Intersection Testing between an Ellipsoid and an Algebraic Surface

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Abstract

This paper presents a new method on the intersection testing problem between an ellipsoid and an algebraic surface. In the new method, the testing problem is turned into a new testing problem whether a univariate polynomial has a positive or negative real root. Examples are shown to illustrate the robustness and efficiency of the new method.

1 Introduction

Intersection testing problem between two surfaces is to judge whether the two surfaces intersect with each other or not, and it doesn't care the details of the intersection. Ellipsoids have a small number of geometric parameters and are widely used for approximating a wide class of convex objects in simulations of physical systems. The intersection testing problem between an ellipsoid and an algebraic surface is thus a important problem with applications in computer graphics, computer animation, virtual reality, robotics, CAD/CAM, computational physics, and geomechanics.

For two quadratic surfaces, the methods using Segres characteristics [10, 11] are useful to classify the intersection curves between them in projective complex space, while it does not work in real affine space.

Conventional methods [1, 3, 8, 15] for finding the intersection of two surfaces could also be used to detect whether an ellipsoid and an algebraic surface intersect; if there are no real intersection points between them, then the two surfaces are either separated or one is contained in the other. However, these methods are designed to compute the structure and parameterization of the intersection curve, rather than the gross relationship between the two surfaces, and are more complicated for the intersection testing problem.

[4] discusses the intersection testing problem between a bicubic bézier patch and a plane.The intersection testing problem between NURBS surfaces is much more difficult to deduce such a simple condition as in [4].

This paper discusses the intersection testing problem between an ellipsoid and an algebraic surface in real space. [12] gives an algebraic condition for the separation of two ellipsoids, which shows that the ellipsoids are separated by a plane if and only if their characteristic equation of degree 4 has two distinct positive roots. When two ellipsoid are not separated, it doesn't answer how to distinguish whether two ellipsoid intersect or one is contained in the other, which may be used in applications such as computer graphics, computer animation, virtual reality, and so on.

This paper introduces the tangent point method, which turns the intersection testing problem into a new testing problem whether a univariate polynomial has a positive or negative real root, which is much easier to be solved with Descartes' method, the Sturm sequence method, Du Gua-Huat-Euler theorem [7], or the improved methods in [17]. The new method is intuitionistic in geometry, and seems less sensitive to rounding error by the float point arithmetic than the method in [12].

The remainder paper is organized as follows. In Section 2, we present the outline of the tangent point method. Some examples are given in Section 3. Some conclusions are drawn at the end of the paper.

2 The tangent point method

Suppose that the ellipsoid E and the algebraic surface S are determined by

$$E(x, y, z) = 0, \tag{1}$$

$$S(x, y, z) = 0. \tag{2}$$

Any point $(x, y, z)^T$ is on the surface **E** or **S** if and only if $E(x, y, z) = 0$ or $S(x, y, z) = 0$.

Lemma 1. If there exist two points $\mathbf{p}, \mathbf{q} \in \mathbf{E}$ such that $S(\mathbf{p}) < 0$ and $S(\mathbf{q}) > 0$, then the two surfaces **E** and **S** intersect with each other.

Proof. Suppose there exist two points $\mathbf{p}, \mathbf{q} \in \mathbf{E}$ such that $S(\mathbf{p}) < 0$ and $S(\mathbf{q}) > 0$. The ellipsoid **E** is a simply connected surface, which means there exists a continuous path $\mathbf{C}(t)$ on **E** such that $\mathbf{C}(0) = \mathbf{p}$ and $\mathbf{C}(1) = \mathbf{q}$. Thus, the function $S(\mathbf{C}(t))$ is a continuous function in t in $[0, 1]$. Since $S(\mathbf{C}(0)) < 0$ and $S(\mathbf{C}(1)) > 0$, so there exists some $t_0 \in (0, 1)$, such that $S(\mathbf{C}(t_0)) = 0$, which means that point $\mathbf{C}(t_0)$ is an intersection point between **E** and **S**. Thus, **S** and $\mathbf{S}_2$ intersect. So we have completed the proof.

One of the two points **p** or **q** can be any point on the ellipsoid **E**, which can be directly obtained. Without loss of generality, suppose **p** has been obtained. For the intersection testing problem between two quadrics, any element in their pencil is a quadric, which is simple-connected. We can construct $\mathbf{p} = (0, 0, 0)^T$, which is on one of the two quadrics. Here is the details. Suppose the two quadrics $\mathbf{s}_1$ and $\mathbf{s}_2$ are determined by $f_i(x, y, z) = 0, i = 1, 2$. If $f_1(0, 0, 0) = 0$ or $f_2(x, y, z) = 0$, then $\mathbf{p} = (0, 0, 0)^T$ is on $\mathbf{s}_1$ or $\mathbf{s}_2$. Otherwise, $\mathbf{p} = (0, 0, 0)^T$ is on a new surface $\mathbf{s}_2^*$, which is determined by $f_1(0, 0, 0)f_2(x, y, z) - f_2(0, 0, 0)f_1(x, y, z) = 0$. The intersection testing problem between $\mathbf{s}_1$ and $\mathbf{s}_2$ is equivalent to the problem between $\mathbf{s}_1$ and $\mathbf{s}_2^*$. Let $\mathbf{s}_2$ be $\mathbf{s}_2^*$, we have constructed the point **p** to be $(0, 0, 0)^T$.

To find the other point **q**, we introduce the following constrained function

$$S^*(\mathbf{t}) = S(\mathbf{t})|_{\mathbf{t} \in \mathbf{E}},$$

which means the definition domain of S^* is surface **E**. The ellipsoid **E** is a compact subset of the real Euclidean space thus any continuous function such as $S(x, y, z)$ would attend its maximum and minimum on the ellipsoid **E**. From the mathematical theory, S^* is bounded, whose upper bound and lower bound are denoted by μ and ν. Furthermore, for any real number $t \in [\nu, \mu]$, there exists point $\mathbf{p}_t \in \mathbf{E}$ such that $S^*(\mathbf{p}_t) = t$. If ν or μ is zero, or ν and μ are of opposite sign, then there is a point $\mathbf{t} \in \mathbf{E}$ such that $S(\mathbf{t}) = 0$, which means that **t** is an intersection point between **E** and **S**. Thus, we have

Theorem 1. The two surfaces **E** and **S** intersect with each other if and only if either $\mu > 0$ and $\nu < 0$, or $\mu\nu = 0$.

Proof. Necessity. Suppose surfaces **E** and **S** intersect with each other. If $S^*(\mathbf{t}) \leq 0$ or $S^*(\mathbf{t}) \geq 0$, then we have $\mu = 0$ or $\nu = 0$, and thus we have $\mu\nu = 0$. Otherwise, there exists two points $\mathbf{p}, \mathbf{q} \in \mathbf{E}$ such that $S(\mathbf{p}) < 0$ and $S(\mathbf{q}) > 0$. Thus, we have $\mu > S(\mathbf{q}) > 0$ and $\nu < S(\mathbf{p}) < 0$.

Sufficiency. If $\mu\nu = 0$, then there is a point $\mathbf{t} \in \mathbf{E}$ such that $S(\mathbf{t}) = 0$, and **t** is an intersection point **E** and **S**. Thus, the two surfaces **E** and **S** intersect with each other. Otherwise, suppose $\mu > 0$ and $\nu < 0$. There exist two points $\mathbf{p}, \mathbf{q} \in \mathbf{E}$ such that $S(\mathbf{p}) = \nu < 0$ and $S(\mathbf{q}) = \mu > 0$. From Lemma 1, the two surfaces **E** and **S** intersect with each other. So we have completed the proof.

From the assumption, point **p** has been found, so we only discuss on the value of μ. Both μ and ν are extreme values of S^*, and μ is also the solution of the following constrained maximization problem

$$\begin{array}{ll} \max & S^*(x, y, z) \\ \mathbf{s.t.} & E(x, y, z) = 0. \end{array}$$

Suppose S^* reaches μ at point $\mathbf{q}^*$. With Lagrange multiple method, we have

$$\left\{ \begin{array}{lcl} E(\mathbf{q}^*) & = & 0, \\ S(\mathbf{q}^*) - \mu & = & 0, \\ S_x(\mathbf{q}^*) - \lambda E_x(\mathbf{q}^*) & = & 0, \\ S_y(\mathbf{q}^*) - \lambda E_y(\mathbf{q}^*) & = & 0, \\ S_z(\mathbf{q}^*) - \lambda E_z(\mathbf{q}^*) & = & 0, \end{array} \right. \quad (3)$$

where S_x, S_y, S_z and E_x, E_y, E_z are partial derivative functions in x, y, z of S and E, respectively. The geometric meaning of the equation system (3) is that surfaces **E** and $\mathbf{S}_\mu$ are tangent with each other at point $\mathbf{q}^*$, where $\mathbf{S}_\mu$ is determined by $S(x, y, z) - \mu = 0$. Thus, the above method is called the tangent point method in this paper.

We eliminate x, y, z and λ from Equation (3) and obtain a univariate polynomial $D(\mu)$ in μ, which is called judging equation in this paper. The resultant theory [2, 13, 14] is useful for eliminating variables from a set of polynomial equations. Given two polynomials $h_1(t) = \sum_{i=0}^{n} c_i t^i$ and $h_2(t) = \sum_{j=0}^{m} d_j t^j$, where $c_n \neq 0$ and $d_m \neq 0$. $Res(h_1, h_2, t)$, the resultant of $h_1(t)$ and $h_2(t)$ in t, is the determinant shown in equation (4).

Any extreme value of S^* is a zero root of $D(\mu)$. To determine whether there exists any positive extreme value or not, we only need to test whether $D(\mu)$ has any positive root, which can be easily solved with Descartes' method, the Sturm sequence method, Du Gua-Huat-Euler theorem [7], or the improved methods in [17],which are to explicitly compute the number of the positive roots of a univariate polynomial equation. On the other hand, there are some methods which are useful to solve the equation system (3) directly, such as the method in [16].

3 Examples

This section provides four examples. The first two examples are extracted from [12], which uses two separated or intersected ellipsoids, respectively. The third and fourth examples are not covered in [12], in which the algebraic

$$Res(h_1,h_2,t)=\det\left(\begin{bmatrix} c_n & c_{n-1} & \cdots & c_1 & c_0 & 0 & 0 & 0 & 0 & 0 \\ 0 & c_n & c_{n-1} & \cdots & c_1 & c_0 & 0 & 0 & 0 & 0 \\ & & \ddots & & & & \ddots & & & \\ 0 & 0 & 0 & 0 & c_n & c_{n-1} & \cdots & c_1 & c_0 & 0 \\ 0 & 0 & 0 & 0 & 0 & c_n & c_{n-1} & \cdots & c_1 & c_0 \\ d_m & d_{m-1} & \cdots & d_1 & d_0 & 0 & 0 & 0 & 0 & 0 \\ 0 & d_m & d_{m-1} & \cdots & d_1 & d_0 & 0 & 0 & 0 & 0 \\ & & \ddots & & & & \ddots & & & \\ 0 & 0 & 0 & 0 & d_m & d_{m-1} & \cdots & d_1 & d_0 & 0 \\ 0 & 0 & 0 & 0 & 0 & d_m & d_{m-1} & \cdots & d_1 & d_0 \end{bmatrix}\right). \tag{4}$$

surface **S** is an ellipsoid and a quartic surface, respectively. It seems that the new method has intuitive geometric meaning and is less sensitive to rounding error by the floating point arithmetic than the method in [12].

All the examples are implemented with the Maple software.

Example 1. Given two separate ellipsoids **E** : $x^2+y^2+z^2-25=0$ and **S** : $(x-9)^2/9+y^2/4+z^2/16-R=0$. Firstly, let $R=1$(see also [12]). With the tangent point method, we have $D(\mu)=43046721\mu^2-927895986\mu+695656269$. $D(\mu)$ have two positive real roots $187/9$ and $7/9$. From Theorem 1, the two ellipsoids are separated. On the other hand, when we first pick a point $\mathbf{t}(0,0,5)^T$ from the first ellipsoid **E**, and have $S(0,0,5)=153/16>0$. Together with the fact that $D(\mu)$ has no negative real roots, the two ellipsoids are separated. Secondly, let $R=1.77777777777<1+7/9$, and the two ellipsoids are also separated. In this case, we have $D(\mu)=43046721\mu^2-860934420.00\mu+0.009$. $D(\mu)$ have two positive real roots 20.00 and $1.0e-11$. From Theorem 1, the two ellipsoids are separated. With the method in [12], because of the rounding error of the floating point arithmetic, the characteristic equation has two real roots and two complex roots, and then to obtain the wrong answer that the two ellipsoids are intersected.

Example 2. Given two intersected ellipsoids **E** : $x^2+y^2+z^2-25=0$ and **S** : $(x-6)^2/9+y^2/4+z^2/16-1=0$. With the tangent point method, we have $D(\mu)=8503056\mu^2-98257536\mu-94058496$. $D(\mu)$ have two real roots $112/9$ and $-8/9$, and one is positive and the other is negative. From Theorem 1, the two ellipsoids intersect with each other. When pick $\mathbf{t}(0,0,5)^T\in$ **E**, and we have $S(0,0,5)=73/16>0$, and $D(\mu)$ has a negative real root. Thus, we also can draw the same conclusion.

Example 3. Given two intersected ellipsoids **E** : $x^2+y^2+z^2-25=0$ and **S** : $(x-1)^2/9+y^2/4+z^2/2-1=0$. With the tangent point method, we have $D(\mu)=6561\mu^2-24786\mu+15309$. $D(\mu)$ have two positive real roots 3 and $7/9$. From Theorem 1, the two ellipsoids don't intersect with each other. When pick $\mathbf{t}(0,0,5)^T\in$ **E**, and we have $S(0,0,5)=209/18>0$, and $D(\mu)$ has no negative real roots. Thus, we say that **S** is contained in **E**.

Example 4. Given an ellipsoid **E** : $2x^2+3y^2+4z^2-25=0$ and a quantic surface **S** : $(x-1)^4/9+y^2/4+4z^2/23-1=0$. These two surfaces have an intersection point $(1,0,\sqrt{23}/2)^T$. With the tangent point method, we have $D(\mu)=(214249-64296\mu+1296\mu^2)(6436343\mu^3+219006\mu^2-83835\mu+5427)(21914624432020321\mu^6+1491354403691364\mu^5-1835296251045516\mu^4+62016971050548\mu^3+44064679219911\mu^2-5127987925710\mu+209582935731)(27648\mu^3-72576\mu^2+60876\mu-15809)(3057647616\mu^6-16052649984\mu^5+33243955200\mu^4-34087108608\mu^3+17707678272\mu^2-4238920368\mu+370833049)$. With the help of some mathematic software, there are four real roots of $D(\mu)$, i.e., -0.154, 0.507, 3.592 and 46.019. When pick $\mathbf{t}(0,0,5)^T\in$ **E**, and we have $S(0,0,5)=716/207>0$, and $D(\mu)$ has a negative real root. Thus, we say that **S** intersect with **E**.

4 Conclusions

We have presented the tangent point method to solve the intersection testing problem between an ellipsoid and an algebraic surface. The new method utilizes the fact that an ellipsoid is a bounded and closed surface, and deduces a conclusion that the ellipsoid and the algebraic surface intersect with each other if and only if their judging equation either has both negative and positive real roots, or zero is its real root. When zero is not a real root of the judging equation, with the help of a point on the ellipsoid, it only needs to judge whether their judging equation has a negative or positive real root. The new method has intuitionistic geometric meaning, and can tell whether the ellipsoid and the algebraic surface are separated, or intersected, or one is contained in the other. In the future, we will extend the tangent point method to the intersection testing problem between two algebraic surfaces.

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