View metadata, citation and similar papers at core.ac.uk

brought to you by CORE

provided by INRIA a CCSD electronic archive server

Exactly Solvable Stochastic Processes for Traffic Modelling

Maxim Samsonov, Cyril Furtlehner, Jean-Marc Lasgouttes

To cite this version:

Maxim Samsonov, Cyril Furtlehner, Jean-Marc Lasgouttes. Exactly Solvable Stochastic Processes for Traffic Modelling. 25th International Symposium on Computer and Information Sciences - ISCIS 2010, Erol Gelenbe, Ricardo Lent, Georgia Sakellari, Sep 2010, Londres, United Kingdom. pp.75-78. inria-00533154

HAL Id: inria-00533154

https://hal.inria.fr/inria-00533154

Submitted on 15 Nov 2010

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Exactly Solvable Stochastic Processes for Traffic Modelling

Maxim Samsonov * *Cyril Furtlehner* †
Jean-Marc Lasgouttes ‡

Abstract

We analyze different available methods in the study of the exactly solvable stochastic models and their application to construction and modeling the road traffic with acceleration/deceleration dynamics.

1 Introduction

In the study of models for traffic [1] the property of the fundamental diagram plays an important role. The fundamental diagram of traffic flow gives a relation between the traffic flux (cars per unit of time) and the traffic density (cars per unit length) and as well the dependence between the speed and the flux, and the speed and the density, all three graphs are related by the relation: flux = density × speed. There is actually controversy wether about the existance a genuine dynamical phase, called the synchronized phase[2] between the free flow and congested phase[3].

Considering the large amount of successful applications of exactly solvable models to problems of non-equilibrium statistical physics we focus on the following questions:

- is there a stochastic model (exclusion process, zero range process) able to account for some or all reasonable features like the asymmetry between breaking and accelerating[4], of the fundamental diagram observed experimentally, and suitable for exact computations.
- Having such a model, can we provide a method to compute the fundamental diagram and study emergence of non-trivial collective behaviors at macroscopic level, caused for example by some spontaneous symmetry breaking among identical vehicles as in the Japanese experiment of traffic jams on a ring [5].

* INRIA-Saclay, LRI Bât. 490, F-91405 Orsay(France)
† INRIA Saclay, Bat. 490, F-91405 Orsay (France)
‡ INRIA Paris-Rocquencourt, Domaine de Voluceau, BP. 105 – 78153 Le Chesnay (France)

2 A Multi-speed exclusion processes

The model we propose is defined by the following set of reactions, involving pairs of neighbour sites:

$$\begin{array}{llll} AO \xrightarrow{\lambda_a} OA; & BO \xrightarrow{\lambda_b} OB; & BO \xrightarrow{\gamma_{bo}} AO; & AO \xrightarrow{\delta_{ao}} BO \\ AB \xrightarrow{\lambda_{ab}} BA; & AA \xrightarrow{\delta_{aa}} BA; & AB \xrightarrow{\delta_{ab}} BB; & BA \xrightarrow{\delta_{ba}} BB \\ AB \xrightarrow{\gamma_{ab}} AA; & BB \xrightarrow{\gamma_{bb}^1} BA; & BA \xrightarrow{\gamma_{ba}} AA; & BB \xrightarrow{\gamma_{bb}^2} AB \end{array}$$

The λ's, γ's and δ's denote the transition rates, each transition corresponding to a Poisson event and we consider the ring geometry with periodic boudary conditions. The question that we mainly address in this paper is how to identify particular settings of the transition rates, with some of them possibly vanishing, such that the model becomes solvable in a broad sense, i.e. for which either the whole dynamics can be mathematically solved when the model is integrable, or at least that the stationary state can be described explicitly with the invariant measure.

The model itself can be seen as a two particle exclusion diffusion model with coagulation/decoagulation dynamics including overtaking. It generalizes several sub-models which are known to be integrable with particular rates. The diffusion part of the model is just the totally asymmetric exclusion process [6, 7] (TASEP) when $\lambda_a = \lambda_b$, which is known to be integrable, its generalization to include multiparticle dynamics with overtaking is the so-called Karimipour model [8, 9] when $\lambda_{ab} = \lambda_a - \lambda_b$, which turns out to be integrable as well. In some cases, the model can be exactly reformulated in terms of generalized queueing processes (or zero range process in the statistical physics parlance), where service rates of each queue follows as well a stochastic dynamics [10]. The mapping work mainly on the ring geometry, by identifying queues either with empty sites, clients are then the vehicles or with cars, clients are then the empty sites.

Based on numerical simulations on the ring geometry, we make some observations concerning the phenomenology of a simplified form of the model, depending on the parameters. This is illustrated on Figure 1. Non-zero rates are λ_a and λ_b, $\gamma_{bo} = \gamma$, $\delta_{aa} = \delta_{ab} = \delta$, while all others are set to zero, in particular overtaking is excluded ($\lambda_{ab} = 0$). The asymmetry between breaking and accelerating is crucial to observe condensation mechanism, which occurs if the apparition of slow vehicle is a sufficiently rare event, resulting e.g. from a cascade of braking events. Let us also remark that Figure 1.a is very reminiscent of coagulation-decoagulation process, as expected from the previous discussion.

3 Conditions for solvability

A important case of solvability is intergrability. This means that we can construct the spectrum and the eigenstates of the corresponding Markov operator

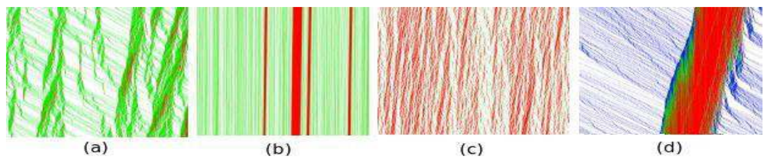

Fig. 1: Spatio-temporal plots for multi-speed exclusion process with 2 speed levels (a) (b) and (c) and with 3 speed levels (d). Time is going downward and particles to the right. red, green and blue represent different speeds in increasing order. The size of the system is 3000 except for (b) where it is 100000. Setting are $\lambda_a = 100$, $\lambda_b = 10$, $\gamma_a = 100$, $\delta_b = 2$ for (a) and (b) and $\delta_b = 10$ for (c), all with density $\rho = 0.2$. In (d) the rates are $\lambda_c = 10$, $\lambda_b = 100$ and $\lambda_a = 200$, $\delta_c = 3$, $\delta_b = 5$, $\gamma_b = 0.1$ and $\gamma_a = 1$ with $\rho = 0.3$.

governing the evolution of the probability distribution $P(\mathcal{C}, t)$ with time:

$$\frac{d}{dt}P(\mathcal{C}, t) = \sum_{\mathcal{C}'} P(\mathcal{C}', t)M(\mathcal{C}', \mathcal{C}) - \sum_{\mathcal{C}'} P(\mathcal{C}, t)M(\mathcal{C}, \mathcal{C}').$$

where the element $M(\mathcal{C}', \mathcal{C})$ of the Markov Matrix is the transitions rate between configuration $\mathcal{C}$ and $\mathcal{C}'$. To get the model which can be analyzed, and possibly not far from being solvable, we impose cancellations conditions for some of the nonlinear terms. Such restrictions between the rates improve solvability of the model, the model becomes close to integrable at specific choice of the rates. In particular it becomes very simple when:

$$\lambda_a = \gamma_{ab} + \gamma_{ba}; \quad \lambda_b = \delta_{ab} + \gamma_{ba} = \delta^2_{aa} + \gamma^2_{bb}; \quad \lambda_{ab} = \lambda_a - \delta^1_{aa} - \gamma^1_{bb}.$$

and the interaction term has the form: $-\lambda_a \sum_i n^a_i - \lambda_b \sum_i n^b_i = hN$. The operator h plays a role of an average hopping rate operator and $n^{a,b}_i$ are local operators counting particles of each sort. Let us consider first only the coagulation/decoagulation part of the model. This model can be dealt with help of the empty interval method [11]. Let us imagine a densely packed road with cars of two types with open boundary conditions. Then we can write a system of equations for the queues of cars

$$P^{a,b}_t(x, x+1, \cdots, x+m-1) = \mathbb{E}^t(n^{a,b}_x n^{a,b}_{x+1} \cdots n^{a,b}_{x+m-1}).$$

Taking it into account the Markov evolution equations of the corresponding process can be written as a system of linear equations when $\gamma_{ab} = \gamma_{ba} = \gamma^2_{bb} = 0$ and $\lambda_{ab} = \gamma^1_{bb}$:

$$\frac{d}{dt}P^a_t(x, y) = \delta_{ba}P^a_t(x-1, y) - (2\lambda_{ab} + (y-x)(\delta^1_{aa} + \delta^2_{aa}) + \delta_{ab} + \delta_{ba})P^a_t(x, y) +$$
$$+ (\lambda_{ab} - \delta^1_{aa} + \delta_{ab})P^a_t(x, y+1) + \lambda_{ab}P^a_t(x+1, y)$$

Such equations are solved in terms of Bessel functions. More general structure of the cluster functions is possible as well.

4 Product form of jams at steady-state

Taking advantage in some cases, of the mapping of the process on a generalized tandem queue process, we propose in this section, to study the conditions for which the stationary state has a product form. The queuing process which is obtained is then a particular case of general model consisting in a network $\mathcal{G} = (\mathcal{N}, \mathcal{L})$ of queues with dynamical (stochastic) service rates. By dynamical service rates, we actually mean that each single queue $i \in \mathcal{N}$ is represented by a vector $z_i(t) = (n_i(t), \mu_i(t)) \in E_i \subset \mathbb{N}^+ \times \mathbb{R}^+$. $n_i(t)$ is the number of clients and $\mu_i(t)$ is a service rate, which represents the global transition rate from z_i to $z_i' = (n_i - 1, \mu_i') \in V_i^-(z_i)$ ($V_i^-(z)$ is the set of points in E_i having one client less than z). Two sets of transition probability matrices $p_i^{\pm}(z, z')$ and one set of transition rates $q_i^0(z, z')$ are introduced to complete the definition of the process. When a client get served in queue i, the state of the departure queue z_i is modified according to the set $p_i^-(z, z')$ (with $z' \in V^-(z)$) and the state z_{i+1} of the destination queue is modified according the set $p_{i+1}^+(z, z')$, $z' \in V^+(z)$. We have the normalizations,

$$\sum_{z' \in z} p_i^{\pm}(z, z') = 1, \qquad \forall z' \in V^{\pm}(z). \tag{1}$$

Additional internal transitions are allowed, where the service rate μ_i of queue i changes independently of any arrival or departure. The intensities of these transitions are given by the set $q_i^0(z, z')$, $z' \in V_i^0(z)$ of transition rates ($V_i^0(z)$ is the set of points in E_i having the same number of clients as z). The combined set of transition rates,

$$q_i(z, z') \stackrel{\text{def}}{=} \lambda p_i^+(z, z') \mathbb{1}_{\{z' \in V_i^+(z)\}} + \mu(z) p_i^-(z, z') \mathbb{1}_{\{z' \in V_i^-(z)\}} + q_i^0(z, z') \mathbb{1}_{\{z' \in V_i^0(z)\}},$$

defines for each $i \in \mathcal{N}$ a continuous time Markov process representing the dynamics of each queue taken in isolation. For this model we can prove the following.

Theorem 4.1. *Let π_i^λ denote the steady state probability corresponding to queue i taken in isolation. If the following partial balance equations are satisfied,*

$$\sum_{z \in V^+(z_i)} \mu(z) p_i^-(z, z_i) \pi_i^\lambda(z) = \lambda \pi_i^\lambda(z_i), \tag{2}$$

$$\begin{aligned} \mu(z_i)\pi_i^\lambda(z_i) + \sum_{z \in V_i^0(z_i)} q_i^0(z_i, z)\pi_i^\lambda(z_i) = \\ \sum_{z \in V^-(z_i)} \lambda p^+(z, z_i)\pi_i^\lambda(z) + \sum_{z \in V_i^0(z_i)} q_i^0(z, z_i)\pi_i^\lambda(z), \end{aligned} \tag{3}$$

the joint probability measure of the network has the following product form at steady state:

$$P(S = \{z_i, i \in \mathcal{N}\}) = \frac{\prod_{i \in \mathcal{N}} \pi_i^\lambda(z_i)}{P(\sum_i n_i = N)} \tag{4}$$

Note that reversible processes are special cases of processes obeying (2,3), and with this respect, our results is an adaptation to our context of the general results of Kelly concerning the dawning of product form in queuing networks [12]. Some non-reversible example have this partial balance property can effectively be found **??**

References

[1] M. Schreckenberg, A. Schadschneider, K. Nagel, and N. Ito. Discrete stochastic models for traffic flow. *Phys. Rev.*, **E51**:2339, 1995.

[2] B.S. Kerner. *The Physics of Traffic*. Springer Verlag, 2005.

[3] M. Schönhof and D. Helbing. Criticism of three-phase traffic theory. *Transportation Research*, 43:784–797, 2009.

[4] K. Nagel and M. Schreckenberg. A cellular automaton model for freeway traffic. *J. Phys. I,2*, pages 2221–2229, 1992.

[5] Y. Sugiyama et al. Traffic jams without bottlenecks: experimental evidence for the physical mechanism of the formation of a jam. *New Journal of Physics*, 10:1–7, 2008.

[6] F. Spitzer. Interaction of markov processes. *Adv. Math.*, **5**:246, 1970.

[7] T. M. Liggett. *Interacting Particle Systems*. Springer, Berlin, 2005.

[8] L. Cantini. Algebraic bethe ansatz for the two species asep with different hopping rates. *J. Phys. A: Math. Theor.*, **41**:095001, 2008.

[9] V. Karimipour. A multi-species asep and its relation to traffic flow. *Phys. Rev.*, **E59**:205, 1999.

[10] C. Furtlehner and J.M. Lasgouttes. A queueing theory approach for a multi-speed exclusion process. In *Traffic and Granular Flow ' 07*, pages 129–138, 2007.

[11] D. ben Avraham and S. Havlin. *Diffusion and Reactions in Fractals and Disordered Systems*. Cambridge University Press, 2000.

[12] F. P. Kelly. *Reversibility and stochastic networks*. John Wiley & Sons Ltd., 1979. Wiley Series in Probability and Mathematical Statistics.