Robust equilibrated a posteriori error estimators for the Reissner-Mindlin system

Emmanuel Creusé, Serge Nicaise, Emmanuel Verhille

Robust equilibrated a posteriori error estimators for the Reissner-Mindlin system

Emmanuel Creusé*, Serge Nicaise†, Emmanuel Verhille ‡

November 3, 2010

Abstract

We consider a conforming finite element approximation of the Reissner-Mindlin system. We propose a new robust a posteriori error estimator based on $H(\mathrm{div}\,)$ conforming finite elements and equilibrated fluxes. It is shown that this estimator gives rise to an upper bound where the constant is one up to higher order terms. Lower bounds can also be established with constants depending on the shape regularity of the mesh. The reliability and efficiency of the proposed estimator are confirmed by some numerical tests.

Key Words Reissner-Mindlin plate, finite elements, a posteriori error estimators.
AMS (MOS) subject classification 74K20, 65M60, 65M15, 65M50.

1 Introduction

The finite element method is often used for the numerical approximation of partial differential equations, see, e.g., [7, 8, 13]. In many engineering applications, adaptive techniques based on a posteriori error estimators have become an indispensable tool to obtain reliable results. Nowadays there exists a vast amount of literature on locally defined a posteriori error estimators for problems in structural mechanics. We refer to the monographs [1, 2, 29, 32] for a good overview on this topic. In general, upper and lower bounds are established in order to guarantee the reliability and the efficiency of the proposed estimator. Most of the existing approaches involve constants depending on the shape regularity of the

*Université des Sciences et Technologies de Lille, Laboratoire Paul Painlevé UMR 8524, and EPI SIMPAF - INRIA Lille Nord Europe, Cité Scientifique, 59655 Villeneuve d'Ascq Cedex email: creuse@math.univ-lille1.fr

†Université de Valenciennes et du Hainaut Cambrésis, LAMAV, FR CNRS 2956, Institut des Sciences et Techniques de Valenciennes, F-59313 - Valenciennes Cedex 9 France, email: Serge.Nicaise@univ-valenciennes.fr

‡Université des Sciences et Technologies de Lille, Laboratoire Paul Painlevé UMR 8524, Cité Scientifique, 59655 Villeneuve d'Ascq Cedex email: verhille@math.univ-lille1.fr

elements; but these dependencies are often not given. Only a small number of approaches gives rise to estimates with explicit constants, see, e.g., [1, 6, 15, 20, 21, 25, 28, 29, 30]. However in practical applications the knowledge of such constants is of great importance, especially for adaptivity.

The finite element approximation of the Reissner-Mindlin system recently became an active subject of research due to its practical importance and its non trivial challenges to overcome. In particular, appropriated finite elements have to be used in order to avoid shear locking. Such elements are in our days well known and different a priori error estimates are available in the literature. On the contrary for a posteriori error analysis only a small number of results exists, we refer to [5, 9, 11, 12, 21, 26, 27, 24]. Most of these papers enter in the first category mentioned before and to our knowledge only the paper [21] proposes an estimator where an upper bound is proved with a constant 1. Hence our goal is to give an estimator that is robust with respect to the thickness parameter t, with an explicit constant in the upper bound, that is also efficient and that is explicitly computable. For these purposes we use an approach based on equilibrated fluxes and $H(\text{div})$–conforming elements. Similar ideas can be found, e.g., in [6, 15, 21, 28, 30]. For an overview on equilibration techniques, we refer to [1, 25].

The outline of the paper is as follows: We recall, in Section 2, the Reissner-Mindlin system, its numerical approximation and introduce some useful quantities. Section 3 is devoted to some preliminary results in order to prove the upper bound. This one directly follows from these considerations and is given in full details in section 4. The lower bound developped in section 5 relies on suitable norm equivalences and by using appropriated $H(\text{div})$ approximations of the solutions. Finally some numerical tests are presented in section 6, that confirm the reliability and the efficiency of our error estimator.

2 The boundary value problem and its discretization

Let Ω be a bounded open domain of $\mathbb{R}^2$ with a Lipschitz boundary Γ that we suppose to be polygonal. We consider the following Reissner-Mindlin problem : Given $g \in L^2(\Omega)$ defined as the scaled transverse loading function and t a fixed positive real number that represents the thickness of the plate, find $(\omega, \phi) \in H_0^1(\Omega) \times H_0^1(\Omega)^2$ such that

$$a(\phi, \psi) + (\gamma, \nabla v - \psi) = (g, v) \text{ for all } (v, \psi) \in H_0^1(\Omega) \times H_0^1(\Omega)^2, \tag{1}$$

where

$$\gamma = \lambda\, t^{-2}\, (\nabla\omega - \phi) \text{ and } a(\phi, \psi) = \int_\Omega \mathcal{C}\varepsilon(\phi)\varepsilon(\psi) dx. \tag{2}$$

Here, $(\cdot\,,\,\cdot)$ stands for the usual inner product in (any power of) $L^2(\Omega)$, the operator : denotes the usual term-by term tensor product and

$$\varepsilon(\phi) = \frac{1}{2}(\nabla\phi + (\nabla\phi)^T).$$

$\mathcal{C}$ is the usual elasticity tensor given by

$$\mathcal{C}\varepsilon(\phi) = 2\,\mu\,\varepsilon(\phi) + \widetilde{\lambda}\,tr(\varepsilon(\phi))\,\mathcal{I}.$$

The parameters μ, $\widetilde{\lambda}$ and λ are some Lamé coefficients defined according to the Young modulus E and the Poisson coefficient ν of the material. In the following, for shortness the $L^2(D)$-norm is denoted by $\|\cdot\|_D$. The usual norm and seminorm of $H^1(D)$ are respectively denoted by $\|\cdot\|_{1,D}$ and $|\cdot|_{1,D}$ and the usual norm on $H^{-1}(D)$ is denoted $\|\cdot\|_{-1,D}$. For all these norms, in the case $D = \Omega$, the index Ω is dropped. The usual Poincaré-Friedrichs constant in Ω is the smallest positive constant c_F such that

$$||\phi|| \leq c_F\,|\phi|_1 \quad \forall \phi \in H_0^1(\Omega)^2.$$

By Korn's inequality [22], a is an inner product on $H_0^1(\Omega)^2$ equivalent to the usual one. Indeed, defining the energy norm $||\cdot||_{\mathcal{C}}$ by

$$\|\psi\|_{\mathcal{C}}^2 = a(\psi,\psi)\,\forall\,\psi \in H_0^1(\Omega)^2,$$

it can be shown (see annex 7.1) that

$$|\psi|_1^2 \leq \frac{1}{\mu}\|\psi\|_{\mathcal{C}}^2\,\forall\,\psi \in H_0^1(\Omega)^2. \tag{3}$$

Consequently, the continuous problem (1)-(2) is well-posed.

Lemma 2.1 *The problem (1)-(2) has a unique solution* $(\omega,\phi) \in H_0^1(\Omega) \times H_0^1(\Omega)^2$.

Proof: Defining the functional $F((\omega,\phi),(v,\psi)) = a(\phi,\psi) + (\gamma, \nabla v - \psi)$ with $\gamma = \lambda\,t^{-2}\,(\nabla\omega - \phi)$, let us establish its coerciveness, namely that there exists $k > 0$ such that

$$F((\omega,\phi),[\omega,\phi]) \geq k\,(|\omega|_1^2 + |\phi|_1^2), \forall(\omega,\phi) \in H_0^1(\Omega) \times H_0^1(\Omega)^2. \tag{4}$$

Fix an arbitrary pair $(\omega,\phi) \in H_0^1(\Omega) \times H_0^1(\Omega)^2$. First of all, (3) and the standard Cauchy-Schwarz inequality lead to

$$F((\omega,\phi),(\omega,\phi)) \;\; \geq \;\; \mu|\phi|_1^2 + \lambda t^{-2}\left((1-\eta)|\omega|_1^2 + \left(1 - \frac{1}{\eta}\right)\|\phi\|^2\right), \forall \eta > 0.$$

Then we directly obtain

$$F((\omega,\phi),(\omega,\phi)) \geq \frac{\mu}{2}|\phi|_1^2 + \lambda t^{-2}(1-\eta)|\omega|_1^2 + \left(\frac{\mu}{2\,c_F^2} + \lambda t^{-2}\left(1 - \frac{1}{\eta}\right)\right)\|\phi\|^2, \forall \eta > 0. \tag{5}$$

Choosing now $\eta = \dfrac{2c_F^2\lambda t^{-2}}{\mu + 2c_F^2\lambda t^{-2}} < 1$ in (5), we have

$$F((\omega,\phi),(\omega,\phi)) \geq \frac{\mu}{2}|\phi|_1^2 + \frac{\mu\,\lambda t^{-2}}{\mu + 2\,c_F^2\lambda t^{-2}}|\omega|_1^2.$$

This shows that (4) holds with $k = \min\left(\frac{\mu}{2}, \frac{\mu\,\lambda t^{-2}}{\mu + 2\,c_F^2 \lambda t^{-2}}\right)$. The conclusion follows from the Lax-Milgram lemma for which the other assumptions to fulfill are obvious. ■

Let us now consider a discretization of (1)-(2) based on a conforming triangulation $\mathcal{T}_h$ of Ω composed of triangles. We assume that this triangulation is regular, i.e., for any element $T \in \mathcal{T}_h$, the ratio h_T/ρ_T is bounded by a constant $\sigma > 0$ independent of T and of the mesh size $h = \max_{T \in \mathcal{T}_h} h_T$, where h_T is the diameter of T and ρ_T the diameter of its largest inscribed ball. We consider on this triangulation the classical conforming $\mathbb{P}_1$ finite element spaces $W_h \times \Theta_h$ defined by

$$W_h = \left\{ v_h \in \mathcal{C}^0(\bar{\Omega}); v_h = 0 \text{ on } \partial\Omega \text{ and } v_{h|T} \in \mathbb{P}_1(T)\ \forall T \in \mathcal{T}_h \right\} \subset H_0^1(\Omega),$$

$$\Theta_h = W_h \times W_h \subset H_0^1(\Omega) \times H_0^1(\Omega).$$

The discrete formulation of the Reissner-Mindlin problem is now to find $(\omega_h, \phi_h) \in W_h \times \Theta_h$ such that

$$a(\phi_h, \psi_h) + (\gamma_h, \nabla v_h - \mathbf{R}_h \psi_h) = (g, v_h) \text{ for all } (v_h, \psi_h) \in W_h \times \Theta_h, \tag{6}$$

with

$$\gamma_h = \lambda t^{-2}(\nabla \omega_h - \mathbf{R}_h \phi_h). \tag{7}$$

Here, $\mathbf{R}_h$ denotes the reduction integration operator in the context of shear-locking with values in the so-called discrete shear force space Γ_h which depends on the finite element involved [3, 4, 18, 19, 31]. We assume moreover that

$$\text{For all } \psi_h \in \Theta_h, \mathbf{R}_h \psi_h \in H_0(rot, \Omega),$$

where $H_0(rot, \Omega) = \{v \in L^2(\Omega)^2;\ rot\ v \in L^2(\Omega) \text{ and } v \cdot \tau = 0 \text{ on } \partial\Omega\}$, equipped with the norm

$$\|v\|^2_{H(rot,\Omega)} = \|v\|^2_\Omega + \|rot\, v\|^2_\Omega.$$

Here, for any $v = (v_1, v_2)^T \in L^2(\Omega)^2$, $rot\, v = \partial v_2/\partial x - \partial v_1/\partial y$ and τ is the unit tangent vector along $\partial\Omega$. In this work, $\mathbf{R}_h$ is defined as the interpolation operator from Θ_h on the $H_0(rot, \Omega)$ conforming lower-order Nedelec finite element space [22].

By the usual Helmholtz decomposition of any $H_0(rot, \Omega)$ vector field [8, p. 299], there exists $w \in H_0^1(\Omega)$ and $\beta \in H_0^1(\Omega)^2$ such that :

$$(\mathbf{R}_h - I)\phi_h = \nabla w - \beta, \tag{8}$$

as well as a constant $C > 0$ such that

$$\|w\|_1 + \|\beta\|_1 \leq C\ \|(\mathbf{R}_h - I)\phi_h\|_{H(rot,\Omega)}.$$

More precisely, we introduce the constant c_R such that

$$|\beta|_1 \leq c_R \, \|rot(\mathbf{R}_h - I)\phi_h\|,$$

which can be evaluated by [22]

$$c_R = \left(\inf_{q \in L^2(\Omega)} \sup_{v \in H_0^1(\Omega)^2} \frac{(div\, v, q)}{\|q\| \, |v|_1} \right)^{-1}.$$

Given the exact solution $(\omega, \phi) \in H_0^1(\Omega) \times H_0^1(\Omega)^2$ as well as the approximated one $(\omega_h, \phi_h) \in W_h \times \Theta_h$, the usual error e_h^{rot} is defined as

$$(e_h^{rot})^2 = |\omega - \omega_h|_1^2 + |\phi - \phi_h|_1^2 + \lambda^{-1} t^2 \|\gamma - \gamma_h\|^2 + \lambda^{-2} t^4 \|rot(\gamma - \gamma_h)\|^2 + \|\gamma - \gamma_h\|_{-1}^2. \quad (9)$$

The residuals are also defined as follows

$$Res_1(v) = (g, v) - (\gamma_h, \nabla v) \quad \text{for all } v \in H_0^1(\Omega), \quad (10)$$

$$Res_2(\psi) = -a(\phi_h, \psi) + (\gamma_h, \psi) \quad \text{for all } \psi \in H_0^1(\Omega)^2. \quad (11)$$

Finally, let us now introduce, in the spirit of [21], the spaces $N_{div}(\Omega)$ and $H_{div}(\Omega)$ respectively defined by

$$H_{div}(\Omega) = \{y \in L^2(\Omega, \mathbb{R}^2) |\, div\, y \in L^2(\Omega)\},$$
$$N_{div}(\Omega) = \{x \in L^2(\Omega, \mathcal{M}_S^2) |\, div\, x \in L^2(\Omega, \mathbb{R}^2)\},$$

where $\mathcal{M}_S^2$ is the space of symmetric tensors of second rank. We now fix an arbitrary $y^* \in H_{div}(\Omega)$ such that $div\, y^* = -\Pi_h g$, where Π_h is the projection operator from $L^2(\Omega)$ to the piecewise constant fonctions on the triangulation. Let us also fix $x^* \in N_{div}(\Omega)$ such that $div\, x^* = -\gamma_h$. Their existence and construction will be explained later on.

We finally need to introduce the following mesh-dependent norm. For all $(\psi, v) \in H_0^1(\Omega) \times H_0^1(\Omega)^2$, we define

$$|||(\psi, v)|||_{1,h}^2 = \|\nabla \psi\|^2 + \sum_{T \in \mathcal{T}_h} \frac{1}{t^2 + h_T^2} \|\nabla v - \psi\|_T^2. \quad (12)$$

For all functional F defined on $H_0^1(\Omega) \times H_0^1(\Omega)^2$, the dual norm associated with (12) is classically defined by

$$|||F|||_{-1,h} = \sup_{(\psi, v) \in H_0^1(\Omega) \times H_0^1(\Omega)^2 \setminus \{0\}} \frac{F(\psi, v)}{|||(\psi, v)|||_{1,h}}. \quad (13)$$

3 Preliminary results

The aim of this section is to prove four lemmas which will be used in the following of the paper.

Lemma 3.1 *Let us consider* $(\alpha, \varepsilon) \in (\mathbb{R}_+^*)^2$. *Then we have*

$$\lambda(t^{-2} - \alpha^2)\|\nabla(\omega - \omega_h) - (\phi - \boldsymbol{R}_h\phi_h)\|^2 + \lambda\alpha^2(1 - 2\varepsilon)\|\nabla(\omega - \omega_h)\|^2$$
$$\leq \lambda^{-1}t^2\|\gamma - \gamma_h\|^2 - \lambda\alpha^2\left(1 - \frac{2}{\varepsilon}\right)\|\phi_h - \boldsymbol{R}_h\phi_h\|^2 - \lambda\alpha^2\left(1 - \frac{1}{\varepsilon} - \varepsilon\right)\|\phi - \phi_h\|^2.$$

Proof: We first write

$$\|\nabla(\omega - \omega_h) - (\phi - \phi_h) - (\phi_h - \mathbf{R}_h\phi_h)\|^2$$
$$= \|\nabla(\omega - \omega_h)\|^2 + \|\phi - \phi_h\|^2 + \|\phi_h - \mathbf{R}_h\phi_h)\|^2$$
$$-2(\nabla(\omega - \omega_h), \phi - \phi_h) - 2(\nabla(\omega - \omega_h), \phi_h - \mathbf{R}_h\phi_h) + 2(\phi - \phi_h, \phi_h - \mathbf{R}_h\phi_h).$$

Consequently, we have

$$\begin{aligned}
\lambda^{-1}t^2\|\gamma - \gamma_h\|^2 &= \lambda(t^{-2} - \alpha^2)\|\nabla(\omega - \omega_h) - (\phi - \mathbf{R}_h\phi_h)\|^2 \\
&+ \lambda\alpha^2(\|\nabla(\omega - \omega_h)\|^2 + \|\phi - \phi_h\|^2 + \|\phi_h - \mathbf{R}_h\phi_h\|^2) \\
&+ 2\,\lambda\alpha^2(\phi - \phi_h, \phi_h - \mathbf{R}_h\phi_h) - 2\lambda\alpha^2(\nabla(\omega - \omega_h), \phi - \phi_h) \\
&- 2\,\lambda\alpha^2(\nabla(\omega - \omega_h), \phi_h - \mathbf{R}_h\phi_h).
\end{aligned}$$

Using the three following Young inequalities

$$\begin{cases}
-2(\phi - \phi_h, \phi_h - \mathbf{R}_h\phi_h) & \leq \varepsilon\|\phi - \phi_h\|^2 + \dfrac{1}{\varepsilon}\|\phi_h - \mathbf{R}_h\phi_h\|^2, \\
2(\nabla(\omega - \omega_h), \phi - \phi_h) & \leq \varepsilon\|\nabla(\omega - \omega_h)\|^2 + \dfrac{1}{\varepsilon}\|\phi - \phi_h\|^2, \\
2(\nabla(\omega - \omega_h), \phi_h - \mathbf{R}_h\phi_h) & \leq \varepsilon\|\nabla(\omega - \omega_h)\|^2 + \dfrac{1}{\varepsilon}\|\phi_h - \mathbf{R}_h\phi_h\|^2,
\end{cases}$$

we get

$$\lambda(t^{-2} - \alpha^2)\|\nabla(\omega - \omega_h) - (\phi - \mathbf{R}_h\phi_h)\|^2$$
$$\leq \lambda^{-1}t^2\|\gamma - \gamma_h\|^2 - \lambda\alpha^2\left(\|\nabla(\omega - \omega_h)\|^2 + \|\phi - \phi_h\|^2 + \|\phi_h - \mathbf{R}_h\phi_h\|^2\right)$$
$$+\lambda\alpha^2\Bigg(\varepsilon\|\phi - \phi_h\|^2 + \frac{1}{\varepsilon}\|\phi_h - \mathbf{R}_h\phi_h\|^2 + \varepsilon\|\nabla(\omega - \omega_h)\|^2 + \frac{1}{\varepsilon}\|\phi - \phi_h\|^2$$
$$+\varepsilon\|\nabla(\omega - \omega_h)\|^2 + \frac{1}{\varepsilon}\|\phi_h - \mathbf{R}_h\phi_h\|^2\Bigg)$$
$$= \lambda^{-1}t^2\|\gamma - \gamma_h\|^2 - \lambda\alpha^2\left(1 - \frac{2}{\varepsilon}\right)\|\phi_h - \mathbf{R}_h\phi_h\|^2 - \lambda\alpha^2\left(1 - \frac{1}{\varepsilon} - \varepsilon\right)\|\phi - \phi_h\|^2$$
$$-\lambda\alpha^2(1 - 2\varepsilon)\|\nabla(\omega - \omega_h)\|^2.$$

This proves the lemma. ■

Lemma 3.2 *we have*

$$\|\gamma-\gamma_h\|_{-1}^2 \quad \le 4\,(\mu+\tilde\lambda)\,\|\phi-\phi_h\|_{\mathcal{C}}^2 + 2\|Res_2\|_{-1}^2. \tag{14}$$

Proof: First, it can be shown that for any $\psi \in (H_0^1(\Omega))^2$,

$$\|\psi\|_{\mathcal{C}}^2 \le 2\,(\mu+\tilde\lambda)|\psi|_1^2,$$

so that

$$\begin{aligned}(\gamma-\gamma_h,\psi) &= a(\phi-\phi_h,\psi)+a(\phi_h,\psi)-(\gamma_h,\psi)\\ &= a(\phi-\phi_h,\psi)-Res_2(\psi)\\ &\le \|\phi-\phi_h\|_{\mathcal{C}}\|\psi\|_{\mathcal{C}}+\|Res_2\|_{-1}|\psi|_1\\ &\le \Big((2\,(\mu+\tilde\lambda))^{1/2}\|\phi-\phi_h\|_{\mathcal{C}}+\|Res_2\|_{-1}\Big)\,|\psi|_1.\end{aligned}$$

Hence we get

$$\begin{aligned}\|\gamma-\gamma_h\|_{-1}^2 &\le \Big((2\,(\mu+\tilde\lambda))^{1/2}\|\phi-\phi_h\|_{\mathcal{C}}+\|Res_2\|_{-1}\Big)^2\\ &\le 4\,(\mu+\tilde\lambda)\,\|\phi-\phi_h\|_{\mathcal{C}}^2+2\,\|Res_2\|_{-1}^2.\end{aligned}$$

■

Lemma 3.3

$$\|\phi-\phi_h\|_{\mathcal{C}}^2+\lambda^{-1}t^2\|\gamma-\gamma_h\|^2 = Res_1(\omega-\omega_h+w)+Res_2(\phi-\phi_h+\beta)-a(\phi-\phi_h,\beta),$$

where w and β are given by the Helmholtz decomposition (8).

Proof: This result is similar to the one given in [11]. First, (1) and (8) lead to

$$\begin{aligned}(\gamma-\gamma_h,(\mathbf{R}_h-I)\phi_h) &= (\gamma-\gamma_h,\nabla w-\beta)\\ &= (\gamma,\nabla w)-(\gamma,\beta)-(\gamma_h,\nabla w-\beta)\\ &= (g,w)-a(\phi,\beta)-(\gamma_h,\nabla w-\beta)\\ &= -a(\phi-\phi_h,\beta)+(g,w)-a(\phi_h,\beta)-(\gamma_h,\nabla w-\beta).\end{aligned}$$

A simple calculation shows that

$$\begin{aligned}&\|\phi-\phi_h\|_{\mathcal{C}}^2+\lambda^{-1}t^2\|\gamma-\gamma_h\|^2\\ &= a(\phi-\phi_h,\phi-\phi_h)+(\gamma-\gamma_h,(\nabla\omega-\nabla\omega_h)-(\phi-\phi_h))+(\gamma-\gamma_h,(\mathbf{R}_h-I)\phi_h)\\ &= (g,\omega-\omega_h)-a(\phi_h,\phi-\phi_h)-(\gamma_h,\nabla(\omega-\omega_h))\\ &\quad +(\gamma_h,\phi-\phi_h)-a(\phi-\phi_h,\beta)+(g,w)-a(\phi_h,\beta)-(\gamma_h,\nabla w-\beta)\\ &= Res_2(\phi-\phi_h+\beta)+(g,\omega-\omega_h+w)-(\gamma_h,\nabla(\omega-\omega_h+w))-a(\phi-\phi_h,\beta)\\ &= Res_2(\phi-\phi_h+\beta)+Res_1(\omega-\omega_h+w)-a(\phi-\phi_h,\beta).\end{aligned}$$

So we get

$$\|\phi - \phi_h\|_{\mathcal{C}}^2 + \lambda^{-1}t^2\|\gamma - \gamma_h\|^2 = Res_1(\omega - \omega_h + w) + Res_2(\phi - \phi_h + \beta) - a(\phi - \phi_h, \beta).$$

■

Lemma 3.4

$$\begin{aligned}
&\frac{1}{2}\|\phi - \phi_h + \beta\|_{\mathcal{C}}^2 + \frac{1}{2}\|\phi - \phi_h\|_{\mathcal{C}}^2 + \frac{1}{2}\lambda^{-1}t^2\|\gamma - \gamma_h\|^2 \\
&\quad + \frac{1}{2}\sum_{T\in\mathcal{T}_h} \frac{\lambda}{t^2 + h_T^2}\|\nabla(\omega - \omega_h + w) - (\phi - \phi_h + \beta)\|_T^2 \\
&\leq Res_1(\omega - \omega_h + w) + Res_2(\phi - \phi_h + \beta) + \frac{1}{2}\|\beta\|_{\mathcal{C}}^2.
\end{aligned}$$

Proof: The proof is once again similar to the one in [11]. Because of (8), we first remark that

$$\gamma - \gamma_h = \lambda t^{-2}(\nabla\omega - \nabla\omega_h - \phi + \phi_h + \nabla w - \beta),$$

so that we have for all $T \in \mathcal{T}_h$

$$\|\nabla(\omega - \omega_h + w) - (\phi - \phi_h + \beta)\|_T^2 \leq \lambda^{-2}t^4\|\gamma - \gamma_h\|_T^2.$$

Then,

$$\begin{aligned}
&\frac{1}{2}\|\phi - \phi_h + \beta\|_{\mathcal{C}}^2 + \frac{1}{2}\|\phi - \phi_h\|_{\mathcal{C}}^2 + \frac{1}{2}\lambda^{-1}t^2\|\gamma - \gamma_h\|^2 \\
&\quad + \frac{1}{2}\sum_{T\in\mathcal{T}_h} \frac{\lambda}{t^2 + h_T^2}\|\nabla(\omega - \omega_h + w) - (\phi - \phi_h + \beta)\|_T^2 \\
&\leq \frac{1}{2}\|\phi - \phi_h + \beta\|_{\mathcal{C}}^2 + \frac{1}{2}\|\phi - \phi_h\|_{\mathcal{C}}^2 + \frac{1}{2}\lambda^{-1}t^2\|\gamma - \gamma_h\|^2 + \frac{1}{2}\lambda^{-1}t^2\sum_{T\in\mathcal{T}_h}\|\gamma - \gamma_h\|_T^2 \\
&\leq \lambda^{-1}t^2\|\gamma - \gamma_h\|^2 + \frac{1}{2}a(\phi - \phi_h + \beta, \phi - \phi_h + \beta) + \frac{1}{2}a(\phi - \phi_h, \phi - \phi_h) \\
&= \lambda^{-1}t^2\|\gamma - \gamma_h\|^2 + \frac{1}{2}\left(\|\phi - \phi_h\|_{\mathcal{C}}^2 + 2a(\phi - \phi_h, \beta) + \|\beta\|_{\mathcal{C}}^2\right) + \frac{1}{2}\|\phi - \phi_h\|_{\mathcal{C}}^2 \\
&= \|\phi - \phi_h\|_{\mathcal{C}}^2 + \lambda^{-1}t^2\|\gamma - \gamma_h\|^2 + \frac{1}{2}\|\beta\|_{\mathcal{C}}^2 + a(\phi - \phi_h, \beta).
\end{aligned}$$

From lemma 3.3, we get

$$\begin{aligned}
&\frac{1}{2}\|\phi - \phi_h + \beta\|_{\mathcal{C}}^2 + \frac{1}{2}\|\phi - \phi_h\|_{\mathcal{C}}^2 + \frac{1}{2}\lambda^{-1}t^2\|\gamma - \gamma_h\|^2 \\
&\quad + \frac{1}{2}\sum_{T\in\mathcal{T}_h} \frac{\lambda}{t^2 + h_T^2}\|\nabla(\omega - \omega_h + w) - (\phi - \phi_h + \beta)\|_T^2 \\
&\leq Res_1(\omega - \omega_h + w) + Res_2(\phi - \phi_h + \beta) - a(\phi - \phi_h, \beta) + \frac{1}{2}\|\beta\|_{\mathcal{C}}^2 + a(\phi - \phi_h, \beta) \\
&= Res_1(\omega - \omega_h + w) + Res_2(\phi - \phi_h + \beta) + \frac{1}{2}\|\beta\|_{\mathcal{C}}^2.
\end{aligned}$$

■

4 Reliability of the estimator

Theorem 4.1 *Let us consider* $0 < \varepsilon < 1/2$*, as well as two parameters* $\nu_1 > 0$ *and* $\nu_2 > 0$*. Moreover, let us define*

$$A(\varepsilon) = \max\left(\frac{3}{\mu} + c_F^2\frac{\frac{1}{\varepsilon}+\varepsilon-1}{\mu(1-2\varepsilon)} + 4(\mu+\tilde{\lambda}); 1 + \frac{t^2}{\lambda(1-2\varepsilon)}\right).$$

Then,

$$\begin{aligned}(e_h^{rot})^2 &\leq A_1|\|Res_1\||^2_{-1,h} + A_2\|Res_2\|^2_{-1} + A_3\|\phi-\phi_h+\beta\|^2_{\mathcal{C}} \\ &\quad + A_4\|\phi_h - \boldsymbol{R}_h\phi_h\|^2_{H(rot,\Omega)} - \sum_{T\in\mathcal{T}_h} A_5^T\|\nabla(\omega-\omega_h+w)-(\phi-\phi_h+\beta)\|^2_T,\end{aligned} \tag{15}$$

with

$$\left|\begin{aligned} A_1 &= \nu_1 A(\varepsilon)^2; \\ A_2 &= \nu_2 A(\varepsilon)^2 + 2; \\ A_3 &= \frac{1}{\mu}\left(\frac{1}{\nu_1}+\frac{1}{\nu_2}\right) - A(\varepsilon); \\ A_4 &= \max\left(\frac{\frac{2}{\varepsilon}-1}{1-2\varepsilon}\ ;\ 2+2A(\varepsilon)(\mu+\tilde{\lambda})c_R^2\right); \\ A_5^T &= \frac{\lambda A(\varepsilon)}{t^2+h_T^2} - \frac{1}{\nu_1(t^2+h_T^2)},\ \forall\, T\in\mathcal{T}_h.\end{aligned}\right.$$

Proof: First of all, by using lemma 3.1 and the fact that $0 < \varepsilon < 1/2$, we get

$$\begin{aligned}(e_h^{rot})^2 &\leq \left(\frac{1}{\mu} + c_F^2\frac{\frac{1}{\varepsilon}+\varepsilon-1}{\mu(1-2\varepsilon)}\right)\|\phi-\phi_h\|^2_{\mathcal{C}} + \left(1+\frac{t^2}{\lambda(1-2\varepsilon)}\right)\lambda^{-1}t^2\|\gamma-\gamma_h\|^2 \\ &\quad + \frac{\frac{2}{\varepsilon}-1}{1-2\varepsilon}\|\phi_h - \mathbf{R}_h\phi_h\|^2 + \lambda^{-2}t^4\|rot(\gamma-\gamma_h)\|^2 + \|\gamma-\gamma_h\|^2_{-1}.\end{aligned}$$

Then, because of lemma 3.2 as well as

$$\lambda^{-2}t^4\|rot(\gamma-\gamma_h)\|^2 \leq \frac{2}{\mu}\,\|\phi-\phi_h\|^2_{\mathcal{C}} + 2\,\|rot(\phi_h-\mathbf{R}_h\phi_h)\|^2,$$

we obtain

$$\begin{aligned}(e_h^{rot})^2 &\leq \left(\frac{3}{\mu} + c_F^2\frac{\frac{1}{\varepsilon}+\varepsilon-1}{\mu(1-2\varepsilon)} + 4(\mu+\tilde{\lambda})\right)\|\phi-\phi_h\|^2_{\mathcal{C}} + 2\|rot(\phi_h-\mathbf{R}_h\phi_h)\|^2 \\ &\quad + \left(1+\frac{t^2}{\lambda(1-2\varepsilon)}\right)\lambda^{-1}t^2\|\gamma-\gamma_h\|^2 + \left(\frac{\frac{2}{\varepsilon}-1}{1-2\varepsilon}\right)\|\phi_h-\mathbf{R}_h\phi_h\|^2 \\ &\quad + 2\|Res_2\|^2_{-1}.\end{aligned}$$

By the definition of $A(\varepsilon)$ as well as lemma 3.4, we get

$$\begin{aligned}(e_h^{rot})^2 &\leq A(\varepsilon)\Big(2Res_1(\omega-\omega_h+w)+2Res_2(\phi-\phi_h+\beta)+\|\beta\|_{\mathcal{C}}^2 \\ &\quad -\|\phi-\phi_h+\beta\|_{\mathcal{C}}^2-\sum_{T\in\mathcal{T}_h}\frac{\lambda}{t^2+h_T^2}\|\nabla(\omega-\omega_h+w)-(\phi-\phi_h+\beta)\|_T^2\Big) \\ &\quad +\left(\frac{\frac{2}{\varepsilon}-1}{1-2\varepsilon}\right)\|\phi_h-\mathbf{R}_h\phi_h\|^2+2\|Res_2\|_{-1}^2+2\|rot(\phi_h-\mathbf{R}_h\phi_h)\|^2.\end{aligned}$$

We notice that

$$Res_1(\omega-\omega_h+w)\leq |||Res_1|||_{-1,h}|||(\psi,\omega-\omega_h+w)|||_{1,h}\ \forall\ \psi\in H_0^1(\Omega)^2,$$

$$Res_2(\phi-\phi_h+\beta)\leq \|Res_2\|_{-1}|\phi-\phi_h+\beta|_1.$$

Introducing now the parameters $\nu_1>0$ and $\nu_2>0$ and using two times Young's inequality lead to

$$\begin{aligned}(e_h^{rot})^2 &\leq \nu_1 A^2(\varepsilon)|||Res_1|||_{-1,h}^2+\frac{1}{\nu_1}|||(\psi,\omega-\omega_h+w)|||_{1,h}^2 \\ &\quad +\nu_2A^2(\varepsilon)\|Res_2\|_{-1}^2+\frac{1}{\nu_2}|\phi-\phi_h+\beta|_1^2 \\ &\quad -A(\varepsilon)\|\phi-\phi_h+\beta\|_{\mathcal{C}}^2+A(\varepsilon)\|\beta\|_{\mathcal{C}}^2 \\ &\quad +\left(\frac{\frac{2}{\varepsilon}-1}{1-2\varepsilon}\right)\|\phi_h-\mathbf{R}_h\phi_h\|^2+2\|Res_2\|_{-1}^2+2\|rot(\phi_h-\mathbf{R}_h\phi_h)\|^2 \\ &\quad -\sum_{T\in\mathcal{T}_h}\left(\frac{\lambda A(\varepsilon)}{t^2+h_T^2}\right)\|\nabla(\omega-\omega_h+w)-(\phi-\phi_h+\beta)\|_T^2.\end{aligned}$$

Finally, choosing $\psi=\phi-\phi_h+\beta$, we get

$$|||(\psi,\omega-\omega_h+w)|||_{1,h}^2=\|\nabla(\phi-\phi_h+\beta)\|^2+\sum_{T\in\mathcal{T}_h}\frac{1}{t^2+h_T^2}\|\nabla(\omega-\omega_h+w)-(\phi-\phi_h+\beta)\|_T^2,$$

and so (15) holds. ■

Corollary 4.2 *Let us assume that* $t\leq\sqrt{3\lambda c_F^2/\mu}$*, and let us define :*

$$\zeta=\max\left\{\frac{1}{\mu},\frac{1}{2\,\lambda}\right\}. \qquad (16)$$

Then,

$$(e_h^{rot})^2\leq 2\zeta\left(\frac{3}{\mu}+\frac{c_F^2}{\mu}(3+2\sqrt{3})+4(\mu+\tilde{\lambda})\right)|||Res_1|||_{-1,h}^2$$

$$+\left(2\zeta\left(\frac{3}{\mu}+\frac{c_F^2}{\mu}(3+2\sqrt{3})+4(\mu+\tilde{\lambda})\right)+2\right)\|Res_2\|_{-1}^2$$

$$+\max\left(7+4\sqrt{3}\ ;\ 2+\left(\frac{3}{\mu}+\frac{c_F^2}{\mu}(3+2\sqrt{3})+4(\mu+\tilde{\lambda})\right)2(\mu+\tilde{\lambda})c_R^2\right)\|\phi_h-\boldsymbol{R}_h\phi_h\|_{H(rot,\Omega)}^2.$$

Proof: Assuming $1-2\varepsilon>0$, the parameters ν_1 and ν_2 arising in the values of A_3 and A_5^T in (15) are first chosen such that $A_3\le 0$ and $A_5^T\ge 0\ \forall T\in\mathcal{T}_h$. Namely we take $\nu_1=\nu_2=2\,\zeta/A(\varepsilon)$. Consequently, we obtain

$$(e_h^{rot})^2\ \le \tilde{A}_1|||Res_1|||_{-1,h}^2+\tilde{A}_2\|Res_2\|_{-1}^2+\tilde{A}_4\|\phi_h-\mathbf{R}_h\phi_h\|_{H(rot,\Omega)}^2, \tag{17}$$

with

$$\left|\begin{array}{l}\tilde{A}_1=2\zeta A(\varepsilon);\\[2mm] \tilde{A}_2=2\zeta A(\varepsilon)+2;\\[2mm] \tilde{A}_4=\max\left(\dfrac{\frac{2}{\varepsilon}-1}{1-2\varepsilon}\ ;\ 2+2A(\varepsilon)(\mu+\tilde{\lambda})c_R^2\right).\end{array}\right.$$

Now, in order to provide a result as sharp as possible, it remains to choose appropriately the parameter ε to make the coefficients $\tilde{A}_1$, $\tilde{A}_2$ and $\tilde{A}_4$ arising in (17) as small as possible. Since we always have $1\le 3/\mu+4(\mu+\tilde{\lambda})$, the assumption $t\le\sqrt{3\lambda c_F^2/\mu}$ leads to

$$A(\varepsilon)=\frac{3}{\mu}+c_F^2\frac{\frac{1}{\varepsilon}+\varepsilon-1}{\mu(1-2\varepsilon)}+4(\mu+\tilde{\lambda}).$$

At this stage we remark that the two functions $A(\varepsilon)$ as well as $\frac{\frac{2}{\varepsilon}-1}{1-2\varepsilon}$ reach their minimum value for the same value of the argument ε, namely for $\varepsilon=2-\sqrt{3}$. So, by a simple calculation, corollary 4.2 holds. ■

Now, it remains to bound each of the two residuals.

Lemma 4.3 *Let $N\in\mathbb{N}^*$ be such that $\max_{T\in\mathcal{T}_h}Y(T)\le N$, with $Y(T)=\#\{T'\in\mathcal{T}_h\mid T'\subset\omega_T\}$ and $\omega_T=\{K\in\mathcal{T}_h|K\cap T\neq\emptyset\}$ is the patch of elements surrounding T (consequence of the mesh regularity). Then there exists $\kappa_2>0$ only depending on the mesh regularity such that*

$$|||Res_1|||_{-1,h}^2\le 2\,N\,\kappa_2^2\sum_{T\in\mathcal{T}_h}(t^2+h_T^2)\|\gamma_h-y^*\|_T^2+osc^2(g), \tag{18}$$

where $osc(g)$ corresponds to an oscillating term.

Proof: For any $v\in H_0^1(\Omega)$, let us consider $v_h=Jv$ where $J:H_0^1(\Omega)\to W_h$ is defined such that (see, for example [14], known as the Clément operator)

$$\exists\,\kappa_1>0\,;\,\forall\,T\in\mathcal{T}_h,\|\nabla v_h\|_T\le\kappa_1\|\nabla v\|_{\omega_T}. \tag{19}$$

Moreover, it can be shown [11] that there exists $\kappa_2 > 0$ and $\kappa_3 > 0$ such that for all $T \in \mathcal{T}_h$ and for any $\psi \in H_0^1(\Omega)^2$,

$$\|\nabla(v - v_h)\|_T \leq \kappa_2 \left(\|\nabla v - \psi\|_{\omega_T} + h_T \|\nabla\psi\|_{\omega_T}\right),$$

$$h_T^{-1}\|v - v_h\|_T \leq \kappa_3 \left(\|\nabla v - \psi\|_{\omega_T} + h_T \|\nabla\psi\|_{\omega_T}\right).$$

Then for all $v \in H_0^1(\Omega)$, we get

$$\begin{aligned} Res_1(v) &= Res_1(v - v_h) \\ &= (g, v - v_h) - (\gamma_h, \nabla(v - v_h)) \\ &= (g + divy^*, v - v_h) - (\gamma_h - y^*, \nabla(v - v_h)) \\ &= \sum_{T\in\mathcal{T}_h} \left((g + divy^*, v - v_h)_T - (\gamma_h - y^*, \nabla(v - v_h))_T\right) \\ &\leq \sum_{T\in\mathcal{T}_h} h_T\sqrt{t^2 + h_T^2}\|g + divy^*\|_T \times \frac{h_T^{-1}}{\sqrt{t^2 + h_T^2}}\|v - v_h\|_T \\ &\quad + \sum_{T\in\mathcal{T}_h} \sqrt{t^2 + h_T^2}\|\gamma_h - y^*\|_T \times \frac{1}{\sqrt{t^2 + h_T^2}}\|\nabla(v - v_h)\|_T. \end{aligned}$$

So, we can write

$$\begin{aligned} Res_1(v) &\leq \sum_{T\in\mathcal{T}_h} h_T\sqrt{t^2 + h_T^2}\|g + divy^*\|_T \\ &\qquad \times \frac{\kappa_3}{\sqrt{t^2 + h_T^2}}\left(\|\nabla v - \psi\|_{\omega_T} + h_T\|\nabla\psi\|_{\omega_T}\right) \\ &\quad + \sum_{T\in\mathcal{T}_h} \sqrt{t^2 + h_T^2}\|\gamma_h - y^*\|_T \\ &\qquad \times \frac{\kappa_2}{\sqrt{t^2 + h_T^2}}\left(\|\nabla v - \psi\|_{\omega_T} + h_T\|\nabla\psi\|_{\omega_T}\right) \\ &\leq \left(\sum_{T\in\mathcal{T}_h} \kappa_3^2 h_T^2(t^2 + h_T^2)\|g + divy^*\|_T^2 \right. \\ &\qquad \left. + \sum_{T\in\mathcal{T}_h} \kappa_2^2(t^2 + h_T^2)\|\gamma_h - y^*\|_T^2\right)^{1/2} \\ &\quad \left(\underbrace{2\sum_{T\in\mathcal{T}_h}\left(\frac{1}{t^2 + h_T^2}\|\nabla v - \psi\|_{\omega_T}^2 + \frac{h_T^2}{t^2 + h_T^2}\|\nabla\psi\|_{\omega_T}^2\right)}_{=S}\right)^{1/2}. \end{aligned}$$

Now recalling that $\max_{T\in\mathcal{T}_h} Y(T) \leq N$ we have

$$\begin{aligned}
S &\leq N \sum_{T\in\mathcal{T}_h} \left(\frac{1}{t^2+h_T^2}\|\nabla v - \psi\|_T^2 + \underbrace{\frac{h_T^2}{t^2+h_T^2}}_{\leq 1} \|\nabla\psi\|_T^2 \right) \\
&\leq N \sum_{T\in\mathcal{T}_h} \left(\frac{1}{t^2+h_T^2}\|\nabla v - \psi\|_T^2 + \|\nabla\psi\|_T^2 \right) \\
&\leq N \left(\|\nabla\psi\|_\Omega^2 + \sum_{T\in\mathcal{T}_h} \frac{1}{t^2+h_T^2}\|\nabla v - \psi\|_T^2 \right) \\
S &\leq N |||(\psi, v)|||_{1,h}^2.
\end{aligned}$$

So we get

$$\begin{aligned}
Res_1(v) \;\; &\leq \left(\kappa_3^2 \sum_{T\in\mathcal{T}_h} h_T^2(t^2+h_T^2)\|g + divy^*\|_T^2 \right. \\
&\qquad \left. + \kappa_2^2 \sum_{T\in\mathcal{T}_h} (t^2+h_T^2)\|\gamma_h - y^*\|_T^2 \right)^{1/2} \times \sqrt{2N}|||(\psi, v)|||_{1,h}.
\end{aligned}$$

Consequently

$$\begin{aligned}
|||Res_1|||_{-1,h}^2 \;\; &\leq 2\,N \left(\kappa_3^2 \sum_{T\in\mathcal{T}_h} h_T^2(t^2+h_T^2)\|g + divy^*\|_T^2 \right. \\
&\qquad \left. + \kappa_2^2 \sum_{T\in\mathcal{T}_h} (t^2+h_T^2)\|\gamma_h - y^*\|_T^2 \right).
\end{aligned}$$

Since $div\, y^* = -\Pi_h\, g$, we get $\|g + divy^*\|_T^2 \leq C\, h_T^2 \|g\|_{\omega_T}^2$ and (18) holds. ■

Lemma 4.4 *For $\Psi \in H_0^1(\Omega)^2$, we have*

$$Res_2(\psi) \;\; \leq \;\; \|\mathcal{C}^{-1/2}(x^* - \mathcal{C}\varepsilon(\phi_h))\| \; \|\psi\|_{\mathcal{C}}. \tag{20}$$

Proof: Using standard Green formula, we easily obtain

$$Res_2(\psi) = \int_\Omega (x^* - \mathcal{C}\varepsilon(\phi_h)) : \varepsilon(\psi)\, dx + \int_\Omega (\gamma_h + div\, x^*)\, \psi\, dx,$$

Since $\mathcal{C}$ is a symmetric positive definite operator, we can define $\mathcal{C}^{1/2}$ and $\mathcal{C}^{-1/2}$ such that $\mathcal{C}^{1/2} \circ \mathcal{C}^{1/2} = \mathcal{C}$ and $\mathcal{C}^{1/2} \circ \mathcal{C}^{-1/2} = \mathcal{I}$. Then the definition of x^* directly yields

$$Res_2(\psi) = \int_\Omega \mathcal{C}^{-1/2}(x^* - \mathcal{C}\varepsilon(\phi_h)) \, : \, \mathcal{C}^{1/2}\varepsilon(\psi)\, dx,$$

and the Cauchy-Schwarz inequality finally leads to (20). ■

Theorem 4.5 (Reliability of the estimator) *Under the assumption of corollary 4.2, we have*

$$(e_h^{rot})^2 \le 4\,\zeta\,N\,\kappa_2^2 \left(\frac{3}{\mu} + \frac{c_F^2}{\mu}(3+2\sqrt{3}) + 4(\mu+\tilde{\lambda})\right) \sum_{T\in\mathcal{T}_h} (t^2 + h_T^2)\|\gamma_h - y^*\|_T^2$$
$$+4\left(\zeta\left(\frac{3}{\mu} + \frac{c_F^2}{\mu}(3+2\sqrt{3}) + 4(\mu+\tilde{\lambda})\right) + 1\right)(\mu+\tilde{\lambda})\|\mathcal{C}^{-1/2}(x^* - \mathcal{C}\varepsilon(\phi_h))\|^2$$
$$+\max\left((7+4\sqrt{3}); 2 + 2\left(\frac{3}{\mu} + \frac{c_F^2}{\mu}(3+2\sqrt{3}) + 4(\mu+\tilde{\lambda})\right)(\mu+\tilde{\lambda})c_R^2\right)\|\phi_h - \boldsymbol{R}_h\phi_h\|_{H(rot,\Omega)}^2$$
$$+osc^2(g).$$

Proof: The theorem is a direct consequence of corollary 4.2, lemma 4.3 and lemma 4.4. ■

Remark 4.6 In theorem 4.5, all constants are explicitly given. Indeed, even if c_F and c_R depend on the domain Ω whereas κ_2 and N depend on the used mesh, they can be evaluated or at least bounded, see [10] and section 6 below devoted to the numerical validations.

Remark 4.7 The assumption $t \le \sqrt{3\lambda c_F^2/\mu}$ needed in corollary 4.2 is not restrictive since, in the Reissner-Mindlin model, t is expected to be a very small parameter, so that this property is naturally obtained.

5 Efficiency of the estimator

In order to prove the efficiency of the estimator, each part of it has now to be bounded by the error e_h^{rot} up to a multiplicative constant. In the following, the notation $a \lesssim b$ and $a \sim b$ means the existence of positive constants c_1 and c_2, which are independent of the mesh size, of the plate thickness parameter t, of the quantities a and b under consideration and of the coefficients of the operators such that $a \lesssim c_2\,b$ and $c_1\,b \lesssim a \lesssim c_2\,b$, respectively. The constants may in particular depend on the aspect ratio σ of the mesh.

Lemma 5.1

$$\|(\boldsymbol{R}_h - I)\phi_h\|_{H(rot,\Omega)}^2 \lesssim \lambda^{-2}t^4\|\gamma-\gamma_h\|_\Omega^2 + |\omega-\omega_h|_1^2 + |\phi-\phi_h|_1^2 + \lambda^{-2}t^4\|rot(\gamma-\gamma_h)\|^2.$$

Proof: Since

$$(\mathbf{R}_h - I)\phi_h = \lambda^{-1}t^2(\gamma-\gamma_h) - \nabla(\omega-\omega_h) + (\phi-\phi_h),$$

we have

$$\|\mathbf{R}_h - I)\phi_h\| \le \lambda^{-1}t^2\|\gamma-\gamma_h\| + |\omega-\omega_h|_1 + \|\phi-\phi_h\|,$$

and with the Poincaré-Friedrichs inequality, we get

$$\|(\mathbf{R}_h - I)\phi_h\|^2 \lesssim \lambda^{-2}t^4\|\gamma-\gamma_h\|^2 + |\omega-\omega_h|_1^2 + |\phi-\phi_h|_1^2.$$

Moreover, we have

$$\|rot(\phi_h - \mathbf{R}_h\phi_h)\|^2 \lesssim \lambda^{-2}t^4\|rot(\gamma-\gamma_h)\|^2 + |\phi-\phi_h|_1^2,$$

so that lemma 5.1 holds. ■

Lemma 5.2 *There exists a relevant choice of x^* such that*

$$\|\mathcal{C}^{-1/2}(x^* - \mathcal{C}\varepsilon(\phi_h))\|^2 \lesssim \|\gamma_h-\gamma\|_{-1}^2 + |\phi_h-\phi|_1^2. \tag{21}$$

Proof: First, there exists only one pair $(\phi_h^*, \phi_h^{**}) \in H_0^1(\Omega)^2 \times \Theta_h$ solution of

$$\begin{cases} a(\phi_h^*, \psi) &= -(\gamma_h, \psi) \quad \forall\, \psi \in H_0^1(\Omega)^2, \\ a(\phi_h^{**}, \psi_h) &= -(\gamma_h, \psi_h) \quad \forall\, \psi_h \in \Theta_h. \end{cases}$$

Then, by Theorem 3.9 of [30] and a relevant construction of x^*, for all T in $\mathcal{T}_h$ we have

$$\|\mathcal{C}^{-1/2}(x^* - \mathcal{C}\varepsilon(\phi_h^{**}))\|_T \lesssim \|\phi_h^* - \phi_h^{**}\|_{\mathcal{C},\omega_T}.$$

Because of the mesh regularity, we also get the global estimate

$$\|\mathcal{C}^{-1/2}(x^* - \mathcal{C}\varepsilon(\phi_h^{**}))\| \lesssim \|\phi_h^* - \phi_h^{**}\|_{\mathcal{C}}. \tag{22}$$

Clearly

$$\mathcal{C}^{-1/2}(x^* - \mathcal{C}\varepsilon(\phi_h)) = \mathcal{C}^{-1/2}(x^* - \mathcal{C}\varepsilon(\phi_h^{**})) + \mathcal{C}^{1/2}\varepsilon(\phi_h^{**}-\phi_h).$$

By (22) and the triangular inequality, we arrive at

$$\begin{aligned} \|\mathcal{C}^{-1/2}(x^* - \mathcal{C}\varepsilon(\phi_h))\| &\lesssim \|\mathcal{C}^{-1/2}(x^* - \mathcal{C}\varepsilon(\phi_h^{**}))\| + \|\phi_h^{**}-\phi_h\|_{\mathcal{C}} \\ &\lesssim \|\phi_h^* - \phi_h^{**}\|_{\mathcal{C}} + \|\phi_h^{**}-\phi_h\|_{\mathcal{C}}. \end{aligned} \tag{23}$$

Now, it remains to bound each of the two terms of the right-hand side of (23). To begin with, let us consider $\psi_h \in \Theta_h$. Thanks to the definition of ϕ_h^{**}, we get

$$\begin{aligned} a(\phi_h - \phi_h^{**}, \psi_h) &= (\gamma_h, \psi_h - \mathbf{R}_h\psi_h) \\ &= (\gamma_h, \psi_h) - (\gamma_h, \mathbf{R}_h\psi_h) \\ &= (\gamma_h - \gamma, \psi_h) + a(\phi_h - \phi, \psi_h) \\ &\lesssim (\|\gamma_h-\gamma\|_{-1} + |\phi_h-\phi|_1)|\psi_h|_1. \end{aligned}$$

By taking $\psi_h = \phi_h - \phi_h^{**}$, we obtain

$$\|\phi_h^{**} - \phi_h\|_{\mathcal{C}} \lesssim \|\gamma_h-\gamma\|_{-1} + |\phi_h-\phi|_1. \tag{24}$$

Then, by the triangular inequality, we get

$$\|\phi_h^* - \phi_h\|_{\mathcal{C}} \leq \|\phi_h^* - \phi\|_{\mathcal{C}} + \|\phi - \phi_h\|_{\mathcal{C}},$$

and by the definition of ϕ_h^*, we have for all $\psi \in H_0^1(\Omega)^2$

$$a(\phi_h^* - \phi, \psi) = (\gamma - \gamma_h, \psi),$$

so that

$$\|\phi_h^* - \phi\|_{\mathcal{C}} \lesssim \|\gamma - \gamma_h\|_{-1}.$$

We then obtain

$$\|\phi_h^* - \phi_h\|_{\mathcal{C}} \leq \|\gamma - \gamma_h\|_{-1} + \|\phi - \phi_h\|_{\mathcal{C}} \lesssim \|\gamma - \gamma_h\|_{-1} + |\phi - \phi_h|_1. \tag{25}$$

Using (24) and (25) in (23), we get (21). ■

Lemma 5.3 *There exists a relevant choice of y^* such that*

$$\sum_{T \in \mathcal{T}_h} (t^2 + h_T^2)\|\gamma_h - y^*\|_T^2 \lesssim t^2\|\gamma - \gamma_h\|^2 + \|\gamma - \gamma_h\|_{-1}^2 + osc^2(g), \tag{26}$$

where $osc^2(g)$ is an oscillating term.

Proof: Because of lemma 3.1 of [15], we have for any $T \in \mathcal{T}_h$ the equivalence

$$\|\gamma_h - y^*\|_T \sim h_T^{1/2} \sum_{E \subset \partial T} \|(\gamma_h - y^*) \cdot \nu_T\|_E,$$

where ν_T is the outward unit normal vector to T. Now we define y^* as in [15], by noticing that (6) implies that

$$(\gamma_h, \nabla v_h) = (g, v_h) \quad \forall v_h \in W_h,$$

hence there exist fluxes $g_E \in \mathcal{P}_1(E)$, for all edges E such that

$$\int_T \gamma_h \cdot \nabla v_h = \int_T g v_h + \int_{\partial T} g_T v_h \quad \forall v_h \in \mathcal{P}_1(T),$$

where $g_T = g_E \nu_E \nu_T$, ν_E being a fixed normal vector to E. According to the definition of the BDM_1 elements there then exists a unique $y_T^* \in \mathcal{P}_1(T)^2$ such that

$$y_T^* \cdot \nu_E = g_E \quad \forall E \subset T.$$

Hence we define y^* such that its restriction to each triangle T is equal to y_T^*. According to its definition y^* belongs to $H_{div}(\Omega)$ and moreover according to [15], we have

$$\operatorname{div} y^* = -\Pi_h g.$$

Then by the use of theorem 6.2 from [1] and the mesh regularity we get

$$\|\gamma_h - y^*\|_T \lesssim h_T^{1/2} \sum_{E\subset\partial T\setminus\partial\Omega} \|[\gamma_h \cdot \nu_E]_E\|_E + \sum_{T'\subset\omega_T} h_{T'}\|div\,\gamma_h + g\|_{T'},$$

where $[v]_E$ denotes the jump of the quantity v through the edge E. Consequently

$$\begin{aligned}
\sum_{T\in\mathcal{T}_h}(t^2+h_T^2)\|\gamma_h - y^*\|_T^2 &\lesssim \sum_{T\in\mathcal{T}_h} h_T(t^2+h_T^2)\sum_{E\subset\partial T\setminus\partial\Omega}\|[\gamma_h\cdot\nu_E]_E\|_E^2 \\
&+ \sum_{T\in\mathcal{T}_h}\sum_{T'\subset\omega_T} h_{T'}^2(t^2+h_T^2)\|div\gamma_h + g\|_{T'}^2 \\
&\lesssim \sum_{E\subset\partial T\setminus\partial\Omega} h_E(t^2+h_E^2)\|[\gamma_h\cdot\nu_E]_E\|_E^2 \\
&+ \sum_{T\in\mathcal{T}_h} h_T^2(t^2+h_T^2)\|div\gamma_h+g\|_T^2.
\end{aligned} \tag{27}$$

Using the classical edge bubble functions as well as elementwise inverse estimates, it is proved in [11], section 4.3 that :

$$\begin{aligned}
\sum_{E\in\mathcal{E}(\Omega)\setminus\partial\Omega} h_E(t^2+h_E^2)\|[\gamma_h\cdot\nu_E]_E\|_E^2 \lesssim \sum_{T\in\mathcal{T}_h} h_T^2(t^2+h_T^2)\|g-\Pi_h g\|_T^2 \\
+\|\gamma-\gamma_h\|_{-1}^2 + t^2\|\gamma-\gamma_h\|^2.
\end{aligned} \tag{28}$$

Moreover, with the use of classical element bubble functions as well as elementwise inverse estimates, it is also proved in [11], section 4.1 that :

$$\begin{aligned}
\sum_{T\in\mathcal{T}_h} h_T^2(t^2+h_T^2)\|div\gamma_h+\Pi_h g\|_T^2 &\lesssim t^2\|\gamma-\gamma_h\|^2 + \|\gamma-\gamma_h\|_{-1}^2 \\
&+ \sum_{T\in\mathcal{T}_h} h_T^2(h_T^2+t^2)\|g-\Pi_h g\|_T^2.
\end{aligned} \tag{29}$$

Now, from (27) associated to the standard triangular inequality :

$$\|div\gamma_h + g\|_T \leq \|div\gamma_h + \Pi_h g\|_T + \|g-\Pi_h g\|_T,$$

the use of (28) and (29) leads to (26).

■

Theorem 5.4 (Efficiency of the estimator) *There exists a relevant choice of* x^* *and of* y^* *such that*

$$\begin{aligned}
&4\,\zeta\,N\,\kappa_2^2\left(\frac{3}{\mu}+\frac{c_F^2}{\mu}(3+2\sqrt{3})+4(\mu+\tilde{\lambda})\right)\sum_{T\in\mathcal{T}_h}(t^2+h_T^2)\|\gamma_h-y^*\|_T^2 \\
&\quad+\left(2\,\zeta\,(\frac{3}{\mu}+\frac{c_F^2}{\mu}(3+2\sqrt{3})+4(\mu+\tilde{\lambda}))+2\right)2(\mu+\tilde{\lambda})\|\mathcal{C}^{-1/2}(x^*-\mathcal{C}\varepsilon(\phi_h))\|^2 \\
&\quad+\max\left((7+4\sqrt{3});2+(\frac{3}{\mu}+\frac{c_F^2}{\mu}(3+2\sqrt{3})+4(\mu+\tilde{\lambda}))2(\mu+\tilde{\lambda})c_R^2\right)\|\phi_h-\boldsymbol{R}_h\phi_h\|_{H(rot,\Omega)}^2 \\
&\lesssim (e_h^{rot})^2 + osc^2(g).
\end{aligned}$$

Proof: The proof is a direct consequence of lemma 5.1, 5.2 and 5.3. ■

6 Numerical validation

Here we illustrate and validate our theoretical results by a simple computational example. Let Ω be the unit square $]0,1[^2$. We consider the exact solution (ω,ϕ) in Ω of the Reissner-Mindlin problem (1)-(2) given by

$$\phi = \begin{pmatrix} \dfrac{1-2x}{x^2(1-x)^2} \\ \dfrac{1-2y}{y^2(1-y)^2} \end{pmatrix} exp\left(-\frac{1}{x(1-x)} - \frac{1}{y(1-y)}\right),$$

and

$$\omega = \left(1 - (2\mu+\tilde{\lambda})\lambda^{-1}t^2\,(a(x)+a(y))\right)\, exp\left(-\frac{1}{x(1-x)} - \frac{1}{y(1-y)}\right),$$

with

$$a(z) = \frac{6z^4 - 12z^3 + 12z^2 - 6z + 1}{z^4(1-z)^4}.$$

The corresponding scaled transverse loading function g is given by

$$g = (2\mu+\tilde{\lambda})\;(c(x)+c(y)+2\,a(x)\,a(y))\; exp\left(-\frac{1}{x(1-x)} - \frac{1}{y(1-y)}\right)$$

with

$$\begin{aligned} c(z) &= \frac{120z^{10} - 600z^9 + 1620z^8 - 2880z^7 + 3504z^6 - 2952z^5}{z^8(1-z)^8} \\ &+ \frac{1708z^4 - 656z^3 + 156z^2 - 20z + 1}{z^8(1-z)^8}. \end{aligned}$$

This analytical solution is extended by 0 on $\partial\Omega$ to obtain $(\omega,\phi) \in H_0^1(\Omega) \times H_0^1(\Omega)^2$. Here we take $t = 1/1024$, $\lambda = 1$, $\mu = 1$ and $\tilde{\lambda} = 1$. The meshes we use are uniform ones composed of n^2 squares, each of them being cut into 8 triangles as displayed on Figure 1 for $n = 4$. The refinement strategy is an uniform one so that the value of the mesh size h between two consecutive meshes is twice smaller. In order to validate the reliability of the estimator, we consider the "discrete error" given by

$$e_{h,dis}^{rot} = \sqrt{|\omega-\omega_h|_1^2 + |\phi-\phi_h|_1^2 + \lambda^{-1}t^2\|\gamma-\gamma_h\|^2 + \lambda^{-2}t^4\|rot(\gamma-\gamma_h)\|^2 + ||P_h\gamma-\gamma_h||_{-1,h}^2},$$

where $P_h\gamma$ stands for the piecewise $\mathbb{P}_1$-discontinuous interpolation of γ on the mesh $\mathcal{T}_h$. This discrete error is defined by approximating the $H^{-1}(\Omega)$ norm of $\gamma-\gamma_h$ arising in e_h^{rot} (see (9)) by its discrete locally computable version defined by

$$||P_h\gamma-\gamma_h||_{-1,h}^2 = \sup_{v_h\in W_h} \frac{|(P_h\gamma-\gamma_h, v_h)|^2}{|v_h|_1^2}. \tag{30}$$

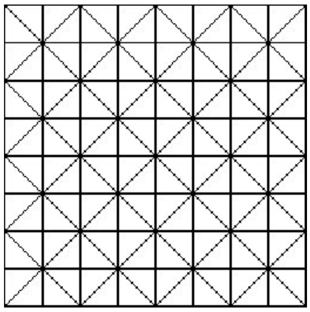

Figure 1: Mesh level corresponding to $n = 4$ and $h = \sqrt{2}/8$.

The computation of $||P_h\gamma - \gamma_h||^2_{-1,h}$ is now an easy task and simply corresponds to the determination of the largest eigenvalue of a classical generalized finite dimensional eigenvalue problem. In order to validate the reliability of the estimator according to theorem 4.5, the error estimator is defined by

$$(\eta_h)^2 = 4\,\zeta\,N\,\kappa_2^2\left(\frac{3}{\mu} + \frac{c_F^2}{\mu}(3+2\sqrt{3}) + 4(\mu+\tilde{\lambda})\right)\sum_{T\in\mathcal{T}_h}(t^2+h_T^2)\|\gamma_h - y^*\|_T^2$$
$$+4\left(\zeta\left(\frac{3}{\mu} + \frac{c_F^2}{\mu}(3+2\sqrt{3}) + 4(\mu+\tilde{\lambda})\right)+1\right)(\mu+\tilde{\lambda})\|\mathcal{C}^{-1/2}(x^* - \mathcal{C}\varepsilon(\phi_h))\|^2$$
$$+\max\left((7+4\sqrt{3}); 2+2\left(\frac{3}{\mu} + \frac{c_F^2}{\mu}(3+2\sqrt{3}) + 4(\mu+\tilde{\lambda})\right)(\mu+\tilde{\lambda})c_R^2\right)\|\phi_h - \mathbf{R}_h\phi_h\|^2_{H(rot,\Omega)},$$

and we plot on Figure 2 the evolution of the computed effectivity index $\eta_h/e^{rot}_{h,dis}$ versus h. Here, the values of $x^\star$ as well as $y^\star$ are respectively computed in the same manner as in [15] and [30], in order to obtain relevant choices as required by theorem 5.4 to ensure the efficiency of the estimator. Practically, some fluxes g_E through the edges E of each triangle of the mesh are needed, and have to be computed by solving local linear problems. In fact, in our tests, these values are explicitely defined. For y^*, we use $g_E = \{\{\gamma_h \cdot \nu_E\}\}$, where $\{\{\gamma_h \cdot \nu_E\}\}$ denotes the averaged value on the triangles on each side of E of $\gamma_h \cdot \nu_E$ evaluated at the middle of E. For x^*, we use $g_E = \sum_{x\in\mathcal{N}(T)}\{\{\mathcal{C}\varepsilon(\phi_h)\}\}(x)\nu_E\lambda_x$. Here, $\{\{\mathcal{C}\varepsilon(\phi_h)\}\}(x)$ is the averaged value over the triangles surrouding the node x of the piecewise constant function on each triangle $\mathcal{C}\varepsilon(\phi_h)$, and λ_x stands for the classical $\mathbb{P}_1$-Lagrange basis function associated with the node x. Moreoever, for the construction of x^*, the Argyris basis functions have to be used (see section 4 of [30] as well as [17] for the practical implementation).

From (16) we have $\zeta = 1$. The Poincaré-Friedrichs constant c_F is here equal to $1/(\sqrt{2}\pi)$ since Ω is the unit square. Because of the kind of meshes used (see Figure 1), we have $N = 8$ and $\kappa_2 = 1 + \dfrac{12}{\sqrt{2}\pi}$ (see annex 7.2). Finally, it can be proved [23] that on the unit square, $c_R \leq 2\sqrt{\frac{1}{2-\sqrt{2}}}$, hence below we take this upper bound for c_R (while it is conjectured that $c_R = \sqrt{\frac{2\pi}{\pi - 2}}$, see [16]). As expected by the theory, it can be observed that the

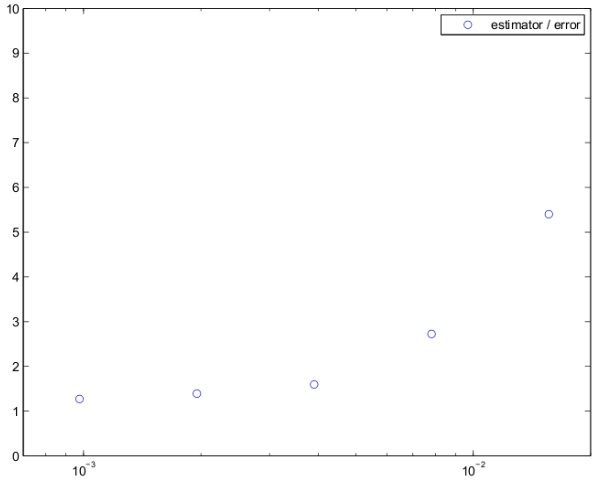

Figure 2: $\eta_h / e^{rot}_{h,dis}$ versus h.

computed effectivity index is larger than one. Moreover, it converges towards a constant close to one when h goes towards zero, so that the proposed estimator is asymptotically exact.

7 Annexes

7.1 Proof of (3)

Let us consider $v \in C_c^\infty(\Omega)^2$. Two integrations by parts yield :

$$\begin{aligned} 2\int_\Omega |\varepsilon(v)|^2 dx &= \int_\Omega |\nabla v|^2 dx + \int_\Omega \nabla v (\nabla v)^T dx \\ &= \int_\Omega |\nabla v|^2 dx + \int_\Omega |div v|^2 dx \\ &\geq \int_\Omega |\nabla v|^2 dx. \end{aligned}$$

Hence by a density argument we obtain

$$\|\nabla v\| \leq \sqrt{2}\, \|\varepsilon(v)\| \quad \forall\, v \in H_0^1(\Omega)^2.$$

Then, we recall

$$\mathcal{C}\varepsilon(\phi) = 2\mu\varepsilon(\phi) + \tilde{\lambda} Tr(\varepsilon(\phi))\mathcal{I},$$

so that

$$\begin{aligned} \|\phi\|_{\mathcal{C}}^2 &= \int_\Omega \mathcal{C}\varepsilon(\phi)\varepsilon(\phi) dx \\ &= 2\mu \int_\Omega \varepsilon(\phi)\varepsilon(\phi) dx + \tilde{\lambda} \int_\Omega Tr(\varepsilon(\phi))\mathcal{I}\varepsilon(\phi) dx \\ &= 2\mu \int_\Omega |\varepsilon(\phi)|^2 dx + \tilde{\lambda} \int_\Omega (Tr\varepsilon(\phi))^2 dx \\ &\geq \mu \|\nabla\phi\|^2. \end{aligned}$$

This proves (3).

7.2 Evaluation of κ_2 for the triangulation of section 6

With the definitions given above, let us consider z an affine function on ω_T, so that $Jz = z$ on T. With v and v_h defined in the proof of lemma 4.3 and the triangular inequality, we get

$$\|\nabla(v - v_h)\|_T \quad \leq \quad \|\nabla(v - z)\|_T + \|\nabla J(v - z)\|_T$$

From (19), we get

$$\|\nabla(v - v_h)\|_T \quad \leq \quad (1 + \kappa_1)\|\nabla(v - z)\|_{\omega_T}.$$

Defining $A = \nabla z$ and considering $\psi \in H_0^1(\Omega)^2$, we have

$$\|\nabla(v - v_h)\|_T \leq (1 + \kappa_1)\|\nabla v - A\|_{\omega_T} \leq (1 + \kappa_1)\left(\|\nabla v - \psi\|_{\omega_T} + \|\psi - A\|_{\omega_T}\right).$$

Now, z is chosen such that

$$A = \frac{1}{|\omega_T|} \int_{\omega_T} \psi dx.$$

By Poincaré inequality, there exists $C_{\omega_T} > 0$, depending on the patch ω_T, such that

$$\|\psi - A\|_{\omega_T} \leq C_{\omega_T}\, h_T\, \|\nabla\psi\|_{\omega_T}\ \forall\ \psi \in H_0^1(\Omega)^2.$$

So,

$$\begin{aligned}
\|\nabla(v - v_h)\|_T &\leq (1+\kappa_1)\|\nabla v - \psi\|_{\omega_T} + (1+\kappa_1)\, C_{\omega_T}\, h_T \|\nabla\psi\|_{\omega_T} \\
&\leq \underbrace{(1+\kappa_1) max\{1; C_{\omega_T}\}}_{=\,\kappa_2} \left(\|\nabla v - \psi\|_{\omega_T} + h_T\|\nabla\psi\|_{\omega_T}\right). \qquad (31)
\end{aligned}$$

Now, it remains to evaluate κ_1 as well as C_{ω_T}. Let η_z be the nodal basis associated to W_h. We have

$$J\, v\ =\ \sum_{z \in \mathcal{N}} v_z\, \eta_z\,,\ \forall\, v\ \in\ H_0^1(\Omega),$$

from what we deduce

$$\nabla\ J\, v\ =\ \sum_{z \in \mathcal{N}} (v_z - v)\, \nabla\, \eta_z\,,\ \forall\, v\ \in\ H_0^1(\Omega).$$

Let us define $\mathcal{N}_T = \mathcal{N} \cap T$. We have

$$\begin{aligned}
\|\nabla\, J\, v\|_T &= \|\sum_{z \in \mathcal{N}_T} (v_z - v)\, \nabla\, \eta_z\|_T \\
&\leq \sum_{z \in \mathcal{N}_T} \|v_z - v\|_T\, \|\nabla\, \eta_z\|_T \\
&\leq \sum_{z \in \mathcal{N}_T} \|v_z - v\|_{\omega_z}\, \|\nabla\, \eta_z\|_T
\end{aligned}$$

But

$$\|\nabla\, \eta_z\|_T \leq \rho_T^{-1},$$

and from [10, (5.12)], we get

$$\|v_z - v\|_{\omega_z} \leq c(\omega_z, 2) \|\nabla\, v\|_{\omega_z}.$$

With the triangulation involved, we have

$$c(\omega_z, 2) \leq \frac{\sqrt{2}\, h_T}{\pi},$$

and

$$\|\nabla\, J\, v\|_T\ \leq\ \frac{3\sqrt{2}}{\pi}\, \frac{h_T}{\rho_T}\, \|\nabla\, v\|_{\omega_z},$$

so that

$$\kappa_1 \leq \frac{3\sqrt{2}}{\pi} \frac{h_T}{\rho_T}.$$

For the involved triangulation $h_T/\rho_T = 2$ and hence

$$\kappa_1 \leq \frac{12}{\sqrt{2}\,\pi}. \tag{32}$$

Since from [10], we have $C_{\omega_T} = \dfrac{3}{\pi}$, (31) and (32) lead to

$$\kappa_2 \leq 1 + \frac{12}{\sqrt{2}\,\pi}.$$

References

[1] M. Ainsworth and J. Oden. *A posteriori error estimation in finite element analysis.* John Wiley and Sons, 2000.

[2] I. Babuška and T. Strouboulis. *The finite element methods and its reliability.* Clarendon Press, Oxford.

[3] K.J. Bathe, F. Brezzi and M. Fortin. *Mixed-interpolated elements for Reissner-Mindlin plates.* Int. J. Num. Meths. Engrg, Volume 28, pp 1787-1801, 1989.

[4] K.J. Bathe and E. Dvorkin. *A four-node plate bending element based on Mindlin-Reissner plate theory and a mixed interpolation.* Int. J. Num. Meths. Engrg., 21, pp 367-383, 1985.

[5] L. Beirão da Veiga, C. Chinosi, C. Lovadina, and R. Stenberg. *A-priori and a-posteriori error analysis for a family of Reissner-Mindlin plate elements.* BIT, 48(2):189–213, 2008.

[6] D. Braess and J. Schöberl. *Equilibrated residual error estimator for edge elements.* Math. Comp., 77(262):651–672, 2008.

[7] S. C. Brenner and L. R. Scott. *The mathematical theory of finite element methods.* Springer, New York, 1994.

[8] F. Brezzi and M. Fortin. *Mixed and hybrid finite element methods.* Springer-Verlag, New-York, 1991.

[9] C. Carstensen. *Residual-based a posteriori error estimate for a nonconforming Reissner-Mindlin plate finite element.* SIAM J. Numer. Anal., 39(6):2034–2044 (electronic), 2002.

[10] C. Cartensen and S.A Funken. *Constants in Clément-interpolation error and residual based a posteriori error estimates in Finite Element Methods*. East-West J. Numer. Math., Vol 8, No 3. pp 153-175, 2000.

[11] C. Cartensen and Jun Hu. *A posteriori error analysis for conforming MITC elements for Reissner-Mindlin plates*. Mathematics of computation, 77, 262, pp 611-632, 2008.

[12] C. Carstensen and J. Schöberl. *Residual-based a posteriori error estimate for a mixed Reißner-Mindlin plate finite element method*. Numer. Math., 103(2):225–250, 2006.

[13] P. G. Ciarlet. *The finite element method for elliptic problems*. North-Holland, Amsterdam, 1978.

[14] Ph. Clement. *Approximation by finite element functions using local regularization*. RAIRO R2, pp 77-84, 1975.

[15] S. Cochez-Dhondt and S. Nicaise. *A posteriori error estimators based on equilibrated fluxes*. Computational Methods in Applied Mathematics, Vol 10, pp 49-68, 2010.

[16] http://anum-maths.univ-rennes1.fr/JourneesEquipe/04JEAN/Transparents/Dauge.pdf.

[17] V. Domínguez and F.-J. Sayas. *A simple Matlab implementation of the Argyris element*. ACM Trans. Math. Software 35, no. 2, Art. 16, 11 pp, 2009.

[18] R. Duran, E. Hernandez, L. Hervella-Nieto, E. Liberman and R. Rodriguez. *Error estimates for lower-order isoparametric quadrilateral finite elements for plates*. SIAM J. Numer. Anal., 41, pp 1751-1772, 2003.

[19] Ricardo Durán and Elsa Liberman. *On mixed finite element methods for the Reissner-Mindlin plate model*. Mathematics of Computation, 58, 198, pp 561-573, 1992.

[20] A. Ern, S. Nicaise, and M. Vohralík. *An accurate* $\mathbf{H}(\mathrm{div})$ *flux reconstruction for discontinuous Galerkin approximations of elliptic problems*. C. R. Math. Acad. Sci. Paris, 345(12):709–712, 2007.

[21] M.E. Frolov, P. Neittaanmäki and S.I. Repin. *Guaranteed functional error estimates for the Reissner-Mindlin plate problem*. Journal of Mathematical Sciences, 132, 4, pp 553-561, 2006.

[22] V. Girault and P.-A. Raviart. *Finite elements methods for Navier-Stokes equations, Theory and Algorithms*. Springer Series in Computational Mathematics, Berlin, 1986.

[23] C. O. Horgan and L. E. Payne. *On inequalities of Korn, Friedrichs and Babuška-Aziz*. Arch. Rational Mech. Anal., 82(2):165–179, 1983.

[24] J. Hu and Y. Huang. *A posteriori error analysis of finite element methods for Reissner-Mindlin plates*. SIAM J. Numer. Anal., 47 (6):4446-4472, 2010.

[25] P. Ladevèze and D. Leguillon. *Error estimate procedure in the finite element method and applications.* SIAM J. Numer. Anal., 20:485–509, 1983.

[26] E. Liberman. *A posteriori error estimator for a mixed finite element method for Reissner-Mindlin plate.* Math. Comp., 70(236):1383–1396 (electronic), 2001.

[27] C. Lovadina and R. Stenberg. *A posteriori error analysis of the linked interpolation technique for plate bending problems.* SIAM J. Numer. Anal., 43(5):2227–2249 (electronic), 2005.

[28] R. Luce and B. Wohlmuth. *A local a posteriori error estimator based on equilibrated fluxes.* SIAM J. Numer. Anal., 42:1394–1414, 2004.

[29] P. Neittaanmaäki and S. Repin. *Reliable methods for computer simulation: error control and a posteriori error estimates.*, volume 33 of Studies in Mathematics and its applications. Elsevier, Amsterdam, 2004.

[30] S. Nicaise, K. Witowski, and B. I. Wohlmuth. *An a posteriori error estimator for the Lamé equation based on equilibrated fluxes.* IMA J. Numer. Anal., 28(2):331–353, 2008.

[31] R. Stenberg and M. Suri. *An hp error analysis of MITC plate elements.* SIAM J. Numer. Anal. 34, pp 544-568, 1997.

[32] R. Verfurth. *A review of a posteriori error estimation and adaptive mesh-refinement techniques.* Teubner Skripten zur Numerik, 1996.