Heat-kernels and maximal Lp – Lq–estimates: the non-autonomous case

Matthias Hieber, Sylvie Monniaux

Heat-kernels and maximal $L^p - L^q$–estimates: the non-autonomous case

Matthias Hieber   Sylvie Monniaux

ABSTRACT. In this paper, we establish maximal $L^p - L^q$ estimates for non autonomous parabolic equations of the type $u'(t) + A(t)u(t) = f(t)$, $u(0) = 0$ under suitable conditions on the kernels of the semigroups generated by the operators $-A(t)$, $t \in [0,T]$. We apply this result on semilinear problems of the form $u'(t) + A(t)u(t) = f(t, u(t))$, $u(0) = 0$.

1 Introduction

Maximal regularity results for linear initial value problems are very effective tools when dealing with nonlinear, in particular quasilinear or fully nonlinear, problems. In fact, it is known that the standard evolution operator approach is in particular not applicable to reaction-diffusion equations where the "diffusion matrices" depend on ∇u. However, involving maximal regularity results and techniques based on the implicit function theorem, one is able to treat problems of the kind described above. For an up-to-date overview of the results and techniques known in this context, we refer to the monographs [2], [19], [16] and [5].

From the point of view of harmonic analysis, it is natural to replace the property of maximal L^p–regularity by the L^p–boundedness of certain Banach space valued singular integrals. More precisely, let $1 < p, q < \infty$, $\Omega \subset \mathbb{R}^N$ open and $-A$ be the generator of an analytic semigroup on $L^q(\Omega)$. Then there is maximal $L^p - L^q$–regularity for the problem

$$(1) \qquad \begin{aligned} u'(t) + Au(t) &= f(t), \quad t \in [0,T] \\ u(0) &= 0 \end{aligned}$$

if and only if the convolution operator R given by

$$(Rf)(t) = \int_0^t Ae^{-(t-s)A} f(s)\, ds$$

acts boundedly on $L^p(0,T;L^q(\Omega))$. Since Mihlin's theorem for operator-valued symbols is applicable in this situation if and only if $q = 2$ one is forced to use other techniques in order to prove L^p-boundedness of R. By using the transference principle, Lamberton [15] proved maximal $L^p - L^q-$regularity for (1) provided the semigroup T generated by $-A$ acts as a contraction on $L^q(\Omega)$ for all $q \in [1,\infty]$. Observe that his approach is necessarily restricted to second order differential operators and does in particular not allow to treat parabolic systems. On the other hand assuming suitable heat-kernel bounds on the semigroup generated by $-A$, maximal $L^p - L^q-$regularity for the solution of (1) was proved recently by Hieber and Prüss in [14]. For generalizations see [7]). It seems that the idea of using heat-kernel bounds in this context was first used in [21] by Strook and Varadhan. The problem of maximal $L^p - L^q-$regularity for arbitrary parabolic evolution equations of the form (1) with $-A$ being the generator of an analytic semigroup on some L^p-space, as formulated by Brézis (see e.g. [6]), seems to remain open, in general.

Considering quasilinear problems of the form $u'(t)+A(t,u(t))u(t) = f(t)$, $u(0) = 0$, maximal regularity results for the non-autonomous linear equation

$$\begin{array}{rcll} u'(t) + A(t)u(t) & = & f(t), & t \in [0,T] \\ u(0) & = & 0 & \end{array} \tag{2}$$

are of great interest. If the domains of $A(t)$ are constant, *i.e.* if $D(A(t)) = D$ for all $t \in [0,T]$ one obtains maximal regularity results for (2) by the one for the autonomous case simply by writing

$$u'(t) + A(0)u(t) = (A(0) - A(t))u(t) + f(t), \quad u(0) = 0$$

and by using perturbation arguments. Observe, however, that the domains $D(A(t))$ vary with t for example when $A(t)$ is the L^q-realization of a second order differential operator subject to co-normal Neumann boundary conditions.

In this paper we do not only prove maximal $L^p - L^q-$regularity results for examples of this kind, but treat the general case where $\{A(t), t \in [0,T]\}$

satisfies the Acquistapace-Terreni commutator condition and the heat-kernels of the semigroups T_t generated by $-A(t)$ satisfy suitable bounds. The method we use is very much inspired by the ones used in [14] and [7]. Essentially it is based on the technique developed in [11] and [10] allowing to prove $L^1 - L^1_w-$boundedness of singular integrals with weaker conditions on the kernel than the classical Hörmander almost L^1-condition.

In order to treat parabolic differential operators acting on Riemannian manifolds we choose spaces of homogeneous type as underlying setting.

This paper is organized as follows. In Section 2, we give the precise assumptions and state our main theorem. The proof of the main result is given in Section 3. In Section 4, we apply of our result to semilinear problems of the form $u'(t) + A(t)u(t) = f(t, u(t))$, $u(0) = 0$, where the domains of $A(t)$ may vary with t.

Throughout this paper, we denote by $\mathcal{L}(X, Y)$ the space of all bounded linear operators from X to Y, whenever X and Y are Banach spaces. If A is a linear operator in X, we denote its domain by $D(A)$, its resolvent set by $\rho(A)$ and its spectrum by $\sigma(A)$. Moreover, for any $\theta \in (0, \pi)$, we set $\Sigma_\theta := \{z \in \mathbb{C} \setminus \{0\}; |\arg(z)| < \theta\}$.

2 Assumptions and the main result

Let (Ω, m, d) be a space of homogeneous type. This means that Ω is a topological space, m is a $\sigma-$finite measure on Ω and d is a quasi-metric (*i.e.* $d(x, z) \leq \gamma_d(d(x, y) + d(y, z))$ for all $x, y, z \in \Omega$, where $\gamma_d \geq 1$) on Ω. We assume the doubling property : there exists a constant $C_D \geq 1$ such that $m(B_\Omega(x, 2r)) \leq C_D\ m(B_\Omega(x, r))$ holds for all $x \in \Omega$ and all $r > 0$, where $B_\Omega(x, r) := \{y \in \Omega; d(x, y) < r\}$. We remark that this property implies the strong homogeneity property (H) given as follows :

(H) there exists two constants $C_H \geq 1$ and $\ell > 0$ such that

$$m(B_\Omega(x, ar)) \leq C_H\ a^\ell m(B_\Omega(x, r))$$

holds for all $x \in \Omega$, $a \geq 1$, $r > 0$.

We consider now $\mathcal{M}$, a measurable subset of Ω, and we let $T > 0$. Let $\{A(t), t \in [0, T]\}$ be a family of linear densely defined operators in $L^2(\mathcal{M}, m)$. We assume that

(A) there exists $\varphi \in (0, \frac{\pi}{2})$ such that $\sigma(A(t)) \subset \Sigma_\varphi$, and for all $\vartheta \in (\varphi, \pi)$ there exists a constant $M_\vartheta > 0$ such that

$$\|(\lambda - A(t))^{-1}\|_{\mathcal{L}(L^2(\mathcal{M},m))} \leq \frac{M_\vartheta}{1+|\lambda|}$$

holds for all $\lambda \in \mathbb{C} \setminus \Sigma_\vartheta$, $t \in [0,T]$.

The condition (A) implies that the operators $-A(t)$ generate uniformly bounded analytic C_0-semigroups $(e^{-\sigma A(t)})_{\sigma \geq 0}$ on $L^2(\mathcal{M}, m)$. We assume that for all $t \in [0,T]$ and for all $\sigma > 0$, there exists $k_t(\sigma, \cdot, \cdot)$, bounded and measurable on $\mathcal{M} \times \mathcal{M}$ such that

$$(e^{-\sigma A(t)} f)(x) = \int_{\mathcal{M}} k_t(\sigma, x, y) f(y) dm(y) \quad m - a.a.\ x \in \mathcal{M}$$

for all $f \in L^2(\mathcal{M}, m)$. We assume moreover that the kernels satisfy a uniform estimate of the following type :

(K) there exist a constant $n > 0$ and a bounded decreasing function g defined on $(0, \infty)$ satisfying $\lim\limits_{r \to \infty} r^{2\ell+\gamma} g(r) = 0$ for some $\gamma > 0$ such that

$$|k_t(\sigma, x, y)| \leq \min\left(\frac{1}{m(B_\Omega(x, \sigma^{\frac{1}{n}}))}\ ,\ \frac{1}{m(B_\Omega(y, \sigma^{\frac{1}{n}}))}\right)\ g\left(\frac{d(x,y)}{\sigma^{\frac{1}{n}}}\right)$$

holds for all $t \in [0,T]$, $\sigma > 0$, and for $m - a.a.\ x, y \in \mathcal{M}$.

It is well-known that condition (K) is satisfied for a large class of differential operators. For details in the context, we refer to [4], [9], [12], [3], [8]. The condition (K) implies that the semigroups $\{(e^{-\sigma A(t)})_{\sigma \geq 0}, t \in [0,T]\}$ act consistently also on $L^q(\mathcal{M}, m)$ for $1 \leq q \leq \infty$. Moreover, they are uniformly bounded and analytic on $L^q(\mathcal{M}, m)$ for $1 < q < \infty$ (see [11], Proposition 2.3). Denote their generators by $-A_q(t)$. We assume moreover that the operators $\{A_q(t), t \in [0,T]\}$ verify the following commutator conditions on $L^q(\mathcal{M}, m)$, for all $q \in (1, \infty)$.

(C_q) There exists $\omega_q \in \left(\varphi, \frac{\pi}{2}\right)$, $\alpha_q, \beta_q \in [0,1]$ with $\alpha_q < \beta_q$ and a constant $c_q > 0$ such that

$$\|A_q(t)(\lambda - A_q(t))^{-1}(A_q(t)^{-1} - A_q(s)^{-1})\|_{\mathcal{L}(L^q(\mathcal{M},m))} \leq \frac{c_q \ |t-s|^{\beta_q}}{(1+|\lambda|)^{1-\alpha_q}}$$

for all $\lambda \in \mathbb{C} \setminus \Sigma_{\omega_q}$ and $s, t \in [0,T]$.

The condition (C_2) and the assumption (A) imply in particular that

(c_2) $\|(\lambda - A(t))^{-1} - (\lambda - A(s))^{-1}\|_{\mathcal{L}(L^2(\mathcal{M},m))} \leq \dfrac{c_2(M_{\omega_2}+1)\ |t-s|^{\beta_2}}{(1+|\lambda|)^{1-\alpha_2}}$

for all $\lambda \in \mathbb{C} \setminus \Sigma_{\omega_2}$ and $s, t \in [0,T]$.

We assume also the following commutator condition on $L^1(\mathcal{M}, m)$ and $L^\infty(\mathcal{M}, m)$. For $q = 1$ and $q = \infty$, it holds

(c_q) There exists $\omega_q \in \left(\varphi, \frac{\pi}{2}\right)$, $\alpha_q, \beta_q \in [0,1]$ with $\alpha_q < \beta_q$ and a constant $c_q > 0$ such that

$$\|(\lambda - A_q(t))^{-1} - (\lambda - A_q(s))^{-1}\|_{\mathcal{L}(L^q(\mathcal{M},m))} \leq \frac{c_q \ |t-s|^{\beta_q}}{(1+|\lambda|)^{1-\alpha_q}}$$

for all $\lambda \in \mathbb{C} \setminus \Sigma_{\omega_q}$ and $s, t \in [0,T]$.

Definition 2.1 *The family of operators $\{A(t), t \in [0,T]\}$ belongs to the class $MR(p,q)$, and we say that there is maximal L^p-regularity on $L^q(\mathcal{M}, m)$ if for all function f in $L^p(0,T; L^q(\mathcal{M}, m))$, there exists a unique function*

$$u \in W^{1,p}(0,T; L^q(\mathcal{M}, m)) \quad \textit{with} \quad A_q(\cdot)u(\cdot) \in L^p(0,T; L^q(\mathcal{M}, m)),$$

verifying (2) in the $L^p(0,T; L^q(\mathcal{M}, m))-$sense.

We are now in the position to state the main result of this paper.

Theorem 2.2 *Assume that the family $\{A(t), t \in [0,T]\}$ satisfies the assumptions (A), (K), (C_q) for all $q \in (1, \infty)$, (c_1) and (c_∞). Let $1 < p, q < \infty$. Then the family $\{A(t), t \in [0,T]\}$ belongs to the class $MR(p,q)$.*

3 Proofs

The results in [1], [18] imply the following facts. If u is a solution of (2) then u fulfills the equation

$$\begin{aligned} A(t)u(t) &= \int_0^t A(t)^2 e^{-(t-s)A(t)}(A(t)^{-1} - A(s)^{-1})A(s)u(s)\ ds \\ &\qquad + \int_0^t A(t)e^{-(t-s)A(t)} f(s)\ ds \end{aligned}$$

for $a.a.$ $t \in (0,T)$. For the time being, let $q \in (1,\infty)$ be fixed. If the constant c_q in (C_q) is sufficiently small, the operator $\mathcal{Q} \in \mathcal{L}(L^p(0,T;L^q(\mathcal{M},m)))$ defined by

$$(\mathcal{Q}g)(t) := \int_0^t A(t)^2 e^{-(t-s)A(t)}(A(t)^{-1} - A(s)^{-1})g(s)\ ds, \quad t \in (0,T),$$

has norm less than $\frac{1}{2}$ for all $p \in (1,\infty)$. Observe, however, (see also [17], Remark before Corollary 2), that the family $\{A(t), t \in [0,T]\}$ belongs to the class $MR(p,q)$ if and only if this holds true for $\{A(t)+\nu, t \in [0,T]\}$, where ν is an arbitrary constant. Hence, there is no loss of generality in choosing the constants c_q in (C_q) as small as we want, by choosing α_q slightly larger. It follows that the operator $1 - \mathcal{Q}$ is then invertible in $\mathcal{L}(L^p(0,T;L^q(\mathcal{M},m)))$. We summarize our observations in the following proposition.

Proposition 3.1 *The family $\{A(t), t \in [0,T]\}$ belongs to the class $MR(p,q)$ if and only if the operator S defined by*

$$(Sf)(t) := \int_0^t A(t)e^{-(t-s)A(t)} f(s)\ ds \quad t \in [0,T]$$

acts as a bounded operator on $\mathcal{L}(L^p(0,T;L^q(\mathcal{M},m)))$.

In [13], Theorem 3.2, we proved that assuming (A) and (C_2), the family $\{A(t), t \in [0,T]\}$ belongs to $MR(2,2)$. Hence the operator

$$S : L^2((0,T)\times\mathcal{M}, \lambda\otimes m) \to L^2((0,T)\times\mathcal{M}, \lambda\otimes m)$$

defined by

$$(Sf)(t,x) = \left(\int_0^t A(t)e^{-(t-s)A(t)} f(s,\cdot)ds\right)(x) \tag{3}$$

for $\lambda \otimes m - a.a.$ $(t,x) \in (0,T) \times \mathcal{M}$ acts boundedly on $L^2((0,T) \times \mathcal{M})$. Here λ denotes the Lebesgue measure on $(0,T)$. Moreover, we proved in [13], Theorem 3.1, that $MR(p,q)$ is independent of $p \in (1,\infty)$. Hence, the Marcinkiewicz interpolation theorem implies the following result.

Proposition 3.2 *Let S be defined as in (3). Then the assertion of Theorem 2.2 holds true provided S, as well as its adjoint, is of weak type* $(1,1)$.

At this point some remarks on our strategy how to prove that S is of weak type $(1,1)$ are in order. For the time being consider the autonomous case, *i.e.* $A(t) = A$ for $t \in [0,T]$ and write $f \in L^1((0,T) \times \mathcal{M})$ in its Calderòn-Zygmund decomposition as $f = g + b$ with $b = \sum b_i$. The strategy used in [14] and [7] to show that R defined as in Section 1 is of weak type $(1,1)$ is to decompose Rb_i as

$$RV_{\tau_i}b_i + (R - RV_{\tau_i})b_i$$

with an appropriate "smoothing term" V_{τ_i}. Observe that in the non-autonomous situation the operator S defined as in (3) seems not to be treatable directly with the semigroup methods developed for the autonomous case. However, semigroup methods do work for $\tilde{S}$ with

$$\tilde{S}f(t) = \int_0^t A(s)e^{-(t-s)A(s)}f(s)\, ds.$$

In fact, we decompose Sb_i in the following as

$$\tilde{S}U_{\tau_i}b_i + (S_{\tau_i} - \tilde{S}U_{\tau_i})b_i + (S - S_{\tau_i})b_i$$

with $\tilde{S}$ defined below associated to the kernel $-\partial_1 k_s(t-s,x,y)1\!\!1_{(0,t)}(s)$. In order to control the above three terms we need assumptions (C_q), (c_1) and (c_∞).

The rest of this section is devoted to the proof of the fact that S is of weak type $(1,1)$. To this end, let $Q = (0,T) \times \mathcal{M}$ and $\mu = \lambda \otimes m$. Let S be the bounded operator on $L^2(Q,\mu)$ defined as above. Denote the kernel of S by p. Then $p : Q \times Q \to \mathbb{R}$ is of the form

$$p(t,x;s,y) = -\partial_1 k_t(t-s,x,y)1\!\!1_{(0,t)}(s) \quad \mu - a.a\ (t,x),(s,y) \in Q$$

with the notation $\partial_1 k_t(\sigma,x,y) = \dfrac{\partial k_t}{\partial \sigma}(\sigma,x,y)$. This means that

$$Sf(t,x) = -\int_0^t \int_{\mathcal{M}} \partial_1 k_t(t-s,x,y) f(s,y)\ ds\ dm(y)$$

holds for each continuous function f with compact support in Q, for $\mu - a.a.$ $(t,x) \notin \mathrm{supp}(f)$.

Next, let $\tilde{p}$ be given by

$$Q \times Q \ni (t,x;s,y) \longmapsto \tilde{p}(t,x;s,y) = -\partial_1 k_s(t-s,x,y) 1\!\!1_{(0,t)}(s)$$

for $\mu - a.a.$ $(t,x),(s,y) \in Q$. Denote by $\tilde{S}$ the operator in $L^2(Q,\mu)$ associated with $\tilde{p}$.

Lemma 3.3 *Under the conditions* (A) *and* (c_2), *the operator* $\tilde{S}$ *is bounded on* $L^2(Q,\mu)$.

Proof. Since S is bounded on $L^2(Q,\mu)$ (Proposition 3.1 and [13], Theorem 3.2), it suffices to prove that $\tilde{S} - S$ is bounded on $L^2(Q,\mu)$. By assumption (A), we have

$$\begin{aligned}(\tilde{S} - S)f(t,\cdot) &= \int_0^t (A(s)e^{-(t-s)A(s)} - A(t)e^{-(t-s)A(t)})f(s,\cdot)\ ds \\ &= \int_0^t \left(\frac{1}{2i\pi} \int_{\Gamma_{\omega_2}} \lambda e^{-(t-s)\lambda}((\lambda - A(s))^{-1} - (\lambda - A(t))^{-1}) f(s,\cdot)\ d\lambda \right) ds\end{aligned}$$

where ω_2 is defined as in (C_2) and $\Gamma_{\omega_2} = (\infty,0)e^{i\omega_2} \cup e^{-i\omega_2}(0,\infty)$. Therefore, taking into account the estimates (A) and (c_2) it holds for all $f \in L^2(Q,\mu)$

$$\begin{aligned}&\|(\tilde{S} - S)f(t,\cdot)\|_{L^2(\mathcal{M},m)} \leq \\ &\leq (M_{\omega_2}+1)c_2 \frac{1}{\pi} \int_0^t \left(\int_0^\infty r e^{-(t-s)r\cos\omega_2} \frac{(t-s)^{\beta_2}}{(1+r)^{1-\alpha_2}}\ dr \right) \|f(s,\cdot)\|_{L^2(\mathcal{M},m)} ds \\ &\leq C(M_{\omega_2}, \cos\omega_2, c_2, \alpha_2, \beta_2) \int_0^t (t-s)^{\beta_2 - \alpha_2 - 1} \|f(s,\cdot)\|_{L^2(\mathcal{M},m)} ds,\end{aligned}$$

where $C(M_{\omega_2}, \cos\omega_2, c_2, \alpha_2, \beta_2)$ is a constant depending only on the listed quantities. Applying Young's inequality, we have

$$\|(\tilde{S} - S)f\|_{L^2(Q,\mu)} \leq C(M_{\omega_2}, \cos\omega_2, c_2, \alpha_2, \beta_2, T)\|f\|_{L^2(Q,\mu)}.$$

$\square$

Consider next for $\tau > 0$, the operator $U_\tau \in \mathcal{L}(L^2(Q,\mu))$ associated to the kernel u_τ given by

$$Q \times Q \ni (t,x;s,y) \longmapsto u_\tau(t,x;s,y) = 1\!\!1_{((t-\tau)_+,t)}(s)\ \frac{1}{\tau}\ k_t(\tau,x,y).$$

Moreover, define $S_\tau \in \mathcal{L}(L^2(Q,\mu))$ to be the operator associated to the kernel p_τ given by

$$p_\tau : Q \times Q \ni (t,x;s,y) \longmapsto -\frac{1}{\tau}\int_s^{s+\tau} \partial_1 k_t(t-\sigma+\tau,x,y)\ 1\!\!1_{\sigma<t}\ d\sigma.$$

Let $D := (0,T) \times \Omega$ and define the quasi-metric δ on D by

$$\delta((t,x);(s,y)) := |t-s| + d(x,y)^n \quad (t,x),(s,y) \in D,$$

where n was defined in (K). Then (D,μ,δ) is a space of homogeneous type. Define $\mathcal{H}_\tau \in \mathcal{L}(L^2(D,\mu))$ associated to the kernel h_τ given by

$$h_\tau(t,x;s,y) = \frac{e}{\tau}\ e^{-\frac{|t-s|}{\tau}} \min\left(\frac{1}{m(B_\Omega(x,\tau^{\frac{1}{n}})}, \frac{1}{m(B_\Omega(y,\tau^{\frac{1}{n}})}\right)\ g\left(\frac{d(x,y)}{\tau^{\frac{1}{n}}}\right)$$

for $(t,x),(s,y) \in D \times D$. Finally, set $B_D(\xi,\rho) = \{\eta \in D; \delta(\eta,\xi) < \rho\}$, the ball in D with center ξ and radius ρ.

Lemma 3.4 *Under the conditions (A), (K), (c_2) and (c_1), the following assertions hold*

(a) $|u_\tau(\xi,\eta)| \le h_\tau(\xi,\eta)$ *for $\mu-a.e.$ $\xi,\eta \in Q$ and for all $\tau > 0$;*

(b) *(Harnack inequality) there exist a constant $c \ge 1$ such that*

$$\sup_{\eta \in B_D(\zeta,\tau)} h_\tau(\xi,\eta) \le c \inf_{\eta \in B_D(\zeta,\tau)} h_{2^n\tau}(\xi,\eta)$$

holds for all $\zeta,\xi \in D$ and all $\tau > 0$;

(c) *there exists a constant $C > 0$ such that for all $\tau > 0$,*

$$\left|\int_D h_\tau(\eta,\xi)v(\eta)\ d\mu(\eta)\right| \le C \sup_{\rho>0} \frac{1}{\mu(B_D(\xi,\rho))}\int_{B_D(\xi,\rho)} |v(\eta)|\ d\mu(\eta)$$

holds for all $\xi \in D$ and all $v \in L^2(D,\mu)$;

(*d*) *for all* $\tau > 0$, *the operator* $S - S_\tau \in \mathcal{L}(L^2(Q,\mu))$ *associated to the kernel* $p - p_\tau$ *verifies*

$$\nu_\varrho := \sup_{\eta \in Q, \tau > 0} \int_{\delta(\xi,\eta) \geq \varrho\tau} |p(\xi,\eta) - p_\tau(\xi,\eta)| d\mu(\xi) < \infty$$

for all $\varrho > 0$;

(*e*) *for all* $\tau > 0$, *the operator* $S_\tau - \tilde{S}U_\tau \in \mathcal{L}(L^2(Q,\mu))$ *extends to a bounded operator on* $L^1(Q,\mu)$. *Moreover,* $\|S_\tau - \tilde{S}U_\tau\|_{\mathcal{L}(L^1(Q,\mu))}$ *is independent of* $\tau > 0$.

Proof. (*a*) Taking into account assumption (K), the assertion follows by the choice of u_τ and h_τ.

(*b*), (*c*) Assertions (*b*) and (*c*) were shown in [14], Lemma 4.2 and Lemma 4.3.

(*d*) The proof of assertion (*d*) follows the lines of [7], Section 3.4. We omit the details.

(*e*) For all $\tau > 0$ and for all $f \in L^2(Q,\mu) \cap L^1(Q,\mu)$, we have

$$\begin{aligned}
&(S_\tau - \tilde{S}U_\tau) f(t,\cdot) \; = \\
= \; & \int_0^T \frac{1}{\tau} \left(\int_s^{s+\tau} (A_1(t) e^{-(t-\sigma+\tau)A_1(t)} - A_1(\sigma) e^{-(t-\sigma+\tau)A_1(\sigma)}) 1\!\!1_{\sigma \leq t} \, d\sigma \right) f(s,\cdot) \, ds \\
= \; & \int_0^T ds \, \frac{1}{\tau} \int_s^{s+\tau} 1\!\!1_{\sigma \leq t} \, d\sigma \left(\frac{1}{2i\pi} \int_{\Gamma_{\omega_1}} \lambda e^{-(t-\sigma+\tau)\lambda} ((\lambda - A_1(t))^{-1} - (\lambda - A_1(\sigma))^{-1}) f(s,\cdot) \, d\lambda \right).
\end{aligned}$$

Taking into account the estimate (c_1), the argument given in the proof of Lemma 3.3 implies

$$\begin{aligned}
& \|(A_1(t) e^{-(t-\sigma+\tau)A_1(t)} - A_1(\sigma) e^{-(t-\sigma+\tau)A_1(\sigma)}) f(s,\cdot) 1\!\!1_{\sigma \leq t}\|_{L^1(\mathcal{M},m)} \; \leq \\
& \leq \; c_1 \, \frac{1}{\pi} \int_0^\infty r e^{-(t-\sigma+\tau) r \cos\omega_1} \frac{(t-\sigma)^{\beta_1}}{(1+r)^{1-\alpha_1}} \; 1\!\!1_{\sigma \leq t} \, dr \|f(s,\cdot)\|_{L^1(\mathcal{M},m)} \\
& \leq \; C(\cos\omega_1, c_1, \alpha_1, \beta_1) \, \frac{(t-\sigma)^{\beta_1}}{(t-\sigma+\tau)^{\alpha_1+1}} \; 1\!\!1_{\sigma \leq t} \|f(s,\cdot)\|_{L^1(\mathcal{M},m)}.
\end{aligned}$$

Therefore, we obtain

$$\|(S_\tau - \tilde{S}U_\tau) f(s,\cdot)\|_{L^1(\mathcal{M},m)} \; \leq$$

$$\leq C(\cos\omega_1, c_1, \alpha_1, \beta_1) \int_0^T \frac{1}{\tau} \int_s^{s+\tau} \frac{(t-\sigma)^{\beta_1}}{(t-\sigma+\tau)^{\alpha_1+1}} \, 1\!\!1_{\sigma\leq t} d\sigma \, \|f(s,\cdot)\|_{L^1(\mathcal{M},m)} ds$$
$$\leq C(\cos\omega_1, c_1, \alpha_1, \beta_1) \int_0^t (t-s)^{\beta_1-\alpha_1-1} \|f(s,\cdot)\|_{L^1(\mathcal{M},m)} ds.$$

Then, applying Young's inequality, we have

$$\|(S_\tau - \tilde{S}U_\tau)f\|_{L^1(Q,\mu)} \leq C(M_{\omega_1}, \cos\omega_1, c_1, \alpha_1, \beta_1, T)\|f\|_{L^1(Q,\mu)},$$

which implies the assertion. □

Proposition 3.5 *Under the conditions* (A), (K), (c_2) *and* (c_1), *the operator* S *is of weak type* $(1,1)$ *on* Q.

Proof. First, remark that the operator S is of weak type $(1,1)$ on Q if and only if the operator R defined by $Rf := 1\!\!1_Q S(1\!\!1_Q f)$ is of weak type $(1,1)$ on $D = (0,T)\times\Omega$. To prove this last assertion, we define the following bounded operators on $L^2(D,\mu)$:

$$\tilde{R}f := 1\!\!1_Q \tilde{S}(1\!\!1_Q f), \; R_\tau f := 1\!\!1_Q S_\tau(1\!\!1_Q f), \; V_\tau f := 1\!\!1_Q U_\tau(1\!\!1_Q f). \tag{4}$$

Consider the Calderón-Zygmund decomposition of a function $f \in L^1(D,\mu)$ (see [20], I4, Theorem 2). Then there exist $N \in \mathbb{N}$ and $\kappa > 0$, depending only on (D,μ,δ), such that for each function $f \in L^1(D,\mu)$ and for all $r > \frac{\|f\|_1}{\mu(D)}$ if $\mu(D) < \infty$ and for all $r > 0$ otherwise, there exist functions g, b_i $(i \in \mathbb{N})$ in $L^1(D,\mu)$ such that we can write f as $f = g + \sum_{i\in\mathbb{N}} b_i$ and with the following properties

- (i) $|g(t,x)| \leq \kappa r$ for $\mu - a.e.$ $(t,x) \in D$;
- (ii) for all $i \in \mathbb{N}$, there exist $(t_i,x_i) \in D$ and $\tau_i > 0$ such that $\mathrm{supp}(b_i) \subset B_i := \{(t,x) \in D; \delta((t,x);(t_i,x_i)) < \tau_i\}$ and
$$\|b_i\|_1 \leq \kappa r \mu(B_i);$$
- (iii) $\displaystyle\sum_{i\in\mathbb{N}} \mu(B_i) \leq \frac{\kappa}{r} \|f\|_1$;
- (iv) each point of D is contained in at most N balls B_i.

In order to prove Proposition 3.5, we have to show that there exists a constant $\mathcal{C} > 0$ such that for all function $f \in L^2(Q,\mu) \cap L^1(Q,\mu)$, and for all $r > 0$,

$$r\ \mu(\{(t,x) \in D; |Rf(t,x)| > r\}) \leq \mathcal{C}\|f\|_1.$$

To this end, let $f \in L^2(D,\mu) \cap L^1(D,\mu)$ be fixed. For $0 < r \leq \frac{\|f\|_1}{\mu(D)}$, we have

$$r\ \mu(\{(t,x) \in D; |Rf(t,x)| > r\}) \leq r\mu(D) \leq \|f\|_1.$$

Let $r > \frac{\|f\|_1}{\mu(D)}$ be fixed. The Calderón-Zygmund decomposition for (f,r) gives then $f = g + \sum_{i\in\mathbb{N}} b_i$ where g, b_i ($i \in \mathbb{N}$) satisfy $(i)-(iv)$. Remark that the condition (i) of the decomposition implies that $g \in L^\infty(D,\mu)$ and $\|g\|_\infty \leq \kappa r$. Moreover, the conditions (ii) and (iii) imply that $\sum_{i\in\mathbb{N}} \|b_i\|_1 \leq \kappa^2\|f\|_1$, and then $\|g\|_1 \leq (1+\kappa^2)\|f\|_1$.

The idea of the following proof is to decompose Rb_i as

$$Rb_i = \tilde{R}V_{\tau_i}b_i + (R_{\tau_i} - \tilde{R}V_{\tau_i})b_i + (R - R_{\tau_i})b_i$$

with operators $\tilde{R}$, R_τ and V_τ defined as in (4) and τ_i chosen suitably.

We subdivide the proof in four steps.

• Step 1

The function g defined as above belongs to $L^1(D,\mu) \cap L^\infty(D,\mu)$ and

$$\|g\|_2^2 \leq \|g\|_1\|g\|_\infty \leq r\ \kappa(1+\kappa^2)\|f\|_1.$$

Since the operator R is bounded on $L^2(D,\mu)$ we have by Chebychev's inequality

$$\begin{aligned}\mu(\{(t,x) \in D; |Rg(t,x)| > \frac{r}{4}\}) &\leq \frac{16}{r^2}\ \|Rg\|_2^2 \leq \frac{16}{r^2}\ \|R\|^2_{\mathcal{L}(L^2(D,\mu))}\|g\|_2^2 \\ &\leq \frac{1}{r}\ 16\ \kappa(1+\kappa^2)\|R\|^2_{\mathcal{L}(L^2(D,\mu))}\|f\|_1.\end{aligned}$$

• Step 2

The function $\sum_{i\in\mathbb{N}} V_{\tau_i}b_i$ belongs to $L^2(D,\mu)$, where τ_i is the radius of the ball B_i and the operators V_τ were defined in (4). This follows from the estimates

(a), (b) and (c) of Lemma 3.4 (see also [14], Section 5, Step II, or [7], proof of Theorem 2.6). Moreover, there exists a constant cst (depending only on N, κ, the constant C appearing in the assertion (c) of Lemma 3.4 and the norm of the maximal Hardy-Littlewood operator on $L^2(D,\mu)$) such that $\left\| \sum_{i\in\mathbb{N}} V_{\tau_i} b_i \right\|_2^2 \leq cst\ r\|f\|_1$.

Therefore, using the the fact that the operator $\tilde{R}$ is bounded on $L^2(D,\mu)$ (Lemma 3.3) and Chebychev's inequality we have, as in the first step

$$\begin{aligned}
& \mu\left(\left\{(t,x)\in D;\left|\tilde{R}\left(\sum_{i\in\mathbb{N}} V_{\tau_i} b_i\right)(t,x)\right| > \frac{r}{4}\right\}\right) \\
\leq\ & \frac{16}{r^2}\left\|\tilde{R}\left(\sum_{i\in\mathbb{N}} V_{\tau_i} b_i\right)\right\|_2^2 \\
\leq\ & \frac{16}{r^2}\ \|\tilde{R}\|^2_{\mathcal{L}(L^2(D,\mu))}\left\|\sum_{i\in\mathbb{N}} V_{\tau_i} b_i\right\|_2^2 \\
\leq\ & \frac{1}{r}\ 16\ cst\|\tilde{R}\|^2_{\mathcal{L}(L^2(D,\mu))}\|f\|_1.
\end{aligned}$$

• Step 3

By Lemma 3.4 (e), there exist a constant $K>0$ such that for all $i\in\mathbb{N}$

$$\|(R_{\tau_i}-\tilde{R}V_{\tau_i})b_i\|_1 \leq K\|b_i\|_1.$$

Therefore, by Chebychev's inequality, it holds

$$\begin{aligned}
& \mu\left(\left\{(t,x)\in D;\left|\sum_{i\in\mathbb{N}}(R_{\tau_i}-\tilde{R}V_{\tau_i})b_i(t,x)\right| > \frac{r}{4}\right\}\right) \\
\leq\ & \frac{4}{r}\ K\sum_{i\in\mathbb{N}}\|b_i\|_1 \\
\leq\ & \frac{1}{r}\ 4\ K(1+\kappa^2)\|f\|_1.
\end{aligned}$$

• Step 4

We have now to estimate the quantity

$$r\ \mu\left(\left\{(t,x)\in D;\left|\sum_{i\in\mathbb{N}}(R-R_{\tau_i})b_i(t,x)\right|>\frac{r}{4}\right\}\right).$$

For that purpose, set $B_i^* := B_D((t_i,x_i),5\gamma_\delta\tau_i)$, where $\gamma_\delta\geq 1$ is the constant appearing in the triangle inequality for the quasi-metric δ. Choose $\gamma_\delta = (2\gamma_d)^n$ if the constant γ_d correspond to the quasi-metric d. We then have using the strong homogeneity property (H), $\mu(B_i^*)\leq C_H(5\gamma_\delta)^\ell\mu(B_i)$. Therefore, using Lemma 3.4 (d) with $\varrho=4$ and the properties of the Calderón-Zygmund decomposition, it holds

$$\begin{aligned}
&\mu\left(\left\{(t,x)\in D;\left|\sum_{i\in\mathbb{N}}(R-R_{\tau_i})b_i(t,x)\right|>\frac{r}{4}\right\}\right)\ \leq\\
&\leq\ \sum_{i\in\mathbb{N}}\mu(B_i^*)+\mu\left(\left\{(t,x)\in D\setminus(\cup_{i\in\mathbb{N}}B_i^*);\left|\sum_{i\in\mathbb{N}}(R-R_{\tau_i})b_i(t,x)\right|>\frac{r}{4}\right\}\right)\\
&\leq\ C_H(5\gamma_\delta)^\ell\sum_{i\in\mathbb{N}}\mu(B_i)+\sum_{i\in\mathbb{N}}\frac{4}{r}\int_{D\setminus B_i^*}|(R-R_{\tau_i})b_i(t,x)|\ dt\ dx\\
&\leq\ C_H(5\gamma_\delta)^\ell\ \frac{\kappa}{r}\ \|f\|_1+\frac{4}{r}\ \left(\sup_{i\in\mathbb{N},\eta\in Q}\int_{\delta(\xi,\eta)\geq 4\tau_i}|p(\xi,\eta)-p_{\tau_i}(\xi,\eta)|d\mu(\xi)\right)\ \sum_{i\in\mathbb{N}}\|b_i\|_1\\
&\leq\ \frac{1}{r}\ \left(C_H(5\gamma_\delta)^\ell\kappa+4\nu_4\kappa^2\right)\|f\|_1.
\end{aligned}$$

We have proved the existence of a constant $\mathcal{C}>0$ such that for all $f\in L^1(D,\mu)\cap L^2(D,\mu)$, $\|Rf\|_{1,w}\leq\mathcal{C}\|f\|_1$, which means that R is of weak type $(1,1)$ on D. Hence S is of weak type $(1,1)$ on Q. □

By interpolation, we know that the operator S acts as a bounded operator on $L^q(Q,\mu)=L^q(0,T;L^q(\mathcal{M},m))$, for all $q\in(1,2]$. In order to prove that this is also the case for $q\in[2,\infty)$, we will show that S', the adjoint of S, is also of weak type $(1,1)$.

Proposition 3.6 *Under the conditions* (A), (K), (c_2) *and* (c_∞), *the operator* S', *the adjoint of* S, *is of weak type* $(1,1)$.

Proof. For $f \in L^2(Q,\mu)$, we have $\mu - a.e.$

$$S'f(t,\cdot) = \int_t^T A(s)' e^{-(s-t)A(s)'} f(s,\cdot)ds$$

where the operators $A(s)'$ ($s \in [0,T]$), acting on $L^2(\mathcal{M},m)$, are the adjoints of $A(s)$ ($s \in [0,T]$). We now set $G(s) := A(T-s)'$ ($s \in [0,T]$) and let φ be given as in (A). Then the family $\{G(s), s \in [0,T]\}$ satisfies the following estimates :

(G) $\sigma(G(s)) \subset \Sigma_\varphi$ and for all $\vartheta \in (\varphi,\pi)$, there exists a constant $M_\vartheta > 0$ such that $\|(\lambda - G(s))^{-1}\|_{\mathcal{L}(L^2(\mathcal{M},m))} \le \dfrac{M_\vartheta}{1+|\lambda|}$ holds for all $\lambda \in \mathbb{C} \setminus \Sigma_\vartheta,\ s \in [0,T]$;

(c_2') $\|(\lambda - G(t))^{-1} - (\lambda - G(s))^{-1}\|_{\mathcal{L}(L^2(\mathcal{M},m))} \le \dfrac{c_2(M_{\omega_2}+1)\ |t-s|^{\beta_2}}{(1+|\lambda|)^{\alpha_2}}$

for all $\lambda \in \mathbb{C} \setminus \Sigma_{\omega_2}$ and $s,t \in [0,T]$.

The semigroups generated by $G(s)$ ($s \in [0,T]$) are associated to kernels $k_s'(\sigma,x,y) = \overline{k_s(\sigma,y,x)}$ $m - a.e.$ satisfying estimate (K). The operators $G_1(s) = A_\infty(T-s)'$ ($s \in [0,T]$) acting on $L^1(\mathcal{M},m)$ verify an assumption analog to the assumption (c_1):

(c_1') $\|(\lambda - G_1(t))^{-1} - (\lambda - G_1(s))^{-1}\|_{\mathcal{L}(L^1(\mathcal{M},m))} \le \dfrac{c_\infty\ |t-s|^{\beta_\infty}}{(1+|\lambda|)^{1-\alpha_\infty}}$

for all $\lambda \in \mathbb{C} \setminus \Sigma_{\omega_\infty}$ and $s,t \in [0,T]$.

With the previous notations, we may express $S'f$, for $f \in L^2(Q,\mu)$, as

$$(S'f)_T(t,\cdot) = \int_0^t G(s)e^{-(t-s)G(s)} f_T(s,\cdot)ds$$

for $a.a.$ $t \in (0,T)$, where for $\phi \in L^2(0,T;L^2(\mathcal{M},m))$ the function ϕ_T is defined by $\phi_T(s) := \phi(T-s)$ for $a.a.$ $s \in (0,T)$. Using the same argument as in the proof of Lemma 3.3, and (c_2') and (c_1'), it follows that the operator $\tilde{S}'$ defined by

$$(\tilde{S}'f)_T(t,\cdot) := \int_0^t G(t)e^{-(t-s)G(t)} f_T(s,\cdot)ds$$

is bounded on $L^2(Q,\mu)$. Hence the operator $S' - \tilde{S}'$ is bounded on $L^1(Q,\mu)$. It remains to show that the operator $\tilde{S}'$ is of weak type $(1,1)$. Since $\tilde{S}'$ has the same form as S, we may apply Proposition 3.5 and the estimates (c_2') and (c_1')to conclude that S' is of weak type $(1,1)$. □

Combining Proposition 3.2 with Proposition 3.5 and 3.6 the proof of Theorem 2.2 is complete.

4 Applications to semilinear problems

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary and $T > 0$. We consider the following parabolic initial value problem

$$(5) \qquad \begin{array}{rcll} \dfrac{\partial u}{\partial t} + A(t,x,D)u & = & f(t,u,\nabla u) & \text{in } (0,T)\times\Omega \\ B(t,x,D)u & = & 0 & \text{on } \ (0,T)\times\partial\Omega \\ u(0,x) & = & 0 & \text{in } \ \Omega \end{array}$$

where

$$(6) \qquad f \in C^{1^-,1^-,1^-}([0,T]\times\mathbb{R}\times\mathbb{R}^N).$$

The operators $A(t,x,D)$ and $B(t,x,D)$ are given by

$$\begin{array}{rcl} A(t,x,D)u & = & -\displaystyle\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{i,j}(t,x)\,\frac{\partial u}{\partial x_j}\right) + a(t,x)u \\ B(t,x,D)u & = & \displaystyle\sum_{i,j=1}^{n} a_{i,j}(t,x)\nu_i(x)\,\frac{\partial u}{\partial x_j}. \end{array}$$

Here $(\nu_1(x),...,\nu_N(x))$ denotes the outer normal vector at a point $x \in \partial\Omega$. We assume moreover that

(*i*) $a_{i,j} \in C([0,T];C^1(\overline{\Omega}))$, $a \in C([0,T];C(\overline{\Omega}))$, $1 \le i,j \le N$

(*ii*) $a_{i,j} = a_{j,i}$, $1 \le i,j \le N$

(iii) there exists a constant $\delta > 0$ such that

$$\sum_{i,j=1}^{N} a_{i,j}(t,x)\xi_i\xi_j \geq \delta|\xi|^2 \quad \text{for all} \ \ \xi \in \mathbb{R}^N.$$

In order to apply our maximal $L^p - L^q -$ regularity result to the semilinear equation (5), we can rewrite (5) as an equation in $L^q(\Omega)$ for $1 < q < \infty$. To this end, define

$$\begin{aligned} D(A_q(t)) &:= \{u \in W^{2,q}(\Omega); B(t,x,D)u(x) = 0, x \in \partial\Omega\} \\ A_q(t)u(x) &:= A(t,x,D)u(x), \quad \text{for } a.e.\ x \in \Omega. \end{aligned}$$

Then, for each $t \in [0,T]$, the operator $A_q(t)$ generates an analytic semigroup $\{T_t(\sigma), \sigma \geq 0\}$ on $L^q(\Omega)$, which is bounded provided a is sufficiently large. Assume in addition to (i) that

(iv) $a_{i,j} \in C^\mu([0,T]; L^\infty(\Omega))$ $(1 \leq i,j \leq N)$ and $a \in C^\mu([0,T]; L^N(\Omega))$ for one $\mu > \frac{1}{2}$.

Then it was proved by Yagi [22], Theorem 4.1 that (C_q) is satisfied, for all $q \in (1,\infty)$ provided properties $(i) - (iv)$ hold. Moreover, it is known (see e.g. [9], [3], [8]) that the heat-kernel $k_t(\sigma,x,y)$ of $T_t(\sigma)$ fulfills

$$|k_t(\sigma,x,y)| \leq \frac{M}{\sigma^{\frac{N}{2}}}\ e^{-b\ \frac{|x-y|^2}{\sigma}}, \quad x,y \in \Omega,\ \sigma > 0,\ t \in [0,T]$$

for suitable constants $M, b > 0$. Finally, assumptions (c_1) and (c_∞) hold by [22], Theorem 4.2.

Summing up, it follows that $\{A_q(t), t \in [0,T]\}$ fulfills the assumptions of Theorem 2.2. Hence, it follows from this theorem that the family $\{A(t), t \in [0,T]\}$ belongs to $MR(p,q)$. Now we rewrite (5) as

$$\begin{aligned} u'(t) + A(t)u(t) &= F(t,u(t)), \quad t \in (0,T) \\ u(0) &= 0 \end{aligned}$$

with $F \in C^{1^-,1^-}([0,T] \times C^1(\overline{\Omega}); L^q(\Omega))$ given by

$$F(t,u)(x) := f(t,u(x),\nabla u(x)) \quad x \in \Omega,\ t \in [0,T].$$

Let $\tau \in (0,T]$ and define $Z_\tau := W^{1,p}(0,\tau;L^q(\Omega)) \cap D_\tau(\mathcal{A}_{p,q})$ for $1 < p,q < \infty$, where $D_\tau(\mathcal{A}_{p,q}) = \{u \in L^p(0,\tau;L^q(\Omega)); u(t) \in D(A_q(t))$ for *a.e.* $t \in (0,\tau)$ and $A_q(\cdot)u(\cdot) \in L^p(0,\tau;L^q(\Omega))\}$. The space Z_τ is a Banach space with the norm

$$\|u\|_{Z_\tau} := \|u\|_{W^{1,p}(0,\tau;L^q(\Omega))} + \|A_q(\cdot)u(\cdot)\|_{L^p(0,\tau;L^q(\Omega))}$$

for $u \in Z_\tau$. By choosing p and q large enough, we find $s \in \left(\frac{1}{p}, 1 - \frac{1}{2} - \frac{N}{2q}\right)$ such that

$$Z_\tau \hookrightarrow W^{s,p}(0,\tau;W^{2(1-s),q}(\Omega)) \hookrightarrow C([0,\tau];C^1(\overline{\Omega})). \tag{7}$$

Let now $u \in Z_\tau$ and consider the mapping $\Phi : u \mapsto v$, where v is the solution of the linear problem

$$\begin{aligned} v'(t) + A(t)v(t) &= F(t,u(t)), \quad t \in (0,\tau) \\ v(0) &= 0. \end{aligned}$$

Since $\{A(t), t \in [0,T]\} \in MR(p,q)$, there exists a constant $M > 0$ (independent of τ) such that

$$\|\Phi(u) - \Phi(\overline{u})\|_{Z_\tau} \leq M\|F(\cdot,u(\cdot)) - F(\cdot,\overline{u}(\cdot))\|_{L^p(0,\tau;L^q(\Omega))}$$

for all $u, \overline{u} \in Z_\tau$ and all $\tau \in (0,T]$. The assumption (6) implies that

$$\|F(\cdot,u(\cdot)) - F(\cdot,\overline{u}(\cdot))\|_{L^p(0,\tau;L^q(\Omega))} \leq L\|u - \overline{u}\|_{L^p(0,\tau;C^1(\overline{\Omega}))}.$$

Hence we have

$$\|\Phi(u) - \Phi(\overline{u})\|_{Z_\tau} \leq ML\tau^{\frac{1}{p}}\|u - \overline{u}\|_{C([0,\tau];C^1(\overline{\Omega}))}$$

for $u, \overline{u} \in Z_\tau$. It follows from (7) that

$$\|\Phi(u) - \Phi(\overline{u})\|_{Z_\tau} \leq ML\tau^{\frac{1}{p}}\|u - \overline{u}\|_{Z_\tau}.$$

By choosing τ small enough, the Banach fixed point theorem implies the following result.

Theorem 4.1 *Assume that $a_{i,j}$ ($1 \leq i,j \leq N$), a and f satisfy the assumptions above. Then there exist $p_1, q_1 \in (1,\infty)$, $T_1 \in (0,T]$ such that for all $p \in [p_1,\infty)$ and all $q \in [q_1,\infty)$ there exists a unique $u \in W^{1,p}(0,T_1;L^q(\Omega))$ with $u(t) \in D(A_q(t))$ for a.e. $t \in (0,T_1)$ and $A_q(\cdot)u(\cdot) \in L^p(0,T_1;L^q(\Omega))$, satisfying (5) on $(0,T_1) \times \Omega$.*

References

[1] P. Acquistapace, B. Terreni. A unified approach to abstract linear nonautonomous parabolic equations. *Rend. Sem. Mat. Univ. Padova*, 78:47–107, 1987.

[2] H. Amann. *Linear and quasilinear parabolic problems.* Birkhäuser Verlag, Basel, 1995.

[3] W. Arendt, A.F.M. ter Elst. Gaussian estimates for second order elliptic operators with boundary conditions. *J. Op. Theory*, 38:87–130, 1997.

[4] D.G. Aronson. Bounds for the fundamental solution of a parabolic equation. *Bull. Amer. Math. Soc.*, 73:890–896, 1967.

[5] P. Clément, J. Prüss. Global existence for a semilinear parabolic Volterra equation. *Math. Z.*, 209:17–26, 1992.

[6] T. Coulhon, D. Lamberton. Régularité L^p pour les équations d'évolution. Communication au *Séminaire d'Analyse Fonctionnelle 84/85*, Publications mathématiques de l'Université Paris VII, 26:155–165, 1987.

[7] T. Coulhon, X.T. Duong. Maximal regularity and kernel bounds : observations on a theorem by Hieber and Prüss. *Adv. in Differential Equations*, 1999 (to appear).

[8] D. Daners. Heat kernel estimates for operators with boundary conditions. *Tech. Report*, University of Sydney, 1997.

[9] E.B. Davies. *Heat kernels and spectral theory.* Cambridge University Press, 1989.

[10] X.T. Duong, A. McIntosh. Singular integral operators with nonsmooth kernels on irregular domains. *Preprint*, 1995.

[11] X.T. Duong, D.W. Robinson. Semigroup kernels, Poisson bounds, and holomorphic functional calculus. *J. Func. Anal.*, 142:89–128, 1996.

[12] E. Fabes. Gaussian upper bounds on fundamental solutions of parabolic equations : the method of Nash. *Springer Lect. Notes in Math. 1565*, 1–20, 1992.

[13] M. Hieber, S. Monniaux. Pseudo-differential operators and maximal regularity results for non-autonomous parabolic equations. *Proc. Amer. Math. Soc.*, 1999 (to appear).

[14] M. Hieber, J. Prüss. Heat kernels and maximal $L^p - L^q$ estimates for parabolic evolution equations. *Commun. in Partial Differential Equations*, 22:1647–1669, 1997.

[15] D. Lamberton. Équations d'évolutions linéaires associées à des semi-groupes de contractions dans les espaces L^p. *J. Func. Anal.*, 72:252–262, 1987.

[16] A. Lunardi. *Analytic semigroups and optimal regularity in parabolic problems.* Birkhäuser Verlag, Basel, 1995.

[17] S. Monniaux, J. Prüss. A theorem of the Dore-Venni type for non-commuting operators. *Trans. Amer. Math. Soc.*, 349:4787–4814, 1997.

[18] S. Monniaux, A. Rhandi. Semigroup methods to solve non-autonomous evolution equations. *Semigroup Forum*, 1999 (to appear).

[19] J. Prüss. *Evolutionary integral equations and applications.* Birhäuser Verlag, Basel, 1993.

[20] E.M. Stein. *Harmonic Analysis.* Princeton University Press, 1993.

[21] D. Stroock, S. Varadhan. *Multidimensional diffusion processes*, Springer Verlag, 1979.

[22] A. Yagi. Parabolic evolution equations in which the coefficients are the generators of infinitely differentiable semigroups, II. *Funkcialaj Ekvacioj*, 33:139–150, 1990.

Mathematisches Institut 1, Universität Karlsruhe, Englerstrasse 2, D-76128 Karlsruhe, Germany

E-mail address: matthias.hieber@math.uni-karlsruhe.de

Laboratoire de Mathématiques - case cour A, Faculté des sciences de Saint-Jérôme, Avenue Escadrille Normandie-Niemen, F-13397 Marseille Cédex 20, France

*E-mail address:*sylvie.monniaux@math.u-3mrs.fr