]=l[1],evalm(R));
```

$$U := \begin{bmatrix} d^2 + \frac{g}{l_1} & 0 & -\frac{g}{l_1} \\ 0 & d^2 + \frac{g}{l_1} & -\frac{g}{l_1} \end{bmatrix}$$

```
> ext:=Exti(Involution(U,A),A,1);
```

$$ext := \left[\begin{bmatrix} d^2\, l_1 + g & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 & 0 \\ 0 & d^2\, l_1 + g & -g \end{bmatrix}, \begin{bmatrix} g \\ g \\ d^2\, l_1 + g \end{bmatrix} \right]$$

The formal adjoint of U is $\theta(U) = U^T$. If $N' = A^{1\times 2}/(A^{1\times 3}\,\theta(U))$ is the Auslander transpose of the A-module $M' = A^{1\times 3}/(A^{1\times 2}\,U)$ finitely presented by U, then the computation of $\mathrm{ext}^1_A(N', A)$ gives the torsion submodule $t(M')$ of M': it is generated by the residue class of the row z of $ext[2]$ in M' which corresponds to the non-trivial entries in $ext[1]$, i.e., $l_1\, d^2 + g$. This means that we have $(l_1\, d^2 + g)\, z = 0$ in M', where $z = (1 \quad -1 \quad 0)\,(x_1 \quad x_2 \quad u)^T = x_1 - x_2$, i.e., the difference of the positions of the pendula (relative to the bar) is a torsion element of M' which generates $t(M') = (D^{1\times 2}\, U')/(D^{1\times 2}\, U)$, where $U' = ext[2]$ (see Algorithm 1.3.1).

We can directly obtain the torsion elements of M' as follows:

```
> TorsionElements(U,[x1(t),x2(t),u(t)],A);
```

$$\left[\left[l_1\, (\tfrac{d^2}{dt^2}\, \theta_1(t)) + g\, \theta_1(t) = 0 \right], \left[\theta_1(t) = x_1(t) - x_2(t) \right] \right]$$

We can explicitly integrate the corresponding autonomous element of $\ker_{\mathcal{F}}(U.)$ as follows

```
> AutonomousElements(U,[x[1](t),x[2](t),u(t)],A)[2];
```

$$\left[\theta_1 = \_C1\, \sin\left(\frac{\sqrt{g}\, t}{\sqrt{l_1}}\right) + \_C2\, \cos\left(\frac{\sqrt{g}\, t}{\sqrt{l_1}}\right) \right]$$

where $\_C1$ and $\_C2$ denote two arbitrary real constants.

As explained in Section 3.3, the existence of an autonomous element of $\ker_{\mathcal{F}}(U.)$ implies that of a first integral of $\ker_{\mathcal{F}}(U.)$. We can compute this first integral as follows:

```
> V:=FirstIntegral(U,[x[1](t),x[2](t),u(t)],A);
```

$$\begin{aligned} V := -(&-(\tfrac{d}{dt}\, x_1(t))\, \_C1\, \sin\left(\frac{\sqrt{g}\, t}{\sqrt{l1}}\right) \sqrt{l1} - (\tfrac{d}{dt}\, x_1(t))\, \_C2\, \cos\left(\frac{\sqrt{g}\, t}{\sqrt{l1}}\right) \sqrt{l1} \\ &+ \sqrt{g}\, x_1(t)\, \_C1\, \cos\left(\frac{\sqrt{g}\, t}{\sqrt{l1}}\right) - \sqrt{g}\, x_1(t)\, \_C2\, \sin\left(\frac{\sqrt{g}\, t}{\sqrt{l1}}\right) \\ &+ (\tfrac{d}{dt}\, x_2(t))\, \_C1\, \sin\left(\frac{\sqrt{g}\, t}{\sqrt{l1}}\right) \sqrt{l1} + (\tfrac{d}{dt}\, x_2(t))\, \_C2\, \cos\left(\frac{\sqrt{g}\, t}{\sqrt{l1}}\right) \sqrt{l1} \\ &- \sqrt{g}\, x_2(t)\, \_C1\, \cos\left(\frac{\sqrt{g}\, t}{\sqrt{l1}}\right) + \sqrt{g}\, x_2(t)\, \_C2\, \sin\left(\frac{\sqrt{g}\, t}{\sqrt{l1}}\right)) \Big/ \sqrt{l1} \end{aligned}$$

We let the reader check that we have $\dot{V}(t) = 0$. For the explicit computations, see [17].

Even if a non-trivial autonomous element exists in $\ker_{\mathcal{F}}(U.)$, we can parametrize all elements of $\ker_{\mathcal{F}}(U.)$ in terms of one arbitrary function $\xi_1 \in \mathcal{F}$ and two arbitrary constants $\_C_1$ and $\_C_2$ using the following Monge parametrization (see Section 2.2):

```
> P:=Parametrization(U,A);
```

$$P := \begin{bmatrix} g\,\xi_1(t) \\ -\_C1\,\sin\left(\frac{\sqrt{g}\,t}{\sqrt{l1}}\right) - \_C2\,\cos\left(\frac{\sqrt{g}\,t}{\sqrt{l1}}\right) + g\,\xi_1(t) \\ l1\,(\frac{d^2}{dt^2}\,\xi_1(t)) + g\,\xi_1(t) \end{bmatrix}$$

Therefore, we have $U\,(x_1 \quad x_2 \quad u)^T = 0 \Leftrightarrow (x_1 \quad x_2 \quad u)^T = P(\_C1, \_C2, \xi_1)$, where ξ_1 is an arbitrary element of $\mathcal{F} = C^\infty(\mathbb{R}_+)$ and $\_C1$ and $\_C2$ two arbitrary real constants. In particular, we can check that P defines elements of $\ker_{\mathcal{F}}(U.)$ (even parametrizes all) since we have:

```
> simplify(ApplyMatrix(U,P,A));
```

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Finally, the constants can easily be computed in terms of the initial conditions of the system.

Example 5.1.2. Let us study an OD time-delay model of a two reflector antenna considered in Example 4.3.5. Let $A = \mathbb{Q}(K_1, K_2, T_e, K_p, K_c)[d, \delta]$ be the commutative polynomial ring of OD time-delay operators, where d (resp., δ) is the OD (resp., time-delay) operator.

```
> A:=DefineOreAlgebra(diff=[d,t],dual_shift=[delta,s],polynom=[t,s],
> comm=[K1,K2,Te,Kp,Kc],shift_action=[delta,t]):
```

We enter the presentation matrix R of the two reflector antenna:

```
> R := evalm([[d, -K[1], 0, 0, 0, 0, 0, 0, 0],
> [0, d+K[2]/T[e], 0, 0, 0, 0, -K[p]/T[e]*delta, -K[c]/T[e]*delta,
> -K[c]/T[e]*delta],[0, 0, d, -K[1], 0, 0, 0, 0, 0],
> [0, 0, 0, d+K[2]/T[e], 0, 0, -K[c]/T[e]*delta, -K[p]/T[e]*delta,
> -K[c]/T[e]*delta],[0, 0, 0, 0, d, -K[1], 0, 0, 0],
> [0, 0, 0, 0, 0, d+K[2]/T[e], -K[c]/T[e]*delta, -K[c]/T[e]*delta,
> -K[p]/T[e]*delta]]);
```

$$R := \begin{bmatrix} d & -K_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & d+\frac{K_2}{T_e} & 0 & 0 & 0 & 0 & -\frac{K_p\,\delta}{T_e} & -\frac{K_c\,\delta}{T_e} & -\frac{K_c\,\delta}{T_e} \\ 0 & 0 & d & -K_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & d+\frac{K_2}{T_e} & 0 & 0 & -\frac{K_c\,\delta}{T_e} & -\frac{K_p\,\delta}{T_e} & -\frac{K_c\,\delta}{T_e} \\ 0 & 0 & 0 & 0 & d & -K_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & d+\frac{K_2}{T_e} & -\frac{K_c\,\delta}{T_e} & -\frac{K_c\,\delta}{T_e} & -\frac{K_p\,\delta}{T_e} \end{bmatrix}$$

The matrix R defines the following linear OD time-delay system:

```
> ApplyMatrix(R,[y[1](t),y[2](t),y[3](t),y[4](t),y[5](t),y[6](t),
> u[1](t),u[2](t),u[3](t)],A)=evalm([[0]$6]);
```

$$\begin{bmatrix} \mathrm{D}(y_1)(t) - K_1\, y_2(t) \\ -\dfrac{-\mathrm{D}(y_2)(t)\, T_e - K_2\, y_2(t) + K_p\, u_1(t-1) + K_c\, u_2(t-1) + K_c\, u_3(t-1)}{T_e} \\ \mathrm{D}(y_3)(t) - K_1\, y_4(t) \\ \dfrac{\mathrm{D}(y_4)(t)\, T_e + K_2\, y_4(t) - K_c\, u_1(t-1) - K_p\, u_2(t-1) - K_c\, u_3(t-1)}{T_e} \\ \mathrm{D}(y_5)(t) - K_1\, y_6(t) \\ \dfrac{\mathrm{D}(y_6)(t)\, T_e + K_2\, y_6(t) - K_c\, u_1(t-1) - K_c\, u_2(t-1) - K_p\, u_3(t-1)}{T_e} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Using the involution $\theta = \mathrm{id}_A$ of A, we can define the adjoint matrix $R\_adj = \theta(R) = R^T$ of R:

```
> R_adj:=Involution(R,A):
```

Let us consider the A-module $M = A^{1\times 9}/(A^{1\times 6}\, R)$ finitely presented by R and let us check whether or not M is a torsion-free A-module by computing the A-module $\mathrm{ext}^1_A(N, A)$, where $N = A^{1\times 6}/(A^{1\times 9}\, R^T)$ is the Auslander transpose of M (see 1 of Theorem 1.3.1):

```
> st:=time(): Ext1:=Exti(R_adj,A,1): time()-st;
```

$$0.920$$

```
> Ext1[1];
```

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The fact that the first matrix $Ext1[1]$ of $Ext1$ is the identity matrix implies $\mathrm{ext}^1_A(N, A) = 0$, i.e., using Corollary 1.3.1, M is a torsion-free A-module. Moreover, according to Algorithm 1.3.1, the third matrix $Ext1[3]$ of $Ext1$ defines a parametrization of M.

```
> Ext1[3];
```

$$\begin{bmatrix} K_c\, K_1\, \delta & K_c\, K_1\, \delta & K_p\, K_1\, \delta \\ K_c\, \delta\, d & K_c\, \delta\, d & K_p\, \delta\, d \\ K_c\, K_1\, \delta & K_p\, K_1\, \delta & K_c\, K_1\, \delta \\ K_c\, \delta\, d & K_p\, \delta\, d & K_c\, \delta\, d \\ K_p\, K_1\, \delta & K_c\, K_1\, \delta & K_c\, K_1\, \delta \\ K_p\, \delta\, d & K_c\, \delta\, d & K_c\, \delta\, d \\ 0 & 0 & T_e\, d^2 + K_2\, d \\ 0 & T_e\, d^2 + K_2\, d & 0 \\ d^2\, T_e + K_2\, d & 0 & 0 \end{bmatrix}$$

If $\mathcal{F}$ is an injective A-module, then, using 1 of Corollary 1.4.1, the system $\ker_{\mathcal{F}}(R.)$ is parametrizable and $Q = Ext1[3]$ defines a parametrization of $\ker_{\mathcal{F}}(R.)$, i.e., $\ker_{\mathcal{F}}(R.) = Q\, \mathcal{F}^3$. This parametrization can be obtained by using the function `Parametrization`:

```
> Parametrization(R,A);
```

$$\left[\begin{array}{c} K_c\,K_1\,\xi_1\,(t-1)+K_c\,K_1\,\xi_2\,(t-1)+K_p\,K_1\,\xi_3\,(t-1)\\ K_c\,\mathrm{D}\,(\xi_1)\,(t-1)+K_c\,\mathrm{D}\,(\xi_2)\,(t-1)+K_p\,\mathrm{D}\,(\xi_3)\,(t-1)\\ K_c\,K_1\,\xi_1\,(t-1)+K_p\,K_1\,\xi_2\,(t-1)+K_c\,K_1\,\xi_3\,(t-1)\\ K_c\,\mathrm{D}\,(\xi_1)\,(t-1)+K_p\,\mathrm{D}\,(\xi_2)\,(t-1)+K_c\,\mathrm{D}\,(\xi_3)\,(t-1)\\ K_p\,K_1\,\xi_1\,(t-1)+K_c\,K_1\,\xi_2\,(t-1)+K_c\,K_1\,\xi_3\,(t-1)\\ K_p\,\mathrm{D}\,(\xi_1)\,(t-1)+K_c\,\mathrm{D}\,(\xi_2)\,(t-1)+K_c\,\mathrm{D}\,(\xi_3)\,(t-1)\\ T_e\,\left(D^{(2)}\right)(\xi_3)\,(t)+K_2\,\mathrm{D}\,(\xi_3)\,(t)\\ T_e\,\left(D^{(2)}\right)(\xi_2)\,(t)+K_2\,\mathrm{D}\,(\xi_2)\,(t)\\ T_e\,\left(D^{(2)}\right)(\xi_1)\,(t)+K_2\,\mathrm{D}\,(\xi_1)\,(t) \end{array}\right]$$

The previous parametrization involves three arbitrary functions ξ_1, ξ_2 and ξ_3 of $\mathcal{F}$.

Let us now check whether or not the A-module M is reflexive. According to 3 of Theorem 1.3.1, we have to check that the second extension A-module $\mathrm{ext}^2_A(N, A)$ vanishes.

```
> Ext2[1];
```

$$\left[\begin{array}{ccc} \delta & 0 & 0\\ T_e\,d^2+K_2\,d & 0 & 0\\ 0 & \delta & 0\\ 0 & T_e\,d^2+K_2\,d & 0\\ 0 & 0 & \delta\\ 0 & 0 & T_e\,d^2+K_2\,d \end{array}\right]$$

Since the first matrix $Ext2[1]$ of $Ext2$ is not equal to the identity matrix, we obtain that the A-module $\mathrm{ext}^2_A(N, A)$ is not reduced to zero, and thus, M is a torsion but not reflexive A-module. In particular, M is not a free A-module, and by duality, the linear system $\ker_{\mathcal{F}}(R.)$ is not flat.

```
> PiPolynomial(R,A,[delta]);
```

$$[\delta]$$

By definition of π-polynomials (see 4 of Algorithm 4.2.1), it means that $L = A_\delta^{1\times 9}/(A_\delta^{1\times 6}\,R) \cong A_\delta \otimes_A M$ is a free $A_\delta = \mathbb{Q}(K_1, K_2, T_e, K_p, K_c)[d, \delta, \delta^{-1}]$-module. If $\mathcal{G}$ is an A_δ-module, then the new system $\ker_{\mathcal{G}}(R.)$ is flat.

Let us compute a basis of the free A_δ-module L, and thus, a flat output of $\ker_{\mathcal{G}}(R.)$. To do that, we apply the function `LocalLeftInverse` to the parametrization $Q = Ext1[3]$ of M but by allowing the invertibility of the polynomial δ in A_δ:

```
> T:=LocalLeftInverse(Ext1[3],[delta],A);
```

$$T := \begin{bmatrix} -\dfrac{K_c}{\%1} & 0 & -\dfrac{K_c}{\%1} & 0 & \dfrac{K_p+K_c}{\%1} & 0 & 0 & 0 & 0 \\ -\dfrac{K_c}{\%1} & 0 & \dfrac{K_p+K_c}{\%1} & 0 & -\dfrac{K_c}{\%1} & 0 & 0 & 0 & 0 \\ \dfrac{K_p+K_c}{\%1} & 0 & -\dfrac{K_c}{\%1} & 0 & -\dfrac{K_c}{\%1} & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\%1 := \delta\, K_1 \left(-2\, {K_c}^2 + {K_p}^2 + K_p\, K_c\right)$$

By construction, the matrix T is a left inverse of Q. Let us check this fact:

```
>  Mult(T,Ext1[3],A);
```

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Then, $(z_1 \quad z_2 \quad z_3)^T = T\,(y_1 \;\dots\; y_6 \quad u_1 \quad u_2 \quad u_3)^T$ is a basis of the free A_δ-module L, and thus, a flat output of the system $\ker_{\mathcal{G}}(R.)$, where $(y_1 \;\dots\; y_6 \quad u_1 \quad u_2 \quad u_3)^T = Q\,(z_1 \quad z_2 \quad z_3)^T$. More precisely, the flat output z_1, z_2 and z_3 of $\ker_{\mathcal{G}}(R.)$ is defined by:

```
>  evalm([seq([z[i](t)],i=1..3)])=ApplyMatrix(T,[seq(x[i](t),i=1..6),
>  seq(u[i](t),i=1..3)],A);
```

$$\begin{bmatrix} z_1(t) \\ z_2(t) \\ z_3(t) \end{bmatrix} = \begin{bmatrix} \dfrac{-K_c\, x_1(t+1) - K_c\, x_3(t+1) + K_p\, x_5(t+1) + K_c\, x_5(t+1)}{K_1\left(-2\,{K_c}^2 + {K_p}^2 + K_p\,K_c\right)} \\ \dfrac{-K_c\, x_1(t+1) + K_p\, x_3(t+1) + K_c\, x_3(t+1) - K_c\, x_5(t+1)}{K_1\left(-2\,{K_c}^2 + {K_p}^2 + K_p\,K_c\right)} \\ \dfrac{K_p\, x_1(t+1) + K_c\, x_1(t+1) - K_c\, x_3(t+1) - K_c\, x_5(t+1)}{K_1\left(-2\,{K_c}^2 + {K_p}^2 + K_p\,K_c\right)} \end{bmatrix}$$

Substituting the previous flat output of $\ker_{\mathcal{G}}(R.)$ into its parametrization $Ext1[3]$, we obtain the identity $(y_1 \;\dots\; y_6 \quad u_1 \quad u_2 \quad u_3) = U\,(y_1 \;\dots\; y_6 \quad u_1 \quad u_2 \quad u_3)$, where U is defined by:

```
>  U:=simplify(evalm(Ext1[3]&*S));
```

$$U := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{d}{K_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{d}{K_1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{d}{K_1} & 0 & 0 & 0 & 0 \\ \frac{(K_p + K_c)\,\%2\,d}{\%1} & 0 & -\frac{K_c \%2\,d}{\%1} & 0 & -\frac{K_c \%2\,d}{\%1} & 0 & 0 & 0 & 0 \\ -\frac{K_c\,\%2\,d}{\%1} & 0 & \frac{(K_p + K_c)\,\%2\,d}{\%1} & 0 & -\frac{K_c\,\%2\,d}{\%1} & 0 & 0 & 0 & 0 \\ -\frac{K_c\,\%2\,d}{\%1} & 0 & -\frac{K_c\,\%2\,d}{\%1} & 0 & \frac{(K_p + K_c)\,\%2\,d}{\%1} & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\%1 := \delta\, K_1 \left(-2\, {K_c}^2 + {K_p}^2 + K_p\, K_c\right)$$

$$\%2 := (T_e\, d + K_2)$$

We note that $(y_1 \; \ldots \; y_6 \quad u_1 \quad u_2 \quad u_3)$ can only be expressed in terms of y_1, y_3 and y_5. Hence, $\{y_1,\, y_3,\, y_5\}$ also defines a basis of the free A_δ-module L (see also [73]). More precisely, we have:

```
> evalm([seq([y[i](t)=ApplyMatrix(U,[seq(y[j](t),j=1..6),
> seq(u[j](t),j=1..3)],A)[i,1]],i=1..6)]);
```

$$\begin{bmatrix} y_1\,(t) = y_1\,(t) \\ y_2\,(t) = \frac{\mathrm{D}(y_1)(t)}{K_1} \\ y_3\,(t) = y_3\,(t) \\ y_4\,(t) = \frac{\mathrm{D}(y_3)(t)}{K_1} \\ y_5\,(t) = y_5\,(t) \\ y_6\,(t) = \frac{\mathrm{D}(y_5)(t)}{K_1} \end{bmatrix}$$

```
> evalm([seq([u[i](t)=ApplyMatrix(U,[seq(x[j](t),j=1..6),
> seq(u[j](t),j=1..3)],A)[6+i,1]],i=1..3)]);
```

$$\left[\begin{array}{ll}
u_1(t) = & \dfrac{K_2\,(K_p+K_c)\,\mathrm{D}\,(y_1)\,(t+1)+T_e\,(K_p+K_c)\,\left(D^{(2)}\right)(y_1)\,(t+1)}{K_1\left(-2\,{K_c}^2+{K_p}^2+K_p\,K_c\right)} \\
& -\dfrac{K_2\,K_c\,\mathrm{D}\,(y_3)\,(t+1)+T_e\,K_c\left(D^{(2)}\right)(y_3)\,(t+1)}{K_1\left(-2\,{K_c}^2+{K_p}^2+K_p\,K_c\right)} \\
& -\dfrac{K_2\,K_c\,\mathrm{D}\,(y_5)\,(t+1)+T_e\,K_c\left(D^{(2)}\right)(y_5)\,(t+1)}{K_1\left(-2\,{K_c}^2+{K_p}^2+K_p\,K_c\right)} \\
u_2(t) = & -\dfrac{K_2\,K_c\,\mathrm{D}\,(y_1)\,(t+1)+T_e\,K_c\left(D^{(2)}\right)(y_1)\,(t+1)}{K_1\left(-2\,{K_c}^2+{K_p}^2+K_p\,K_c\right)} \\
& +\dfrac{K_2\,(K_p+K_c)\,\mathrm{D}\,(y_3)\,(t+1)+T_e\,(K_p+K_c)\,\left(D^{(2)}\right)(y_3)\,(t+1)}{K_1\left(-2\,{K_c}^2+{K_p}^2+K_p\,K_c\right)} \\
& -\dfrac{K_2\,K_c\,\mathrm{D}\,(y_5)\,(t+1)+T_e\,K_c\left(D^{(2)}\right)(y_5)\,(t+1)}{K_1\left(-2\,{K_c}^2+{K_p}^2+K_p\,K_c\right)} \\
u_3(t) = & -\dfrac{K_2\,K_c\,\mathrm{D}\,(y_1)\,(t+1)+T_e\,K_c\left(D^{(2)}\right)(y_1)\,(t+1)+K_2\,K_c\,\mathrm{D}\,(y_3)\,(t+1)}{K_1\left(-2\,{K_c}^2+{K_p}^2+K_p\,K_c\right)} \\
& -\dfrac{T_e\,K_c\left(D^{(2)}\right)(y_3)\,(t+1)-K_2\,(K_p+K_c)\,\mathrm{D}\,(y_5)\,(t+1)}{K_1\left(-2\,{K_c}^2+{K_p}^2+K_p\,K_c\right)} \\
& +\dfrac{T_e\,(K_p+K_c)\,\left(D^{(2)}\right)(y_5)\,(t+1)}{K_1\left(-2\,{K_c}^2+{K_p}^2+K_p\,K_c\right)}
\end{array}\right]$$

Finally, the previous expressions of the inputs u_i's in terms of the flat outputs y_1, y_3 and y_5 can be used to solve motion planning problems in which the outputs of the system are exactly the previous flat outputs. For more details, see [73] and the references therein.

Example 5.1.3. Let us consider Example 1.2.10, namely, the linear PD system formed by the infinitesimal transformations of the Lie pseudogroup defining the contact transformations ([84]).

We first introduce the first Weyl algebra $A = A_3(\mathbb{Q})$ of PD operators in d_1, d_2 and d_3 and with coefficients in the commutative polynomial ring $\mathbb{Q}[x_1, x_2, x_3]$.

```
> A:=DefineOreAlgebra(diff=[d[1],x[1]],diff=[d[2],x[2]],diff=[d[3],x[3]],
> polynom=[x[1],x[2],x[3]]):
```

The linear PD system is then defined by the following presentation matrix R of PD operators:

```
> R:=evalm([[(x[2]/2)*d[1],x[2]*d[2]+1,x[2]*d[3]+d[1]/2],
> [-(x[2]/2)*d[2]-3/2,0,d[2]/2],[-d[1]-(x[2]/2)*d[3],-d[2],-d[3]/2]]);
```

$$R := \begin{bmatrix} \frac{x_2\,d_1}{2} & x_2\,d_2+1 & x_2\,d_3+\frac{d_1}{2} \\ -\frac{x_2\,d_2}{2}-\frac{3}{2} & 0 & \frac{d_2}{2} \\ -d_1-\frac{x_2\,d_3}{2} & -d_2 & -\frac{d_3}{2} \end{bmatrix}$$

Let us compute a finite free resolution of the left A-module $M = A^{1\times 3}/(A^{1\times 3}\,R)$:

```
> F:=FreeResolution(R,A);
```

$$F := \mathrm{table}([1 = \begin{bmatrix} \frac{x_2\,d_1}{2} & x_2\,d_2+1 & x_2\,d_3+\frac{d_1}{2} \\ -\frac{x_2\,d_2}{2}-\frac{3}{2} & 0 & \frac{d_2}{2} \\ -d_1-\frac{x_2\,d_3}{2} & -d_2 & -\frac{d_3}{2} \end{bmatrix},$$
$$2 = \begin{bmatrix} d_2 & -d_1-x_2\,d_3 & 2+x_2\,d_2 \end{bmatrix}, 3 = \mathrm{INJ}(1)])$$

Let us check whether or not the left A-module M admits a shorter free resolution.

```
> G:=ShorterFreeResolution(F,A);
```

$$G := \mathrm{table}([1 = \begin{bmatrix} \frac{x_2\,d_1}{2} & x_2\,d_2+1 & x_2\,d_3+\frac{d_1}{2} & -x_2 \\ -\frac{x_2\,d_2}{2}-\frac{3}{2} & 0 & \frac{d_2}{2} & 0 \\ -d_1-\frac{x_2\,d_3}{2} & -d_2 & -\frac{d_3}{2} & 1 \end{bmatrix}, 2 = \mathrm{INJ}(3)])$$

We obtain that the first matrix G_1 of G defines a shorter free resolution of the left A-module M, namely, we have $M \cong A^{1\times 4}/(A^{1\times 3}\,G_1)$. We note that this shorter free resolution of M can be directly obtained as folllows:

```
> ShortestFreeResolution(R,A);
```

$$\mathrm{table}([1 = \begin{bmatrix} \frac{x_2\,d_1}{2} & x_2\,d_2+1 & x_2\,d_3+\frac{d_1}{2} & -x_2 \\ -\frac{x_2\,d_2}{2}-\frac{3}{2} & 0 & \frac{d_2}{2} & 0 \\ -d_1-\frac{x_2\,d_3}{2} & -d_2 & -\frac{d_3}{2} & 1 \end{bmatrix}, 2 = \mathrm{INJ}(3)])$$

According to Proposition 1.3.3, the left A-module M is a stably free iff the matrix G_1 admits a right inverse:

```
> RightInverse(G[1],A);
```

$$\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & x_2 \\ 0 & -x_2 & 0 \\ d_2 & -d_1-x_2\,d_3 & 2+x_2\,d_2 \end{bmatrix}$$

We obtain that the left A-module M is stably free of rank $4-3=1$. This result can also be obtained by checking that $\mathrm{lpd}_D(M)=0$ as it then implies that M is a projective left A-module, i.e., stably free by Proposition 1.2.7:

```
> ProjectiveDimension(R,A);
```

$$0$$

Let us compute the rank of the finitely generated left A-module M:

```
> OreRank(R,A);
```

$$1$$

The fact that $\mathrm{rank}_A(M) < 2$ implies that we cannot use Stafford's theorem which asserts that every stably free left A-module of rank at least 2 is free (3 of Theorem 1.1.2) to conclude that M is a free left A-module of rank 1. However, we can try to find if there exists an injective minimal parametrization of M (see Definition 1.4.3):

```
> Q:=MinimalParametrization(R,A);
```

$$Q := \begin{bmatrix} -d_2 \\ d_1 + x_2\, d_3 \\ -2 - x_2\, d_2 \end{bmatrix}$$

```
> T:=LeftInverse(Q,A);
```

$$T := \begin{bmatrix} \dfrac{x_2}{2} & 0 & \dfrac{-1}{2} \end{bmatrix}$$

```
> Mult(T,Q,A);
```

$$\begin{bmatrix} 1 \end{bmatrix}$$

Hence, we obtain that M is a free left A-module of rank 1 and a basis z of M is defined by the residue class of T in the left A-module M. Moreover, the set of generators $\{y_j = \pi(f_j)\}_{j=1,2,3}$ of M satisfies $(y_1 \quad y_2 \quad y_3)^T = Q\, z$, i.e., Q is an injective parametrization of M. Finally, if $\mathcal{F}$ is a left A-module (e.g., $\mathcal{F} = \mathbb{Q}[x_1, x_2]$), then the underdetermined linear PD system $\ker_{\mathcal{F}}(R.)$ admits the following injective parametrization

```
> evalm([seq([eta[i](x)],i=1..3)])=Parametrization(R,A);
```

$$\begin{bmatrix} \eta_1(x_1,\, x_2,\, x_3) \\ \eta_2(x_1,\, x_2,\, x_3) \\ \eta_3(x_1,\, x_2,\, x_3) \end{bmatrix} = \begin{bmatrix} -(\frac{\partial}{\partial x_2}\, \xi_1(x_1,\, x_2,\, x_3)) \\ (\frac{\partial}{\partial x_1}\, \xi_1(x_1,\, x_2,\, x_3)) + x_2\, (\frac{\partial}{\partial x_3}\, \xi_1(x_1,\, x_2,\, x_3)) \\ -2\, \xi_1(x_1,\, x_2,\, x_3) - x_2\, (\frac{\partial}{\partial x_2}\, \xi_1(x_1,\, x_2,\, x_3)) \end{bmatrix}$$

i.e., $\ker_{\mathcal{F}}(R.) = Q\,\mathcal{F}$ and $T\,Q = 1$, and $\xi_1 = T\,\eta$ is defined by:

```
> xi[1](x)=ApplyMatrix(T,[seq(eta[i](x),i=1..3)],A)[1,1];
```

$$\xi_1(x_1,\, x_2,\, x_3) = \frac{1}{2}\, x_2\, \eta_1(x_1,\, x_2,\, x_3) - \frac{1}{2}\, \eta_3(x_1,\, x_2,\, x_3)$$

5.2 The Jacobson package

Example 5.2.1. Let us consider the first Weyl algebra $A = A_1(\mathbb{Q})$:

```
> A:=DefineOreAlgebra(diff=[d,t],polynom=[t])
```

and the following matrix R with entries in A:

```
> R:=evalm([[-t*d+1,t^2*d,-1,0],[-d,-t*d+1,0,-1]]);
```

$$R := \begin{bmatrix} -t\,d+1 & t^2\,d & -1 & 0 \\ -d & -t\,d+1 & 0 & -1 \end{bmatrix}$$

Let us compute the Hermite form of the matrix R over the principal left ideal domain $B_1(\mathbb{Q})$ of OD operators with rational coefficients containing A:

```
> H:=OreHermite(R,A,"monic");
```

$$H := \left[\begin{bmatrix} 1 & -t \\ d & -t\,d \end{bmatrix}, \begin{bmatrix} 1 & 2\,t^2\,d-t & -1 & t \\ 0 & 2\,d^2\,t^2+2\,t\,d & -d & -t\,d \end{bmatrix} \right]$$

The second matrix H_2 of H is the Hermite form of R and the relation $H_2 = H_1\,R$ holds, where H_1 is the first matrix of H. Let us check this point:

```
> Mult(H[1],R,A);
```

$$\begin{bmatrix} 1 & 2\,t^2\,d-t & -1 & t \\ 0 & 2\,d^2\,t^2+2\,t\,d & -d & t\,d \end{bmatrix}$$

Let us check that the matrix H_1 is unimodular, i.e., $H_1 \in \mathrm{GL}_2(B)$:

```
> LeftInverseRat(H[1],A);
```

$$\begin{bmatrix} -t\,d+1 & t \\ -d & 1 \end{bmatrix}$$

Let us now compute the Jacobson normal form of the matrix R:

```
> J:=OreJacobson(R,A);
```

$$J := \left[\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & -t\,d+1 & t^2\,d \\ 0 & 1 & -d & -t\,d+1 \end{bmatrix} \right]$$

The Jacobson form J_2 of R satisfies $J_2 = J_1\,R\,J_3$, where J_i is the i^{th} matrix of J:

```
> Mult(J[1],R,J[3],A);
```

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

The matrix J_1 is unimodular and its inverse is defined by:

```
> LeftInverseRat(J[1],A);
```

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

Similarly, the matrix J_3 is unimodular and its inverse is defined by:

```
> LeftInverseRat(J[3],A);
```

$$\begin{bmatrix} t\,d-1 & -t^2\,d & 1 & 0 \\ d & t\,d-1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Example 5.2.2. Let us consider the skew polynomial ring $A = \mathbb{Q}[n]\langle\sigma\rangle$ of forward shift operators with polynomial coefficients, namely, for all $a \in \mathbb{Q}[n]$, $\sigma(a(n)) = a(n+1)\,\sigma$:

```
> A:=DefineOreAlgebra(shift=[sigma,n],polynom=[n]):
```

Let R be the matrix with entries in A obtained by substituting d by σ and t by n in Example 5.2.1:

```
> R:=evalm([[-n*sigma+1,n^2*sigma,-1,0],[-sigma,n*sigma+1,0,-1]]);
```

$$R := \begin{bmatrix} -n\,\sigma+1 & n^2\,\sigma & -1 & 0 \\ -\sigma & n\,\sigma+1 & 0 & -1 \end{bmatrix}$$

Let us compute the Hermite normal form of R over the principal left ideal domain $B = \mathbb{Q}(n)\langle\sigma\rangle$ containing the ring A:

```
> H:=OreHermite(R,A,"monic");
```

$$H := \left[\begin{bmatrix} 1 & -n \\ \sigma & 1-n\,\sigma-\sigma \end{bmatrix}, \begin{bmatrix} 1 & -n & -1 & n \\ 0 & 1-\sigma & -\sigma & n\,\sigma+\sigma-1 \end{bmatrix}\right]$$

The matrix H_2 satisfies the relation $H_2 = H_1\,R$, where H_i is the i^{th} matrix of H:

```
> Mult(H[1],R,A);
```

$$\begin{bmatrix} 1 & -n & -1 & n \\ 0 & 1-\sigma & -\sigma & n\,\sigma+\sigma-1 \end{bmatrix}$$

The matrix H_1 is unimodular and its inverse is defined by:

```
> LeftInverseRat(H[1],A);
```

$$\begin{bmatrix} -n\,\sigma+1 & n \\ -\sigma & 1 \end{bmatrix}$$

Let us compute the Jacobson normal form of the matrix R:

```
> J:=OreJacobson(R,A);
```

$$J := \left[\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & -n\,\sigma+1 & n^2\,\sigma \\ 0 & 1 & -\sigma & n\,\sigma+1 \end{bmatrix}\right]$$

The Jacobson normal form J_2 of R satisfies $J_2 = J_1\,R\,J_3$, where J_i is the i^{th} matrix of J:

```
> Mult(J[1],R,J[3],A);
```

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

The matrix J_1 is clearly unimodular and we can check that J_3 is unimodular:

```
> LeftInverseRat(J[3],A);
```

$$\begin{bmatrix} n\,\sigma-1 & -n^2\,\sigma & 1 & 0 \\ \sigma & -n\,\sigma-1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

5.3 The QuillenSuslin package

Example 5.3.1. Let us consider the following row vector R with entries in $A = \mathbb{Q}[x, y]$:

```
> var:=[x,y]:
```

```
> R:=[x-4*y+2,x*y+x,x+4*y^2-2*y+1];
```

$$R := [x - 4\,y + 2,\; x\,y + x,\; x + 4\,y^2 - 2\,y + 1]$$

Let us check that the ideal generated by the entries of R is equal to A:

```
> IsUnimod(R,var,true);
```

$$true$$

Therefore, the matrix R admits a right inverse defined by:

```
> RightInverse(R,var,true);
```

$$[y,\; -1,\; 1]$$

Using Corollary 1.3.3, the A-module $M = A^{1\times 3}/(A\,R)$ is stably free and thus free by the Quillen-Suslin theorem (2 of Theorem 1.1.2). Let us now compute a basis of the free A-module M:

```
> U:=QSAlgorithm(R,var,true);
```

$$U := \left[\begin{array}{ccc} y & -2\,y + 4\,y^2 - x\,y + 1 & -y\,(x + 4\,y^2 - 2\,y + 1) \\ -1 & x - 4\,y + 2 & x + 4\,y^2 - 2\,y + 1 \\ 1 & -x + 4\,y - 2 & -x - 4\,y^2 + 2\,y \end{array}\right]$$

We can check that the first row of the inverse of U, denoted by U_{inv}, is exactly R:

```
> U_inv:=CompleteMatrix(R,var,true);
```

$$U\_inv := \left[\begin{array}{ccc} x - 4\,y + 2 & x\,y + x & x + 4\,y^2 - 2\,y + 1 \\ 1 & y & 0 \\ 0 & 1 & 1 \end{array}\right]$$

Therefore, the residue classes of the last two rows of U_{inv} in M form a basis of the free A-module M of rank 2. This result can directly be obtained by using the function `BasisOfCokernelModule`:

```
> BasisOfCokernelModule(Matrix(R),var,true);
```

$$\left[\begin{array}{ccc} 1 & y & 0 \\ 0 & 1 & 1 \end{array}\right]$$

Finally, an injective parametrization of the A-module M is given by the last two columns of U:

```
> InjectiveParametrization(Matrix(R),var,false);
```

$$\left[\begin{array}{cc} -2\,y + 4\,y^2 - x\,y + 1 & -y\,(x + 4\,y^2 - 2\,y + 1) \\ x - 4\,y + 2 & x + 4\,y^2 - 2\,y + 1 \\ -x + 4\,y - 2 & -x - 4\,y^2 + 2\,y \end{array}\right]$$

Example 5.3.2. Let us consider the linear OD time-delay system (1.71). The presentation matrix R of (1.71) is defined by

```
> R:=Matrix([[d-delta+2, 2,-2*delta],[d,d,-d*delta-1]]);
```

$$R := \begin{bmatrix} d-\delta+2 & 2 & -2\,\delta \\ d & d & -d\,\delta-1 \end{bmatrix}$$

where d (resp., δ) is the OD (resp., time-delay) operator. We consider the commutative polynomial ring $A = \mathbb{Q}[d, \delta]$ of OD time-delay operators and the A-module $M = A^{1\times 3}/(A^{1\times 2}\, R)$.

```
> var:=[d,delta];
```

$$var := [d,\, \delta]$$

Let us check whether or not the matrix R admits a right inverse:

```
> IsUnimod(R,var);
```

$$true$$

Since R admits a right inverse, the A-module M is stably free, and thus, free by the Quillen-Suslin theorem (2 of Theorem 1.1.2). Therefore, using Corollary 1.5.2, there exists $U \in \mathrm{GL}_3(D)$ such that $R\,U = (I_2 \quad 0)$. Let us compute such a matrix U:

```
> U:=QSAlgorithm(R,var);
```

$$U := \begin{bmatrix} 0 & 0 & -2 \\ \dfrac{d\,\delta}{2}+\dfrac{1}{2} & -\delta & d^2\,\delta+d-d\,\delta^2-\delta+2 \\ \dfrac{d}{2} & -1 & d^2-d\,\delta \end{bmatrix}$$

We can check again that the matrix U satisfies $R\,U = (I_2 \quad 0)$

```
> simplify(R.U);
```

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

and U is a unimodular matrix since the all the entries of its inverse U^{-1} belong to A:

```
> LinearAlgebra[MatrixInverse](U);
```

$$\begin{bmatrix} d-\delta+2 & 2 & -2\,\delta \\ d & d & -d\,\delta-1 \\ -1/2 & 0 & 0 \end{bmatrix}$$

The residue class of the last row T of the matrix U^{-1} in M defines a basis of the free A-module M. In particular, the free A-module M admits the following injective parametrization

```
> Q:=InjectiveParametrization(R,var,true);
```

$$Q := \begin{bmatrix} -2 \\ d^2\,\delta+d-d\,\delta^2-\delta+2 \\ d^2-d\,\delta \end{bmatrix}$$

i.e., we have $\ker_A(.Q) = A^{1\times 2}\, R$ and $T\,Q = I_2$. Moreover, the linear OD time-delay system $\ker_{\mathcal{F}}(R.)$ is flat and Q is an injective parametrization of $\ker_{\mathcal{F}}(R.)$, where $\mathcal{F}$ is a A-module (e.g., $C^\infty(\mathbb{R})$), i.e., every element $\eta \in \ker_{\mathcal{F}}(R.)$ has the form $\eta = Q\,\xi$ for a unique element $\xi \in \mathcal{F}$.

Moreover, using Corollary 1.5.3, the flat linear OD time-delay system $\ker_{\mathcal{F}}(R(d,\delta).)$ is equivalent to the linear controllable OD system $\ker_{\mathcal{F}}(R(d,1).)$. Let us compute an invertible transformation which sends the elements of $\ker_{\mathcal{F}}(R(d,1).)$ to those of $\ker_{\mathcal{F}}(R(d,\delta).)$:

```
> V:=SetLastVariableA(R,var,1,true);
```

$$V := \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2}\,d\,\delta^2 - \frac{1}{2}\,d\,\delta + \frac{1}{2}\,\delta - \frac{1}{2} & 1 & \delta - 1 \\ \frac{d\,(\delta-1)}{2} & 0 & 1 \end{bmatrix}$$

Let us check that the relation $R(d,\delta)\,V = R(d,1)$ holds:

```
> S:=simplify(R.V);
```

$$S := \begin{bmatrix} d+1 & 2 & -2 \\ d & d & -1-d \end{bmatrix}$$

Then, for all $\zeta \in \ker_{\mathcal{F}}(R(d,1).)$, we have $\eta = V\,\zeta \in \ker_{\mathcal{F}}(R(d,\delta).)$. The inverse transformation, i.e., the transformation sending $\ker_{\mathcal{F}}(R(d,\delta).)$ to $\ker_{\mathcal{F}}(R(d,1).)$, is defined by V^{-1}:

```
> LinearAlgebra[MatrixInverse](V);
```

$$\begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2}\,d\,\delta - \frac{1}{2}\,\delta + \frac{1}{2} + \frac{1}{2}\,d & 1 & -\delta + 1 \\ -\frac{d\,(\delta-1)}{2} & 0 & 1 \end{bmatrix}$$

Now, since the $E = \mathbb{Q}[d]$-module $N = E^{1\times 3}/(E^{1\times 2}\,S)$ is free, there exists $W \in \mathrm{GL}_3(E)$ such that $S\,W = (I_2 \quad 0)$. For instance, we can take the matrix

```
> W:=QSAlgorithm(S,var);
```

$$W := \begin{bmatrix} 0 & 0 & -2 \\ \frac{1}{2} + \frac{d}{2} & -1 & d^2+1 \\ \frac{d}{2} & -1 & d^2 - d \end{bmatrix}$$

whose determinant equals 1. Hence, the matrix W defines a one-to-one correspondence between the elements of $\ker_{\mathcal{F}}((I_2 \quad 0).) = \mathcal{F}$ and those of $\ker_{\mathcal{F}}(R(d,1).)$. Composing the transformations defined by V and W, we get a one-to-one correspondence between the elements of $\ker_{\mathcal{F}}((I_2 \quad 0).) = \mathcal{F}$ and those of $\ker_{\mathcal{F}}(R(d,\delta).)$. More precisely, for all $\theta \in \mathcal{F}$, we have $(0 \quad 0 \quad \theta)^T \in \ker_{\mathcal{F}}((I_2 \quad 0).)$ and, using the relation $U = V\,W$ and the fact that the last row of U is defined by the matrix Q, we finally get $\eta = U\,(0 \quad 0 \quad \theta)^T = Q\,\theta \in \ker_{\mathcal{F}}(R(d,\delta))$. Hence, we find again that Q defines an injective parametrization of $\ker_{\mathcal{F}}(R.)$.

Example 5.3.3. Let us consider the OD time-delay model of a flexible rod with a force applied on one end defined in Example 1.5.3. Let $A = \mathbb{Q}[d,\delta]$ be the commutative polynomial ring of OD time-delay operators, where d (resp., δ) is the OD (resp., time-delay) operator, and the presentation matrix $R \in A^{2\times 3}$ of (1.77) defined by

```
> var:=[d,delta];
```

$$var := [d,\ \delta]$$

```
> R:=Matrix([[d,-d*delta,-1],[2*delta*d,-d*delta^2-d,0]]);
```

$$R := \begin{bmatrix} d & -d\,\delta & -1 \\ 2\,d\,\delta & -d\,\delta^2 - d & 0 \end{bmatrix}$$

Let us check whether or not the A-module $M = A^{1\times 3}/(A^{1\times 2}\,R)$ is stably free, and thus, free by the Quillen-Suslin theorem:

```
> IsUnimod(R,var);
```

$$\mathit{false}$$

We obtain that R does not admit a right inverse, and thus, the A-module M is not free by Corollary 1.3.3. In particular, there is no matrix $U \in \mathrm{GL}_3(A)$ such that $R\,U = (I_2 \quad 0)$ or, equivalently, R cannot be completed to a matrix $V \in \mathrm{GL}_3(A)$. Let us compute the set of all maximal minors of R:

```
> m:=MaxMinors(R);
```

$$m := [d^2\,\delta^2 - d^2,\, 2\,d\,\delta,\, -d\,\delta^2 - d]$$

The ideal I of A defined by the maximal minors is generated by

```
> Involutive[InvolutiveBasis](m,var);
```

$$[d]$$

i.e., $I = (d)$. Thus, d is the greatest common divisor of the maximal minors of R. In particular, we obtain that the torsion A-submodule $t(M)$ of M is not reduced to 0. A solution of the first Lin-Bose's problem (see Section 1.5) can be obtained by means of `LinBose1` as follows:

```
> F:=LinBose1(R,var);
```

$$F := \left[\begin{bmatrix} -1 & 0 \\ 0 & -d \end{bmatrix}, \begin{bmatrix} -d & d\,\delta & 1 \\ -2\,\delta & \delta^2 + 1 & 0 \end{bmatrix} \right]$$

We then have $R = R''\,R'$ and $\det R'' = d$ and R' admits a right inverse:

```
> simplify(F[1].F[2]);
```

$$\begin{bmatrix} d & -d\,\delta & -1 \\ 2\,d\,\delta & -d\,\delta^2 - d & 0 \end{bmatrix}$$

```
> LinearAlgebra[Determinant](F[1]);
```

$$d$$

```
> IsUnimod(F[2],var);
```

$$\mathit{true}$$

Let us now solve the second Lin-Bose's problem (see Section 1.5).

```
> P:=LinBose2(R,var);
```

$$P := \begin{bmatrix} d & -d\,\delta & -1 \\ 2\,d\,\delta & -d\,\delta^2 - d & 0 \\ -1 & \dfrac{\delta}{2} & 0 \end{bmatrix}$$

Hence, we have embedded R in the square matrix P whose determinant is:

```
> LinearAlgebra[Determinant](C);
```

$$d$$

5.4 The Stafford package

Example 5.4.1. Let us consider Example 2 of [57], namely, the left ideal I of the first Weyl algebra $A = A_3(\mathbb{Q})$ defined by the following three PD operators

```
> A:=DefineOreAlgebra(diff=[d[1],x[1]],diff=[d[2],x[2]],diff=[d[3],x[3]],
> polynom=[x[1],x[2],x[3]]):
> P[1]:=d[1]*d[3]^2; P[2]:=d[1]*d[2]; P[3]:=d[2]*d[3]^2;
```

$$P_1 := d_1\, d_3^2$$
$$P_2 := d_1\, d_2$$
$$P_3 := d_2\, d_3^2$$

i.e., $I = A\, P_1 + A\, P_2 + A\, P_3$. Using Stafford's theorem (see Theorem 1.5.2), the left ideal I can be generated by two elements of A. Let us compute such pairs of PD operators:

```
> G:=TwoGenerators(P[1],P[2],P[3],A);
```

$$G := [d_1\, d_3^2,\ d_1\, d_2 + (x_1\, x_3^2 + x_1^2\, x_3 + x_1^3)\, d_2\, d_3^2,\ [0,\ x_1\, x_3^2 + x_1^2\, x_3 + x_1^3]]$$

Thus, the left ideal I is also generated by the first two entries G_1 and G_2 of G. Let us check again this result by computing Gröbner bases of I and the left ideal $J = A\, G_1 + A\, G_2$:

```
> Gbasis([P[1],P[2],P[3]],A); Gbasis([G[1],G[2]],A);
```

$$[d_1\, d_2,\ d_2\, d_3^2,\ d_1\, d_3^2]$$
$$[d_1\, d_2,\ d_2\, d_3^2,\ d_1\, d_3^2]$$

The left ideal I can also be generated by the first two entries H_1 and H_2 of H defined by:

```
> H:=TwoGenerators(P[3],P[1],P[2],A);
```

$$H := [d_2\, d_3^2,\ d_1\, d_3^2 + (x_3^2\, x_2 + x_3 + x_3^4)\, d_1\, d_2,\ [0,\ x_3^2\, x_2 + x_3 + x_3^4]]$$

Let us check again this result by computing a Gröbner basis of the left ideal of A generated by the first two entries H_1 and H_2 of H:

```
> Gbasis([H[1],H[2]],A);
```

$$[d_1\, d_2,\ d_2\, d_3^2,\ d_1\, d_3^2]$$

Finally, I can also be generated by the first two following entries K_1 and K_2 of K defined by

```
> K:=TwoGeneratorsRat(P[2],P[3],P[1],A);
```

$$K := [d_1\, d_2,\ d_2\, d_3^2 + (x_1\, x_2 + x_2^2)\, d_1\, d_3^2,\ [0,\ x_1\, x_2 + x_2^2]]$$

i.e., $I = A\, K_1 + A\, K_2$, since we also have:

```
> Gbasis([K[1],K[2]],A);
```

$$[d_1\, d_2,\ d_2\, d_3^2,\ d_1\, d_3^2]$$

Example 5.4.2. Let us consider the first Weyl algebra $A = A_3(\mathbb{Q})$ of PD operators with coefficients in the commutative polynomial ring $\mathbb{Q}[x_1, x_2, x_3]$:

```
> A:=DefineOreAlgebra(diff=[d[1],x[1]],diff=[d[2],x[2]],diff=[d[3],x[3]],
> polynom=[x[1],x[2],x[3]]):
```

We consider the following system matrix of PD operators:

```
> R:=evalm([[d[1]+x[3],d[2],d[3]]]);
```

$$R := \begin{bmatrix} d_1 + x_3 & d_2 & d_3 \end{bmatrix}$$

The corresponding PD linear system is $\vec{\nabla} \,.\, \vec{y} + x_3\, y_1 = 0$, namely:

```
> x :=x[1],x[2],x[3]:
> ApplyMatrix(R,[seq(y[i](x),i=1..3)],A)[1,1]=0;
```

$$x_3\, y_1(x_1,\, x_2,\, x_3) + (\tfrac{\partial}{\partial x_1}\, y_1(x_1,\, x_2,\, x_3)) + (\tfrac{\partial}{\partial x_2}\, y_2(x_1,\, x_2,\, x_3)) + (\tfrac{\partial}{\partial x_3}\, y_3(x_1,\, x_2,\, x_3)) = 0$$

Let us check whether or not the finitely presented left A-module $M = A^{1\times 3}/(A\, R)$ is stably free:

```
> S:=RightInverse(R,Alg);
```

$$S := \begin{bmatrix} -d_3 \\ 0 \\ d_1 + x_3 \end{bmatrix}$$

Hence, the matrix R admits a right inverse S and

```
> Mult(R,S,Alg);
```

$$\begin{bmatrix} 1 \end{bmatrix}$$

and thus, using Corollary 1.3.3, the left A-module M is stably free. Let us compute its rank:

```
> OreRank(R,Alg);
```

$$2$$

Using Stafford's theorem (see 3 of Theorem 1.1.2), the left A-module M is a free of rank 2. Let $\mathcal{F}$ be a left A-module (e.g., $\mathcal{F} = C^\infty(\mathbb{R}^3)$) and let us consider the linear PD system $\ker_{\mathcal{F}}(R.)$. Since M is a free left A-module, the linear system $\ker_{\mathcal{F}}(R.)$ admits an injective parametrization. Let us compute an injective parametrization of $\ker_{\mathcal{F}}(R.)$:

```
> Q:=InjectiveParametrization(R,A);
```

$$\begin{aligned} Q := [&-d_3^2\, d_1 - d_3^2\, x_3 - 2\, d_3 + d_3^2 + d_3^2\, d_2\,, -3\, d_1 - d_1^2\, d_3 - 2\, d_1\, d_3\, x_3 \\ &+ d_3\, d_1 + d_3\, d_1\, d_2 - 3\, x_3 - d_3\, x_3^2 + d_3\, x_3 + 2 + x_3\, d_3\, d_2 + d_2] \\ [&d_3\,,\, d_1 + x_3] \\ [&1 + d_1^2\, d_3 + 2\, d_1\, d_3\, x_3 + d_3\, x_3^2 - d_3\, d_1 - d_3\, x_3 - d_3\, d_1\, d_2 - x_3\, d_3\, d_2\,, \\ &d_1^3 + 3\, d_1^2\, x_3 + 3\, d_1\, x_3^2 - d_1^2 - 2\, d_1\, x_3 - d_1^2\, d_2 - 2\, d_1\, d_2\, x_3 + x_3^3 - x_3^2 - d_2\, x_3^2] \end{aligned}$$

Let us first check that the matrix Q defines a parametrization of M, and thus, of $\ker_{\mathcal{F}}(R.)$:

```
> SyzygyModule(Q,A);
```

$$\begin{bmatrix} d_1 + x_3 & d_2 & d_3 \end{bmatrix}$$

Since $\ker_A(.Q) = A\,R$, the matrix Q is a parametrization of M. Let us now check whether or not this parametrization is injective:

```
> T:=LeftInverse(Q,A);
```

$$T := \begin{bmatrix} 0 & -d_1^2 + d_2\,d_1 - 2\,d_1\,x_3 + d_2\,x_3 - x_3^2 + d_1 + x_3 & 1 \\ 1 & d_3\,d_1 - d_3\,d_2 + d_3\,x_3 - d_3 + 2 & 0 \end{bmatrix}$$

Therefore, $M \cong A^{1\times 3}\,Q = A$, which proves again that M is a free left A-module of rank 2. Moreover, the residue classes of the rows of T in M define a basis of the free left A-module M. This result can directly be obtained by using the function `BasisOfModule`:

```
> BasisOfModule(R,A);
```

$$\begin{bmatrix} 0 & -d_1^2 + d_2\,d_1 - 2\,d_1\,x_3 + d_2\,x_3 - x_3^2 + d_1 + x_3 & 1 \\ 1 & d_3\,d_1 - d_3\,d_2 + d_3\,x_3 - d_3 + 2 & 0 \end{bmatrix}$$

The functions `InjectiveParametrization` and `BasisOfModule` are based on Algorithm 1.5.3. But, they also use extra methods to speed up the consuming computations by avoiding as much as possible to compute two generators of left ideals of A appearing in Algorithm 1.5.3.

5.5 The PurityFiltration package

Example 5.5.1. Let us first introduce the commutative polynomial ring A of PD in d_1 and d_2 with rational constant coefficients

```
> A:=DefineOreAlgebra(diff=[d[1],x[1]],diff=[d[2],x[2]],polynom=[x[1],x[2]]):
```

and the system matrix R of the linear PD system defined by:

```
> R:=matrix(3,3,[0,d[2]-d[1],d[2]-d[1],d[2],-d[1],-d[2]-d[1],d[1],-d[1],-2*d[1]]);
```

$$\begin{bmatrix} 0 & d_2 - d_1 & d_2 - d_1 \\ d_2 & -d_1 & -d_2 - d_1 \\ d_1 & -d_1 & -2\,d_1 \end{bmatrix}$$

This example is first due to Janet (see [84]). Let us study the purity filtration of the A-module $M = A^{1\times 3}/(A^{1\times 3}\,R)$.

```
> F:=PurityFiltration(R,A);
```

$$F := \left[\begin{bmatrix} 0 & d_2 - d_1 & d_2 - d_1 \\ d_2 & -d_1 & -d_2 - d_1 \\ d_1 & -d_1 & -2\,d_1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & d_2 - d_1 \\ d_2 & -d_1 \\ d_1 & -d_1 \end{bmatrix}, \begin{bmatrix} 0 & d_2 - d_1 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & -d_2 \\ 0 & -d_1 \end{bmatrix} \right]$$

If we denote by F_i the i^{th} matrix of F, then we have:

$$\begin{cases} M = A^{1\times 3}/(A^{1\times 3}\,F_1), \\ M/t(M) \cong A^{1\times 3}/(A^{1\times 2}\,F_2), \\ t(M) = (A^{1\times 2}\,F_2)/(A^{1\times 3}\,F_1) \cong A^{1\times 2}/(A^{1\times 3}\,F_3), \\ \operatorname{ext}_A^1(\operatorname{ext}_A^1(M,A),A) \cong A^{1\times 2}/(A^{1\times 2}\,F_4), \\ \operatorname{ext}_A^2(\operatorname{ext}_A^2(M,A),A) \cong A^{1\times 2}/(A^{1\times 3}\,F_5). \end{cases}$$

The matrix F_1 defines a finite free resolution of the A-module $M = A^{1\times 3}/(A^{1\times 3}\,R)$ of length at most two. For this example, we have $F_1 = R$. Let us check that $\dim_A(\mathrm{ext}^1_A(\mathrm{ext}^1_A(M,A),A)) = 1$:

```
> DimensionRat(F[4],A);
```

$$1$$

Moreover, let us check that $\dim_A(\mathrm{ext}^2_A(\mathrm{ext}^2_A(M,A),A)) = 0$:

```
> DimensionRat(F[5],A);
```

$$0$$

Let now us compute an equivalence presentation of the A-module $t(M) \cong A^{1\times 2}/(A^{1\times 3}\,F_3)$:

```
> U:=PurityFiltrationTorsion(R,A);
```

$$U := \left[\begin{bmatrix} 0 & d_2 - d_1 \\ d_2 & -d_1 \\ d_1 & -d_1 \end{bmatrix}, \begin{bmatrix} 0 & d_2 - d_1 & -1 & 0 \\ -1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -d_2 \\ 0 & 0 & 0 & -d_1 \end{bmatrix} \right]$$

Hence, we have $t(M) \cong A^{1\times 2}/(A^{1\times 3}\,U_1) \cong A^{1\times 4}/(A^{1\times 5}\,U_2)$. Let us check whether or not we can simplify again the presentation matrix U_2 by uncoupling the two diagonal blocks of U_2:

```
> B:= BaerExtensionTorsionConstCoeff(R,A);
```

$$B := \left[\begin{bmatrix} 0 & d_2 - d_1 & d_2 - d_1 \\ d_2 & -d_1 & -d_2 - d_1 \\ d_1 & -d_1 & -2\,d_1 \end{bmatrix}, \begin{bmatrix} 0 & d_2 - d_1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -d_2 \\ 0 & 0 & 0 & -d_1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & d_2 - d_1 \\ -1 & 1 \end{bmatrix} \right]$$

We obtain

$$\begin{aligned} t(M) = A^{1\times 3}/(A^{1\times 3}\,B_1) &\cong A^{1\times 4}/(A^{1\times 5}\,B_2) \\ &\cong A^{1\times 2}/(A^{1\times 2}\,F_4) \oplus A^{1\times 2}/(A^{1\times 3}\,F_5) \\ &\cong \mathrm{ext}^1_A(\mathrm{ext}^1_A(M,A),A) \oplus \mathrm{ext}^2_A(\mathrm{ext}^2_A(M,A),A), \end{aligned}$$

where the third and fourth matrices B_3 and B_4 of B define the first A-isomorphism.

Let us now compute an equivalent presentation of the A-module $M = A^{1\times 3}/(A^{1\times 3}\,R)$:

```
> Q:=BaerExtensionConstCoeff(R,A);
```

$$Q := \left[\begin{bmatrix} 0 & d_2 - d_1 & d_2 - d_1 \\ d_2 & -d_1 & -d_2 - d_1 \\ d_1 & -d_1 & -2\,d_1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & d_2 - d_1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -d_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & -d_1 \end{bmatrix}, \right.$$

$$\left. \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & -1 \\ 1 & 0 & -1 \\ 0 & d_2 - d_1 & d_2 - d_1 \\ -1 & 1 & 2 \end{bmatrix} \right]$$

We obtain

$$\begin{aligned} M = A^{1\times 3}/(A^{1\times 3}\,Q_1) \cong L \triangleq A^{1\times 7}/(A^{1\times 7}\,Q_2) \\ \cong A^{1\times 3}/(A^{1\times 2}\,F_2) \oplus A^{1\times 2}/(A^{1\times 2}\,F_4) \oplus A^{1\times 2}/(A^{1\times 3}\,F_5) \\ \cong M/t(M) \oplus \mathrm{ext}^1_A(\mathrm{ext}^1_A(M,A),A) \oplus \mathrm{ext}^2_A(\mathrm{ext}^2_A(M,A),A), \end{aligned}$$

and the third and fourth matrices of E define the first A-isomorphism. We can use the OreMorphisms package (see Section 5.6 and [20]) to check again this A-isomorphism:

```
> with(OreMorphisms):
```

Following Proposition 3.1.1, we first need to compute $X \in A^{3\times 7}$ satisfying $Q_1\,Q_3 = X\,Q_2$, where $Q_1 = R$:

```
> X:=Factorize(Mult(Q[1],Q[3],A),Q[2],A);
```

$$X := \begin{bmatrix} 0 & d_2 - d_1 & 1 & 0 & 1 & -1 & 1 \\ d_2 & -d_1 & 1 & 0 & 0 & 0 & 1 \\ d_1 & -d_1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Then, using the command TestIso of OreMorphisms, we can test whether or not the pair of matrices (Q_3, X) defines an A-isomorphism from M to L:

```
> TestIso(Q[1],Q[2],Q[3],X,A);
```

true

Let us check that the matrix Q_4 defines an A-isomorphism from P to L. We first compute $Y \in A^{7\times 3}$ satisfying $Q_2\,Q_4 = Y\,Q_1$, where $Q_1 = R$:

```
> Y:=Factorize(Mult(Q[2],Q[4],A),Q[1],A);
```

$$Y := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Then, we can check again that the matrices Q_4 and Y define an A-isomorphism from L to M:

```
> TestIso(Q[2],Q[1],Q[4],Y,A);
```

$$\mathit{true}$$

The main interest of the presentation Q_2 (resp., representation) of M (resp., $\ker_{\mathcal{F}}(R.)$) is that the different i^{th}-dimensional layers of the linear PD system $\ker_{\mathcal{F}}(Q_2.)$ are uncoupled. Hence, the integration of $\ker_{\mathcal{F}}(Q_2.)$ is highly simplified:

```
> Eqs:=convert(convert(ApplyMatrix(E[2],[zeta[1](x[1],x[2]),zeta[2](x[1],x[2]),
> zeta[3](x[1],x[2]),tau[1](x[1],x[2]),tau[2](x[1],x[2]),upsilon[1](x[1],x[2]),
> upsilon[2](x[1],x[2])],A),vector),list):
```

```
> eqs:=map(a->a=0,Eqs);
```

$$[\zeta_1(x_1,x_2)-\zeta_3(x_1,x_2)=0, \zeta_2(x_1,x_2)+\zeta_3(x_1,x_2)=0, -\frac{\partial}{\partial x_1}\tau_2(x_1,x_2)+\frac{\partial}{\partial x_2}\tau_2(x_1,x_2)=0,$$
$$-\tau_1(x_1,x_2)+\tau_2(x_1,x_2)=0, \upsilon_1(x_1,x_2)=0, \upsilon_1(x_1,x_2)-\frac{\partial}{\partial x_2}\upsilon_2(x_1,x_2)=0, -\frac{\partial}{\partial x_1}\upsilon_2(x_1,x_2)=0]$$

If $\mathcal{F}=C^\infty(\mathbb{R}^2)$, then a generic element of $\ker_{\mathcal{F}}(Q_2.)$ has the form $(\zeta_1 \quad \zeta_2 \quad \zeta_3 \quad \tau_1 \quad \tau_2 \quad \upsilon_1 \quad \upsilon_2)^T$, where:

```
> S:=pdsolve(eqs,{zeta[1](x[1],x[2]),zeta[2](x[1],x[2]),zeta[3](x[1],x[2]),
> tau[1](x[1],x[2]),tau[2](x[1],x[2]),upsilon[1](x[1],x[2]),upsilon[2](x[1],
> x[2])});
```

$$S := \{\upsilon_2(x_1,x_2)=\_C1, \zeta_1(x_1,x_2)=\zeta_3(x_1,x_2), \zeta_2(x_1,x_2)=-\zeta_3(x_1,x_2),$$
$$\zeta_3(x_1,x_2)=\zeta_3(x_1,x_2), \tau_2(x_1,x_2)=\_F1(x_2+x_1), \tau_1(x_1,x_2)=\_F1(x_2+x_1), \upsilon_1(x_1,x_2)=0\}$$

Then, $\eta = Q_3(\zeta_1 \quad \zeta_2 \quad \zeta_3 \quad \tau_1 \quad \tau_2 \quad \upsilon_1 \quad \upsilon_2)^T$, namely,

```
> sols:=convert(S,list):
```

```
> eta:=ApplyMatrix(Q[3],[rhs(sols[2]),rhs(sols[3]),rhs(sols[4]),rhs(sols[6]),
> rhs(sols[5]),rhs(sols[7]),rhs(sols[1])],A);
```

$$\eta := \begin{bmatrix} \zeta_3(x_1,x_2) \\ -\zeta_3(x_1,x_2)+2\_F1(x_2+x_1)+\_C1 \\ \zeta_3(x_1,x_2)-\_F1(x_2+x_1) \end{bmatrix}$$

is the general solution of the linear PD system $\ker_{\mathcal{F}}(R.)$:

```
> ApplyMatrix(R,eta,A);
```

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Finally, we point out that the computer algebra system Maple cannot compute the above closed-form solution of the linear PD system $R\,\eta = 0$, a fact illustrating the interest of the results obtained in Section 2.4 based on the purity filtration and of the PurityFiltration package.

Example 5.5.2. Let us study the purity filtration of the left $A = A_2(\mathbb{Q})$-module $M = A^{1\times 3}/(A^{1\times 4}\,R)$, where R is the matrix of PD operators defined by:

```
> R:=evalm([[d[1],x[2],d[2]],[x[1],d[2],0],[d[1],x[2],d[1]],
> [x[1]*d[1]+1,d[1]*d[2],d[2]]]);
```

$$R := \begin{bmatrix} d_1 & x_2 & d_2 \\ x_1 & d_2 & 0 \\ d_1 & x_2 & d_1 \\ x_1 d_1 + 1 & d_1 d_2 & d_2 \end{bmatrix}$$

Let us compute the purity filtration of the left A-module M:

```
> A:=DefineOreAlgebra(diff=[d[1],x[1]],diff=[d[2],x[2]],polynom=[x[1],x[2]]):
> F:=PurityFiltration(R,A);
```

$$F := [\begin{bmatrix} d_1 & x_2 & d_2 \\ x_1 & d_2 & 0 \\ d_1 & x_2 & d_1 \\ 1 + x_1 d_1 & d_1 d_2 & d_2 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} d_1 & x_2 & d_2 \\ x_1 & d_2 & 0 \\ d_1 & x_2 & d_1 \\ 1 + x_1 d_1 & d_1 d_2 & d_2 \end{bmatrix},$$

$$\begin{bmatrix} d_1 & x_2 & d_2 \\ -x_1 & -d_2 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 0 & -d_1 + d_2 \\ 0 & -d_1 & -d_2 \end{bmatrix}]$$

We get $M/t(M) = A^{1\times 3}/(A^{1\times 3}\,F_2) = 0$, $t(M) = A^{1\times 3}/(A^{1\times 4}\,F_3) = M$, i.e., M is a torsion left A-module, $\mathrm{ext}^1_A(\mathrm{ext}^1_A(M,A),A) \cong A^{1\times 3}/(A^{1\times 3}\,F_4)$ and $\mathrm{ext}^2_A(\mathrm{ext}^2_A(M,A),A) \cong A^{1\times 3}/(A^{1\times 4}\,F_5)$. Looking at the matrices F_4 and F_5, we can check that $\mathrm{ext}^1_A(\mathrm{ext}^1_A(M,A),A) \cong A^{1\times 2}/(A^{1\times 2}\,F_4')$, where the matrix F_4' is defined by

$$F_4' = \begin{pmatrix} d_1 & x_2 \\ x_1 & d_2 \end{pmatrix},$$

and $\mathrm{ext}^2_A(\mathrm{ext}^2_A(M,A),A) \cong A/(A\,d_1 + A\,d_2)$.

Let us compute $\dim_A(\mathrm{ext}^1_A(\mathrm{ext}^1_A(M,A),A))$ and $\dim_A(\mathrm{ext}^2_A(\mathrm{ext}^2_A(M,A),A))$:

```
> Dimension(F[4],A);
```

$$3$$

```
> Dimension(F[5],A);
```

$$2$$

We have $\dim_A(\mathrm{ext}_A^1(\mathrm{ext}_A^1(M,A),A)) = 3$ and $\dim_A(\mathrm{ext}_A^2(\mathrm{ext}_A^2(M,A),A)) = 2$.

Let us check whether or not M is the direct sum of $\mathrm{ext}_A^1(\mathrm{ext}_A^1(M,A),A)$ and $\mathrm{ext}_A^2(\mathrm{ext}_A^2(M,A),A)$.

```
> B:=BaerExtensionTorsion(R,A,0,1);
```

$$B := \left[\begin{bmatrix} d_1 & x_2 & d_2 \\ x_1 & d_2 & 0 \\ d_1 & x_2 & d_1 \\ 1+x_1 d_1 & d_1 d_2 & d_2 \end{bmatrix}, \begin{bmatrix} d_1 & x_2 & d_2 & 0 & 0 & 0 \\ -x_1 & -d_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -d_1+d_2 \\ 0 & 0 & 0 & 0 & -d_1 & -d_2 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{bmatrix}, \right.$$

$$\begin{bmatrix} 1-x_1 & -d_2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ d_1 & x_2 & d_2 \\ -x_1 & -d_2 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

$$[Ore\_algebra, [\text{"}diff\text{"}, \text{"}diff\text{"}], [x_1, x_2], [d_1, d_2], [x_1, x_2], [], 0, [], [], [x_1, x_2], [], [],$$
$$[diff = [d_1, x_1], diff = [d_2, x_2]], [\_a \mapsto \_a\, d_1, \_a \mapsto \_a\, d_2]]]$$

Since $B_2 = \mathrm{diag}(F_4, F_5)$, we obtain that $M \cong \mathrm{ext}_A^1(\mathrm{ext}_A^1(M,A),A) \oplus \mathrm{ext}_A^2(\mathrm{ext}_A^2(M,A),A)$. Moreover, the third matrix B_3 of B defines a left A-isomorphism $\phi : M \longrightarrow L = A^{1\times 6}/(A^{1\times 7}\, B_2)$, and the fourth matrix B_4 defines its inverse ϕ^{-1}.

Using the OreMorphisms package (see Section 5.6 and [20]), let us check this result:

```
> TestIso(B[1],B[2],B[3],Factorize(Mult(B[1],B[3],A),B[2],A),A);
```

true

```
> TestIso(B[2],B[1],B[4],Factorize(Mult(B[2],B[4],A),B[1],A),A);
```

true

Hence, we have $M \cong L \cong A^{1\times 2}/(A^{1\times 2}\, F_4') \oplus A/(A\, d_1 + A\, d_2)$, and thus we obtain:

$$\ker_{\mathcal{F}}(R.) = B_3\, \ker_{\mathcal{F}}(B_2.) = B_3\, (\ker_{\mathcal{F}}(F_4.) \oplus \ker_{\mathcal{F}}(F_5.)).$$

Example 5.5.3. Let us consider a linear OD time-delay system describing a model of a tank containing a fluid and subjected to a one-dimensional horizontal move studied in Example 2.2.4.

Let us introduce the commutative polynomial ring $A = \mathbb{Q}(\alpha)[d, \delta]$ of OD time-delay operators

```
> A:=DefineOreAlgebra(diff=[d,t],dual_shift=[delta,s],polynom=[t,s],comm=[alpha]):
```

where $d\,y(t) = \dot{y}(t)$, $\delta\,y(t) = y(t-1)$ and α is a system parameter, and the matrix system R:

```
> R:=matrix(2,3,[d,-d*delta^2,alpha*d^2*delta,d*delta^2,-d,alpha*d^2*delta]);
```

$$R := \begin{bmatrix} d & -d\,\delta^2 & \alpha\,d^2\,\delta \\ d\,\delta^2 & -d & \alpha\,d^2\,\delta \end{bmatrix}$$

Let $M = A^{1\times 3}/(A^{1\times 2}\,R)$ be the A-module finitely presented by R. Let us compute the purity filtration of the A-module $M = A^{1\times 3}/(A^{1\times 2}\,R)$:

```
> Q:=PurityFiltration(R,A);
```

$$Q := [\begin{bmatrix} d & -d\,\delta^2 & \alpha\,d^2\,\delta \\ d\,\delta^2 & -d & \alpha\,d^2\,\delta \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1-\delta^2 & \alpha\,\delta\,d \end{bmatrix}, \begin{bmatrix} d & d \\ d\,\delta^2 & d \end{bmatrix}, \begin{bmatrix} d & d \\ d\,\delta^2 & d \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}]$$

Then, we have:

$$\begin{cases} M = A^{1\times 3}/(A^{1\times 2}\,Q_1), \\ M/t(M) \cong A^{1\times 3}/(A^{1\times 2}\,Q_2), \\ t(M) = (A^{1\times 3}\,Q_2)/(A^{1\times 2}\,Q_1) \cong A^{1\times 2}/(A^{1\times 2}\,Q_3), \\ \mathrm{ext}^1_A(\mathrm{ext}^1_A(M,A),A) \cong A^{1\times 2}/(A^{1\times 2}\,Q_4) \cong t(M), \\ \mathrm{ext}^2_A(\mathrm{ext}^2_A(M,A),A) \cong A^{1\times 2}/(A^{1\times 3}\,Q_5) = 0. \end{cases}$$

Using the purity filtration of the A-module M, let us compute a linear OD time-delay system which is equivalent to $\ker_{\mathcal{F}}(R.)$:

```
> P:=BaerExtensionConstCoeff(R,A);
```

$$P := [\begin{bmatrix} d & -d\,\delta^2 & \alpha\,d^2\,\delta \\ d\,\delta^2 & -d & \alpha\,d^2\,\delta \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1-\delta^2 & \alpha\,\delta\,d & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & d & d & 0 & 0 \\ 0 & 0 & 0 & d\,\delta^2 & d & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & -1-\delta^2 & \alpha\,\delta\,d \\ d & -d\,\delta^2 & \alpha\,d^2\,\delta \\ d\,\delta^2 & -d & \alpha\,d^2\,\delta \end{bmatrix}]$$

We obtain that $\ker_{\mathcal{F}}(P_1.) \cong \ker_{\mathcal{F}}(P_2.)$, where $P_1 = R$, and the corresponding A-isomorphism and its inverse are defined by the matrices P_3 and P_4. In particular, on the matrix P_2, we can easily check that M is not the direct sum of $M/t(M)$ and $t(M)$. Following Example 2.2.4, we can easily integrate $\ker_{\mathcal{F}}(P_2.)$ and thus $\ker_{\mathcal{F}}(R.) = P_3\,\ker_{\mathcal{F}}(P_2.)$.

Finally, let us consider the second model of a tank containing a fluid and subjected to a one-dimensional horizontal move studied in Example 2.2.5 and defined by the following matrix:

```
> R:=evalm([[delta^2,1,-2*d*delta],[1,delta^2,-2*d*delta]]);
```

$$R := \begin{bmatrix} \delta^2 & 1 & -2\,d\,\delta \\ 1 & \delta^2 & -2\,d\,\delta \end{bmatrix}$$

Let us compute the purity filtration of the finitely presented A-module $M = A^{1\times 3}/(A^{1\times 2}\,R)$:

```
> Q:=PurityFiltration(R,A);
```

$$Q := [\begin{bmatrix} \delta^2 & 1 & -2\,d\,\delta \\ 1 & \delta^2 & -2\,d\,\delta \end{bmatrix}, \begin{bmatrix} 1 & -1 & 0 \\ 0 & -1-\delta^2 & 2\,d\,\delta \end{bmatrix}, \begin{bmatrix} \delta^2 & -1 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} \delta^2 & -1 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}]$$

Then, we have:

$$\begin{cases} M = A^{1\times 3}/(A^{1\times 2}\,Q_1), \\ M/t(M) \cong A^{1\times 3}/(A^{1\times 2}\,Q_2), \\ t(M) = (A^{1\times 3}\,Q_2)/(A^{1\times 2}\,Q_1) \cong A^{1\times 2}/(A^{1\times 2}\,Q_3), \\ \mathrm{ext}^1_A(\mathrm{ext}^1_A(M,A),A) \cong A^{1\times 2}/(A^{1\times 2}\,Q_4) \cong t(M), \\ \mathrm{ext}^2_A(\mathrm{ext}^2_A(M,A),A) \cong A^{1\times 2}/(A^{1\times 3}\,Q_5) = 0. \end{cases}$$

Using the purity filtration of the A-module M, let us compute a linear OD time-delay system which is equivalent to $\ker_{\mathcal{F}}(R.)$:

```
> P:=BaerExtensionConstCoeff(R,A);
```

$$P := [\begin{bmatrix} \delta^2 & 1 & -2\,d\delta \\ 1 & \delta^2 & -2\,d\delta \end{bmatrix}, \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1-\delta^2 & 2\,d\delta & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \delta^2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 1 & 0 & -1/2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1/2+1/2\,\delta^2 & -d\delta \\ 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & -1-\delta^2 & 2\,d\delta \\ \delta^2 & 1 & -2\,d\delta \\ 1 & \delta^2 & -2\,d\delta \end{bmatrix}]$$

We obtain:

$$\begin{aligned} M = A^{1\times 3}/(A^{1\times 2}\,P_1) \cong L \triangleq A^{1\times 7}/(A^{1\times 6}\,P_2) \\ \cong A^{1\times 3}/(A^{1\times 2}\,Q_2) \oplus A^{1\times 2}/(A^{1\times 2}\,Q_3) \\ \cong M/t(M) \oplus t(M). \end{aligned}$$

The A-homomorphism $\phi : M \longrightarrow L$ defined by $\phi(\pi(\lambda)) = \varrho(\lambda\, P_3)$, where $\varrho : A^{1\times 7} \longrightarrow L$ is the canonical projection and $\lambda \in A^{1\times 3}$, is an A-isomorphism. Moreover, $\phi^{-1} : L \longrightarrow M$ is defined by $\phi^{-1}(\varrho(\mu)) = \pi(\mu\, P_4)$ for all $\mu \in A^{1\times 7}$. These results can be checked using the OreMorphisms package (see Section 5.6):

```
> with(OreMorphisms):
> TestIso(P[1],P[2],P[3],Factorize(Mult(P[1],P[3],A),P[2],A),A);
```

$$true$$

```
> TestIso(P[2],P[1],P[4],Factorize(Mult(P[2],P[4],A),P[1],A),A);
```

$$true$$

Thus, we have $\ker_{\mathcal{F}}(R.) \cong \ker_{\mathcal{F}}(P_2.) \cong \ker_{\mathcal{F}}(Q_2.) \oplus \ker_{\mathcal{F}}(Q_3.)$ and we can easily integrate $\ker_{\mathcal{F}}(Q_2.)$ as explained in Example 2.2.5. Finally, since $P_3. : \ker_{\mathcal{F}}(P_2.) \longrightarrow \ker_{\mathcal{F}}(R.)$ is an A-isomorphism, we obtain the Monge parametrization $\ker_{\mathcal{F}}(R.) = Q_3 \ker_{\mathcal{F}}(B_2.)$.

5.6 The OreMorphisms package

Example 5.6.1. The Dirac equations for a massless particle is defined by the matrix

```
> R:=matrix(4,4,[d[4],0,-i*d[3],-(i*d[1]+d[2]),0,d[4],-i*d[1]+d[2],i*d[3],
> i*d[3],i*d[1]+d[2],-d[4],0,i*d[1]-d[2],-i*d[3],0,-d[4]]);
```

$$R := \begin{bmatrix} d_4 & 0 & -i\,d_3 & -i\,d_1 - d_2 \\ 0 & d_4 & -i\,d_1 + d_2 & i\,d_3 \\ i\,d_3 & i\,d_1 + d_2 & -d_4 & 0 \\ i\,d_1 - d_2 & -i\,d_3 & 0 & -d_4 \end{bmatrix}$$

with entries in the Ore algebra $A = \mathbb{Q}(i)[d_1, d_2, d_3, d_4]$ of PD operators with coefficients in $\mathbb{Q}(i)$:

```
> A:=DefineOreAlgebra(diff=[d[1],x[1]],diff=[d[2],x[2]],diff=[d[3],x[3]],
> diff=[d[4],x[4]],polynom=[x[1],x[2],x[3],x[4]],comm=[i],
> alg_relations=[i^2+1]):
```

See Example 3.6.1. Let us consider the A-module $M = A^{1\times 4}/(A^{1\times 4}\, R)$ finitely presented by the matrix R and let us compute its endomorphism ring $E = \mathrm{end}_A(M)$:

```
> Endo:=MorphismsConstCoeff(R,R,A):
```

The A-module structure of the ring E can be generated by

```
> nops(Endo[1]);
```

$$18$$

generators which satisfy

```
> rowdim(Endo[2]);
```

$$22$$

A-linear relations. Let us compute idempotents of E defined by matrices with entries in $\mathbb{Q}(i)$:

```
> Idem:=IdempotentsMatConstCoeff(R,Endo[1],A,0);
```

$$Idem := [[\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1/2 & 0 & -1/2 & 0 \\ 0 & 1/2 & 0 & -1/2 \\ -1/2 & 0 & 1/2 & 0 \\ 0 & -1/2 & 0 & 1/2 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix} 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \end{bmatrix}],$$

$$[Ore\_algebra, [\text{“}diff\text{''}, \text{“}diff\text{''}, \text{“}diff\text{''}, \text{“}diff\text{''}], [x_1, x_2, x_3, x_4], [d_1, d_2, d_3, d_4], [x_1, x_2, x_3, x_4], [i], 0,$$
$$[], [i^2+1], [x_1, x_2, x_3, x_4], [], [], [diff = [d_1, x_1], diff = [d_2, x_2], diff = [d_3, x_3], diff = [d_4, x_4]]]]$$

We obtain the trivial idempotents 0 and id_M of E as well as two non-trivial idempotents e_1 and e_2 respectively defined by the matrices $Idem[1,2]$ and $Idem[1,4]$. Let us consider $P = Idem[1,2]$ and $Q \in A^{4\times 4}$ such that $R\,P = Q\,R$:

```
>  P:=Idem[1,2]; Q:=Factorize(Mult(R,P,A),R,A);
```

$$P := \begin{bmatrix} 1/2 & 0 & -1/2 & 0 \\ 0 & 1/2 & 0 & -1/2 \\ -1/2 & 0 & 1/2 & 0 \\ 0 & -1/2 & 0 & 1/2 \end{bmatrix} \quad Q := \begin{bmatrix} 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \end{bmatrix}$$

Since the entries of the matrices P and Q belong to the field $\mathbb{Q}$ and $P^2 = P$ and $Q^2 = Q$, using linear algebraic techniques, we can easily compute bases of the free $\mathbb{Q}$-modules $\ker_{\mathbb{Q}}(.P)$, $\mathrm{im}_{\mathbb{Q}}(.P) = \ker_{\mathbb{Q}}(.(I_4 - P))$, $\ker_{\mathbb{Q}}(.Q)$ and $\mathrm{im}_{\mathbb{Q}}(.Q) = \ker_{\mathbb{Q}}(.(I_4 - Q))$ as follows:

```
>  U1:=SyzygyModule(P,A): U2:=SyzygyModule(evalm(1-P),A):
>  U:=stackmatrix(U1,U2);
>  V1:=SyzygyModule(Q,A): V2:=SyzygyModule(evalm(1-Q),A):
>  V:=stackmatrix(V1,V2);
```

$$U := \begin{bmatrix} -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \\ 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \quad V := \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 \end{bmatrix}$$

In particular, the previous matrices define bases of the free A-modules $\ker_A(.P)$, $\mathrm{im}_A(.P)$, $\ker_A(.Q)$ and $\mathrm{im}_A(.Q)$. Hence, the unimodular matrices U and V, i.e., $U \in \mathrm{GL}_4(A)$ and $V \in \mathrm{GL}_4(A)$, are such that the matrices $U\,P\,U^{-1}$ and $V\,Q\,V^{-1}$ are block-diagonal formed by the diagonal matrices 0 and I_2:

```
>  VERIF1:=Mult(U,P,LeftInverse(U,A),A);
>  VERIF2:=Mult(V,Q,LeftInverse(V,A),A);
```

$$VERIF1 := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad VERIF2 := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

By Theorem 3.8.1, R is equivalent to the block-diagonal matrix $S = V\,R\,U^{-1}$ defined by:

```
> S:=Mult(V,R,LeftInverse(U,A),A);
```

$$S := \begin{bmatrix} -i\,d_3 + d_4 & -i\,d_1 - d_2 & 0 & 0 \\ i\,d_1 - d_2 & -d_4 - i\,d_3 & 0 & 0 \\ 0 & 0 & d_4 + i\,d_3 & -i\,d_1 - d_2 \\ 0 & 0 & -i\,d_1 + d_2 & -i\,d_3 + d_4 \end{bmatrix}$$

This result can directly be obtained by using the function `HeuristicDecomposition`:

```
> HeuristicDecomposition(R,P,A)[1];
```

$$\begin{bmatrix} -i\,d_3 + d_4 & -i\,d_1 - d_2 & 0 & 0 \\ -i\,d_1 + d_2 & d_4 + i\,d_3 & 0 & 0 \\ 0 & 0 & d_4 + i\,d_3 & i\,d_1 + d_2 \\ 0 & 0 & -i\,d_1 + d_2 & -d_4 + i\,d_3 \end{bmatrix}$$

Since $\mathrm{coim}_A(.P) \cong \mathrm{im}_A(.P)$ and $\mathrm{coim}_A(.Q) \cong \mathrm{im}_A(.Q)$, the A-modules $\mathrm{coim}_A(.P)$ and $\mathrm{coim}_A(.Q)$ are free. Hence, using Theorem 3.6.1, the matrix R is equivalent to a block-triangular matrix. It can be obtained by computing bases of the free A-modules $\ker_A(.P)$, $\mathrm{coim}_A(.P)$, $\ker_A(.Q)$ and $\mathrm{coim}_A(.Q)$ as follows:

```
> Y2:=LeftInverse(Exti(Involution(Y1,A),A,1)[3],A): Y:=stackmatrix(U1,Y2);
> Z2:=LeftInverse(Exti(Involution(Z1,A),A,1)[3],A): Z:=stackmatrix(V1,Z2);
```

$$Y := \begin{bmatrix} -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \quad Z := \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The matrices $Y \in \mathrm{GL}_4(A)$ and $Z \in \mathrm{GL}_4(A)$, respectively formed by the bases of $\ker_A(.P)$ and $\mathrm{coim}_A(.P)$ and by the bases of $\ker_A(.Q)$ and $\mathrm{coim}_A(.Q)$, are such that $T = Z\,R\,Y^{-1}$ is a block-triangular matrix defined by:

```
> T:=Mult(Z,R,LeftInverse(Y,A),A);
```

$$T := \begin{bmatrix} d_4 - i\,d_3 & i\,d_1 + d_2 & 0 & 0 \\ i\,d_1 - d_2 & d_4 + i\,d_3 & 0 & 0 \\ i\,d_3 & -i\,d_1 - d_2 & d_4 + i\,d_3 & -i\,d_1 - d_2 \\ -i\,d_1 + d_2 & -i\,d_3 & -i\,d_1 + d_2 & d_4 - i\,d_3 \end{bmatrix}$$

This last result can directly be obtained by using the function `HeuristicReduction`:

```
> HeuristicReduction(R,P,A)[1];
```

$$\begin{bmatrix} d_4 - i\,d_3 & i\,d_1 + d_2 & 0 & 0 \\ i\,d_1 - d_2 & d_4 + i\,d_3 & 0 & 0 \\ i\,d_3 & -i\,d_1 - d_2 & d_4 + i\,d_3 & -i\,d_1 - d_2 \\ -i\,d_1 + d_2 & -i\,d_3 & -i\,d_1 + d_2 & d_4 - i\,d_3 \end{bmatrix}$$

Example 5.6.2. Let us consider a model of a tank containing a fluid and subjected to a one-dimensional horizontal move (see Example 3.8.3). The presentation matrix is defined by:

```
> A:=DefineOreAlgebra(diff=[d,t],dual_shift=[delta,s],polynom=[t,s],
> comm=[alpha]):
> R:=matrix(2,3,[d,-d*delta^2,alpha*d^2*delta,d*delta^2,-d,alpha*d^2*delta]);
```

$$R := \begin{bmatrix} d & -d\,\delta^2 & \alpha\,d^2\delta \\ d\,\delta^2 & -d & \alpha\,d^2\delta \end{bmatrix}$$

We consider the $A = \mathbb{Q}(\alpha)[d, \delta]$-module $M = A^{1\times 3}/(A^{1\times 2}\,R)$ finitely presented by the matrix R. Let us compute the endomorphism ring $E = \mathrm{end}_A(M)$ of M:

```
> Endo:=MorphismsConstCoeff(R,R,A):
```

The A-module E is generated by the endomorphisms f_i's defined by $f_i(\pi(\lambda)) = \pi(\lambda\,P_i)$ for all $\lambda \in A^{1\times 3}$, where $\pi : A^{1\times 3} \longrightarrow M$ is the canonical projection and the P_i's are defined by:

```
> Endo[1];
```

$$\left[\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \delta^2 & -1 & \alpha\,d\,\delta \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1-\delta^2 & 1-\delta^2 & 0 \end{bmatrix},\right.$$

$$\left.\begin{bmatrix} 0 & 0 & 0 \\ -1+\delta^2 & -1+\delta^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ \alpha\,d & \alpha\,d & 0 \\ \delta & \delta & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & \alpha\,d\,\delta \\ 1 & -\delta^2 & 0 \\ 0 & 0 & -\delta^2-1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & -\delta^2 & \alpha\,d\,\delta \\ 0 & 0 & 0 \end{bmatrix}\right]$$

The generators f_i's of E satisfy the following A-linear relations

```
> Endo[2];
```

$$\begin{bmatrix} -d & 0 & d\,\delta^2 & 0 & 0 & 0 & d & 0 \\ d\,\delta^2 & 0 & -d & 0 & 0 & 0 & -d & 0 \\ 0 & d & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & d & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \delta & 0 & -1+\delta^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & d & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & d \end{bmatrix}$$

i.e., if $F = (f_1 \ \ldots \ f_8)^T$, then we have $Endo[2]\,F = 0$.

The multiplication table $Endo[3]$ of the generators f_i's gives us a way to rewrite the composition $f_i \circ f_j$ in terms of A-linear combinations of the f_k's or, in other words, if $\otimes$ is the Kronecker product, namely, $F \otimes F = ((f_1 \circ F)^T \ \ldots \ (f_8 \circ F)^T)^T$, then the multiplication table T of the generators f_j's satisfies $F \otimes F = T\, F$, where T is the matrix $Endo[3]$ without the first column which corresponds to the indices (i, j) of the product $f_i \circ f_j$. We do not print here this matrix as it belongs to $A^{64\times 8}$. We can use it for rewriting any polynomial in the f_i's with coefficients in A in terms of a A-linear combination of the generators f_j's.

Let us now try to compute idempotents of E defined by idempotent matrices, namely, elements $e \in E$ satisfying $e^2 = e$ and defined by two matrices $P \in A^{3\times 3}$ and $Q \in A^{2\times 2}$ satisfying the relations $R\,P = Q\,R$, $P^2 = P$ and $Q^2 = Q$:

```
> Idem:=IdempotentsMatConstCoeff(R,Endo[1],A,0);
```

$$Idem := [[\begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 \\ -c51\,(-1+\delta^2) & -c51\,(-1+\delta^2) & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 0 & 0 \\ -\delta^2 & 1 & -\alpha\,\delta\,d \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ \delta^2 & 0 & \alpha\,\delta\,d \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1/2 & -1/2 & 0 \\ -1/2 & 1/2 & 0 \\ -c51\,(-1+\delta^2) & -c51\,(-1+\delta^2) & 1 \end{bmatrix}],$$

$$[Ore\_algebra, [\text{“}diff\text{''}, dual\_shift], [t, s], [d, \delta], [t, s], [\alpha, c51], 0, [], [], [t, s], [], [], [diff = [d, t], dual\_shift = [\delta, s]]]]$$

Let us consider the first entry P_1 of $Idem[1]$ where we have set the arbitrary constant $c51$ to 0 for simplicity reason and let us compute a matrix $Q_1 \in A^{2\times 2}$ such that $R\,P_1 = Q_1\,R$:

```
> P[1]:=subs(c51=0,evalm(Idem[1,1])); Q[1]:=Factorize(Mult(R,P[1],A),R,A);
```

$$P_1 := \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad Q_1 := \begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{bmatrix}$$

Since the entries of the matrices P_1 and Q_1 belong to $\mathbb{Q}$, using linear algebraic techniques, we can easily compute bases of the free A-modules $\ker_A(.P_1)$, $\ker_A(.Q_1)$, $\operatorname{im}_A(.P_1) = \ker_A(.(I_3 - P_1))$ and $\operatorname{im}_A(.Q_1) = \ker_A(.(I_2 - Q_1))$:

```
> U1:=SyzygyModule(P[1],A): U2:=SyzygyModule(evalm(1-P[1]),A):
> U:=stackmatrix(U1,U2);
> V1:=SyzygyModule(Q[1],A): V2:=SyzygyModule(evalm(1-Q[1]),A):
> V:=stackmatrix(V1,V2);
```

$$U := \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \quad V := \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

We can check that $J_1 = U\,P_1\,U^{-1}$ and $J_2 = V\,Q_1\,V^{-1}$ are block-diagonal matrices formed by the matrices 0 and I_m:

```
> VERIF1:=Mult(U,P,LeftInverse(U,A),A);
> VERIF2:=Mult(V,Q,LeftInverse(V,A),A);
```

$$VERIF1 := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad VERIF2 := \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Using Theorem 3.8.1, R is then equivalent to the following block-diagonal matrix $V\,R\,U^{-1}$:

```
> R_dec:=map(factor,simplify(Mult(V,R,LeftInverse(U,A),A)));
```

$$R\_dec := \begin{bmatrix} d\,(\delta^2+1) & 2\,\alpha\,d^2\delta & 0 \\ 0 & 0 & -d\,(\delta-1)\,(\delta+1) \end{bmatrix}$$

This last result can directly be obtained by means of the function `HeuristicDecomposition`:

```
> map(factor,HeuristicDecomposition(R,P[1],A)[1]);
```

$$\begin{bmatrix} d\,(\delta^2+1) & 2\,\alpha\,d^2\,\delta & 0 \\ 0 & 0 & -d\,(\delta-1)\,(\delta+1) \end{bmatrix}$$

We can use another idempotent matrix P_2 listed in $Idem[1]$ to obtain another decomposition of the A-module M. Let us consider the fourth one and the corresponding idempotent matrix Q_2:

```
> P[2]:=Idem[1,4]; Q[2]:=Factorize(Mult(R,P[2],A),R,A);
```

$$P_2 := \begin{bmatrix} 0 & 0 & 0 \\ -\delta^2 & 1 & -\alpha\,\delta\,d \\ 0 & 0 & 0 \end{bmatrix} \quad Q_2 := \begin{bmatrix} 0 & \delta^2 \\ 0 & 1 \end{bmatrix}$$

Since $P_2^2 = P_2$ and $Q_2^2 = Q_2$, the A-modules $\ker_A(.P_2)$, $\ker_A(.Q_2)$, $\mathrm{im}_A(.P_2) = \ker_A(.(I_3 - P_2))$ and $\mathrm{im}_A(.Q_2) = \ker_A(.(I_2 - Q_2))$ are projective (see Remark 3.8.1), and thus, free by the Quillen-Suslin theorem (see 2 of Theorem 1.1.2). Let us compute bases of those A-modules:

```
> U11:=SyzygyModule(P[2],A): U21:=SyzygyModule(evalm(1-P[2]),A):
> UU:=stackmatrix(U11,U21);
> V11:=SyzygyModule(Q[2],A): V21:=SyzygyModule(evalm(1-Q[2]),A):
> VV:=stackmatrix(V11,V21);
```

$$UU := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ \delta^2 & -1 & \alpha\,\delta\,d \end{bmatrix} \quad VV := \begin{bmatrix} -1 & \delta^2 \\ 0 & 1 \end{bmatrix}$$

As previously, we can check that the idempotent matrices P_2 and Q_2 are equivalent to block-diagonal matrices formed by the matrices 0 and I_m:

```
> VERIF1:=Mult(UU,P[1],LeftInverse(UU,A),A);
> VERIF2:=Mult(VV,Q[1],LeftInverse(VV,A),A);
```

$$VERIF1 := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad VERIF2 := \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

According to Theorem 3.8.1, the matrix R is then equivalent to the block-diagonal matrix:

```
> R_dec1:=map(factor,simplify(Mult(VV,R,LeftInverse(UU,A),A)));
```

$$R\_dec1 := \begin{bmatrix} d\,(\delta-1)\,(\delta+1)\,(\delta^2+1) & \alpha\, d^2\,\delta\,(\delta-1)\,(\delta+1) & 0 \\ 0 & 0 & d \end{bmatrix}$$

We can check this last result by means of the function `HeuristicDecomposition`:

```
> map(factor,HeuristicDecomposition(R,P[2],A)[1]);
```

$$\begin{bmatrix} d\,(\delta-1)\,(\delta+1)\,(\delta^2+1) & \alpha\, d^2\,\delta\,(\delta-1)\,(\delta+1) & 0 \\ 0 & 0 & d \end{bmatrix}$$

Thus, we obtain another decomposition of the matrix R. If we denote by

$$\begin{cases} T_1 = (d\,(\delta^2+1) \quad 2\,\alpha\, d^2\,\delta), \\ T_2 = d\,(\delta^2-1), \\ T_3 = (d\,(\delta^2-1)\,(\delta^2+1) \quad \alpha\, d^2\,\delta\,(\delta^2-1)), \\ T_4 = d, \end{cases} \qquad \begin{cases} M_1 = A^{1\times 2}/(A\,T_1), \\ M_2 = A/(A\,T_2), \\ M_3 = A^{1\times 2}/(A\,T_3), \\ M_4 = A/(A\,T_4), \end{cases}$$

then we have the two following decompositions of the A-module M:

$$M \cong M_1 \oplus M_2, \quad M \cong M_3 \oplus M_4.$$

5.7 The Serre package

Example 5.7.1. Let us consider the model (4.5) of a string with an interior mass studied in Example 4.2.2. Let $A = \mathbb{Q}(\eta_1,\eta_2)[d,\sigma_1,\sigma_2]$ be the commutative polynomial ring of OD incommensurable time-delay operators, where $d\,y(t) = \dot{y}(t)$ and $\sigma_i\,y(t) = y(t-h_i)$ for $i = 1, 2$.

```
> A:=DefineOreAlgebra(diff=[d,t],dual_shift=[sigma[1],x[1]],
> dual_shift=[sigma[2],x[2]],polynom=[t,x[1],x[2]],comm=[eta[1],eta[2]]):
```

The presentation matrix $R \in A^{4\times 6}$ of (4.5) is defined by:

```
> R:=matrix(4,6,[1,1,-1,-1,0,0,d+eta[1],d-eta[1],-eta[2],eta[2],0,0,
> sigma[1]^2,1,0,0,-sigma[1],0,0,0,1,sigma[2]^2,0,-sigma[2]]);
```

$$R := \begin{bmatrix} 1 & 1 & -1 & -1 & 0 & 0 \\ d+\eta_1 & d-\eta_1 & -\eta_2 & \eta_2 & 0 & 0 \\ {\sigma_1}^2 & 1 & 0 & 0 & -\sigma_1 & 0 \\ 0 & 0 & 1 & {\sigma_2}^2 & 0 & -\sigma_2 \end{bmatrix}$$

Let us illustrate Algorithm 4.2.1 with this example. As explained in Section 4.2, the hypothesis of Theorem 4.2.2 can be completely checked when the A-module $\mathrm{ext}^1_A(M,A) \cong A^3/(R\,A^4)$ is 0-dimensional, i.e., is a finite-dimensional $\mathbb{Q}(\eta_1,\eta_2)$-vector space. Let us check whether or not this hypothesis is fulfilled using the function `DimensionRat` of OreModules:

```
> DimensionRat(transpose(R),A);
```

$$0$$

Now, we can compute a finite basis of the $\mathbb{Q}(\eta_1,\eta_2)$-vector space $A^3/(R\,A^4) \cong A^{1\times 3}/(A^{1\times 4}\,R^T)$ using the command `KBasis` of OreModules:

```
> KBasis(transpose(R),A);
```

$$[\lambda_4]$$

We obtain that the A-module $A^3/(R\,A^4)$ is a 1-dimensional $\mathbb{Q}(\eta_1, \eta_2)$-vector space and a basis is defined by the residue class $\tau(\Lambda)$ of the column vector $\Lambda = (0 \quad 0 \quad 1)^T$ in $A^3/(R\,A^4)$. Hence, let us consider the column vector $\Lambda = (0 \quad 0 \quad 1)^T$

```
> Lambda:=evalm([[0],[0],[0],[1]]);
```

$$\Lambda := \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

the matrix $P = (R \quad -\Lambda)$ defined by

```
> P:=augment(R,-evalm([[0],[0],[0],[1]]));
```

$$P := \begin{bmatrix} 1 & 1 & -1 & -1 & 0 & 0 & 0 \\ d+\eta_1 & d-\eta_1 & -\eta_2 & \eta_2 & 0 & 0 & 0 \\ {\sigma_1}^2 & 1 & 0 & 0 & -\sigma_1 & 0 & 0 \\ 0 & 0 & 1 & {\sigma_2}^2 & 0 & -\sigma_2 & -1 \end{bmatrix}$$

and the A-module $E = A^{1\times 7}/(A^{1\times 4}\,P)$. Let us now check whether or not the A-module E is free. According to Theorem 4.2.1, the full row rank matrix P presents a stably free A-module E iff P admits a right inverse. Let us check this point:

```
> RightInverse(P,A);
```

$$\begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ -1/2 & -1/2\,{\eta_2}^{-1} & -\frac{\eta_1}{\eta_2} & 0 \\ -1/2 & 1/2\,{\eta_2}^{-1} & \frac{\eta_1}{\eta_2} & 0 \\ 0 & 0 & -\sigma_1 & 0 \\ -1/2\,\sigma_2 & 1/2\,\frac{\sigma_2}{\eta_2} & \frac{\eta_1\sigma_2}{\eta_2} & 0 \\ -1/2 & -1/2\,{\eta_2}^{-1} & -\frac{\eta_1}{\eta_2} & -1 \end{bmatrix}$$

We obtain that E is a stably free A-module, and thus, is free of rank 2 by the Quillen-Suslin theorem (2 of Theorem 1.1.2). Let us compute a minimal parametrization of the A-module E:

```
> Q:=MinimalParametrization(P,A);
```

$$Q := \begin{bmatrix} -2\,\eta_2 & \eta_2\,\sigma_1 & 0 \\ 0 & -\eta_2\,\sigma_1 & 0 \\ -d-\eta_1-\eta_2 & \sigma_1\,\eta_1 & 0 \\ \eta_1-\eta_2+d & -\sigma_1\,\eta_1 & 0 \\ -2\,\eta_2\,\sigma_1 & -\eta_2+\eta_2\,{\sigma_1}^2 & 0 \\ \eta_1\,\sigma_2-\sigma_2\,\eta_2+\sigma_2 d & -\sigma_1\,\eta_1\,\sigma_2 & 1 \\ -d-\eta_1-\eta_2 & \sigma_1\,\eta_1 & -\sigma_2 \end{bmatrix}$$

Hence, we get $\ker_A(.Q) = A^{1\times 4}\,P$ or equivalently $E \cong A^{1\times 7}\,Q$. Let us check whether or not this parametrization is injective:

```
> T:=LeftInverse(Q,A);
```

$$T := \begin{bmatrix} 0 & 0 & -1/2\,{\eta_2}^{-1} & -1/2\,{\eta_2}^{-1} & 0 & 0 & 0 \\ 0 & -\frac{\sigma_1}{\eta_2} & \frac{\sigma_1}{\eta_2} & \frac{\sigma_1}{\eta_2} & -{\eta_2}^{-1} & 0 & 0 \\ 0 & 0 & 0 & -\sigma_2 & 0 & 1 & 0 \end{bmatrix}$$

We get $T\,Q = I_3$, i.e., $A^{1\times 7}\,Q = A^{1\times 3}$, which proves that Q is an injective parametrization of E. Let us now write $Q = (Q_1^T \quad Q_2^T)^T$, where the submatrix $Q_1 \in A^{6\times 3}$ is defined by

```
> Q_1:=submatrix(Q,1..6,1..3);
```

$$Q_1 := \begin{bmatrix} -2\,\eta_2 & \eta_2\,\sigma_1 & 0 \\ 0 & -\eta_2\,\sigma_1 & 0 \\ -d-\eta_1-\eta_2 & \sigma_1\,\eta_1 & 0 \\ \eta_1-\eta_2+d & -\sigma_1\,\eta_1 & 0 \\ -2\,\eta_2\,\sigma_1 & -\eta_2+\eta_2\,{\sigma_1}^2 & 0 \\ \eta_1\,\sigma_2-\sigma_2\,\eta_2+\sigma_2\,d & -\sigma_1\,\eta_1\,\sigma_2 & 1 \end{bmatrix}$$

and the matrix $Q_2 \in A^{1\times 3}$ is defined by:

```
> Q_2:=submatrix(Q,7..7,1..3);
```

$$Q_2 := \begin{bmatrix} -d-\eta_1-\eta_2 & \sigma_1\,\eta_1 & -\sigma_2 \end{bmatrix}$$

Using Theorem 4.2.2, we obtain $M \cong A^{1\times 3}/(A\,Q_2)$, which using Corollary 4.2.2, proves again that the linear system $\ker_{\mathcal{F}}(R.)$ is equivalent to $\ker_{\mathcal{F}}(Q_2.)$, namely, (4.14).

Since the column vector Λ admits a left inverse defined by

```
> LeftInverse(Lambda,A);
```

$$\begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}$$

the Quillen-Suslin theorem (2 of Theorem 1.1.2) implies that there exist $V \in \mathrm{GL}_4(A)$ and $W \in \mathrm{GL}_6(A)$ such that $V\,R\,W = \mathrm{diag}(I_3, Q_2)$. For more details, see Corollary 4.3.1. Let us compute such matrices V and W following Corollary 4.3.1. Let us first check that $\ker_A(.Q_1)$ is a free A-module of rank 3:

```
> K:=SyzygyModule(Q_1,A);
```

$$K := \begin{bmatrix} 1 & 1 & -1 & -1 & 0 & 0 \\ 0 & -2\,\eta_1 & \eta_1 - \eta_2 + d & d + \eta_2 + \eta_1 & 0 & 0 \\ 0 & -1 + {\sigma_1}^2 & -{\sigma_1}^2 & -{\sigma_1}^2 & \sigma_1 & 0 \end{bmatrix}$$

Then, we get $\ker_A(.Q_1) = A^{1\times 3}\,K$. Moreover, K has full row rank since:

```
> SyzygyModule(K,A);
```

$$INJ\,(3)$$

Hence, we get $A^{1\times 3}\,K \cong A^{1\times 3}$, a fact proving that $\ker_A(.Q_1)$ is a free A-module of rank 3. Let us now compute a matrix $Q_3 \in A^{6\times 3}$ such that $W = (Q_3 \quad Q_1) \in \mathrm{GL}_6(A)$. We can take:

```
> Q_3:=RightInverse(K,A);
```

$$Q_3 := \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & -1/2\,{\eta_2}^{-1} & \frac{\eta_1}{\eta_2} \\ 0 & 1/2\,{\eta_2}^{-1} & -\frac{\eta_1}{\eta_2} \\ 0 & 0 & \sigma_1 \\ 0 & 0 & 0 \end{bmatrix}$$

Then, the matrix $W = (Q_3 \quad Q_1)$ defined by

```
> W:=augment(Q_3,Q_1);
```

$$W := \begin{bmatrix} 1 & 0 & 1 & -2\,\eta_2 & \eta_2\,\sigma_1 & 0 \\ 0 & 0 & -1 & 0 & -\eta_2\,\sigma_1 & 0 \\ 0 & -1/2\,{\eta_2}^{-1} & \frac{\eta_1}{\eta_2} & -d - \eta_1 - \eta_2 & \sigma_1\,\eta_1 & 0 \\ 0 & 1/2\,{\eta_2}^{-1} & -\frac{\eta_1}{\eta_2} & \eta_1 - \eta_2 + d & -\sigma_1\,\eta_1 & 0 \\ 0 & 0 & \sigma_1 & -2\,\eta_2\,\sigma_1 & -\eta_2 + \eta_2\,{\sigma_1}^2 & 0 \\ 0 & 0 & 0 & \eta_1\,\sigma_2 - \sigma_2\,\eta_2 + \sigma_2\,d & -\sigma_1\,\eta_1\,\sigma_2 & 1 \end{bmatrix}$$

is invertible, i.e., $W \in \mathrm{GL}_6(A)$, and its inverse $W^{-1} \in A^{6\times 6}$ is defined by:

```
> W_inv:=inverse(W);
```

$$W\_inv := \begin{bmatrix} 1 & 1 & -1 & -1 & 0 & 0 \\ 0 & -2\,\eta_1 & \eta_1 - \eta_2 + d & d + \eta_2 + \eta_1 & 0 & 0 \\ 0 & -1 + {\sigma_1}^2 & -{\sigma_1}^2 & -{\sigma_1}^2 & \sigma_1 & 0 \\ 0 & 0 & -1/2\,{\eta_2}^{-1} & -1/2\,{\eta_2}^{-1} & 0 & 0 \\ 0 & -\frac{\sigma_1}{\eta_2} & \frac{\sigma_1}{\eta_2} & \frac{\sigma_1}{\eta_2} & -{\eta_2}^{-1} & 0 \\ 0 & -\frac{{\sigma_1}^2\,\eta_1\,\sigma_2}{\eta_2} & 1/2\,\frac{\sigma_2\left(2\,{\sigma_1}^2\,\eta_1 + \eta_1 - \eta_2 + d\right)}{\eta_2} & 1/2\,\frac{\sigma_2\left(2\,{\sigma_1}^2\,\eta_1 + \eta_1 - \eta_2 + d\right)}{\eta_2} & -\frac{\sigma_1\,\eta_1\,\sigma_2}{\eta_2} & 1 \end{bmatrix}$$

Finally, if we introduce the matrix $X = (R\,Q_3 \quad \Lambda)$, namely,

```
> X:=augment(Mult(R,Q_3,A),Lambda);
```

$$X := \begin{bmatrix} 1 & 0 & 0 & 0 \\ d+\eta_1 & 1 & 0 & 0 \\ {\sigma_1}^2 & 0 & -1 & 0 \\ 0 & 1/2\,\frac{-1+{\sigma_2}^2}{\eta_2} & -\frac{\eta_1\left(-1+{\sigma_2}^2\right)}{\eta_2} & 1 \end{bmatrix}$$

then X is invertible, i.e., $V \in \mathrm{GL}_4(A)$, and its inverse $V = X^{-1} \in A^{4\times 4}$ is defined by:

```
> V:=inverse(X);
```

$$V := \begin{bmatrix} 1 & 0 & 0 & 0 \\ -d-\eta_1 & 1 & 0 & 0 \\ {\sigma_1}^2 & 0 & -1 & 0 \\ 1/2\,\frac{\left(-1+{\sigma_2}^2\right)\left(d+\eta_1+2\,{\sigma_1}^2\,\eta_1\right)}{\eta_2} & -1/2\,\frac{-1+{\sigma_2}^2}{\eta_2} & -\frac{\eta_1\left(-1+{\sigma_2}^2\right)}{\eta_2} & 1 \end{bmatrix}$$

Finally, by Corollary 4.3.1, the matrix R is then equivalent to the matrix $V\,R\,W = \mathrm{diag}(I_3, Q_2)$:

```
> Mult(V,R,W,A);
```

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -d-\eta_1-\eta_2 & \sigma_1\,\eta_1 & -\sigma_2 \end{bmatrix}$$

Example 5.7.2. Let us consider the conjugate Beltrami equations (4.8) studied in Examples 4.2.4, 4.2.7 and 4.3.3. We first introduce the first Weyl algebra $A = A_2(\mathbb{Q})$ of PD operators in dx and dy with coefficients in the commutative polynomial ring $\mathbb{Q}[x, y]$:

```
> A:=DefineOreAlgebra(diff=[dx,x],diff=[dy,y],polynom=[x,y],comm=[a,b]):
```

The presentation matrix (4.8) is defined by:

```
> R:=evalm([[dx, -x*dy],[dy, x*dx]]);
```

$$R := \begin{bmatrix} dx & -x\,dy \\ dy & x\,dx \end{bmatrix}$$

Let us introduce the following column vector

```
> Lambda:=evalm([[a],[b]]);
```

$$\Lambda := \begin{bmatrix} a \\ b \end{bmatrix}$$

where a and b are two arbitrary constants, and the matrix $P = (R \quad -\Lambda)$ defined by:

```
> P:=augment(R,-Lambda);
```

$$P := \begin{bmatrix} dx & -x\,dy & -a \\ dy & x\,dx & -b \end{bmatrix}$$

Let us check whether or not the matrix P admits a right inverse:

```
> RightInverse(P,A);
```

$$\begin{bmatrix} \frac{x(a\,x\,dx+x\,dy\,b+a)}{a} & -\frac{x(a\,x\,dx+x\,dy\,b+a)}{b} \\ -\frac{a\,dy\,x-2\,b-dx\,b\,x}{a} & \frac{a\,dy\,x-2\,b-dx\,b\,x}{b} \\ \frac{x\left(x\,dx^2+3\,dx+x\,dy^2\right)}{a} & -\frac{1+x^2\,dx^2+3\,x\,dx+x^2\,dy^2}{b} \end{bmatrix}$$

We obtain that P admits the previous right inverse when $a \neq 0$ and $b \neq 0$, which shows that P generically admits a right inverse. In what follows, we shall suppose that $a \neq 0$ and $b \neq 0$. Then, the left A-module $E = A^{1\times 3}/(A^{1\times 2}\,P)$ is stably free of rank 1.

Let us compute minimal parametrizations of E, namely, matrices $L_i \in A^3$ such that the left A-modules $N_i = A/(A^{1\times 3}\,L_i)$ are torsion and $\ker_A(.L_i) = A^{1\times 2}\,R$, i.e., $E \cong A^{1\times 3}\,L_i$.

```
> L:=map(collect,MinimalParametrizations(P,A),{x,y,dx,dy},distributed):
> nops(L);
```

$$2$$

The OreModules command MinimalParametrizations returns 2 minimal parametrizations. The first one is

```
> L[1];
```

$$\begin{bmatrix} a\,x\,dy^2 b - a\,dx^2 b x + a\,dx\,b + dy\,b^2 + \left(a^2 - b^2\right) dy\,x\,dx \\ -a^2\,dy^2 + 2\,a\,dy\,dx\,b - dx^2 b^2 \\ a\,dx^2 x\,dy + a\,dy\,dx + a\,dy^3 x - dx^3 b x + dy^2 b - dx\,dy^2 b x \end{bmatrix}$$

and the second one is:

```
> L[2];
```

$$\begin{bmatrix} -b a^2 - x\,dy\,a^3 + dx\,b a^2 x - a\,\left(a^2+b^2\right) x^2 dy\,dx - b\,\left(a^2+b^2\right) x^2 dy^2 \\ a\,\left(a^2+b^2\right) x\,dy^2 + dx\,b^2 a - b\,\left(3\,a^2+2\,b^2\right) dy - b\,\left(a^2+b^2\right) dy\,x\,dx \\ a\,x\,dy^2 b + a\,dx^2 b x - a^2\,dy - \left(a^2+b^2\right) dx^2 x^2 dy - \left(a^2+b^2\right) x^2 dy^3 - 3\,\left(a^2+b^2\right) dy\,x\,dx \end{bmatrix}$$

Let us check whether or not they are injective, i.e., whether or not they admit a left inverse:

```
> map(LeftInverse,L,A);
```

$$[[], []]$$

None of them is injective. The left A-module $N_1 = A/(A^{1\times 3}\,L_1)$ is then defined by

```
> J_1:=map(collect,Exti(Involution(Min[1],A),A,1),{dx,dy,x,y},distributed);
```

$$J_1 := [\begin{bmatrix} dx^2 b^2 - 2\,a\,dy\,dx\,b + a^2\,dy^2 \\ \left(-b^2 a - a^3\right) x\,dy^2 - dx\,b^2 a - dy\,b^3 + \left(b a^2 + b^3\right) x\,dy\,dx \end{bmatrix}, \begin{bmatrix} 1 \end{bmatrix}, SURJ\,(1)]$$

i.e., the two entries of the first matrix $J_1[1]$ of J_1 annihilate the generator $\sigma_1(1)$ of N_1, where $\sigma_1(1)$ is the residue class of the standard basis 1 of A in N_1.

```
> J_2:=map(collect,Exti(Involution(Min[2],A),A,1),{dx,dy,x,y},distributed);
```

$$J_2 := \left[\begin{bmatrix} -dx\,b^2a + \left(2\,b^3 + 3\,ba^2\right) dy + \left(ba^2 + b^3\right) x\,dy\,dx + \left(-b^2a - a^3\right) x\,dy^2 \\ a^2b^2 + \left(-2\,a^3b - 2\,ab^3\right) x\,dy + \left(2\,a^2b^2 + a^4 + b^4\right) x^2\,dy^2 \end{bmatrix}, \begin{bmatrix} 1 \end{bmatrix}, SURJ\,(1)\right]$$

Similarly, the two entries of the first matrix $J_2[1]$ of J_2 annihilate the generator $\sigma_2(1)$ of N_2, where $\sigma_2(1)$ is the residue class of 1 in the left A-module $N_2 = A/(A^{1\times 3}\,L_2)$, i.e., $\sigma_2(1)$ satisfies $d_i\,\sigma_2(1) = 0$, for $i = 1, 2$, where $d_1 \in A$ is defined by

```
> N2[1][1,1];
```

$$-dx\,b^2a + \left(2\,b^3 + 3\,ba^2\right) dy + \left(ba^2 + b^3\right) x\,dy\,dx + \left(-b^2a - a^3\right) x\,dy^2$$

and d_2 is defined by:

```
> N2[1][2,1];
```

$$a^2b^2 + \left(-2\,a^3b - 2\,ab^3\right) x\,dy + \left(2\,a^2b^2 + a^4 + b^4\right) x^2\,dy^2$$

Since the two entries of $J_1[1]$ do not contain constant terms, they cannot be equal to non-zero constants for particular values of a and b. The same comment holds for d_1. But, the coefficients of d_2 in dx and dy are:

```
> l:=[coeffs(%,{dx,dy})]:  coefs:=map(factor,map(coeffs,l,x));
```

$$coefs := [a^2b^2, \left(a^2 + b^2\right)^2, -2\,ba\left(a^2 + b^2\right)]$$

Let us find a and b such that d_2 becomes the non-zero constant -1:

```
> Eqs:={coefs[1]=-1,seq(coefs[i]=0,i=2..nops(coefs))};
```

$$Eqs := \left\{\left(a^2 + b^2\right)^2 = 0, a^2b^2 = -1, -2\,ba\left(a^2 + b^2\right) = 0\right\}$$

```
> Sols:=solve(Eqs,{a,b});
```

$$Sols := \left\{a = RootOf\left(\_Z^2 + 1\right), b = 1\right\}, \left\{a = RootOf\left(\_Z^2 + 1\right), b = -1\right\},$$
$$\left\{a = 1, b = RootOf\left(\_Z^2 + 1\right)\right\}, \left\{a = -1, b = RootOf\left(\_Z^2 + 1\right)\right\}$$

For instance, if we take $a = 1$ and $b = i$, then the coefficients of d_2 become:

```
> subs({a=1,b=I},coefs);
```

$$[-1, 0, 0]$$

Hence, let us consider the new ring $B = A_2(\mathbb{Q}(i))$ of PD operators in dx and dy with coefficients in the field $\mathbb{Q}(i) = \mathbb{Q}[i]/(i^2 + 1)$:

```
> B:=DefineOreAlgebra(diff=[dx,x],diff=[dy,y],polynom=[x,y],comm=[i,a,b],
> alg_relations=[i^2=-1]):
```

The column vector Λ is then

```
> Lambda_2:=subs({a=1,b=i},evalm(Lambda));
```

$$\Lambda_2 := \begin{bmatrix} 1 \\ i \end{bmatrix}$$

and the matrix P becomes:

```
> P_2:=simplify(subs({i^2=-1,i^3=-i},subs({a=1,b=i},evalm(P))));
```

$$P_2 := \begin{bmatrix} dx & -x\,dy & -1 \\ dy & x\,dx & -i \end{bmatrix}$$

Substituting $a = 1$ and $b = i$ into L_2, we obtain the matrix Q defined by:

```
> Q:=simplify(subs({i^2=-1,i^3=-i},subs({a=1,b=i},evalm(L[2]))));
```

$$Q := \begin{bmatrix} -i - x\,dy + dx\,ix \\ -dx - dy\,i \\ x\,dy^2 i + dx^2 ix - dy \end{bmatrix}$$

We can check that the last matrix defines a minimal parametrization of $B^{1\times 3}/(B^{1\times 2}\,P_2)$:

```
> MinimalParametrizations(P_2,B);
```

$$\left[\begin{bmatrix} -dx\,ix + i + x\,dy \\ dx + dy\,i \\ -dx^2 ix + dy - x\,dy^2 i \end{bmatrix}\right]$$

Moreover, the minimal parametrization Q admits a left inverse defined by:

```
> T:=LeftInverse(Q,B);
```

$$T := \begin{bmatrix} -i^{-1} & -x & 0 \end{bmatrix}$$

Hence, the left B-module $F = B^{1\times 3}/(B^{1\times 2}\,P_2)$ is free of rank 1 and Theorem 4.2.2 shows that F is isomorphic to the cyclic left B-module $B/(B\,Q_2)$, where Q_2 is defined by:

```
> Q_2:=submatrix(Q,3..3,1..1);
```

$$Q_2 := \begin{bmatrix} x\,dy^2 i + dx^2 ix - dy \end{bmatrix}$$

Moreover, the column vector Γ admits the following left inverse Γ:

```
> Gamma:=LeftInverse(Lambda_2,B);
```

$$\Gamma := \begin{bmatrix} 0 & i^{-1} \end{bmatrix}$$

If $Q_1 \in B^2$ is the first two components of Q

```
> Q_1:=submatrix(Q,1..2,1..1);
```

$$Q_1 := \begin{bmatrix} -i - x\,dy + dx\,ix \\ -dx - dy\,i \end{bmatrix}$$

then Corollary 4.3.1 shows that $\ker_B(.Q_1)$ is a stably free left B-module of rank 1. Moreover, we have $\ker_B(.Q_1) = B\,K$, where the matrix K is defined by

```
> K:=SyzygyModule(Q_1,B);
```

$$K := \begin{bmatrix} -dx\,i + dy & dy\,ix + x\,dx \end{bmatrix}$$

i.e., $\ker_B(.Q_1)$ is a free left B-module of rank 1. Corollary 4.3.1 then shows that the matrices R and $\mathrm{diag}(1, Q_2)$ are equivalent, where $Q_2 = i\,x\,(dx^2 + dy^2) - dy$. Let us compute two matrices V, $W \in \mathrm{GL}_2(B)$ such that $V\,R\,W = \mathrm{diag}(1, Q_2)$.

The right inverse Q_3 of K, defined by

```
> Q_3:=RightInverse(K,B);
```

$$Q_3 := \begin{bmatrix} -\frac{x}{i} \\ -1 \end{bmatrix}$$

is such that the following matrix $W = (Q_3 \quad Q_1)$ defined by

```
> W:=augment(Q_3,Q_1);
```

$$W := \begin{bmatrix} -\frac{x}{i} & -i - x\,dy + dx\,ix \\ -1 & -dx - dy\,i \end{bmatrix}$$

is unimodular, i.e., $W \in \mathrm{GL}_2(B)$, and its inverse is defined by:

```
> W_inv:=LeftInverse(W,B);
```

$$W\_inv := \begin{bmatrix} -dx\,i + dy & dy\,ix + x\,dx \\ i & -x \end{bmatrix}$$

Moreover, the matrix $X = (R\,Q_3 \quad \Lambda)$ defined by

```
> X:=augment(Mult(R,Q_3,B),Lambda_2);
```

$$X := \begin{bmatrix} \frac{-x\,dx - 1 + dy\,ix}{i} & 1 \\ -\frac{x(dy + dx\,i)}{i} & i \end{bmatrix}$$

i.e., after simplifications, defined by

```
> map(expand,subs(i=I,evalm(X)));
```

$$\begin{bmatrix} ix\,dx + i + x\,dy & 1 \\ i\,dy\,x - x\,dx & i \end{bmatrix}$$

is also unimodular, i.e., $X \in \mathrm{GL}_2(B)$. Its inverse $V = X^{-1}$ is defined by

```
> V:=LeftInverse(X,B);
```

$$V := \begin{bmatrix} i^{-1} & 1 \\ -x\,dx + dy\,ix & -i - x\,dy - dx\,ix \end{bmatrix}$$

or, equivalently, after simplifications, defined by

```
> map(expand,subs(i=I,evalm(V)));
```

$$\begin{bmatrix} -i & 1 \\ i\,dy\,x - x\,dx & -i - x\,dy - ix\,dx \end{bmatrix}$$

Finally, we obtain that $V\,R\,W = \mathrm{diag}(1, Q_2)$:

```
> map(collect,subs(i=I,Mult(V,R,W,B)),x);
```

$$\begin{bmatrix} 1 & 0 \\ 0 & ix\left(dx^2 + dy^2\right) - dy \end{bmatrix}$$

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