FeedNetBack-D04.03 - Design of Robust Variable Rate Controllers

Emilie Roche, Olivier Sename, Alexandre Seuret, Daniel Simon, Sébastien Varrier

FeedNeback
co-design for networked control systems

GRANT AGREEMENT N°223866

Deliverable	D04.03
Nature	Report
Dissemination	Public

D04.03 - Design of Robust Variable Rate Controllers

Report Preparation Date	09/JUL/2010 Project month: 18
Authors	Émilie Roche NeCS-INRIA Olivier Sename NeCS Grenoble-INP Alexandre Seuret NeCS-CNRS Daniel Simon NeCS-INRIA Sébastien Varrier NeCS-INRIA
Report Version	V2
Doc ID Code	INRIACO01_D04.03_31MAR2010_V2
Contract Start Date	01/09/2008
Duration	36 months
Project Coordinator :	Carlos CANUDAS DE WIT, INRIA, France

SEVENTH FRAMEWORK PROGRAMME

Theme 3:

Information and Communication Technologies

SUMMARY

A consequence of the execution of control algorithms on digital distributed platforms is inducing delays, jitter and various limitations in sampling rate from different sources in the control loops. These disturbances should be taken into account in the control algorithms design and tuning. Control systems are often cited as examples of "hard real-time systems" where jitter and deadline violations are strictly forbidden. In fact experiments show that this assumption may be false for closed-loop control. Any practical feedback system is designed to obtain some stability margin and robustness w.r.t. the plant parameters uncertainty. This also provides robustness w.r.t. timing uncertainties: closed-loop systems are able to tolerate some amount of sampling period and computing delays deviations, jitter and occasional data loss without loss of stability or integrity.

Hence the design of dependable distributed control systems may rely on robust controllers, i.e. controllers which are slightly sensitive to both process model and execution resource uncertainties, or on controllers which are made adaptive w.r.t. the variations of the control intervals and other implementation induced disturbances.

Section 2 provides new results concerning the control of systems with delays. A novel analysis of linear systems under asynchronous sampling is provided. This approach is based on the discrete-time Lyapunov Theorem applied to the continuous-time model of the sampled-data systems. Tractable conditions are derived to ensure asymptotic stability and also to obtain an estimate of the exponential rate of the solutions. Examples show the efficiency of the method and the reduction of the conservatism compared to other results from the literature. Moreover the methodology addresses the stability analysis of systems under several sampling periods. We show that a sampled-data system can be stable even if one of the sampling period leads to instability. This has been treated by a continuous-time approach and allows considering uncertain or time-varying systems. An extension of the method includes transmission delays in the control loop.

As the variations of the control intervals can be both a consequence of network induced delays and a control variable to manage the CPU and/or network load, robust variable sampling control design is investigated in section 3. Here it is assumed that the control interval is itself a control parameter, e.g. which can be adapted at run-time by a feedback scheduler to cope with operating conditions in a varying environment. The control design is stated using the formulation of Linear Parameters Varying (LPV) systems, where the sampling interval is considered as a varying and measurable parameters of the system. Previous results using a polytopic model of a discretized plant are recalled. A new design using a Linear Fractional Transform (LFT) is developed, where the control interval is considered as a system's uncertainty. This new approach is expected to be more tractable that the polytopic one when the system has several varying parameters. Both designs are assessed and compared using as testbed the control of Autonomous Underwater Vehicles using scheduled ultrasonic sensors for control and navigation.

Contents

1 Problem statement

Control and real-time computing have been associated for a long time, for the control of industrial plants and in embedded or mobile systems, e.g. automotive and robotics. However both parts, control and computing, are often designed with poor interaction and mutual understanding. From the control design point of view, a constant and unique period is usually assumed. Delays are supposed negligible or constants, and jitter is ignored. The implementation design then follows, trying to meet these assumptions.

To implement a controller, the basic idea consists in running the whole set of control equations in a unique periodic real-time task whose clock gives the controller sampling rate. In fact, all parts of the control algorithm do not have an equal weight and urgency w.r.t. the control performance. To minimize the latency, a control law can be basically implemented as two real-time blocks, the urgent one sends the control signal directly computed from the sampled measures, while updating the state estimation or parameters can be delayed or even more computed less frequently (Åström and Wittenmark 1997).

In fact, a complex system involves sub-systems with different dynamics which must be further coordinated (Törngren 1998). Assigning different periods and priorities to different blocks according to their relative weight allows for a better control of critical latencies and for a more efficient use of the computing resource (Simon, Castillo and Freedman 1998). However in such cases finding adequate periods for each block is out of the scope of current control theory and must be done through case studies, simulation and experiments.

Real-time scheduling has mainly focused on how to dimension resources to meet deadlines, or equivalently, on the schedulability analysis for a given resource. Indeed the real-time community has usually considered that control tasks have fixed periods, hard deadlines and worst-case execution times. This assumption has served the separation of control and scheduling designs, but has led to under utilization of CPU resources and inflexible design.

The hard and costly way consists in building a highly deterministic system, from the hardware, operating system and communication protocols sides, so that the actual implementation parameters meet the ideal ones. This extreme solution is used if, for instance, determinism is requested for formal verification and/or certification purpose, e.g. as in the synchronous programming approach (Benveniste and Berry 1991) or in the Time Triggered paradigm (Kopetz and Bauer 2003). However trying to nullify (even virtually) latencies and jitter generally leads to worst case based resources provisioning and tends to needlessly over-constraint the system's design and implementation.

In fact the hard real-time constraints can be often relaxed in a controlled way, e.g. considering the intrinsic robustness provided by the closed-loop paradigm. In a real control implementation, latencies and sampling jitter inevitably exist, in particular when actuators, sensors and controllers are distributed over a network. A smart organization and use of network and processor resources, with control features in mind, may lead to serious improvements in both the control performance, resources usage and overall cost.

Latencies have several sources: the first one comes from the computation duration itself, and worst case execution times are difficult to get. In multi-tasking systems they come from pre-emption due to concurrent tasks with higher priority, from precedence constraints and from synchronization. Another source of delays is the communication medium and protocols when the control system is distributed on a network of connected devices. In particular it has been observed that in synchronous multi-rate systems the value of sampling-induced delays show complex patterns and can be surprisingly long, e.g. (Chen, Armstrong, Fearing and Burdick 1988, Wittenmark 2001).

Control systems are often cited as examples of "hard real-time systems" where jitter and deadline violations are strictly forbidden. In fact experiments show that this assumption may be false for closed-loop control. Any practical feedback system is designed to obtain some stability margin and robustness w.r.t. the plant parameters uncertainty. This also provides robustness w.r.t. timing uncertainties: closed-loop systems are able to tolerate some amount of sampling period and computing delays deviations, jitter and occasional data loss without loss of stability or integrity. For example in (Cervin 2003) the loss of control performance has been checked experimentally using an inverted pendulum, for which a Linear Quadratic (LQ) controller has been designed according to a nominal sampling period and null delay and jitter. Figure 1 (borrowed from (Cervin 2003)) shows the output performance (position error variance) when respectively the period, the I/O latency and the output jitter are increased : the controller behavior can still be considered as correct as long as the sample-induced disturbances stay inside the performance specification bounds.

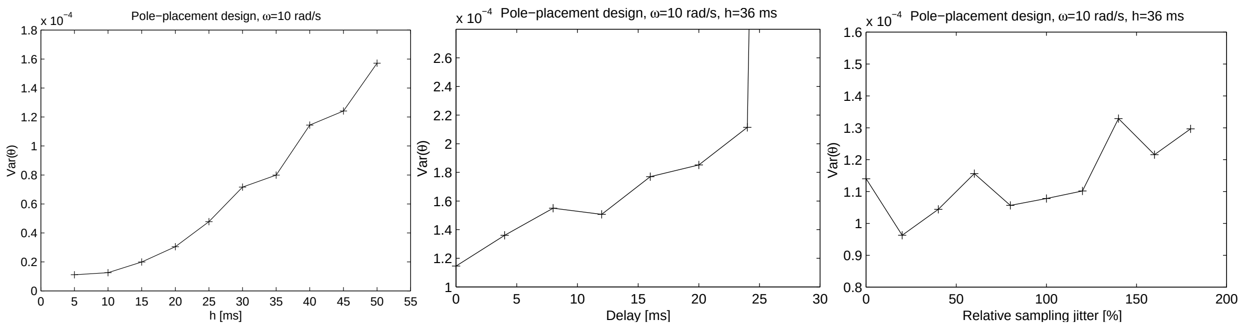


Figure 1: Performance loss w.r.t. timing deviations

The following sections deal with computing-aware robust control, where several control methods dealing with robustness and/or adaptation w.r.t. implementation induces timing uncertainties are successively exposed.

A general consequence of the execution of control algorithms on digital distributed platforms is inducing delays from different sources in the control loops, which should be taken into account in the control algorithms tuning. A first idea in the design of dependable control systems consists in using robust controllers, i.e. controllers which are slightly sensitive to both process model and execution resource uncertainties. Therefore section 2 provides new results concerning the control of systems with delays. As the variations of the control intervals can be both a consequence of network induced delays and a control variable to manage the CPU and/or network load, robust variable sampling control design is investigated in section 3.

2 Stability of sampled-data systems under asynchronous sampling and delays

In the last decades, a large attention has been taken to Networked Control Systems (NCS) (see (Hespanha, Naghshtabrizi and Xu 2007), or (Zampieri 2008)). Such systems are controlled systems containing several distributed plants which are connected through a communication network. In such applications, a heavy temporary load of computation in a processor can corrupt the sampling period of a certain controller. On the other side, the sampling period can be scheduled in the design in order to avoid this load. In both cases, the variations of the sampling period will affect the stability properties. Another phenomena, which has been widely investigated concerns stability under packet losses. In wireless networks, a transmission of data packets is not always guaranteed. Some packets can be lost during the transmission. The objective is thus to guarantee the stability even if some packet are lost in the communication. It is thus an important issue to develop robust stability conditions with respect to the variations of sampling period.

Sampled-data systems have extensively been studied in the literature (Chen and Francis 1995, Fridman, Seuret and Richard 2004, Fujioka 2009, Zhang and Branicky 2001, Zhang, Branicky and Phillips 2001) and the references therein. It is now reasonable to design controllers which guarantee the robustness of the solutions of the closed-loop system under periodic samplings. However the case of asynchronous samplings still leads to several open problems such that the guarantee of stability whatever the sampling period lying in an interval. Recently, several articles drive the problem of time-varying periods based on a discrete-time approach, (Yue, Han and Lam 2008, Oishi and Fujioka 2009, Hetel, Daafouz and Iung 2006). Note that discrete-time approaches do not fit with the case of uncertain systems or systems with time-varying parameters. Recent papers considered the modelling of continuous-time systems with sampled-data control in the form of continuous-time systems with delayed control input. In (Fridman et al. 2004), a Lyapunov-Krasovskii approach is introduced. Improvements are provided in (Fujioka 2009, Mirkin 2007), using the small gain theorem and in (Naghshtabrizi, Hespanha and Teel 2008b) based on the analysis of impulsive systems. These approaches are very relevant because they deal with time-varying sampling periods and with uncertain systems (see (Fridman et al. 2004) and (Naghshtabrizi et al. 2008b)). Nevertheless, these sufficient conditions are still conservative. This means that the sufficient conditions obtained by continuous time approaches are not able to guarantee asymptotic stability whereas the systems is stable. Recently several authors (Fridman 2010, Liu and Fridman 2009a, Seuret 2009) refines those approaches and obtain tighter conditions.

When transmission delays are introduced in the control loop, the problem becomes more complex. It is indeed well known that delays require a more accurate analysis since the time-delay systems are of infinite dimension (Gu, Kharitonov and Chen 2003, Richard 2003). Several articles have been provided to cope with the stability of NCS under sampling and transmission delays. In (Fridman et al. 2004, Liu and Fridman 2009b, Naghshtabrizi, Hespanha and Teel 2008a), stability conditions of systems under asynchronous sampling and transmission delays are presented. However those conditions are still conservative and require improvements. In the present article, we provide a novel method to assess asymptotic stability of such systems. The conditions are presented as an extension of (Seuret 2010b) to the case of time-varying transmission delays. This problem of NCS proposed here is hybrid since we consider a continuous time model of the plant and a discrete-time communication. Thus another important improvement presented here consists in employing the discrete-time Lyapunov theorem to continuous-time modelling of sampled-data systems with delays. The main idea in this report is to consider separately the two types of delays. This method provides a larger upper-bound of the allowable sampling period than the existing ones (based on the continuous-time approach).

The report proposes a novel method to assess stability applicable to linear time-varying sampled-data systems. Those conditions are based on the discrete-time Lyapunov theorem and continuous-time modelling of sampled-data systems. This method provides a larger upper-bound of the allowable sampling period than the existing ones (based on the continuous-time approach). In this paper, asymptotic and exponential stability are considered for the case of constant and time-varying sampling period. The problem of ensuring stability of control systems with multiple rate sampling is also addressed. To do so, an assumption on the probability of the use of a sampling period is introduced. This problem is adapted to the case of systems under a known sequence of two constant periods and to the case of packet loss. The proposed theorem provides stability conditions for such systems even if some periods are greater than the maximal allowable sampling periods for

This section is organized as follows. The next section formulates the problem and presents lemmas that will be used in the sequel. Section 2.2 exposes the novel method and two stability criteria ensuring the $\alpha-$stability of sampled-data systems. Section 2.3 concerns the stability of sampled data systems under several sampling periods and packet losses. Some examples and simulations are provided in Section 2.4 and show the efficiency of the method. Section 2.5 proposes an extension to the case where the control loop also includes transmission delays

Notation: Throughout the article, for a n-dimensional state vector x and a non-negative delay τ, x_t denotes a function such that $x_t(\theta) = x(t-\theta)$ for all $\theta \in [-\tau,\ 0]$. The sets $\mathbb{R}^+$, $\mathbb{R}^{n\times n}$ and $\mathbb{S}^n$ denote respectively the set of positive scalar, the set of $n \times n$ matrices and the set of symmetric matrices of $\mathbb{R}^{n\times n}$. The superscript 'T' stands for the matrix transposition. The notation $P > 0$ for $P \in \mathbb{S}^n$ means that P is positive definite. The symbols I and 0 represent the identity and the zero matrices of appropriate dimension. Recall that a function $\gamma : \mathbb{R}^+ \to \mathbb{R}^+$ is a $\mathcal{K}$-function if it is continuous, strictly increasing and $\gamma(0) = 0$. A function γ' belongs to the set $\mathcal{K}_\infty$ if $\gamma' \in \mathcal{K}$ and $\gamma(s) \to \infty$ as $t \to \infty$. A function $\beta : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$

is a $\mathcal{KL}$-function if, for each fixed $t \geqslant 0$, the function $\beta(\cdot, t)$ is a $\mathcal{K}$-function, and for each fixed $s \geqslant 0$, the function $\beta(s, \cdot)$ is decreasing and $\beta(s,t) \to 0$ as $t \to \infty$.

2.1 Problem formulation

2.1.1 System definition

Consider the linear system with a sampled-data input:

$$\dot{x}(t) = Ax(t) + Bu(t_k), \tag{1}$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$ represent the state variable and the input vector. The matrices A and B are constant and of appropriate dimension. We are looking for a piecewise-constant control law of the form $u(t) = u_d(t_k),\ t_k \leqslant t < t_{k+1}$, where u_d is a discrete-time control signal and $0 = t_0 < t_1 < ... < t_k < ...$ are the sampling instants. Our objective is to ensure the stability of the system together with a state-feedback controller of the form

$$u(t) = Kx_k,\ t_k \leqslant t < t_{k+1}. \tag{2}$$

where $x_k = x(t_k)$ and the gain K in $\mathbb{R}^{n\times m}$ is given. Assume that there exists a positive scalar T such that the difference between two successive sampling instants $T_k = t_{k+1} - t_k$ satisfies

$$\forall k \geqslant 0, \quad 0 < T_k \leqslant \bar{T}. \tag{3}$$

Several authors investigated in guaranteeing the stability of such systems. In (Fridman et al. 2004), a continuous-time approach to model sampled-data systems was developed. It allows assimilating sampling effects as the ones of a particular delay. Substituting (2) into (1), we obtain the following closed-loop system:

$$\begin{array}{l}\dot{x}(t) = Ax(t) + A_d x(t - \tau(t)),\\ \tau(t) = t - t_k,\ t_k \leqslant t < t_{k+1}.\end{array} \tag{4}$$

where $A_d = BK$. From (3), it follows that $\tau(t) \leqslant T$ since $\tau(t) \leqslant t_{k+1} - t_k$. We will further consider (4) as a linear system with uncertain and bounded delay. Examples of time-varying delay, $\tau(t)$, representing a sampling process with a time-varying period are shown in Figure 2: (a) represents the case where the period is arbitrarily varying during the experiment. This phenomena appears for instance in the case of unreliable controller softwares. In such a situation, the sampling period may be corrupted by a large amont of computations. This behavior inevitably appears when the sampling process is coupled with time-varying transmission delays. The second type of sampling delays showed in (b) represents the relevant situation where the control is designed especially with a time-varying sampling period. This situation occurs for instance when a controller is not only dedicated to a single but several tasks or processes to control simultaneously. Thus, the controller is scheduled in a certain manner to optimize the control performances of all the processes. Another example of (b) is the consideration of packet loss. In such a situation, a controller periodically provides control data to be implemented in the actuators but some of them are lost in the communication. Assuming that a control data is held in the actuator until an update with a new control data, the sampling period can dramatically vary during the experiment.

Concerning the analysis of sampled-data systems, sufficient stability conditions were firstly designed in (Fridman et al. 2004) to deal with all kind of delay functions including sampled delays. However, as sampled-data systems are systems subject to a particular delay, this approach was conservative with respect to the maximum allowable sampling period. In (Naghshtabrizi et al. 2008b), the authors introduce a new type of Lyapunov-Krasovskii functionals which depends explicitly and linearly on the delay function. In this approach, a additional characteristic of sampled delay which is $\dot{\tau} = 1$ has been included and leads to

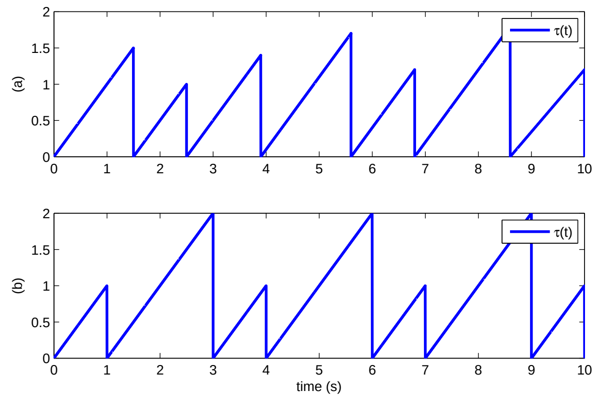


Figure 2: Examples of asynchronous sampling delays: (a) shows the case of a sampling period which varies arbitrarily, (b) represents the case of a repetitive sequence of two known and exact periods.

more accurate stability conditions. The present article establishes a novel approach to cop with the stability analysis of continuous-time sampled-data systems based on the discrete-time Lyapunov Theorem. This approach leads to necessary conditions on a class of functionals which are not required to be of Lyapunov-Krasovskii type. For the sake of simplicity, the notation τ stands for the time-varying sampling delay $\tau(t)$.

2.1.2 Preliminary lemmas

In order to clarify the presentation, the following lemmas on a property of convex linear matrix inequalities, the asymptotic stability of discrete-time systems and the positivity of functionals are stated.

Lemma 1. *(Naghshtabrizi et al. 2008b) Consider three matrices X_1, X_2 and $X_3 \in \mathbb{S}^n$ and a time-varying parameter $\lambda : \mathbb{R}^+ \to [\lambda_m\ \lambda_M]$, for some given λ_m and λ_M. If the following inequality is guaranteed*

$$\forall t \geqslant 0, \quad X_1 + (\lambda_M - \lambda(t))X_2 + (\lambda(t) - \lambda_m)X_3 < 0, \tag{5}$$

then, it is equivalent to

$$X_1 + (\lambda_M - \lambda_m)X_2 < 0, \quad X_1 + (\lambda_M - \lambda_m)X_3 < 0. \tag{6}$$

Lemma 2. *(Jiang and Wang 2002) System* (4) *is uniformly globally asymptotically stable (UGAS) if and only if there exists a $\mathcal{KL}$-function β such that, for all $x_0 \in \mathbb{R}^n$,*

$$|x(t_k, x_0)| \leqslant \beta(|x_0|, t_k), \quad \forall k \geqslant 0 \tag{7}$$

2.2 Main result

This section is motivated by the difference between the discrete and continuous-time Lyapunov theorems. As the problem of sampled-data systems is at the boundary of the discrete and the continuous-time theories, it is important to put in clear the difference between them. More especially, the main idea of this section consists in developing novel stability conditions for sampled-data systems, modelled in continuous-time, using the discrete-time Lyapunov theorem.

Theorem 1. *Let $\alpha \geqslant 0$ and $V : \mathbb{R}^n \to \mathbb{R}^+$ be a function for which there exist real numbers $0 < \mu_1 < \mu_2$ and $p > 0$ such that*

$$\forall (k, x_k) \in \mathbb{N} \times \mathbb{R}^n, \quad \mu_1 |x_k|^p \leqslant V(x_k) \leqslant \mu_2 |x_k|^p. \tag{8}$$

Consider the continuous extension of V along the trajectories of system (1) with the control law (2) , i.e. $V(x(t))$, for all $t \in [t_k,\ t_{k+1}]$. The two following statements are equivalent.

(i) $\forall k \geqslant 0, \quad \Delta_\alpha V(k) = e^{2\alpha T_k} V(x_{k+1}) - V(x_k) < 0$*;*

(ii) *There exists a continuous functional $\mathcal{V}_\alpha : \mathbb{R} \times \mathbb{K} \to \mathbb{R}$, differentiable over all intervals of the form $[t_k\ t_{k+1}[$ which satisfies*

$$\forall k \geqslant 0, \quad e^{2\alpha T_k} \mathcal{V}_\alpha(T_k, \chi_k) = \mathcal{V}_\alpha(0, \chi_k). \tag{9}$$

and such that, for all $k > 0$ and for all t in $[t_k\ t_{k+1}]$, the following inequality holds

$$\mathcal{W}_\alpha(\tau(t), \chi_k) < 0, \tag{10}$$

where $\mathcal{W}_\alpha(\tau(t), \chi_k) = \frac{d}{dt}\left\{e^{2\alpha\tau(t)}[V(x(t)) + \mathcal{V}_\alpha(\tau(t), \chi_k)]\right\}$.

Moreover, if one of these two statements is satisfied, the solutions of system (1) with the control law (2) are exponentially stable with the rate α.

Proof. Consider a given $\alpha \geqslant 0$ and a positive integer k and $t \in [t_k,\ t_{k+1}]$. For the sake of simplicity and when there is no possible confusion, the notation x and τ stand for the state variable $x(t)$ and for the time-varying sampling delay $\tau(t)$ at time t. Assume (ii) is satisfied. Integrating $\mathcal{W}_\alpha$ over the interval $[t_k\ t_{k+1}[$ and assuming that (9) holds, this directly implies $\Delta_\alpha V(k) < 0$ and that (i) holds.

Assume now that (i) is satisfied. Inspired by Lemma 2 in (Peet, Papachristodoulou and Lall 2009), consider the functional $\mathcal{V}_\alpha(\tau, \chi_k) = -V(x) + g_\alpha(\tau)\Delta_\alpha V(k)$, where $g_\alpha(\tau) = T_k/(e^{2\alpha T_k} - 1)$ if $\alpha \neq 0$ and $g_0(\tau) = \tau/T_k$. Indeed, $\mathcal{V}_\alpha$ is a functional since it is expressed with respect to $\Delta_\alpha V(k)$ which depends on the function $\chi_k(0) = x_k$, $\chi_k(T_k) = x_{k+1}$ and $\chi_k(\tau) = x$ for all $t \in [t_k,\ t_{k+1}]$. By simple computations, it is easy to obtain that this functional satisfies (9) and that $\mathcal{W}_\alpha(\tau, \chi_k) = \frac{2\alpha T_k}{e^{2\alpha T_k}-1} e^{2\alpha\tau}\Delta_\alpha V(k)$ if $\alpha \neq 0$ and $W_0(\tau, \chi_k) = \Delta_0 V(k)$. Noting that $(e^{2\alpha T_k} - 1)/2\alpha T_k$ is always positive, for all α, $\mathcal{W}_\alpha$ has the same sign as $\Delta_\alpha V(k)$. This proves the equivalence between (i) and (ii).

From the discrete-time Lyapunov theorem, the equilibrium of the discrete-time system is exponentially stable with a guaranteed rate α by noting that the previous inequality implies $V(x_k) < e^{2\alpha t_k} V(x_0)$ which satisfies Definition 7.

The end of the proof consists in ensuring that the solutions of the continuous-time system are not diverging within a sampling period. From (8), the inequality $V(x) \leqslant \mu_2 |\tilde{A}(\tau) x_k|^p$ is ensured, for all integer k, and $\tau \in [0,\ T_k]$. Noting that the function $\tilde{A}(.)$: $[0,\ T_k] \to \mathbb{R}^{n\times n}$ is continuous and consequently bounded over $[0,\ T_k]$, there exists a positive scalar μ_m such that, $|\tilde{A}(.)| \leqslant \mu_m$. This ensures $V(x) \leqslant \mu_2 \mu_m^p |x_k|^p$. This proves that the continuous Lyapunov function uniformly and exponentially tends to zero. □

In the literature several articles have introduced functionals satisfying the requirements of Theorem 1 (see for instance (Naghshtabrizi et al. 2008b, Fridman 2010, Seuret 2009)). However, those results use of the Lyapunov-Krasovskii theorem which requires necessarily their positive definiteness. Moreover no direct relation between the discrete-time Lyapunov theorem ($\Delta_\alpha V < 0$) and the Lyapunov-Kravoskii theorem ($\dot{V} + \dot{\mathcal{V}}_\alpha < 0$) is provided. Theorem 1 proves that they are equivalent and, moreover, it relaxes the constraint on the positivity of the functional. The only requirement to ensure stability is on the discrete-time Lyapunov function $\mathcal{V}$, and (9).

If $\alpha = 0$, condition (10) becomes exactly the asymptotic stability condition in discrete-time. If $\alpha < 0$, Theorem 1 still makes sense. It thus means that the system can be unstable but the divergence rate of the solutions is not greater than α. To avoid any confusion in the sequel, a system is called $\alpha-$stable if there is an $\alpha \in \mathbb{R}$ such that Theorem 1 is verified.

In the following, several theorems provide sufficient stability conditions for the cases of synchronous and asynchronous samplings including as well the case of uncertain systems.

2.3 Exponential stability of sampled data systems

2.3.1 Case of a constant sampling period

In this section, a study on the $\alpha-$stability of the solutions of sampled-data systems under a constant period is provided. The objective is to design a functional which satisfies the conditions proposed in Theorem 1. Assume that, $T_k = T$, for all $k > 0$. The following theorem is derived.

Theorem 2. *For a given $\alpha \in \mathbb{R}$ and a positive scalar T, assume that there exist, $P > 0$, $R > 0$, S_1 and $X \in \mathbb{S}^n$ and two matrices $S_2 \in \mathbb{R}^{n\times n}$ and $N \in \mathbb{R}^{2n\times n}$ that satisfy*

$$\Psi^1(T) = h^1_\alpha(0)\Pi_1 + h^2_\alpha(T,0)\Pi_2 + h^4_\alpha(T,0)\Pi_3 < 0, \tag{11}$$

$$\Psi^2(T) = \begin{bmatrix} h^1_\alpha(T)\Pi_1 + h^4_\alpha(T,T)\Pi_3 & h^3_\alpha(T,T)N \\ * & -h^3_\alpha(T,T)R \end{bmatrix} < 0, \tag{12}$$

where

$$\begin{array}{l} \Pi_1 = 2M_1^T P(M_0 + \alpha M_1) - M_{12}^T S_1 M_{12} \\ \qquad -2M_2^T S_2 M_{12} - 2NM_{12}, \\ \Pi_2 = M_0^T(RM_0 + 2S_1M_{12} + 2S_2M_2), \\ \Pi_3 = M_2^T X M_2, \end{array} \tag{13}$$

and $M_0 = \begin{bmatrix} A & BK \end{bmatrix}$, $M_1 = \begin{bmatrix} I & 0 \end{bmatrix}$, $M_2 = \begin{bmatrix} 0 & I \end{bmatrix}$, $M_{12} = M_1 - M_2$. where the functions h^i_α for $i = 1,\dots,4$ are defined for all $T > 0$ and $\tau \in [0\ T]$ by

$$\begin{array}{l} h^1_\alpha(\tau) = \begin{cases} e^{2\alpha\tau}, & \text{if } \alpha \neq 0 \\ 1, & \text{if } \alpha = 0 \end{cases} \\ h^2_\alpha(T,\tau) = \begin{cases} \left(e^{2\alpha T} - e^{2\alpha\tau}\right)/2\alpha, & \text{if } \alpha \neq 0 \\ T-\tau, & \text{if } \alpha = 0 \end{cases} \\ h^3_\alpha(T,\tau) = \begin{cases} e^{2\alpha T}\left(e^{2\alpha\tau} - 1\right)/2\alpha, & \text{if } \alpha > 0 \\ \left(e^{2\alpha\tau} - 1\right)/2\alpha, & \text{if } \alpha < 0 \\ \tau, & \text{if } \alpha = 0 \end{cases} \\ h^4_\alpha(T,\tau) = \begin{cases} \left[2\alpha T + e^{2\alpha\tau}(1 - e^{2\alpha T})\right]/2\alpha^2 T, & \text{if } \alpha \neq 0 \\ T - 2\tau, & \text{if } \alpha = 0 \end{cases} \end{array}$$

System (1) *with the control law* (2) *is $\alpha-$stable for the constant sampling period T.*

Proof. Introduce a quadratic Lyapunov function candidate defined by $V(x_k) = x_k^T P x_k$. The objective is then to prove that $\Delta_\alpha V(k) = e^{2\alpha T_k}V(x_{k+1}) - V(x_k) < 0$. Thanks to Theorem 1, the objective is to design functionals $\mathcal{V}_\alpha$ which satisfy (9). A candidate of such type of functionals is defined for all $\tau \in [0\ T_k]$

$$\begin{array}{l} \mathcal{V}_\alpha(\tau,\chi_k) = f_\alpha(T,\tau)\zeta_k^T[S_1\zeta_k + 2S_2x_k] \\ \qquad + f_\alpha(T,\tau)\int_{t_k}^t \dot{x}^T(\theta)R\dot{x}(\theta)d\theta + H^4_\alpha(T,\tau)x_k^T X x_k, \end{array} \tag{14}$$

where $\zeta_k = x - x_k$ and $t \in [t_k,\ t_{k+1}]$

$$\begin{array}{ll} f_\alpha(T,\tau) = e^{-2\alpha\tau} h_\alpha^2(T,\tau), & \\ H_\alpha^4(T,\tau) = \left((1-e^{-2\alpha\tau}) - \frac{\tau}{T}(1-e^{-2\alpha T})\right)/(2\alpha^2), & \text{if } \alpha \neq 0, \\ H_0^4(T,s) = (T-\tau)\tau, & \text{if } \alpha = 0, \end{array} \quad (15)$$

The first step of the proof focuses on the fact that the function V and the functional $\mathcal{V}_\alpha$ satisfy Theorem (1). The function V satisfies (8) since it is a quadratic Lyapunov function. The next part of the proof consists in ensuring (9). Consider a positive scalar $0 < \epsilon < T$. Since $\zeta_k(\epsilon)$ and $f_\alpha(T, T-\epsilon)$ tend to 0 as $\epsilon \to 0$ for all $\alpha \in \mathbb{R}$, it follows that the limits of $\mathcal{V}_\alpha(T-\epsilon, \chi_k)$ and of $\mathcal{V}_\alpha(\epsilon, \chi_k)$ as ϵ tends to 0 are zero. The functional $\mathcal{V}_\alpha$ thus satisfies conditions (9) and is moreover continuous at all sampling instants and differentiable for all $t \in [t_k\ t_{k+1}[$. The rest of the proof consists in ensuring inequality (10). From the definition (15), it yields, for all $\alpha \in \mathbb{R}$ and for all $s \in [0, T]$, $\dot{f}_\alpha(T,s) + 2\alpha f_\alpha(T,s) = -1$. Using the definitions of the scalar functions h_α^i, $i = 1, \ldots, 4$, the following equality is obtained

$$\begin{array}{rl} \frac{\mathcal{W}_\alpha(\tau,\chi_k)}{h_\alpha^1(\tau)} = & 2x^T P(\dot{x} + \alpha x) + \dot{H}_\alpha^4(T,\tau) x_k^T X x_k \\ & -\zeta_k^T[S_1\zeta_k + 2S_2 x_k] - \int_{t_k}^t \dot{x}^T(\theta) R \dot{x}(\theta) d\theta \\ & + f_\alpha(T,\tau)\dot{x}^T[R\dot{x} + 2S_1\zeta_k + 2S_2 x_k]. \end{array} \quad (16)$$

Consider the extended vector $\xi = [x^T\ x_k^T]^T$, for all $t \in [t_k,\ t_{k+1}[$, and a matrix $N \in \mathbb{R}^{2n\times n}$. Since R is assumed to be positive definite and thus non singular, the product $(\dot{x}(\theta) - R^{-1}N^T\xi(t))^T R(\dot{x}(\theta) - R^{-1}N^T\xi(t))$ is positive for all $t \in [t_k,\ t_{k+1}[$ and all $\theta \in [t_k,\ t[$. Integrating its development over $[t_k,\ t]$, the following inequality is obtained

$$\begin{array}{r} \int_{t_k}^t \dot{x}^T(\theta) R \dot{x}(\theta) d\theta - 2\xi^T(t) N(x(t) - x_k) \\ + \tau \xi^T(t) N R^{-1} N^T \xi(t) \geqslant 0, \end{array} \quad (17)$$

Noting that $\dot{x} = M_0\xi$, $x = M_1\xi$, $x_k = M_2\xi$, $\zeta_k = M_{12}\xi$ and that $h_\alpha^1(\tau)\dot{H}_\alpha^4(T,\tau)$ is equal to $h_\alpha^4(T,\tau)$, adding (17) to (16) leads to the inequality for all $t \in [t_k,\ t_{k+1}[$ by

$$\begin{array}{rl} \mathcal{W}_\alpha(\tau,\chi_k) \leqslant & \xi^T[h_\alpha^1(\tau)\Pi_1 + h_\alpha^2(T,\tau)\Pi_2 \\ & + \tau h_\alpha^1(\tau) N R^{-1} N^T + h_\alpha^4(T,\tau)\Pi_3]\xi. \end{array} \quad (18)$$

In the sequel, several cases of α are considered to prove that $\tau h_\alpha^1 \leqslant h_\alpha^3$, for all $\alpha \in \mathbb{R}$. The case $\alpha = 0$ is straightforward since, by definition, $\tau h_0^1 = h_0^3$. If $\alpha \neq 0$, the inequality (18) does not depend linearly on τ but on both τ and a non linear term $e^{2\alpha\tau}$. A first possibility to obtain sufficient conditions for exponential stability would be to consider these two terms independent. However this would add complexity and conservatism since there will be four LMI's to solve in which the cases $(0, e^{2\alpha T})$ and $(T, 1)$ never occur. The solution proposed here is to use the convexity property of the exponential function ensuring that $e^{2\alpha\tau} \geqslant 1 + 2\alpha\tau$ if $\alpha > 0$ and $e^{-2\alpha\tau} \geqslant 1 - 2\alpha\tau$ if $\alpha < 0$. Consequently, the following upper-bounds are obtained

$$\begin{array}{ll} \forall \alpha > 0, & \tau h_\alpha^1(\tau) \leqslant e^{2\alpha\tau}(e^{2\alpha\tau} - 1)/2\alpha \leqslant h_\alpha^2(T,\tau), \\ \forall \alpha < 0, & \tau h_\alpha^1(\tau) \leqslant e^{2\alpha\tau}(1 - e^{-2\alpha\tau})/2\alpha = h_\alpha^2(T,\tau). \end{array}$$

Noting that R, and consequently R^{-1}, are positive definite, it yields

$$\begin{array}{rl} \mathcal{W}_\alpha(\tau,\chi_k) \leqslant & \xi^T[h_\alpha^1(\tau)\Pi_1 + h_\alpha^2(T,\tau)\Pi_2 \\ & + h_\alpha^3(T,\tau) N R^{-1} N^T + h_\alpha^4(T,\tau)\Pi_3]\xi. \end{array} \quad (19)$$

To prove that $\mathcal{W}_\alpha$ is negative definite for all τ, Lemma 1 is applied to (19) with $\lambda(t) = e^{2\alpha\tau}$ (or $\lambda(t) = \tau$ if $\alpha = 0$) which leads to (11) and (12). By vertu of Theorem 1, the α-stability of system (1) with the control law (2) is guaranteed. □

Remark 1. *As suggested in Theorem 2, no additional constraint is introduced on S_1 and S_2, V_α is not necessarily positive definite within two sampling instants. This corresponds to the improvement with respect to the previous approaches exposed in (Fridman 2010, Liu and Fridman 2009a, Naghshtabrizi et al. 2008b, Seuret 2009, Seuret 2010a). The results based on these papers are based on Lyapunov-Krasovskii functionals. In the present paper, only the positive definiteness of a quadratic function is necessary.*

2.3.2 Case of multiple constant sampling periods

The objective is to prove that system (1) with the control law (2) under several sampling periods is stable even if one of them is greater than the maximum allowable sampling period. Exponential stability conditions from Theorem 2 allow quantifying the convergence and divergence of the solutions within each sampling periods. As suggested in (Sun, Liu, Rees and Wang 2008) or in (Zhang and Yu 2010), the combinaison of these convergence and divergence rates more appropriate stability conditions are derived.

In this section, the system is subject to several constant sampling periods T_l, $l = 1, \dots, L$. This problem has been already pointed out in (Zhang et al. 2001) and dealt in (Li, Cela, Niculescu and Reama 2009). Instead of considering only the case of an exact sequence of several samplings, the probabilities p_l are introduced for all $l = 1, \dots, L$. This probability corresponds to the sampling period T_l is employed in the controller. Assume that for all $\beta_l > 0$ there exists $k_0 > 0$ such that the probabilities p_l satisfies

$$\forall k > k_0, \quad \forall l = 1, \dots, l, \quad \left| p_l - \tfrac{K_l(k)}{k} \right| < \beta_l. \tag{20}$$

where $K_l(k)$ represents the number of times that the sampling period T_l was employed before the k^{th} sampling instant. This assumption simply means that the ratio between the times that T_l is employed and the total number of sampling instants tends to the probability p_l. Based on these assumptions, the following theorem is proposed.

Theorem 3. *Consider system* (1) *with the control law* (2) *under several constant sampling periods T_l associated with the probabilities p_l, $l = 1, \dots, L$. Assume that there exist $P > 0 \in \mathbb{S}^n$ and, for $l = 1, \dots, L$, $\alpha_l \in \mathbb{R}$, some matrices , X^l, R^l and $S_1^l \in \mathbb{S}^n$, $S_2^l \in \mathbb{R}^{n\times n}$ and $N^l \in \mathbb{R}^{2n\times n}$ that satisfy for $l = 1, \dots, L$:*

$$h^1_{\alpha_l}(0)\Pi_1^l + h^2_{\alpha_l}(T_l, 0)\Pi_2^l + h^4_{\alpha_l}(T_l, 0)\Pi_3^l < 0, \tag{21}$$

$$\begin{bmatrix} h^1_{\alpha_l}(T_l)\Pi_1^l + h^4_{\alpha_l}(T_l, T_l)\Pi_3^l & h^3_{\alpha_l}(T_l, T_l)N^l \\ * & -h^3_{\alpha_l}(T_l, T_l)R^l \end{bmatrix} < 0, \tag{22}$$

and such that

$$c = \sum_{l=1}^{L} p_l \alpha_l T_l > 0, \tag{23}$$

where Π_1^l, Π_2^l and Π_3^l are defined as in (13) *but with the matrices X^l, R^l S_1^l, S_2^l and N^l. Then, system* (1) *with the control law* (2) *under sampling periods T_l associated with the probability p_l is asymptotically stable.*

Proof. Consider the functionals V_α^l of the same form as in (14) and associated with each sampling period T^l. All the functionals V_α^l are defined with the same matrix P. This ensures that at each sampling instant t_k and for all $l = 1, \dots, L$, it yields $V_\alpha^l(0, \chi_k) = 0$. Denote the function $\sigma : \mathbb{N} \to \{1, \dots, L\}$ such that, for each k, $T_{\sigma(k)} = t_{k+1} - t_k$ is the sampling period associated to the sample k. If conditions (11) and (12) are satisfied, then Theorem 2 ensures that

$$\begin{aligned} x_{k+1}^T P x_{k+1} = V(x_{k+1}) &\leqslant V(x_k) e^{-2\alpha_{\sigma(k)} T_{\sigma(k)}} \\ &\leqslant x_k^T P x_k e^{-2\alpha_{\sigma(k)} T_{\sigma(k)}} \end{aligned}$$

This leads to $x_k^T P x_k \leqslant x_0^T P x_0 e^{-2\sum_{i=1}^{k}\alpha_{\sigma(i)}T_{\sigma(i)}}$. Reordering the term in the exponential function and introducing the probabilities p_l, it yields

$$\begin{aligned}\sum_{i=1}^{k}\alpha_{\sigma(i)}T_{\sigma(i)} &= \left[\sum_{l=1}^{L}(K_l(k) - p_l k + p_l k)\alpha_l T_l\right] \\ &= k\left[c + \sum_{l=1}^{L}(K_l(k)/k - p_l)\alpha_l T_l\right].\end{aligned}$$

Consider $\beta = c/(2L\max_{l=1,\ldots,L}|\alpha_l T_l|)$. From assumption (20), there exists k_0 such that, for all $k > k_0$, $(K_l(k)/k - p_l) < \beta$. This ensures that for all $l = 1, \ldots, L$

$$\begin{aligned}|(K_l(k)/k - p_l)\alpha_l T_l| &\leqslant |K_l(k)/k - p_l||\alpha_l|T_l \\ &\leqslant \frac{c|\alpha_l|T_l}{2L\max_{l=1,\ldots,L}|\alpha_l T_l|} \\ &\leqslant c/2L,\end{aligned}$$

Consequently, this leads to for all $k > 0$, $x_k^T P x_k \leqslant x_0^T P x_0 e^{-ck}$. Then the proof is concluded seeing that when k tends to $+\infty$, $x_k^T P x_k$ tends to 0 since c is positive. This ensures that the UGAS conditions from Definition 7 is ensured and x_k tends to 0 for all initial conditions. □

Corollary 1. *Condition* (23) *can be easily adapted to other sequences of sampling. For example, if the sampling is a repeated sequence of the form* $\{T_1, T_2, T_1, T_2, \ldots\}$, *the conditions to ensure stability becomes* $c' = \alpha_1 T_1 + \alpha_2 T_2 > 0$.

Remark 2. *Consider two sampling periods* T_1 *and* T_2, *the associated convergence rates and the probabilities,* α_i *and* p_i, $i = 1, 2$. *Since the convergence rate of discrete-time systems is* $\lambda_i = e^{-2\alpha_i T_i}$, *the* λ_i*'s are upper-bounds of* $\tilde{A}(T_i)$. *Rewriting* (23) *in terms of the* λ_i*'s, one has* $p_1 ln\lambda_1 + p_2 ln\lambda_2 = 2c$. *Together with the definition of the probabilities* $p_1 + p_2 = 1$, *this leads to that* $p_2/p_1 = (-2c - ln\lambda_1)/(ln\lambda_2 + 2c)$. *The condition* $c > 0$ *implies the existence of a* $\lambda^* \in [\lambda_1,\ 1]$ *such that* $2c = -ln\lambda^*$. *Then Theorem 1 from (Lin, Zhai and Antsaklis 2006) is retrieved.*

2.3.3 Case of asynchronous sampling period

The objective of this section is to provide a $\alpha-$stability for a system with an asynchronous sampling. This means that the difference between two successive sampling instants may vary during the experiment. It is assume that this difference lies in an interval of the form $[0,\ \bar{T}]$.

Theorem 4. *For a given* $\alpha \in \mathbb{R}$ *and a positive scalar* $\bar{T}$, *assume that there exist,* $P > 0$, $R > 0$, $S_1 \in \mathbb{S}^n$ *and* $X = 0$ *and two matrices* $S_2 \in \mathbb{R}^{n\times n}$ *and* $N \in \mathbb{R}^{2n\times n}$ *that satisfy*

$$h_\alpha^1(0)\Pi_1 + h_\alpha^2(\bar{T},0)\Pi_2 < 0, \quad \begin{bmatrix} h_\alpha^1(\bar{T})\Pi_1 & h_\alpha^3(\bar{T},\bar{T})N \\ * & -h_\alpha^3(\bar{T},\bar{T})R \end{bmatrix} < 0, \tag{24}$$

then system (1) *with the control law* (2) *is* $\alpha-$*stable for all asynchronous sampling whose periods are lying in* $[0,\ \bar{T}]$.

Proof. Let k be a positive integer and $\alpha \in \mathbb{R}$. The sampling is asynchrone and consequently, the distance between two successive sampling instant T_k lies in $[0,\ \bar{T}]$. From (11) with $X = 0$, it yields $h_\alpha^1(0)\Pi_1 + h_\alpha^2(T_k, 0)\Pi_2 < 0$. This inequality is linear with respect to $\lambda(t) = e^{2\alpha T_k}$ if $\alpha \neq 0$ or to $\lambda(t) = T_k$ if $\alpha \neq 0$. Then applying Lemma 1, this is equivalent to $\Pi_1 < 0$ and to $h_\alpha^1(0)\Pi_1 + h_\alpha^2(\bar{T}, 0)\Pi_2 < 0$. Consider (11) with $X = 0$. The only difficulty is that $h_\alpha^3(T_k, T_k)$ is not linear with respect to $e^{2\alpha T_k}$ as α is positive. However, this problem can be solved by $h_3\alpha(T_k, T_k) \leqslant e^{2\alpha\bar{T}}(e^{2\alpha T_k} - 1)/2\alpha$. Applying once more Lemma 1 proves that the system is $\alpha-$stable if the LMI's (24) and $\Pi_1 < 0$. As the negative definiteness of Π_1 is already in the second LMI of (24), it can be removed. □

Remark 3. *Theorem 4 proposes relaxed conditions of Theorem 2. A necessary condition of Theorem 4 is* $\Pi_1 < 0$*. The proof shows that this condition is related to the stability of the system if* $T = 0$*. Thus even if a constraint on the minimum sampling period is introduced in the stability conditions, the case* $T = 0$ *is already included. Consequently, if the LMIs* (24) *are satisfied for some* $\bar{T}$*, it directly implies that the system is stable for all asynchronous sampling whose periods are lying in* $[0\ \bar{T}]$*. The same property is also obtained in (Fridman 2010).*
In Theorem 2, the introduction of the matrix X *relaxes this contraints. Consequently can tackle the stability problem of systems which are not stable for small sampling periods but become stable when the sampling period is sufficiently large, as it is shown in the examples.*

2.3.4 Precisions on the previous theorems

The following theorem addresses the problem of asymptotic stability under an asynchronous sampling.

Theorem 5. *Consider two positive scalars* $T_1 < T_2$ *and* $\epsilon \in \mathcal{R}$*. If there exist* $P > 0 \in \mathbb{S}^n$*, some matrices* X^l*,* R^l *and* $S_1^l \in \mathbb{S}^n$*,* $S_2^l \in \mathbb{R}^{n\times n}$ *and* $N^l \in \mathbb{R}^{2n\times n}$ *that satisfy for* $l = 1, 2$*:*

$$\Theta_1^l = \Pi_1^l + T_l\Pi_2^l + T_l\Pi_3^l < 0, \quad \Theta_2^l = \begin{bmatrix} \Pi_1^l - T_l\Pi_3^l & T_lN^l \\ * & -T_lR^l \end{bmatrix} < 0, \tag{25}$$

where

$$S_1^2 = \epsilon S_1^1,\ S_2^2 = \epsilon S_2^1,\ N^2 = \epsilon N^1 \tag{26}$$

and Π_1^l*,* Π_2^l *and* Π_3^l *are defined as in Theorem 3. The system* (1) *with the control law* (2) *under asynchronous sampling whose period is lying in* $[T_1,\ T_2]$ *is asymptotically stable.*

Proof. Let k be a positive integer and $T_k \in [T_1,\ T_2]$. The objective is then to find some matrices $X(k)$, $R(k)$ and $S_1(k) \in \mathbb{S}^n$, $S_2(k) \in \mathbb{R}^{n\times n}$ and $N(k) \in \mathbb{R}^{2n\times n}$, such that the LMI conditions from Theorem 2, $\Psi^j(T_k)$ are satisfied for all $T_k \in [T_1,\ T_2]$ and $j = 1, 2$. There exists a scalar $\lambda_k = [0,\ 1]$ such that $T_k = \lambda_k T_1 + (1-\lambda_k)T_2$. Introduce $\Theta_j(k) = \lambda_k\Theta_j^1 + (1-\lambda_k)\Theta_j^2$. If the conditions of Theorem 5 are ensured, then it implies that $\Theta_j(k) < 0$, for $j = 1, 2$ and all k. Define the new variables $U(k) = (\lambda_k + \epsilon(1-\lambda_k))U^1$, where $U(k)$ stands for the matrix variables $S_1(k)$, $S_2(k)$, $N(k)$ and $\tilde{U}(k) = (\lambda_k\tilde{U}^1 + (1-\lambda_k)\tilde{U}^2)/T_k$ where $\tilde{U}(k)$ stands for $R(k)$ and $X(k)$. This choice makes $\Theta_j(k) = \Psi^j(T_k) < 0$ as in Theorem 2, which ensures the asymptotic stability of the closed-loop system under an asynchronous sampling lying in $[T_1,\ T_2]$. □

The next paragraph is concerned by the case of uncertain systems. Consider now that the matrices A and B are assumed to be uncertain lying in a polytope defined by $[A\ B] = \sum_{k=1}^{M}\lambda_i[A_i\ B_i]$, $\sum_{i=1}^{M}\lambda_i = 1$ and for all $i = 1, .., M$, $0 \leqslant \lambda_i \leqslant 1$. Define the associated polytope $[A\ BK] = \sum_{k=1}^{M}\lambda_i[A_i\ B_iK] = \sum_{k=1}^{M}\lambda_i M_{0i}$. The objective is to prove that the previous theorems and corollaries also hold for uncertain systems represented by a polytope as was seen above. Only an extension of Theorem 2 is proposed in the sequel but the same method holds for the other results proposed in this paper.

Theorem 6. *For a given* $\alpha \in \mathbb{R}$ *and a positive scalar* T*, assume that there exist,* $P > 0$*,* $R > 0$*,* S_1 *and* $X_i \in \mathbb{S}^n$ *and two matrices* $S_2 \in \mathbb{R}^{n\times n}$ *and* $N_i \in \mathbb{R}^{2n\times n}$*, where* $i = 1, \ldots, M$ *that satisfy for all* $i = 1, \ldots, M$

$$\Psi_i^1(T) = h_\alpha^1(0)\Pi_{1i} + h_\alpha^2(T,0)\Pi_{2i} + h_\alpha^4(T,0)\Pi_{3i} < 0, \tag{27}$$

$$\Psi_i^2(T) = \begin{bmatrix} h_\alpha^1(T)\Pi_{1i} + h_\alpha^4(T,T)\Pi_{3i} & h_\alpha^3(T,T)N_i \\ * & -h_\alpha^3(T,T)R \end{bmatrix} < 0, \tag{28}$$

where Π_{1i}*,* Π_{2i} *and* Π_{3i} *have the same definition as in* (13) *but with the matrices* N_i*,* X_i *and* M_{0i}*. Then the uncertain system* (1) *with the control law* (2) *is* $\alpha-$*stable for the constant sampling period* T*.*

Theorems	Ex.1	Ex.2	Ex.3
(Fridman et al. 2004)	$[0,\ 0.869]$	$[0,\ ]$	-
(Naghshtabrizi et al. 2008b)	$[0,\ 1.113]$	$[0,\ 1.99]$	-
(Fridman 2010)	$[0,\ 1.695]$	$[0,\ 2.03]$	-
(Liu and Fridman 2009a)	$[0,\ 1.695]$	$[0,\ 2.53]$	-
Th.2*	$[0,\ 1.723]^*$	$[0,\ 2.62]^*$	$[0.201,\ 1.623]^*$
Th.4[1]	$[0,\ 1.720]$	$[0,\ 2.51]$	-
Th.5	$[0,\ 1.721]$ $(\epsilon = 0.8)$	$[0,\ 2.62]$ $(\epsilon = 0.5)$	$[0.210,\ 0.431]$ $(\epsilon = 0.8)$
	$[0.922,\ 1.723]$ $(\epsilon = 0.96)$		$[0.400,\ 1.251]$ $(\epsilon = 0.55)$
			$[1.200,\ 1.578]$ $(\epsilon = 0.76)$

Table 1: Interval of allowable asynchronous samplings for examples 1, 2 and 3.

Proof. Assume that the conditions from Theorem 6 are satisfied. Define the matrices $\bar{\Psi}^j(T) = \sum_{i=1}^M \lambda_i \Psi_i^j(T)$ for $j = 1, 2$. Since Π_{1i} and Π_{3i} are linear with respect to the matrices M_{0i}, it yields $\bar{\Psi}^2(T) = \Psi^2(T)$, where $\Psi_2(T)$ is given in condition (12) with the polytope $M_0 = \sum_{k=1}^M \lambda_i M_{0i}$ and the matrices $X = \sum_{i=1}^M \lambda_i X_i$ and $N = \sum_{i=1}^M \lambda_i N_i$. The case $\bar{\Psi}^1(T)$ is treated similarly. It leads to some additional difficulties due to the term $M_{0i}^T R M_{0i}$. However applying the Schur complement to this term rewritten as $(RM_{0i})^T R^{-1}(RM_{0i})$, one obtains a condition which is linear with respect to M_{0i}. The same method as for $\bar{\Psi}^2$ is thus applied. Then applying the inverse Schur complement, this leads to $\bar{\Psi}^1(T) = \Psi^1(T)$. Since for all $i = 1, \ldots, M$ and $j = 1, 2$, the matrices Ψ_i^j are negative definite, and since the parameters λ_i satisfy a convex property, this leads to $\Psi^j(T) < 0$. Thus, according to Theorem 2, the solutions of the closed loop systems are $\alpha-$stable for the periodic sampling period T. □

As all the previous stability criteria are expressed with the same matrices Π_1, Π_2 and Π_3, extensions of all the theorems presented in this article to uncertain systems can also be derived.

2.4 Examples

Consider system (1) with

Ex.1(Fridman et al. 2004, Naghshtabrizi et al. 2008b):

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -0.1 \end{bmatrix}, \qquad BK = \begin{bmatrix} 0 & 0 \\ -0.375 & -1.15 \end{bmatrix},$$

Ex.2(Fridman 2010):

$$A = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix}, \qquad BK = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix},$$

Ex.3(Gu et al. 2003, Michiels, Niculescu and Moreau 2004):

$$A = \begin{bmatrix} 0 & 1 \\ -2 & 0.1 \end{bmatrix}, \qquad BK = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},$$

Ex.4(Fridman et al. 2004, Naghshtabrizi et al. 2008b):

$$A = \begin{bmatrix} 1 & 0.5 \\ g_1 & -1 \end{bmatrix}, \qquad BK = \begin{bmatrix} 1 + g_2 \\ -1 \end{bmatrix} \begin{bmatrix} -2.688 \\ -0.664 \end{bmatrix}^T;$$

where $|g_1| \leqslant 0.1$, and $|g_2| \leqslant 0.3$.

Table 1 summarizes the results obtained in the literature and using the theorems provided in the present paper for examples $1, 2$ and 3. For Theorem (2), the notation '*' highlights the fact that the results only hold for the asymptotic stability (i.e. $\alpha = 0$) of the system under constant sampling periods. Concerning Theorem 4, the notation '¹' refers to the case $\alpha = 0$. It can be seen that the results from Theorem 2 and 5 are less conservative than the one from the literature for all examples.

Consider Example 1 with two different sampling periods. The smallest period T_1 is fixed equal to 1. The objective is to obtain the greatest T_2 (greater than 1.72) such that the sequence of sampling $\{T_1, T_2\}$ leads to the stability of the solutions. Solving the conditions from Theorem 3, the maximum sampling period T_2 obtained solving Theorem 3 is $T_2 = 2.01$ with the exponential rates $\alpha_1 = 0.4121$, $\alpha_2 = -0.204$. From Theorem 2 (see Table 1), the system is not stable if the sampling is periodic of period $T_2 = 2.01$. However Corollary 1 leads to $c' = 2.10^{-5}$, which ensures the stability of the system with one stabilizing sampling period and an unstable one. In (Li et al. 2009), the authors obtain less conservative result based on a discrete-time approach. They prove the system can be stable, for instance, when $T_1 = 1$ and $T_2 = 2.5$. In simulations with $T_1 = 1$, one can see that this system remains stable for $T_2 \leqslant 6.2$. Even if Theorem 3 does not ensure the stability of the system under such samplings, it is still interesting since it can deal with systems subject to parameter uncertainties as it will be shown in the sequel.

Consider Example 1 with a periodic sampling T. Assume now that the transmission of the control input is subject to packet losses. Packet losses means that some control values which should be implemented in the actuator are lost in the transmission and thus not employed. The assume that the previous control value holds until a new control packet is received. The loss is associated to a probability p which satisfies (20). In other words, there is a probability $1 - p$ that a packet is implemented in the actuator and a probability p to have a loss. If the constant sampling period is T, the probability to implement a control values during rT, where r is a positive integer is $p^{r-1}(1 - p)$. To avoid a too large number of packet losses, we assume not more than r_m successive losses are permitted. Consider the same example with the sampling periods $T = 0.8,\ 1$ and 1.5s and $r_m = 6$. We want to find the maximal probability p for which the system is still stable. Applying Theorem 4 to this case we obtain $p = 0.421,\ 0.378$ and 0.130 respectively. This confirms that the probability to have a loss is decreasing when the sampling period is increasing. Note that in the literature, most of the result dealing with packet loss ensures that the worst case (i.e. with the sampling period r_mT) is stable. Here it is clear that the worst case is not stable by it self. We only ensure that the contribution of the largest sampling periods to instability is less important than the contribution of the smallest periods to stability.

Example 3 is well known in the time-delay system theory (Gu et al. 2003, Michiels et al. 2004, Seuret, Edwards, Spurgeon and Fridman 2007) because of its particular behavior. For a small delay (or sampling period) the system is unstable. It is easy to see that one of the eigenvalue of the matrix $A + BK$ is positive. However when the sampling period becomes sufficient large, the solutions of the closed loop system are stable. The results obtained solving Theorems 2, 4 (with $\alpha = 0$) and 5 are presented in Table 1. Note that until now, no stability conditions based on the continuous-time modelling for such sampled-data systems have been provided in the literature.

Concerning Example 4, in (Fridman et al. 2004), (Naghshtabrizi et al. 2008b) and (Seuret 2009), it was respectively proven that this system is stable for any time-varying sampling periods smaller than $0.35s$, $0.4476s$, $0.602s$. Theorem 4 adapted to the case of uncertain systems ensures that the closed-loop system is stable for all time-varying sampling period less than $0.827s$. It is clear that the stability conditions are less conservative than the others.

Figure 4 shows the relation between the maximal exponential rate α and the sampling period T based on Theorems 2 and 4 for examples 1,2 and 3. It shows that the conditions from Theorem 2 provides a light improvement of those from Theorem 4, especially for negative values of α. A relevant comment has to do with the stability conditions obtained by Theorem 2 for examples 2 and 3 in Figure 4(b) and (c). According

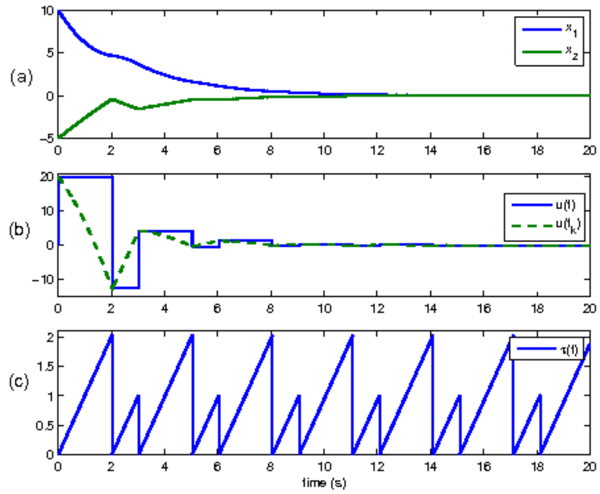


Figure 3: Simulation of the state (a) and the input (continuous and sampled) (b) variables. The interval between two sampling instants is time-varying and equal to $T_1 = 1$ or $T_2 = 2.01$ and such that it follows the sequence $\{T_2, T_1, T_2, T_1, \dots\}$. The sampling delay $\tau(t)$ is showed in (c).

to Theorem 2, the greatest convergence rates for this two examples is not achieved when the sampling period is zero. This means that the solutions of the sampled-data systems are converging more quickly to the equilibrium when a sampling is employed in the controller compared to a continuous controller.

Figure 3 shows a simulation of the states, input (continuous and sampled) and the sampling delay $\tau(t)$ for the initial conditions are $x_0 = [10 - 5]^T$. The sampling is chosen of the following sequence which satisfies $p_1 = p_2 = 0.5$ and of the form $\{T_2, T_1, T_2, T_1, \dots\}$. Then, the delay is given by $t - t_k$ where $t_0 = 0$, $t_1 = T_2$, $t_2 = T_1 + T_2$, $t_3 = T_1 + 2T_2$...

2.5 Extension to consider transmission delays

Delays appears unavoidably in the communication channel. Then it is an important issue to include robustness properties of a controller with respect to both asynchronous samplings and transmission delays. In this section, we present some on going works which concerns the asymptotic stability of such systems.

Consider the linear system with a sampled and delayed input as shown in Figure 5:

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{29}$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$ represent the state variable and the input vector. The matrices A and B are constant and of appropriate dimension. As in the situation of networked control systems, the control input u is affected by the networked communication. In this paper it is only assumed that the network induces a time-varying transmission delay h and a sampling of the transmitted signal. We are looking for a piecewise-constant static state-feedback control law:

$$u(t) = Kx(s_k),\ s_k + h(s_k) \leqslant t < s_{k+1} + h(s_{k+1})$$

where $0 = s_0 < s_1 < ... < s_k < ...$ represent the sampling instants. The sequence of $\{s_k\}_{k \geqslant 0}$ is strictly increasing and goes to infinity as k increases. The transmission delay $h(t)$ is assumed to be constant or time-varying and such that

$$\forall t, \quad h(t) \in [h_1\ h_2], \quad \epsilon_1 \leqslant \dot{h}(t) \leqslant \epsilon_2 < 1 \tag{30}$$

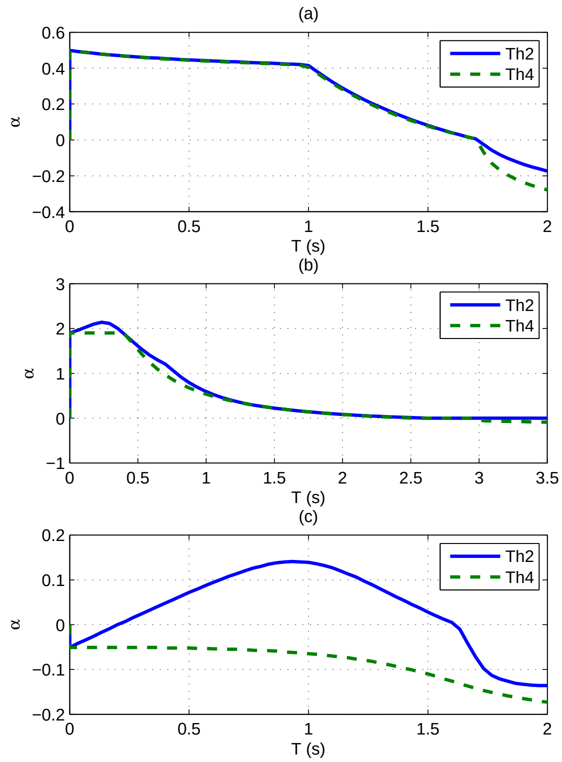


Figure 4: Relation between the exponential decay rate α and the sampling period T with the cases of constant and time-varying periods for Examples 1, (a), 2, (b) and 3, (c).

where $0 \geqslant h_1 < h_2$ and $\epsilon_1 < \epsilon_2$. In order to simplify the notation, $h_k = h(s_k)$ is introduced. Denote $t_k = s_k + h_k$. These instants t_k represent the instants where the control input is updated. We are looking for a piecewise-constant control law of the form $u(t) = u_d(t_k - h_k),\ t_k \leqslant t < t_{k+1}$, where u_d is a discrete-time control signal and . Our objective is to ensure the stability of the system together with a state-feedback controller of the form

$$u(t) = Kx(t_k - h_k),\ t_k \leqslant t < t_{k+1}. \tag{31}$$

where the gain K in $\mathbb{R}^{n\times m}$ is given. Assume that there exists a positive scalar T such that the difference between two successive sampling instants $T_k = s_{k+1} - s_k$ satisfies

$$\forall k \geqslant 0, \quad 0 \leqslant T_1 \leqslant T_k \leqslant T_2. \tag{32}$$

Then the sampling instants t_k of the control input satisfies

$$\bar{T}_k = t_{k+1} - t_k = s_{k+1} - s_k + h_{k+1} - h_k.$$

In order to keep a chronological order of the control values, the value $\bar{T}_k$ is necessary positive, which leads to the condition

$$\forall k \geqslant 0, \quad T_k > h_k - h_{k+1}.$$

Several authors investigated in guaranteeing the stability of such systems. In (Fridman et al. 2004), a continuous-time approach to model sampled-data systems was developed. It allows assimilating sampling effects as the ones of a particular delay. An example of such delays is presented in Figure 6.

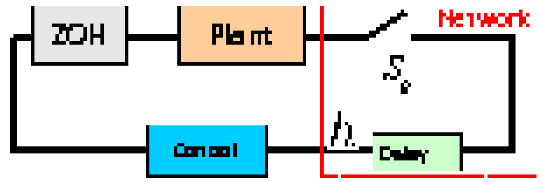


Figure 5: Control loop of Networked control systems under transmission and sampling delays

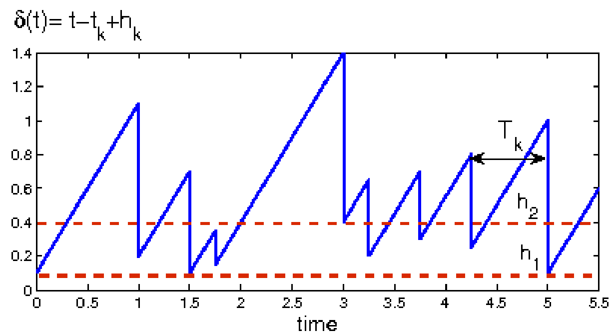


Figure 6: Examples of a delay generated by a transmission delay h_k bounded by h_1 and h_2 and an asynchronous sampling of periods T_k

In (Fridman et al. 2004) or (Millán, Orihuela, Vivas and Rubio 2009), the authors propose an aggregated delay formulation. They develop stability criteria which take into account the delay δ. However they did not consider the different natures of the transmission and the sampling delays. More especially the additional characteristic of sampled delay which is $\dot{\tau} = 1$ has not been included and thus leads to conservative conditions.

In this paper, the aggregated delay δ representing the effect of the transmission and the sampling delays is split into two parts. The main idea is to consider separately the two types delays. This allows having δ greater than the upper bound h_2. A novel method to assess stability of systems subject to this type of delay is proposed. The present article establishes a novel approach to cope with the stability analysis of continuous-time systems under delayed and sampled inputs. This method is based on the discrete-time Lyapunov Theorem and leads to less conservative necessary conditions. Those conditions concern a class of functionals which are not required to be of the Lyapunov-Krasovskii type.

The on going work consists in developing a theorem guaranteeing the stability of such systems (including input delays and sampling) similar to Theorem 1. This leads to additional difficulties and requires a more accurate attention since time-delay systems are infinite dimensional systems. The main issue concerns here the use of functionals for the discrete-time systems instead of the quadratic Lyapunov function employed in Theorem 1.

2.6 Conclusion

In this section, an novel analysis of linear systems under asynchronous sampling is provided. This approach is based on the discrete-time Lyapunov Theorem applied to the continuous-time model of the sampled-data systems. Tractable conditions are derived to ensure asymptotic stability and also to obtained an estimate of the exponential rate of the solutions. The examples show the efficiency of the method and the reduction of the conservatism compared to other results from the literature. Moreover the article addresses the stability analysis of systems under several sampling periods. We show that a sampled-data system can be stable even if one of the sampling period leads to instability. This has been treated by a continuous-time approach and allows considering uncertain or time varying systems. The second part of the section exposes the potential extensions of this method including transmission delays in the control loop.

3 Linear Parameter Varying varying sampling control

The processing load induced by a real-time control task is defined by $U = \frac{c}{h}$ where c and h are the execution time and period of the task. The impact of control messages over a network occupied bandwidth can be evaluated similarly by $B = \frac{d}{h}$ where d is the message size. Reducing the computing cost of a control task can be done in two ways:

- Increasing the sampling interval h can be easily done in real-time, providing that the control interval remains high enough to preserve the system's stability and that interval switching is also handled from the stability viewpoint;
- Reducing its execution time c, e.g. by using a simplified version of the control algorithm: this requires a control tasks switching, which may be complex to be safely performed in real-time.

Therefore, as stated in (Simon, Seuret, Hokayem, Lygeros and Camacho 2009), the key actuator to be used for CPU utilization or network bandwidth control is the control interval. A first idea is to design a bank of controllers, each of them being designed and tuned for a specific sampling frequency, and to switch between them according to the decisions of the feedback scheduler. However, it has been observed that switching without caution between such controllers may lead to instability although each controller in isolation is stable (Schinkel, Chen and Rantzer 2002).

To ensure the system's stability for a measured variable input delay, (Sala 2005) proposes for example a gain scheduled approach based on time-varying observers and state feedback controllers, synthesized using linear matrix inequalities (LMI) and quadratic Lyapunov functions. Indeed, gain scheduling is a popular approach to control non-linear plants, allowing some extent to re-use the well-known linear control theory and tools (Leith and Leithead 2000). The model of the plant is linearized around a set of operating points, and the control responsibility is switched to the controller whose specification is the closest to the actual operating conditions. LPV (Linear Parameter Varying) approaches were then developed to enforce the overall control system stability at switching time.

In (Robert, Sename and Simon 2010) the LPV polytopic approach is used to design a control law with adaptation of the sampling period to account for the available computing resources for an inverted pendulum. Indeed the main drawback of the polytopic method could be the large number of LMIs to solve as the number of varying parameters increases. This is not the case in the Linear Fractional Transform (LFT) method and, as emphasized in (Apkarian and Gahinet 1995), it leads to a LMI problem whose solution can directly be implemented. Moreover, this approach allows to consider in the same way varying parameters and uncertainties (adaptation to the parameters and robustness with respect to uncertainties).

3.1 Discrete time model with varying sampling period

The first point is the problem formulation so that it can be solved following the Linear Parameters Varying (LPV) design of (Apkarian, Gahinet and Becker 1995). Remember that this design ensures the stability and performance robustness of the closed loop parameter-varying system whatever the variations of the parameters inside their predefined allowed range. A state space representation of a continuous linear time invariant (LTI) plant is:

$$G : \begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases} \tag{33}$$

The exact discretization of this system with a zero order hold at the sampling period h can be computed, e.g. see (Åström and Wittenmark 1997), leading to the discrete-time LPV system described in equation (35):

$$G_d : \begin{cases} x_{k+1} = A_d x_k + B_d u_k \\ y_k = C_d x_k + D_d u_k \end{cases} \tag{34}$$

with $A_d \in \mathbb{R}^{ns*ns}$ and

$$\begin{aligned} A_d &= e^{Ah} \quad B_d = \int_0^h e^{A\tau} d\tau B \\ C_d &= C \qquad D_d = D \end{aligned} \tag{35}$$

with h ranging in $[h_{min}; h_{max}]$. The corresponding sampling and hold scheme is depicted by Figure 7: the control signal $U[t(k)]$ computed at the k^{th} instant from measure $X[t(k)]$ is hold until instant $t(k+1)$, which is known and given by a controlled scheduler, e.g. by a feedback scheduler as implemented in (Simon, Robert and Sename 2005), or by a scheduling policy as used for communication between AUVs cooperating inside a swarm (Opderbecke 2009).

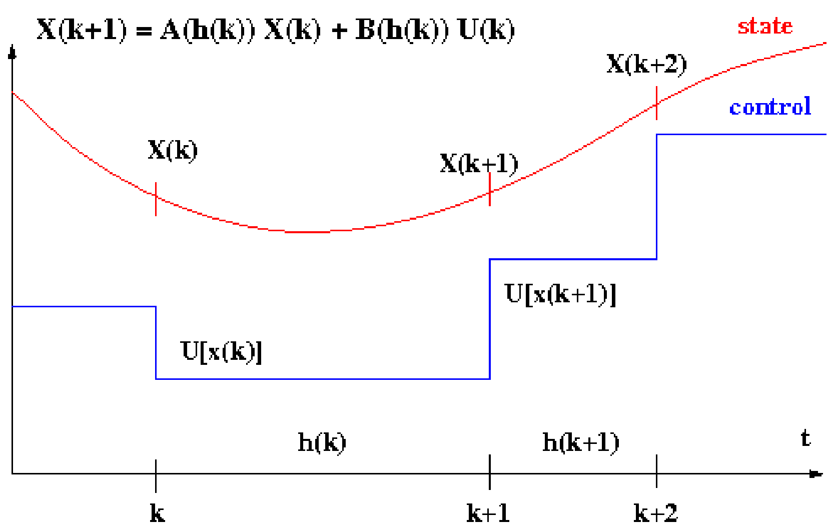


Figure 7: Scheduled varying sampling scheme

The usual numerical method uses the exponential of a matrix M:

$$\begin{pmatrix} A_d(h) & B_d(h) \\ 0 & I \end{pmatrix} = exp\left(\begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix} h\right) \tag{36}$$

The sampling period is assumed to belong to the interval $[h_{min}, h_{max}]$ with $h_{min} > 0$, the sampling period is approximated around the nominal value h_0 as:

$$h = h_0 + \delta \qquad \text{with} \qquad h_{min} - h_0 \leqslant \delta \leqslant h_{max} - h_0 \tag{37}$$

As explained in (Robert et al. 2010) the equation (36) in this case becomes:

$$\begin{pmatrix} A_d(h) & B_d(h) \\ 0 & I \end{pmatrix} = \begin{pmatrix} A_{h_0} & B_{h_0} \\ 0 & I \end{pmatrix} \begin{pmatrix} A_\delta & B_\delta \\ 0 & I \end{pmatrix} \tag{38}$$

3.1.1 Approximation using a Taylor's expansion

For real-time implementation purpose it may be desirable to reduce the computing cost of discretized model, and to use approximations rather than the exact matrix exponential computation. Also for some control design approaches it may be needed to have an affine model of the plant w.r.t. the variations of the sampling period δ. A Taylor series expansion of order k leads to:

$$A_\delta \approx I + \sum_{i=1}^{k} \frac{A^i}{i!}\, \delta^i \tag{39}$$

$$B_\delta \approx \sum_{i=1}^{k} \frac{A^{i-1}B}{i!}\, \delta^i \tag{40}$$

To evaluate the approximation error due to the Taylor approximation, a criterion based on the H_∞ norm is chosen here to express the worst case error between G_{d_e} and G_d, both discrete-time models using respectively the exact method and the approximated one (i.e. the Taylor series approximation of order N).

$$J_N = \max_{h_{min}<h<h_{max}} \parallel G_{d_e}(h,z) - G_d(h,z) \parallel_\infty \tag{41}$$

Taking as example a linear second order system, figure 8 shows the criterion (41) evaluated for different sampling periods ($h \in [40, 120]ms$) and different orders of the Taylor expansions ($k \in [1, 5]$). It shows that this error may be large only if the order 1 is used.

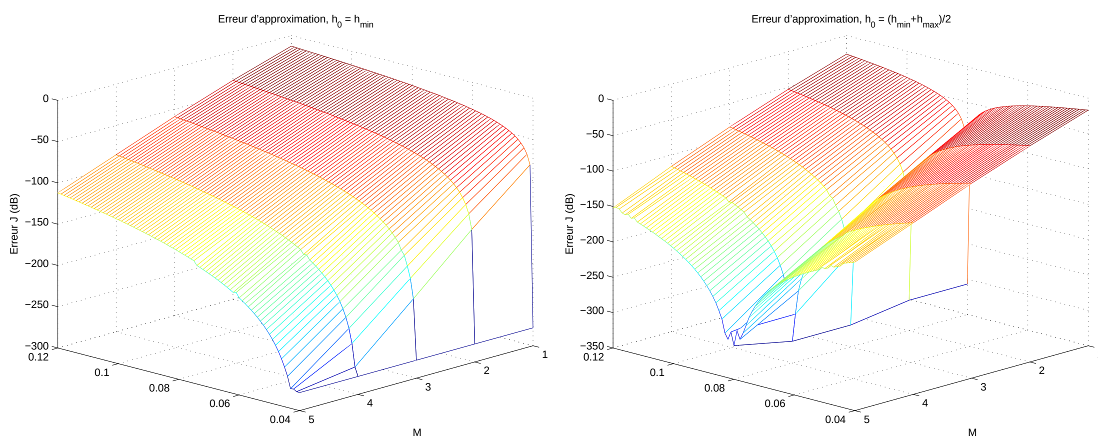


Figure 8: Approximation error: left) $h_0 = h_{min}$ – right) $h_0 = (h_{min} + h_{max})/2$

Finally the matrices of the discretized plant are:

$$\begin{aligned} A_d &= A_{h_0} A_\delta \\ B_d &= B_{h_0} + A_{h_0} B_\delta \end{aligned} \tag{42}$$

with A_d and B_d in an exact or approximate form, depending on the context.

3.2 Preliminary results recalls: the Polytopic approach

Here a parametrized discretization of the continuous time plant and of weighting functions leads to a discrete-time, sampling period dependent, augmented plant. In particular, the plant discretization approximates the exponential matrix appearing in the discretized model by a N^{th} order Taylor series. The original LPV design builds a discrete-time sampling period dependent controller through the convex combination of 2^N controllers, which may be conservative and complex to implement. In the particular case where the control interval is in only varying parameter, the dependency between the variable parameters, which are the successive powers of the sampling period $h, h^2, ..., h^N$, is used to reduce the number of controllers to be combined down to $N + 1$. This reduction of the polytopic set drastically decreases the conservatism of the original design and makes the solution easier to implement. The approach is described in details in (Robert et al. 2010).

3.2.1 Polytopic model

In order to get a polytopic model, a Taylor series of order N is used to approximate the matrix exponential in (36), equations (39) and (40) lead to:

$$A_d(h) = A_{h_0} A_\delta \quad \text{and} \quad B_d(h) = B_{h_0} + A_{h_0} B_\delta \tag{43}$$

Let us define $H = [\delta, \delta^2, \ldots, \delta^N]$ the vector of parameters, that belongs to a polytope (convex polyhedron) $\mathcal{H}$ with 2^N vertices.

$$\mathcal{H} = \left\{ \sum_{i=1}^{2^N} \alpha_i(\delta)\omega_i : \alpha_i(\delta) \geqslant 0, \sum_{i=1}^{2^N} \alpha_i(\delta) = 1 \right\} \tag{44}$$

$$\{\delta, \delta^2, \ldots, \delta^N\}, \ \delta^i \in \{\delta^i_{min}, \delta^i_{max}\} \tag{45}$$

Each vertex is defined by a vector $\omega_i = [\nu_{i_1}, \nu_{i_2}, \ldots, \nu_{i_N}]$ where ν_{i_j} can take the extrema values $\{\delta^j_{min}, \delta^j_{max}\}$ with $\delta_{min} = h_{min} - h_0$ and $\delta_{max} = h_{max} - h_0$.

Matrices $A_d(\delta)$ and $B_d(\delta)$ are therefore affine in H and given by the polytopic forms:

$$A_d(H) = \sum_{i=1}^{2^N} \alpha_i(\delta) A_{d_i}, \quad B_d(H) = \sum_{i=1}^{2^N} \alpha_i(\delta) B_{d_i}$$

where the matrices at the vertices, i.e. A_{d_i} and B_{d_i}, are obtained by the calculation of $A_d(\delta)$ and $B_d(\delta)$ at each vertex of the polytope $\mathcal{H}$. The polytopic coordinates α_i which represent the position of a particular parameter vector $H(\delta)$ in the polytope $\mathcal{H}$ are given solving:

$$H(\delta) = \sum_{i=1}^{2^N} \alpha_i(\delta)\omega_i \ , \ \ \alpha_i(\delta) \geqslant 0 \ , \ \ \sum_{i=1}^{2^N} \alpha_i(\delta) = 1 \tag{46}$$

As the gain-scheduled controller will be a convex combination of 2^N "vertex" controllers, the choice of the N^{th} order series gives a trade-off between the approximation accuracy and the controller complexity.

To decrease the volume and number of vertices of the matrices polytope the dependency between the successive powers of the parameter h is exploited. Remember that the vertices ω_i of $\mathcal{H}$ are defined by $h, h^2, \ldots, h^N$ with $h^i \in \{h^i_{min}, h^i_{max}\}$. Indeed, the representative point of the parameters set is constrained to be on a one dimensional curve, so that the polytope of interest can be reduced to the “lower” $N + 1$ vertices, as illustrated in figure 9 for the cases N = 2 and 3.

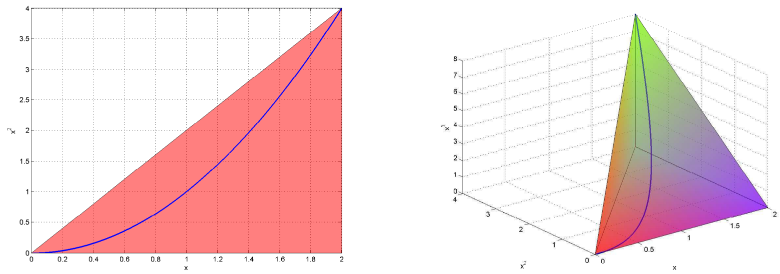


Figure 9: Polytope reduction for N=2 and N=3

3.2.2 Performance specification

In the H_∞ framework, the general control configuration of figure 10 is considered, where W_i and W_o are weighting functions specifying closed-loop performances (see (Skogestad and Postlethwaite 1996)). The objective here is to find a controller K so that internal stability is achieved and $\|\tilde{z}\|_2 < \gamma \|\tilde{w}\|_2$, where γ represents the H_∞ attenuation level.

Classical control design assumes constant performance objectives and produces a controller with a unique sampling period. This sampling period is chosen according to the controller bandwidth, the noise sensitivity and the availability of computation resources. When the sampling period varies, the usable controller bandwidth also varies and the closed-loop objectives should logically be adapted ; therefore, the bandwidth of the weighting functions is adapted. In this aim, W_i and W_o are split into two parts:

- a constant part with constant poles and zeros. This allows, for instance, to compensate for oscillations or flexible modes which are, by definition, independent of the sampling period. This part is merged with the plant before its discretization.
- the variable part contains poles and zeros whose pulsations are expressed as an affine function of the frequency $f = 1/h$, which allow the bandwidth of the weighting functions to be adapted. These poles and zeros are here constrained to be *real* by the discretization step. Finally, opportune cancellations makes the *discretized* templates independent from the sampling period h, facilitating further interconnections.

Indeed preliminary experiences with varying sampling control (Robert, Sename and Simon 2005) pointed out the advantages of performance adaptation w.r.t. to the sampling rate to preserve stability margins and to keep the control size inside wise bounds.

3.2.3 LPV/H_∞ control design

The interconnection between the discrete-time polytopic model of the plant $\tilde{P}$ (now including the constant part of the weighting functions) and the variable weighting functions W_i and W_o leads to the discrete-time LPV augmented plant $P(H)$ is depicted in figure 10.

The H_∞ control design for linear parameter-varying systems detailed in (Apkarian et al. 1995) is used here. The method states that under some mild conditions, there exists a gain-scheduled controller:

$$\begin{cases} x_{K_{k+1}} &= A_K(H)x_{K_k} + B_K(H)y_k \\ u_k &= C_K(H)x_{K_k} + D_K(H)y_k \end{cases} \tag{47}$$

where $x_K \in \mathbb{R}^n$, ensuring overall parameter trajectories, for the closed-loop system:

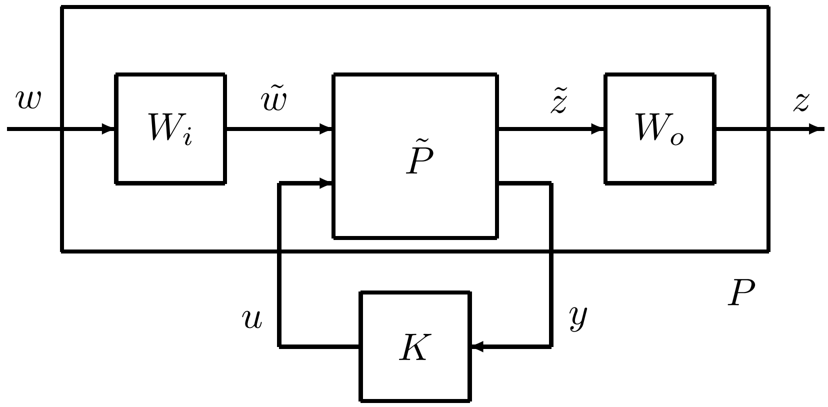


Figure 10: Focused interconnection

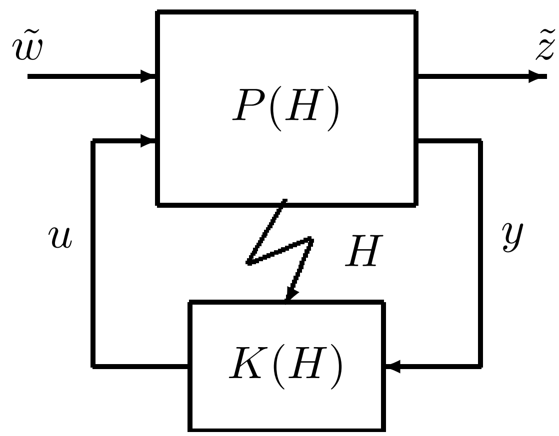


Figure 11: Closed-loop of the LPV system

- closed-loop quadratic stability
- $\mathscr{L}_2$-induced norm of the operator mapping w into z bounded by γ, i.e. $\|z\|_2 < \gamma\|w\|_2$

$N+1$ controllers are reconstructed at each vertex of the parameter polytope (corresponding with the extreme values of the parameters). The gain-scheduled controller $K(H)$ is then the convex combination of these controllers

$$K(H) : \begin{pmatrix} A_K(H) & B_K(H) \\ C_K(H) & D_K(H) \end{pmatrix} = \sum_{i=1}^{r} \alpha_i(h) \begin{pmatrix} A_{K_i} & B_{K_i} \\ C_{K_i} & D_{K_i} \end{pmatrix} \tag{48}$$

$$\text{with } \alpha_i(h) \text{ so that } H = \sum_{i=1}^{r} \alpha_i(h)\omega_i \tag{49}$$

Note that on-line scheduling of the controller needs the computation of $\alpha_i(h)$ knowing h. Considering a Taylor's expansion around h_0 with

$$\delta_{min} = h_{min} - h_0 \text{ and } \delta_{max} = h_{max} - h_0$$

and the case of the reduced polytope, explicit solutions are easily recursively computed using:

$$\begin{cases} \alpha_1 = \frac{\delta_{max}-\delta}{\delta_{max}-\delta_{min}} \\ \alpha_n = \frac{\delta^n_{max}-\delta^n}{\delta^n_{max}-\delta^n_{min}} - \sum_1^{n-1} \alpha_i \quad , \quad n = [2, ..., N] \\ \alpha_{N+1} = 1 - \sum_1^N \alpha_i \end{cases} \tag{50}$$

3.3 Experimental assessment

The latter approach has been experimentally assessed using a "T" inverted pendulum, as extensively described in (Robert 2007).

As such a T pendulum system is difficult to be controlled, our main objective here is to get a closed-loop stable system, to emphasis the practical feasibility of the proposed methodology for real-time control.

The sampling interval is assumed to be in the interval $[1, 3]\ msec$. Note that the sampling rate seems to be very fast compared with the closed-loop desired performance: it appears from previous studies and experiments that such fast sampling is necessary to achieve closed-loop stability for this non-linear device, whatever the control algorithm (Natale, Sename and Canudas de Wit 2004).

After some trials and comparisons (Robert et al. 2010) the control synthesis has been implemented using the reduced polytope model and a Taylor's expansion truncated at the 2^{nd} order. Hence 3 vertex controllers must be combined for every new value of the control interval.

The plant is controlled through Matlab/Simulink using the Real-time Workshop and xPC Target. Two cases are displayed. First, in figure 12 the sampling period variation is continuous and follows a sinusoidal signal of frequency $0.15 rad/s$. The left plot represents simulation results and the right one a real experiment. As the control interval varies continuously, the controller is adapted at each sample: therefore the polytopic coordinates computation 50 and convex combination 48 must be computed previously to the control signal calculation using the state feedback controller 47. Anyway the overall on-line computation remains bounded and simple enough to be easily implemented in real-time.

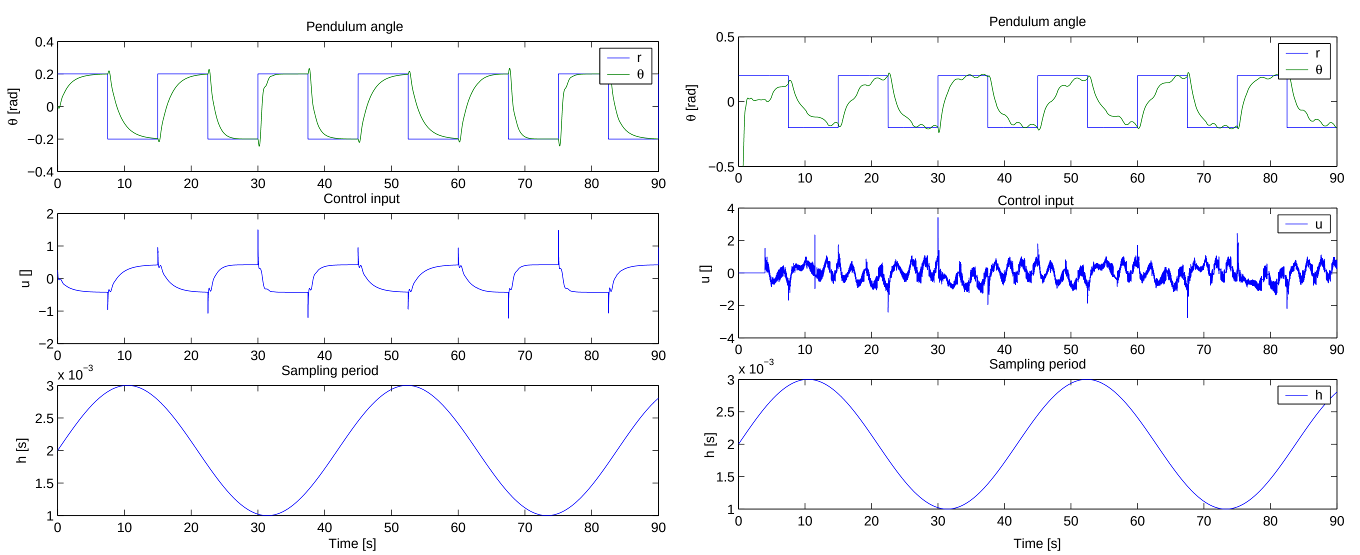


Figure 12: Motions of the T pendulum under a sinusoidal sampling period

The position reference for the pendulum is a square pattern. Note that the settling time varies with the sampling frequency, accordingly to the definition of the variable weighting functions used for the performance specification. On the right part, the experimental data additional spikes and noise which are a consequence of dry friction and elasticity (or stick-slip) in the pendulum actuation. However the experimental plant is still stable and the control signals remain bounded despite these unmodeled mechanical defects whose effect is known to be difficult to compensate for (Olsson, Aström, de Wit, Gäfvert and Lischinsky 1998).

Then, in Figure 13 step changes of the sampling rate are experimented. Note that now the controller's gains computation only need to be performed at the control interval switches, i.e. in this case the on-line overhead of the method compared with a classical H_∞ controller becomes negligible.

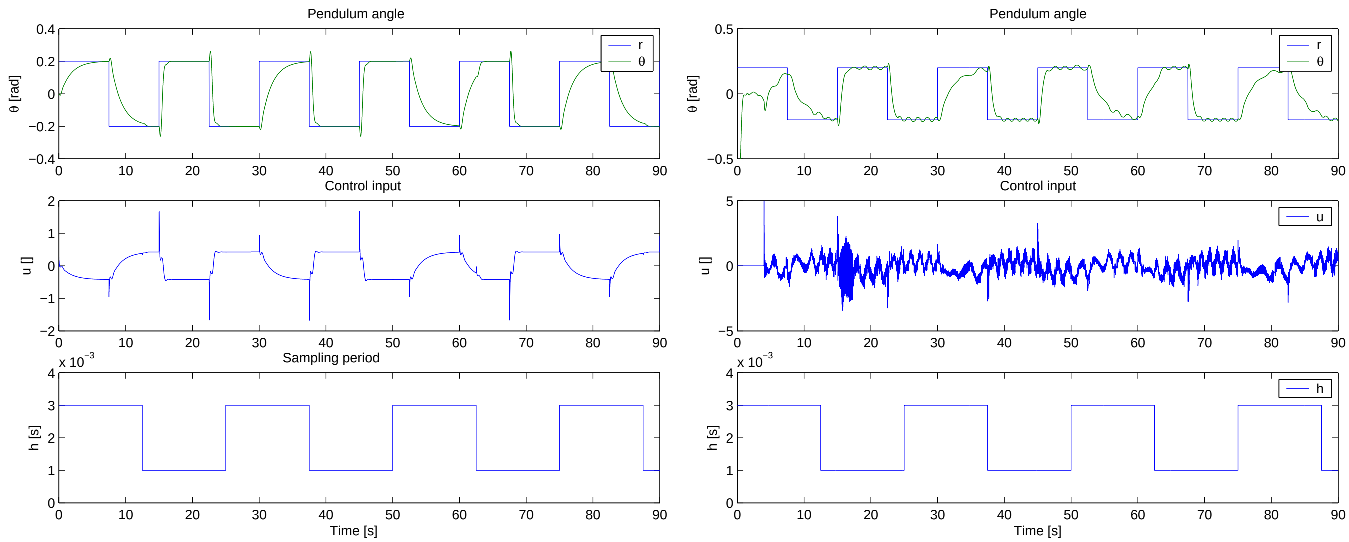


Figure 13: Motions of the T pendulum under a square sampling period

As expected from the sampling dependent performance objectives, the settling time is minimal when the sampling period is maximal, and conversely. There are no abrupt changes in the control signal, even when the sampling period suddenly varies from 1 to 3 ms. Finally similar results are obtained in simulation and experimental tests, which illustrates the inherent robustness property of the H_∞ design.

Indeed few assumptions about sampling have been made for this control design: the main point is that the control interval is known and lies between the predefined bounds $[h_{min}; h_{max}]$, whatever the origin of the control interval variations, its speed and its frequency. Two cases may be considered:

- the control interval is a control variable which can be used by a feedback scheduler to manage the CPU load share, e.g. as in (Aubrun, Simon and Song 2010): in that case the desired control interval is computed by a scheduling controller and sent to the real-time scheduler which manages the control tasks execution ;
- the control interval variations may be due to sensor scheduling, e.g. induced by a communication channel between the sensor and controller nodes: in that case the interval between the successive expected appearance of data at the controller input are delivered by the scheduling policy controller, e.g. a (m, k)-firm scheduler as defined in (Felicioni, Jia, Simonot-Lion and Song 2010).

Some further simulations, still using the same case study with the "T" inverted pendulum and associated LPV/H_∞ based controller, have been made to illustrate the capabilities and robustness of the method. In the simulation depicted by Figure 14 the control interval has been randomly varied at every sample, with values in the set $\{1, 2, 3\}$ ms. Indeed this case mimics data dropping between the sensor and control sites, where the network interface is fed by the sensor at a 1 ms rate, and randomly drops up-to 2 packets out of

3 with a maximum interval of 3 ms between successful transmission. This transmission pattern may be the result of a feedback scheduler using a (1,3)-firm data dropping policy to manage the network bandwidth.

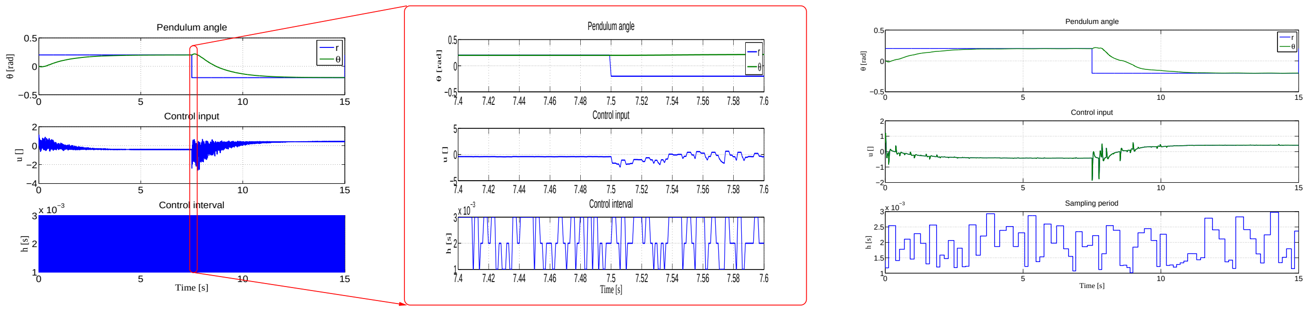


Figure 14: LPV with: i) (1-3)-firm input scheduling ii) arbitrary variations of h

Despite the fast and random changes of the control interval, the system remains stable. The settling time conforms with the performance specification, its actual value lies between the performance templates defined with the variable weighting functions for h_{min} and h_{max}. The control size remains bounded and reasonably small even with the appearance of chattering. Similar simulation results has been obtained when the control interval varies at every sample with random float values in the specified interval.

This LPV/H_∞ based variable sampling control synthesis assumes that the variable parameter, i.e. the control interval in this particular case, is known before the synthesis is performed. Indeed this is true for the desired interval sent to the operating system by a feedback scheduler, or for the scheduled packets incoming interval on the control node, if jitter is small enough and negligible. However, during control processing and communications between nodes, delays appear due to both the control computation on a real CPU and to pre-emption due to higher priority tasks sharing the control node, and to networking between the sensor, control and actuator's node if any. These added delays are not known in real-time when the controller's gains computation starts, as they cannot be accounted in the polytopic parameters computation and in the elementary vertex controllers convex combination.

A possible way to cope with these unmodeled and unmeasured delays is to compute at each controller execution a discrete set of control signals (corresponding with the range of expected latencies), send all the set to the actuator's node and apply the control signal which best fit the actual measured interval at the actuator node. This is for example the approach of (Sala, Cuenca and Salt 2009). A drawback of such an approach is overloading the CPU and network with unused control signals computation and transmission.

In a safely designed and schedulable real-time control system running in nominal conditions, the computing and pre-emption induced latencies are typically smaller than a control period. It is expected that the effect of these quite small latencies can be absorbed by the robustness of the on-board controller. Indeed the fact that the control experiments depicted in Figures 12 and 13 are successful, despite nothing was done to model, identify and compensate for the computation and scheduling latencies, indicates that the proposed control design provides such capabilities.

The simulation plot in Figure 15 further explores the already used case study robustness w.r.t. unmodeled control delays. The control interval is constant and set to 2 ms, and latencies of increasing values up-to 12 ms are added in the control path to simulate unmodeled compound computation and scheduling latencies. It can be observed that the control stability is kept despite quite large added latencies (unstability appears beyond 15 ms delays), while the control quality decreases with the appearance of a classical oscillatory behavior.

Beyond this case study the robustness of this LPV/H_∞ variable sampling control method deserves to be further formally explored, for example using the results exposed in section 2. Anyway, this design method

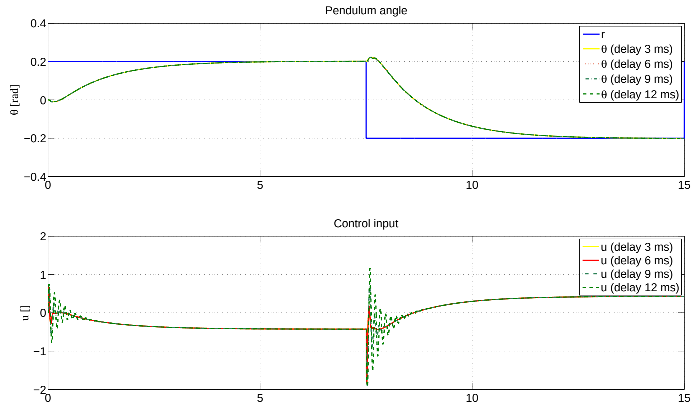


Figure 15: LPV with unmodeled output delay

already appears to be effective to preserve the plant's stability and performance objectives during arbitrarily fast control interval variations.

3.4 A LFT model for varying sampling period systems

In this section a LFT formulation for varying sampling systems is proposed, to express the system's discrete time model by keeping the sampling period as a parameter (i.e. modeled as a system uncertainty).

3.4.1 Background on LFT formulation

The LFT transform is important to represent systems' uncertainties. It allows to add various uncertainties, under frequency or parametric representation, to systems in transfert functions or in state space representations. The following basic definitions and statements of LFT transforms come from (Skogestad and Postlethwaite 1996).

Let P a $(n_1 + n_2) \times (m_1 + m_2)$ matrix, partitioned under the form:

$$P = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix} \tag{51}$$

Let Δ and K be matrices of dimensions $m_1 \times n_1$ and $m_2 \times n_2$. The lower and upper linear fractionnal transforms of P are defined respectively by:

$$F_u(P, \Delta) := P_{22} + P_{21}\Delta(I - P_{11}\Delta)^{-1}P_{12} \tag{52}$$
$$F_l(P, K) := P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21} \tag{53}$$

These transforms are equivalent to the interconnection diagrams depicted by Figure 16. Traditionnally the (a) interconnection is used to represent an uncertain system where Δ models uncertainties, the structure of Δ actually represents the structure of the uncertainties. Conversely (b) shows the interconnection between a process P and a controller K, it is better used for control synthesis purpose.

Relations (52) and (53) are static ones. Let us note that a tranfert function is a LFT:

$$\forall p \in C^*, F_u\left(\begin{pmatrix} A & B \\ C & D \end{pmatrix}, \frac{1}{p}I\right) = D + C(pI - A)^{-1}B := M(p) \tag{54}$$

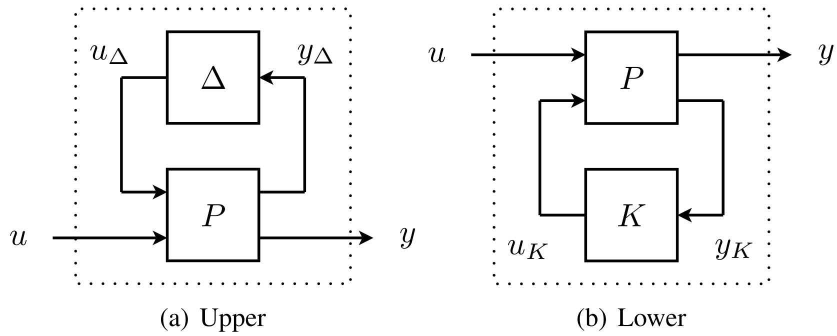


Figure 16: Linear Fractionnal Transforms

Therefore the following relations can be written:

$$F_u\left(M(p),\Delta\right) = F_u\left(F_u(M,\frac{1}{p}I),\Delta\right) \tag{55}$$

Hence Δ and K can be:

- A memoryless linear time invariant (LTI) operator: Δ,
- A LTI operator with memory: $\Delta(p)$,
- A memoryless linear time varying (LTV) operator: $\Delta(t)$.

LFTs can be interconnected with the following properties:

- The sum of n LFT is an LFT.
- The product (cascading) of n LFT is an LFT.
- When it exists, the inverse of an LFT is an LFT.

The LFT has been first used in the framework of LPV systems by (Packard 1994). It is used below to model a varying sampling interval as an uncertainty.

3.4.2 LFT model of a varying sampling system

First, an LPV/LFT model of a system, considering the sampling period as varying parameter is presented, based on the methodology developed in (Robert 2007). The objective is to transform the system under the form described in Figure 17. The matrix Δ represents the uncertainties depending on the sampling period h.

Two cases are considered below in order to account for varying sampling period:

Unstructured uncertainty case From equations (42) two uncertainties blocks are derived:

$$\begin{aligned}\Delta_1 &= A_\delta - I \\ \Delta_2 &= A_{h_0}B_\delta\end{aligned} \tag{56}$$

Matrices of the discrete time system (34) are rewritten with these uncertainties :

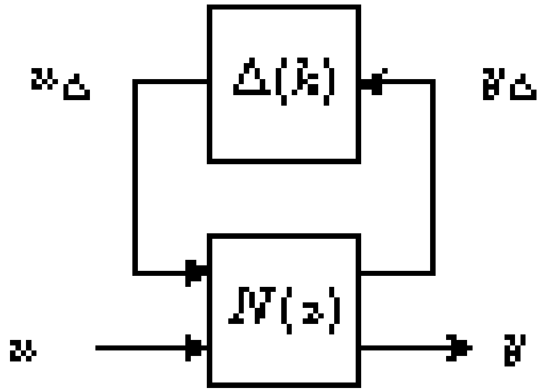


Figure 17: System under LFT representation

$$\begin{cases} x_{k+1} = [A_{h_0} + A_{h_0}\underbrace{(A_\delta - I)}_{\Delta_1}]x_k + [B_{h_0} + \underbrace{A_{h_0}B_\delta}_{\Delta_2}]u \\ y = C_d x_k + D_d u \end{cases}$$

which can be also represented as a transfer:

$$\begin{aligned} G(z) &= D_d + C_d(zI - A_d)^{-1}B_d \\ &= D_d + C_d(zI - A_{h_0}(I + \Delta_1))^{-1}(B_{h_0} + \Delta_2) \end{aligned} \tag{57}$$

$$\begin{aligned} (zI - A_d)^{-1} &= [zI - A_{h_0}(I + \Delta_1)]^{-1} \\ &= [(I - A_{h_0}\Delta_1(zI - A_{h_0})^{-1})\,(zI - A_{h_0})]^{-1} \\ &= (I - A_{h_0}\Delta_1(zI - A_{h_0})^{-1})^{-1}\,(zI - A_{h_0})^{-1} \end{aligned} \tag{58}$$

Following equations (57) and (58), the discretization induced uncertainty on A can be represented by an inverse multiplicative uncertainty, and the one on B discretization by an additive uncertainty as depicted in Figure 18.

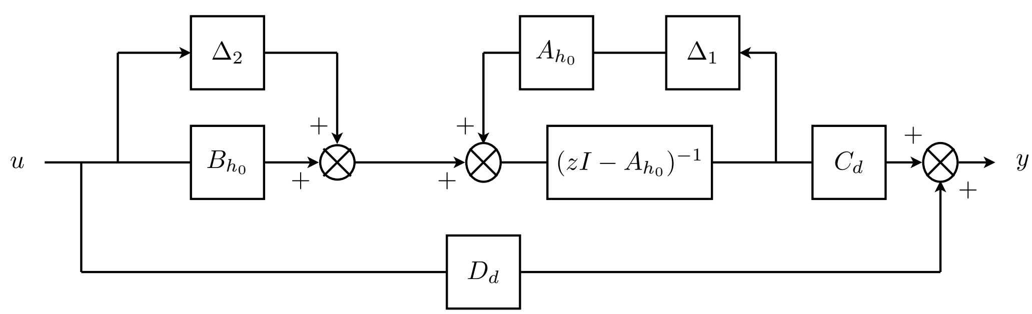


Figure 18: Uncertain representation of the discretization

The system is transformed into the form described in Figure 17 by defining:
$u_\Delta = [u_{\Delta_1},\ u_{\Delta_2}]^T$, $y_\Delta = [y_{\Delta_1},\ y_{\Delta_2}]^T$ and $\Delta = \begin{pmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{pmatrix}$

Finally, the LFT form of the model with unstructured uncertainties (with no approximation) is as follows:

$$\begin{cases} \dot{x}_{k+1} = \mathcal{A}x_k + \mathcal{B}\begin{pmatrix} u_\Delta \\ u \end{pmatrix} \\ \begin{pmatrix} y_\Delta \\ y \end{pmatrix} = \mathcal{C}x_k + \mathcal{D}\begin{pmatrix} u_\Delta \\ u \end{pmatrix} \end{cases} \tag{59}$$

$$\begin{gathered} \Delta = \begin{pmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{pmatrix} \\ \mathcal{A} = A_{h_0} \quad \mathcal{B} = \begin{pmatrix} A_{h_0} & I & B_{h_0} \end{pmatrix} \\ \mathcal{C} = \begin{pmatrix} I \\ 0 \\ C_d \end{pmatrix} \quad \mathcal{D} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & I \\ 0 & 0 & D_d \end{pmatrix} \end{gathered} \tag{60}$$

The LPV/LFT model of the system is then given by the upper LFT interconnection

$$T_{u,y} = F_u(P(z), \Delta) = P_{22} + P_{21}\Delta(I - P_{11}\Delta)^{-1}P_{12} \tag{61}$$

Variant This approach considers Δ as a full block diagonal with full matrices Δ_1 and Δ_2. However matrices Δ_1 and Δ_2 depend on a single parameter δ, so this uncertainty definition may be too conservative.

If the matrix A is not singular, the equation (35) leads to:

$$B_\delta = A^{-1}(A_\delta - I)B \tag{62}$$

and Δ_2 can be written as:

$$\begin{aligned} \Delta_2 &= A_{h_0}B_\delta = A_{h_0}A^{-1}(A_\delta - I)B \\ &= A_{h_0}A^{-1}\Delta_1 B \\ &= W_2\Delta_1 W_1 \end{aligned} \tag{63}$$

Then the LFT model of Figure 17 can be defined as: :

$$\begin{gathered} \Delta = \begin{pmatrix} \Delta_1 & 0 \\ 0 & \Delta_1 \end{pmatrix} \\ \mathcal{A} = A_{h_0} \quad \mathcal{B} = \begin{pmatrix} A_{h_0} & W_2 & B_{h_0} \end{pmatrix} \\ \mathcal{C} = \begin{pmatrix} I \\ 0 \\ C_d \end{pmatrix} \quad \mathcal{D} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & W_1 \\ 0 & 0 & D_d \end{pmatrix} \end{gathered} \tag{64}$$

Remark 1. *To analyse or synthesize a controller from $N\Delta$, the maximum singular value of Δ is used. Here it is defined by $\bar{\sigma}(\Delta) = \max\{\bar{\sigma}(\Delta_1), \bar{\sigma}(\Delta_2)\}$. The variant of eq.* (64) *allows for reducing the conservatism if the difference $\bar{\sigma}(\Delta_1), \bar{\sigma}(\Delta_2)$ is high.*

Structured uncertainty case In order to reduce the conservatism the Taylor series expansion of order k (presented in (39) and (40)) for matrices A_δ and B_δ is used to obtain a structured uncertainty matrix.

$$\begin{aligned}\Delta_1 &\approx A_\delta - I = \sum_{i=1}^{k} \frac{A^i}{i!}\,\delta^i \\ &\approx A\delta\,(I + \frac{A\delta}{2}\,(I + \frac{A\delta}{3}\,(I + \dots))\end{aligned} \tag{65}$$

$$\begin{aligned}\Delta_2 &\approx A_{h_0} B\,\delta = A_{h_0} \sum_{i=1}^{k} \frac{A^i B}{i!}\,\delta^i \\ &\approx A_{h_0}\,(I + \frac{A\delta}{2}\,(I + \frac{A\delta}{3}\,(I + \dots) B\,\delta\end{aligned} \tag{66}$$

Figures 19 and 20 show the corresponding interconnection for a Taylor's expansion truncated at order 3.

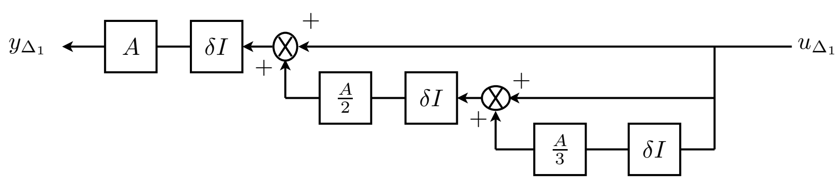


Figure 19: Approximation of Δ_1 for $k = 3$

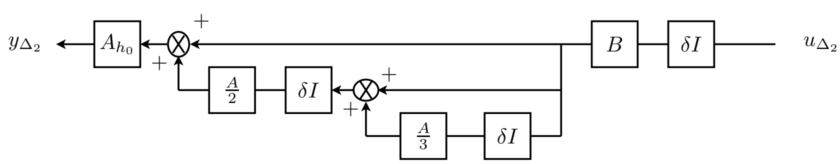


Figure 20: Approximation of Δ_2 for $k = 3$

With this structure of uncertainty, the LFT representation of the system becomes:

$$\begin{gathered}\Delta = \delta I_{2\times k\times n_s} \\ \mathcal{A} = A_{h_0} \quad \mathcal{B} = \begin{pmatrix}\mathcal{B}_1 & \mathcal{B}_2 & B_{h_0}\end{pmatrix}\end{gathered} \tag{67}$$

$$\mathcal{C} = \begin{pmatrix}\mathcal{C}_1 \\ \mathcal{C}_2 \\ C_d\end{pmatrix} \quad \mathcal{D} = \begin{pmatrix}\bar{\mathcal{D}} & 0 & 0 \\ 0 & \bar{\mathcal{D}} & \mathcal{D}_{2u} \\ 0 & 0 & D_d\end{pmatrix} \tag{68}$$

$$\mathcal{B}_1 = \begin{pmatrix}A_a A & 0_{n_s*[n_s*(k-1)]}\end{pmatrix} \quad \mathcal{B}_2 = \begin{pmatrix}A_a & 0_{n_s*[n_s*(k-1)]}\end{pmatrix} \tag{69}$$

$$\mathcal{C}_1 = \left.\begin{pmatrix}I_{n_s} \\ \vdots \\ \vdots \\ I_{n_s}\end{pmatrix}\right\}\, k\ times\ ; \quad \mathcal{C}_2 = \left.\begin{pmatrix}0_{n_s} \\ \vdots \\ \vdots \\ 0_{n_s}\end{pmatrix}\right\}\, k\ times \tag{70}$$

$$\bar{\mathcal{D}} = \begin{pmatrix} 0_{n_s} & \frac{A}{2} & \dots & 0_{n_s} \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \frac{A}{k} \\ 0_{n_s} & \dots & \dots & 0_{n_s} \end{pmatrix} \quad \mathcal{D}_{2u} = \begin{pmatrix} B \\ \vdots \\ \vdots \\ B \end{pmatrix} \tag{71}$$

3.4.3 Recall on $\mathcal{H}_\infty$ / LFT control design

The objective is to get an LPV controller taking into account the set of parameter Δ in the same way as the model, by a lower LFT interconnection:

$$T_{zw} = F_l(K, \Delta) := K_{11} + K_{12}K(I - K_{22}\Delta)^{-1}K_{21} \tag{72}$$

The LFT control scheme is presented in Figure 21, where the dependency on the parameters Δ of the generalized plant P and of the controller K is described in an LFT form.

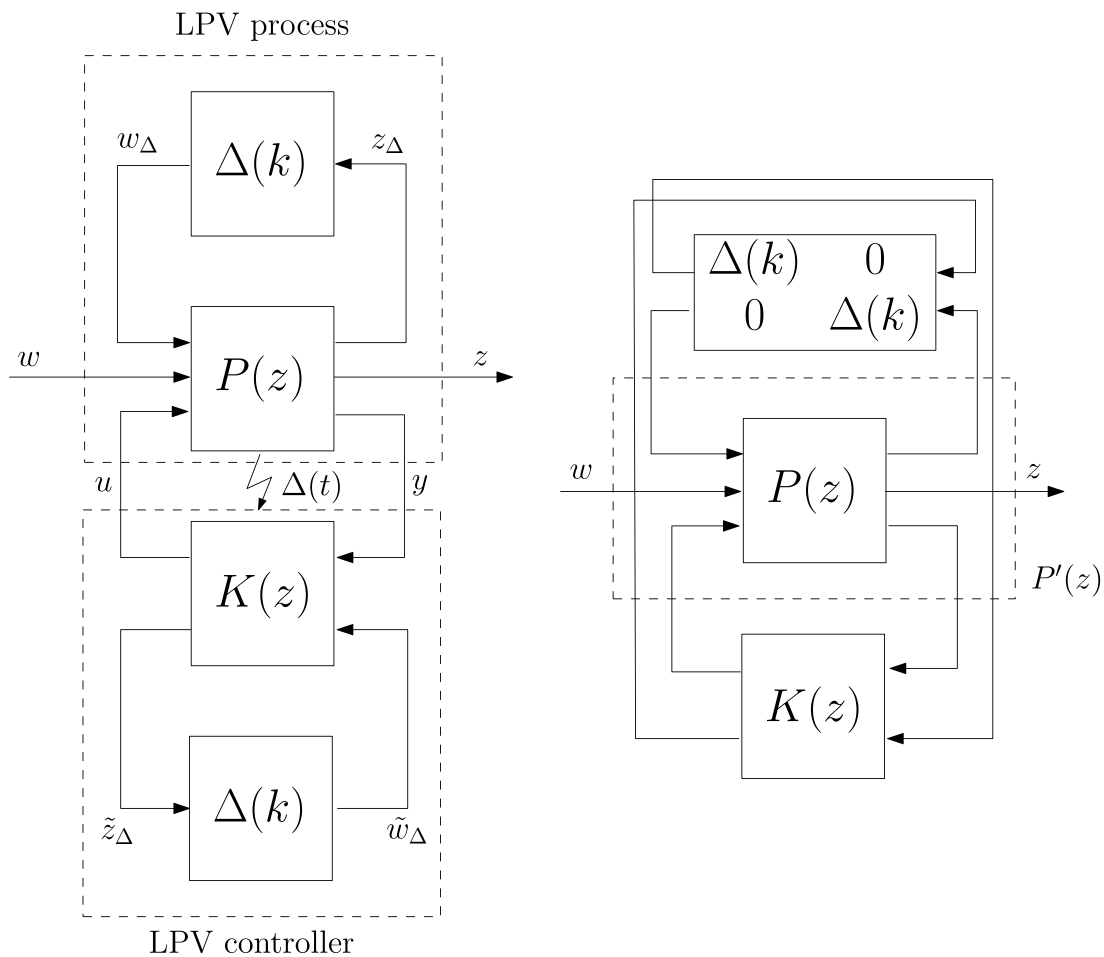


Figure 21: LFT control scheme

The problem is now to find a discrete controller K for which the system described in Figure 21 is stable and respects some specifications defined by weighting functions set in the control loop ($\mathcal{H}_\infty$ design).

The methodology used for the discrete-time controller synthesis is based on the $\mathcal{H}_\infty$ framework, by the resolution of LMIs (derived from the bounded real Lemma), as described in (Apkarian and Gahinet 1995) and (Gauthier, Sename, Dugard and Meissonnier 2007).

To reduce the conservatism due to the presence of uncertainties, the problem is reformulated using some scaling variables L_Δ.

The solution of this problem relies on the bounded real Lemma, as follow:

Lemma 3. Scaled bounded real lemma: *Consider a parameter structure Θ, the associated scaling set $L_\Theta = \{L > 0 : L\Delta = \Delta L, \forall \Delta \in \Theta\}$, and a discrete-time square transfer function $T(z)$ of realization $T(z) = D_{cl} + C_{cl}(zI - A_{cl})^{-1} B_{cl}$, then the following statements are equivalent.*

i) A_{cl} is stable and there exists $L \in L_\Theta$ such that $||L^{1/2}\left(D_{cl} + C_{cl}(zI - A_{cl})^{-1} B_{cl}\right) L^{-1/2}||_\infty < 1$.

ii) There exist positive definite solutions X_{cl} and $L \in L_\Theta$ to the matrix inequality

$$\begin{pmatrix} -X_{cl}^{-1} & A_{cl} & B_{cl} & 0 \\ A_{cl}^T & -X_{cl} & 0 & C_{cl}^T \\ B_{cl}^T & 0 & -L & D_{cl}^T \\ 0 & C_{cl} & D_{cl} & -L^{-1} \end{pmatrix} < 0 \tag{73}$$

The main result is as follows.

Theorem 7. *Consider a discrete-time LPV plant $P_\Delta(z)$ under LFT form. Let $\mathcal{N}_r$ and $\mathcal{N}_s$ denote bases of null spaces of $(B_2^T, D_{\theta 2}^T, D_{12}^T, 0)$ and $(C_2, D_{2\theta}, D_{21}, 0)$, respectively.*

With this notation, the gain-scheduled $\mathcal{H}_\infty$ LFT control problem is solvable if and only if there exist pairs of symmetric matrices $(R, S) \in \mathbb{R}^{n_a \times n_a}$ and $(L_3, J_3) \in \mathbb{R}^{n_\theta \times n_\theta}$ and a scalar $\gamma > 0$ such that

$$\mathcal{N}_r^T \begin{pmatrix} ARA^T - R + B_\theta J_3 B_\theta^T & \star & \star & \star \\ C_\theta R A^T + D_{\theta\theta} J_3 B_\theta^T & C_\theta R C_\theta^T + D_{\theta\theta} J_3 D_{\theta\theta}^T - J_3 & \star & \star \\ C_1 R A^T + D_{1\theta} J_3 B_\theta^T & C_1 R C_\theta^T + D_{1\theta} J_3 D_{\theta\theta}^T & C_1 R C_1^T + D_{1\theta} J_3 D_{1\theta}^T - \gamma I & \star \\ B_1^T & D_{\theta 1}^T & D_{11}^T & -\gamma I \end{pmatrix} \mathcal{N}_r < 0$$

$$\mathcal{N}_s \begin{pmatrix} A^T S A - S + C_\theta^T L_3 C_\theta & \star & \star & \star \\ B_\theta^T S A + D_{\theta\theta}^T L_3 C_\theta & B_\theta^T S B_\theta + D_{\theta\theta}^T L_3 D_{\theta\theta} - L_3 & \star & \star \\ B_1^T S A + D_{\theta 1}^T L_3 C_\theta & B_1^T S B_\theta + D_{\theta 1}^T L_3 D_{\theta\theta} & B_1^T S B_1 + D_{\theta 1}^T L_3 D_{\theta 1} - \gamma I & \star \\ C_1 & D_{1\theta} & D_{11} & -\gamma I \end{pmatrix} \mathcal{N}_s < 0$$

$$\begin{pmatrix} R & I \\ I & S \end{pmatrix} > 0$$

$$L_3\Theta = \Theta L_3,\ J_3\Theta = \Theta J_3,\ \begin{pmatrix} L_3 & I \\ I & J_3 \end{pmatrix} > 0$$

3.5 Application to AUV control

The $\mathcal{H}_\infty$ approach for LPV systems is applied to an Autonomous Underwater Vehicle (AUV) for altitude control. The use of AUV for the exploration of the seabed, and the control of these vehicles has been of large interest for researcher in the past two decades. Many different control laws were studied along the years: decoupling steering, diving, and speed control by PID ((Jalving 1994)), coupled PID and anti-windup control ((Miyamaoto, Aoki, Maeda, Hirokawa, Ichikawa, Saitou, Kobayashi, Kobayashi and Iwasaki 2001)), sliding mode control ((Healey and Lienard 1993, Salgado-Jimenez and Jouvencel 2003)) and $\mathcal{H}_\infty$ control ((Silvestre and Pascoal 2004, Feng and Allen 2004)).

The AUV considered here is an $Aster^X$ like vehicle, developed by the Ifremer (French Research Institute for Exploitation of the Sea)[1]. In this preliminary study only motions in the vertical plane are considered: the control of the yaw angle and speed are not taken into account.

[1] http://www.ifremer.fr/fleet/r&projets.htm

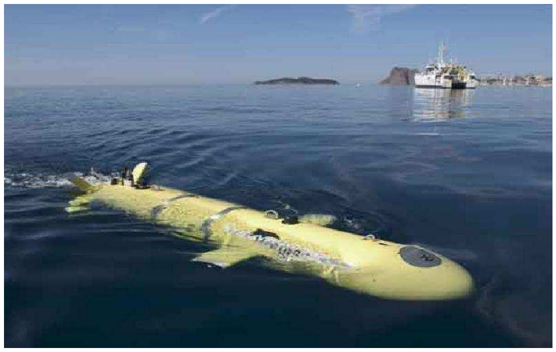


Figure 22: The $Aster^X$ AUV operated by Ifremer

The measurement of the altitude with respect to the sea floor is made using an ultrasonic sensor. Even if the measurement requests are made periodically, the signal flying time and the time at which the measures are received depend on the distance between the AUV and the sensor's target. This could be viewed as a delay in the reception of the measure, or as a need to apply the control samples in an asynchronous way. Another source of sampling intervals variations may come from sensors scheduling, needed to avoid cross-talking between several ultrasonic devices working in a close area (Opderbecke 2009). In the sequel this time uncertainty is considered as a gain scheduling parameter for a LPV controller, as in Figure 23. Then the idea is to adapt the control interval with the distance between two samples, so that the controller only acts when a new measurement is available and processed. This approach allows for large measurement time variations because the controller is not only robust to a delay, but also scheduled according to the sampling interval variation.

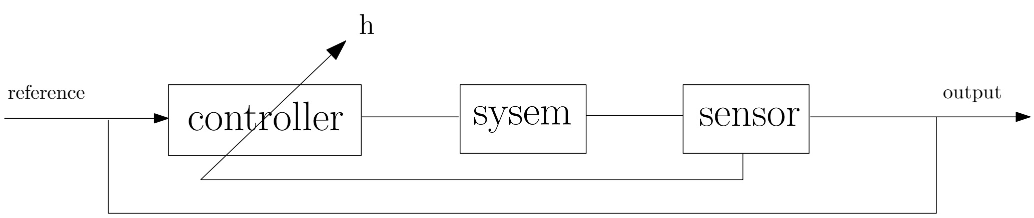


Figure 23: Sensor acting on the sampling period of the controller

The control of underwater vehicles is made difficult by numerous non-linearities, due to cross-coupled dynamics and hydrodynamic forces subject to large uncertainties. Therefore robust control is necessary to safely perform autonomous missions: the ability of LPV based control to combine the performance specification in the $\mathcal{H}_\infty$ framework, and the adaptation of the controller to parameters variations and uncertainties inside a specified range of values provided by the LPV approach is used here.

3.5.1 AUV Models

The model of the vehicle is directly inherited from (Roche, Sename and Simon 2009).

For the description of the vehicle behavior, we consider a 12 dimensional state vector: $X = \begin{bmatrix}\eta(6) & \nu(6)\end{bmatrix}^T$.

$\eta(6)$ is the position, in the inertial referential $\mathcal{R}_0$, describing the linear position η_1 and the angular position η_2: $\eta = \begin{bmatrix}\eta_1 & \eta_2\end{bmatrix}^T$ with $\eta_1 = \begin{bmatrix}x & y & z\end{bmatrix}^T$ and $\eta(2) = \begin{bmatrix}\phi & \theta & \psi\end{bmatrix}^T$ where x, y and z are the positions of the vehicle , and ϕ, θ and ψ are respectively the roll, pitch and yaw angles.

$\nu(6)$ represents the velocity vector, in the local referential $\mathcal{R}$ (linked to the vehicle) describing linear and angular velocities (first derivative of the position, considering the referential transform, see equation (75)):

$\nu = \begin{bmatrix} \nu_1 & \nu_2 \end{bmatrix}^T$ with $\nu_1 = \begin{bmatrix} u & v & w \end{bmatrix}^T$ and $\nu_2 = \begin{bmatrix} p & q & r \end{bmatrix}^T$

3.5.2 Non Linear Model

As given in (Fossen 1994), (Santos 1995), (Jalving and Storkersen 1994), the physical model is given by the following dynamical equation:

$$M\dot{\nu} = G(\nu)\nu + D(\nu)\nu + \Gamma_g + \Gamma_u \tag{74}$$

$$\dot{\eta} = J_c(\eta_2)\nu \tag{75}$$

where:
- M is the mass matrix which represents the real mass of the vehicle augmented by the "water-added-mass" part,
- $G(\nu)$ represents the action of Coriolis and centrifugal forces,
- $D(\nu)$ is the matrix of hydrodynamics damping coefficients,
- Γ_g correspond to the gravity effort and hydrostatic forces,
- $J_c(\eta_2)$ is the referential transform matrix from $\mathcal{R}(C, xyz)$ towards $\mathcal{R}_0(O, X_0Y_0Z_0)$,
- Γ_u represent the forces and moments due to the vehicle's actuators. The considered AUV has an axial propeller to control the velocity in Ox direction (forward force Q_c) and 5 independent mobile fins:

- 2 horizontals fins in the front part of the vehicle (controlled with angles β_1 and β_1').
- 1 vertical fin at the tail of the vehicle (controlled with angle δ).
- 2 fins at the tail of the vehicle (controlled with angles β_2 and β_2') inclined with angle $\pm\pi/3$ w.r.t the vertical fin.

The nonlinear model includes 12 state variables and 6 control inputs. For the computation of the controller, a linear model is proposed. The equilibrium point is chosen as $[u\ v\ w\ p\ q\ r] = [1\ 0\ 0\ 0\ 0\ 0]$: all velocities are taken equal to 0, except the longitudinal velocity taken equal to 1m/s, the cruising speed chosen by the operator according to the payload requirements.

Tangential linearization around the chosen equilibrium point yields to a model of the form:

$$\begin{cases} \dot{x} = Ax(t) + Bu(t) \\ y = Cx(t) + Du(t) \end{cases}$$

where

- x stand for the state: $x = [x\ u\ y\ v\ z\ w\ \phi\ p\ \theta\ q\ \psi\ r]^T$
- u for the control input $u = [\beta_1\ \beta_1'\ \beta_2\ \beta_2'\ \delta_1\ Q_c]^T$
- y for the measured output

All the matrices A, B, C and D depend on the model parameters: hydrodynamical parameters, mass of the vehicle, dimension of fins…Note that most of these parameters are uncertain, and the control design is proposed here for the nominal plant case only.

3.5.3 Model Reduction

The complete control of the vehicle is made difficult due to the large size of the system. A usual solution is to separate the whole model into three different sub-models with reduced size. This allows for a "decoupling" control synthesis for the three directions Ox, Oy and Oz.

To control the altitude z, the model is reduced to 4 state variables: z, θ (pitch angle) and the corresponding velocity w and q. For the actuation, only 4 fins are needed :the 2 horizontals fins in the front part of the vehicle (β_1 and β_1') and the 2 oblique fins at the tail (β_2 and β_2'). As the AUV has to stay in the vertical plan, the 2 pairs of fins have to be actuated in the same way (with the same angle) so only 2 control variables are kept :$\beta_1 = \beta_1'$ and $\beta_2 = \beta_2'$.

Remark: The current study focuses on the control of the altitude z with adaptation to the sampling period w.r.t. the measurement time, following the bottom referenced altitude control scenario. The other degrees of freedom are controlled using basic (i.e. constant sampling) feedback to keep the vehicle in the vertical plane with at the predefined forward velocity. Note that the model built for the control of the longitudinal speed u contains all the dynamics, but can only act on the propeller of the vehicle. To control the yaw angle ψ, the states variables are v, ψ and r. Actuators are the tree fins at the tail of the AUV (corresponding actions δ, β_2 and β_2').

3.5.4 LFT model of the AUV

An LPV/LFT model of the AUV considering the sampling period as varying parameter, using the methodology previously described is built. The Taylor series expansion described in equation (65) and (66) is made at order $k = 2$. Here the Bode diagram of this LPV/LFT system is presented on Figure 24, using the structured case uncertainties:

$$\Delta = \delta I_{2\times k\times n_s} \tag{76}$$

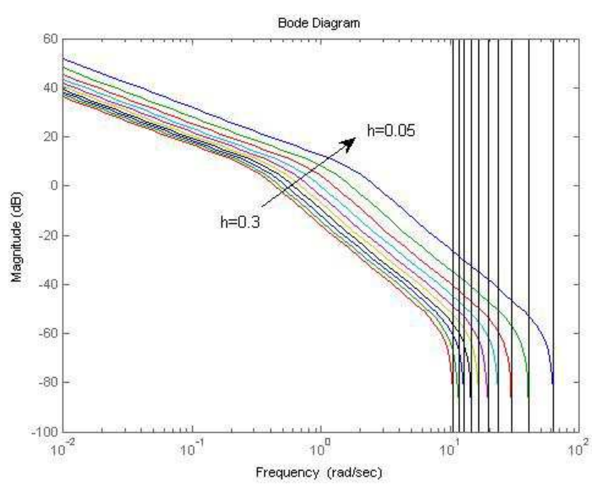


Figure 24: Bode diagram of the system

According to the sampling period variation, this Bode Diagram shows a variation on the system gain and also on the bandwidth.

3.5.5 Structure and weighting function

The method is based on the $\mathcal{H}_\infty$ control design. The first step is to choose a structure and weighting functions that will be placed in the control loop for setting some specifications (response time in closed loop, tracking error...).

The following classical structure is chosen, with:

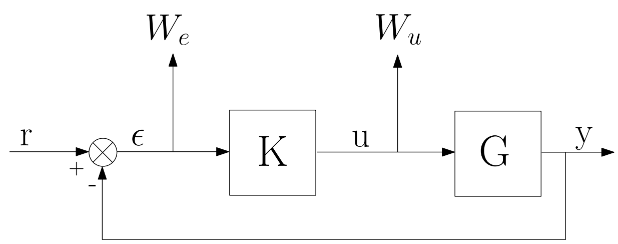


Figure 25: Structure chosen for the control design

- W_e a weight on the tracking error, for fixing specifications on the controlled outputs z. The varying sampling period considered for the controller synthesis imposes a parametrized discretization of weights. This allows the adaptation of the performances with respect to the current sampling period, as explained in (Robert et al. 2010). So W_e is defined with the following state space system, with the frequency $f = 1/h$:

$$\begin{cases} \dot{x} = a \times fx + (a \times f - b \times f)u \\ y = x + u \end{cases} \tag{77}$$

  This weight is then discretized to obtain discrete-time representation:

$$W_d(z) : \begin{cases} x_{k+1} = A_d\, x_k + B_d\, u_k \\ y_k = x_k + u_k \end{cases} \tag{78}$$

$$\begin{cases} A_d = e^{af\,h} = e^{a} \\ B_d = (af)^{-1}(A_d - I)bf = a^{-1}(A_d - I)b \end{cases} \tag{79}$$

  The simplification between h and f leads to discrete LTI representation of the weight.

  The elements a, and b are chosen for obtaining:

  – a good robustness margin.
  – a tracking error less that 1%.
  – a response time of 5 seconds.

- W_u is chosen to account for actuator limitations. As the objective is to control the AUV on the vertical plan, the actions given by the controller to the system should be equal by pair ($\beta_1 = \beta_1'$ and $\beta_2 = \beta_2'$) so only 2 actions are kept for the control design. As all actions are normalized, we choose the identity matrix of size 2 for W_u).

Then the augmented plant (P in Figure 21) is built containing the model of the system and the weighting functions.

3.5.6 LFT controller

A LPV/LFT controller is computed for the control of the altitude z, by considering the previous structure and weights. The interval of variation of the sampling period for the controller design is: $h \in [0.05; 0.3]s$.

Equivalently to the Bode diagram of the system, the controller Bode diagram varies according to the sampling period, as seen in Figure 26.

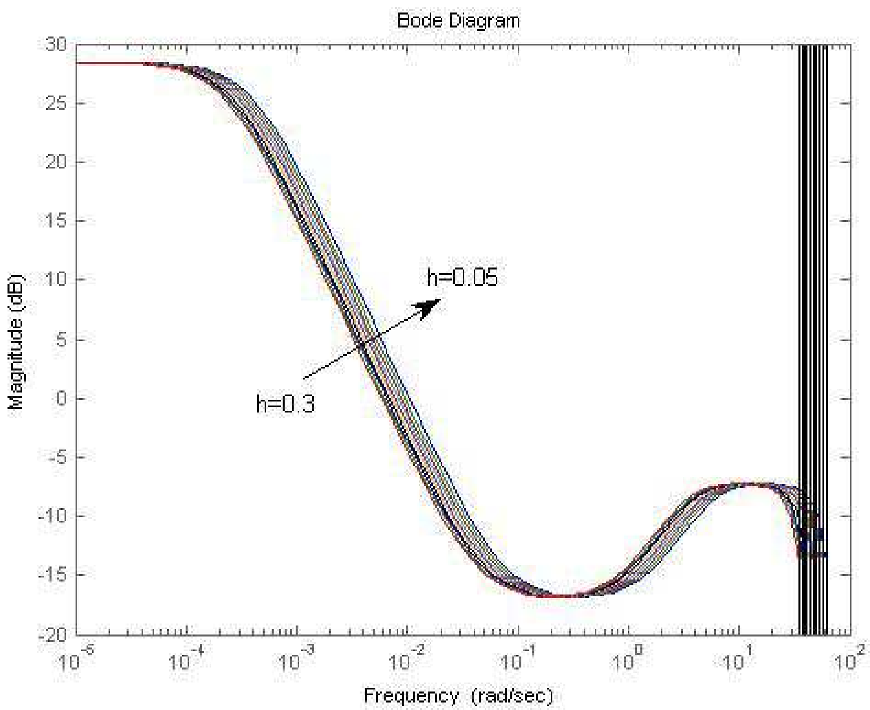


Figure 26: Bode diagram of the LPV/LFT controller

The sampling period clearly affects the controller bandwidth, this in accordance with the system bandwidth variation.

The sensitivity function $S = \frac{1}{1+GK}$ is presented in Figure 27. The robustness margin is not affected by the sampling period variation, but the response time varies. This shows the adaptation of performances w.r.t to current sampling period (because of the varying discrete-time weight on ϵ_z).

3.5.7 Simulation Results

We consider as a mission to follow the sea bottom at constant height and move at constant speed (require for a good interpretation of the results).

The complete linearized model of the AUV is used for the following simulations. We focus in the sequel on the control of the altitude z. An independent discrete-time controller controls the cruising speed u which start at 0 and stays constant and equal to $1m/s$ during all the simulation (its design will not be detailed here; it is a simple $\mathcal{H}_\infty$ discrete-time controller, with a sampling period of $0.1s$).

$\mathcal{H}_\infty$ Discrete controller First, a $\mathcal{H}_\infty$ controller is synthesized and serves as reference (using *dhinflmi* function of the LMI Control Toolbox). This controller is designed for a single sampling period (not gain

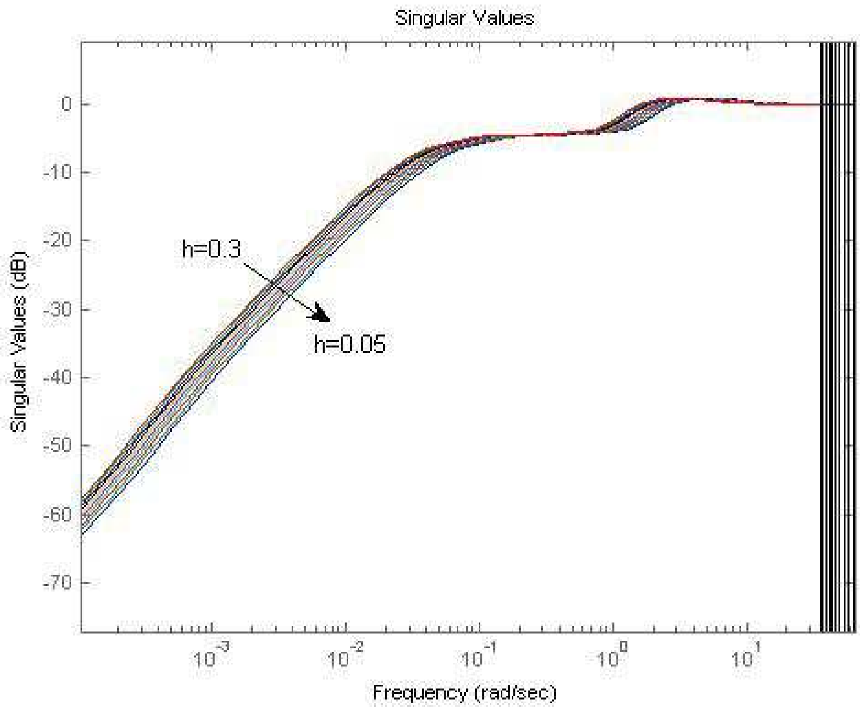


Figure 27: Sensitivity function

scheduled) so it could not be adapted to sampling period variations. However the robustness brought by the method should allow little variation around $h = 0.1s$.

LPV controller with sampling period as varying parameter The controller synthesized in the previous section is used for the control of the system. Different sampling period variation scenarii will be considered.

Both controllers are tested on the AUV model, with no variation of the sampling period. The simulation is done with $h = 0.1s$, namely the period considered for the $\mathcal{H}_\infty$ controller design. The LFT controller is adapted to this period via the Δ matrix. Simulation results are presented in figure 28

The $\mathcal{H}_\infty$ controller gives better results than the LFT one in term of response time but the LFT controller presents a smaller overshoot. Both controllers have no tracking error.

Sinusoidal variation of the sampling period h For this simulation, the sampling period varies sinusoidally. The step on altitude reference is done at minimal and maximal value of the sampling period , but h changes (sinusoidally) during the step on z. Results are presented in Figures 29 and 30.

In the same case the discrete-time controller leads to worst results $h = 0.3$:

The LFT controller gives good results: the system remains stable whatever the sampling period h is. Moreover the adaptation of the performances w.r.t h is also visible: when the sampling period is high ($h = h_{max} = 0.3s$) the response time is slow, and inversely. The discrete-time $\mathcal{H}_\infty$ controller leads to poor performances especially when the sampling period is too far from the one used for the synthesis ($h = 0.3$). However the response time with the LPV/LFT is larger than the one predicted at step design ($5s$), whereas the sensitivity function S (see Figure 27) predicts the respect of the specifications.

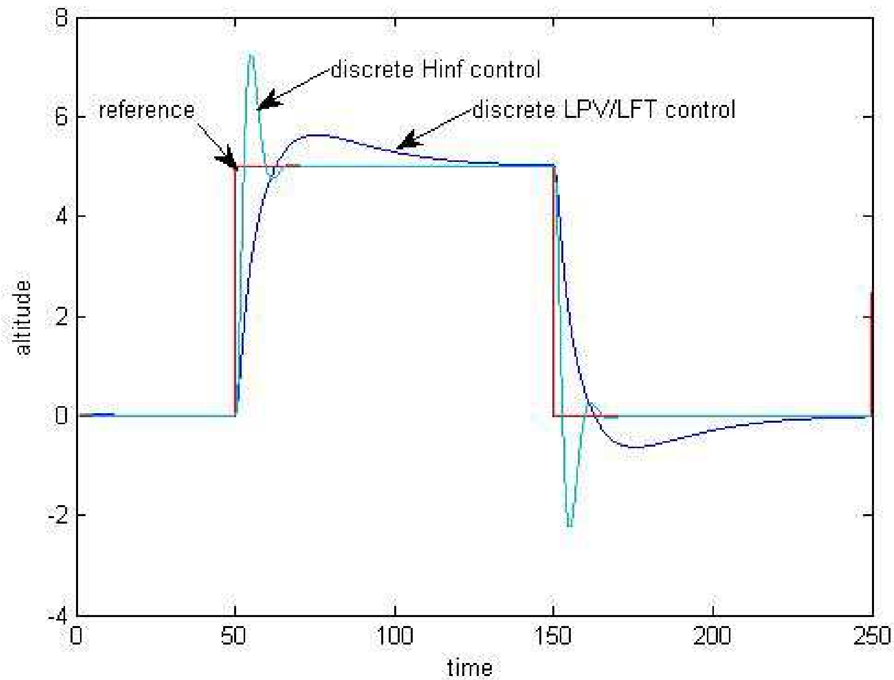


Figure 28: altitude z with discrete time $\mathcal{H}_\infty$ and LFT controllers (Te=0.1s)

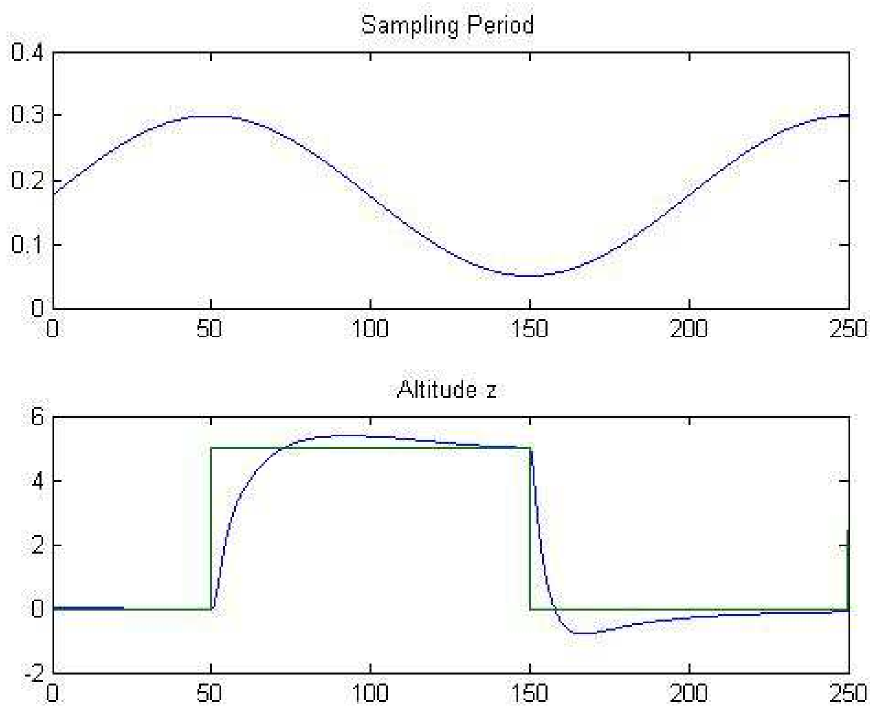


Figure 29: altitude z with LPV/LFT controller, h sinusoidal

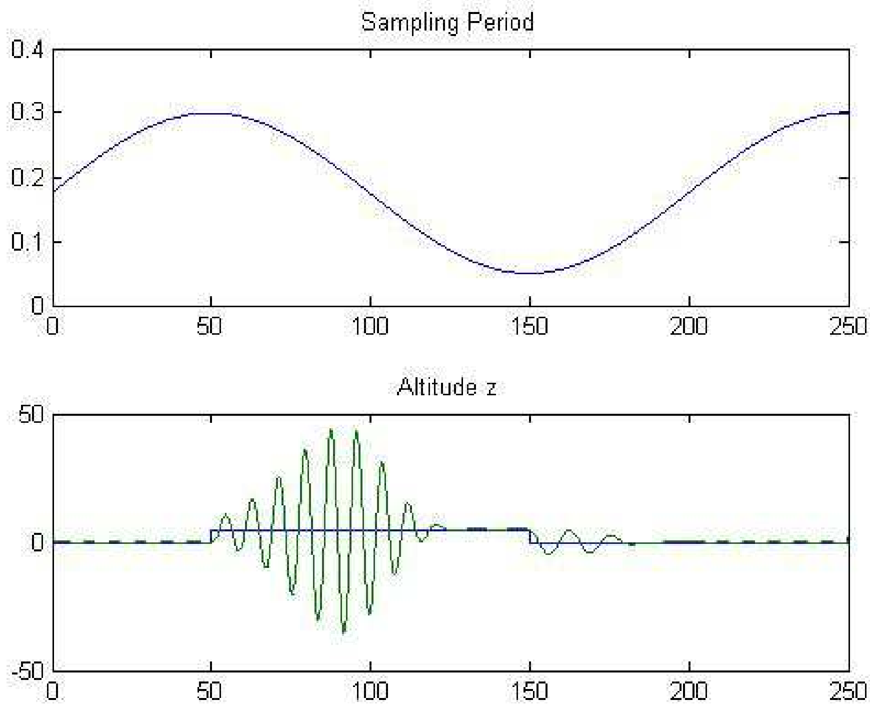


Figure 30: altitude z with discrete-time $\mathcal{H}_\infty$ controller, h sinusoidal

Step variation of the sampling period h Now we consider two successive step variations of the sampling period on extreme value of the sampling period variation interval. A step on the altitude reference is done at the same time, to see the influence of the sampling period on the response. Results are presented on Figures 31 and 32.

In the same case the discrete-time controller leads to instability of the system when $h = 0.3$:

The LPV/LFT controller still leads to good results. On the other hand, the discrete-time $\mathcal{H}_\infty$ controller leads to instability of the system when the sampling period it too far from the one used at design time.

4 Conclusions and perspectives

In the second part of the report, a new formulation for LPV systems, considering the sampling period as varying parameter into an LFT form, has been presented. During the control design step, the definition of a varying weighting function allows performance adaptation w.r.t the current sampling period. The main advantage is the robustness of the LPV/LFT method with respect to the variation of the sampling period, considered here as the varying parameter. By adaptation of the controller to the current sampling period, the LFT controller leads to good results whereas the simple discrete-time controller could lead to instability of the system.

Up to now a unique varying parameter as been considered (the sampling period). In the future, thanks to the LFT formulation, other parameter could be easily added, as for example the forward speed u (to consider different mission scenarii). In the same way, uncertainties can also be added in the Δ matrix.

The control of the AUV remains a difficult task (eg : see the large response time on figures 29 and 31) so a new structure for the control is studied : 2 cascade controllers, one for the pitch angle θ acting directly

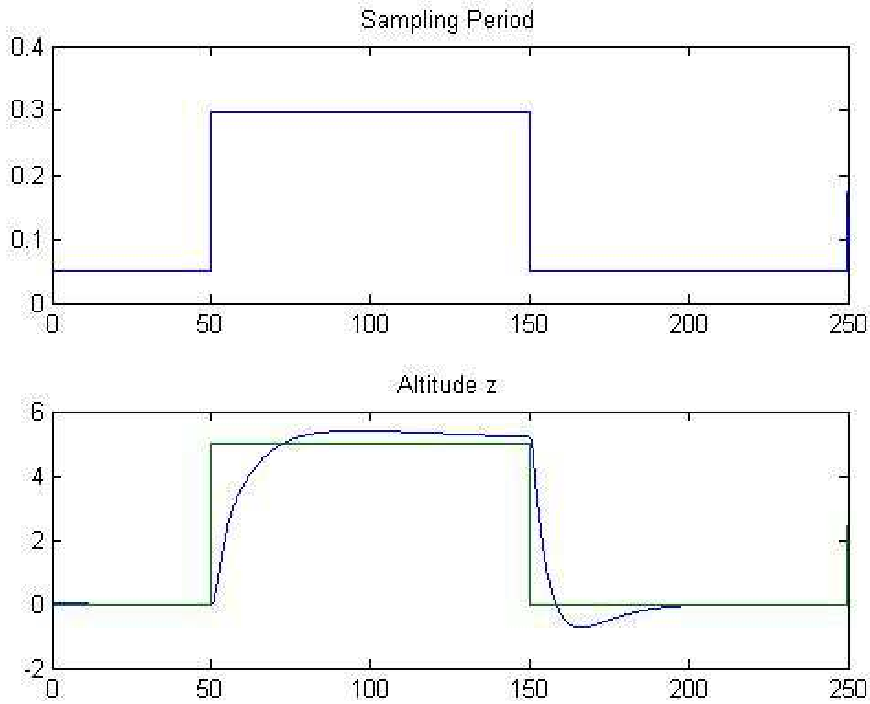


Figure 31: altitude z with LPV/LFT controller, step variation of h

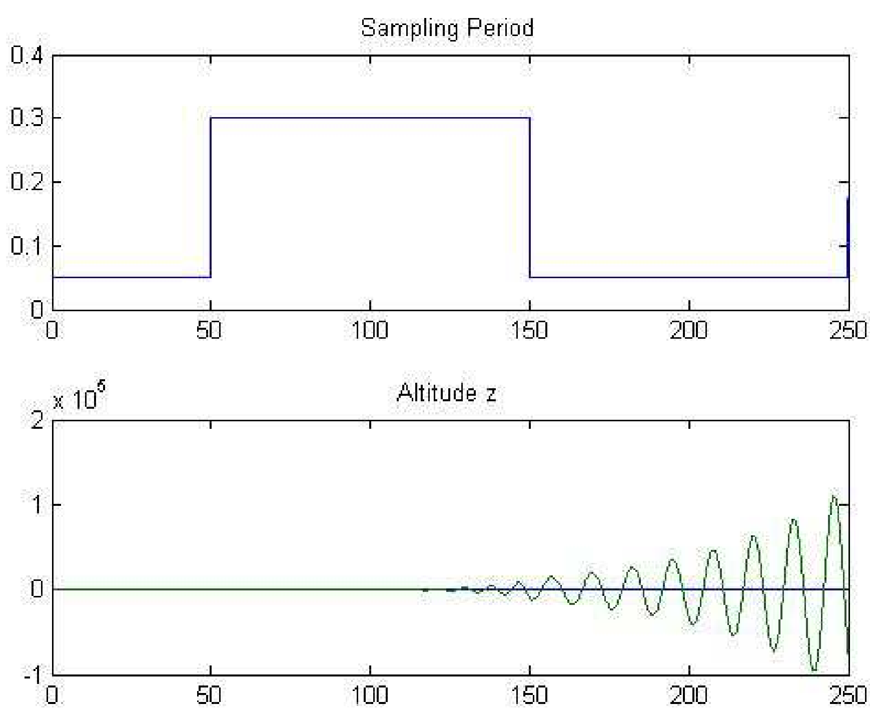


Figure 32: altitude z with discrete-time $\mathcal{H}_\infty$ controller, step variation of h

on the system, and the altitude controller for this subsystem (see figure33).

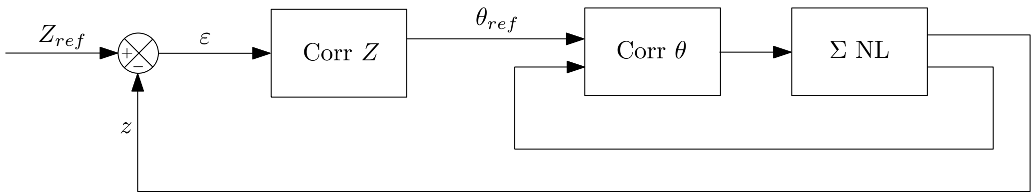


Figure 33: New cascade structure for the AUV control

So the altitude control is achieved first through the control of the pitch angle θ, which is firstly controlled : fins act on the incidence, which changes the altitude. This structure is more adapted to the dynamic of the vehicle.

In effet, some promising results have been extracted from simulations on the complete non linear submarine model (figure 34). This result is obtained for a sampling period $h = 0.125$s.

It is remarkable that this control structure permits to take into account both the altitude z and the incidence θ control.

In a general altitude control, with this new structure, both remarks can be made :

1. The inner incidence controller can be a simple controller, meaning it is not necessary that it ensures a good tracking. However, a fast settling time is as possible required.

2. The outer altitude controller has to be accurate in the sense that it has to minimize the error between the output z and the reference z_{ref}.

In the framework of the sample time varying control, several choices can be made.

In effect, data about the incidence θ are obtained from the inertial central, in the body of the submarine. In this case, the sampling period can be scheduled with the inertial central one (fixed sampling period). Signals for the altitude controller depend on the altitude, so it is necessary to use some time varying sampling period for the outer controller. With those hypotheses, in order to get a suitable control, it is necessary that the sampling inner controller period is faster than the outer one.

On the other hand, both controllers can also be controlled within the same time varying sample period.

Both types of control can be applied. What is sure is that the LPV/LFT techniques presented above, applied on this structure should improve the performances of the control.

Finally this controller can be integrated in a global structure of control, to study the interaction between the three decoupled controller in the three dimension.

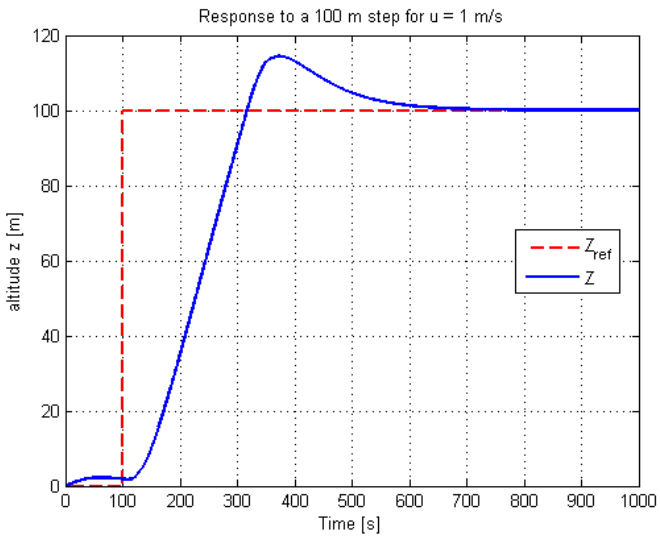
References

Apkarian, P. and Gahinet, P.: 1995, A convex characterisation of gain-scheduled $\mathcal{H}_\infty$ controllers, *IEEE Transaction on Automatic Control*, Vol. 40, pp. 853 – 864.

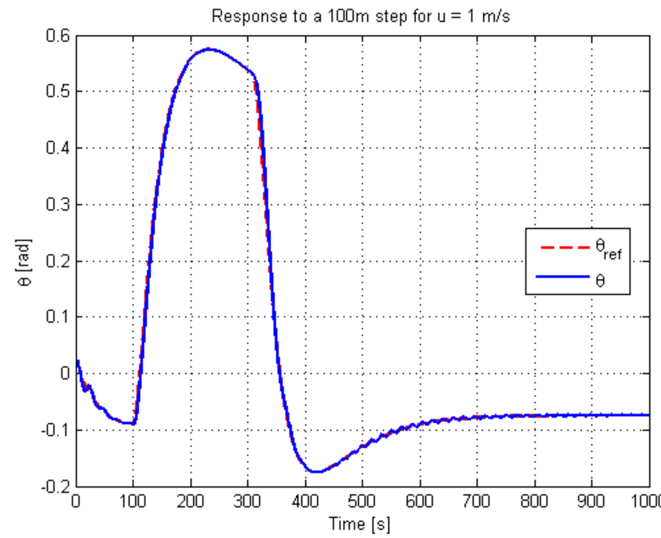
Apkarian, P., Gahinet, P. and Becker, G.: 1995, Self-scheduled H_∞ control of linear parameter-varying systems: A design example, *Automatica* **31**(9), 1251–1262.

Åström, K. J. and Wittenmark, B.: 1997, *Computer-Controlled Systems*, Prentice Hall.

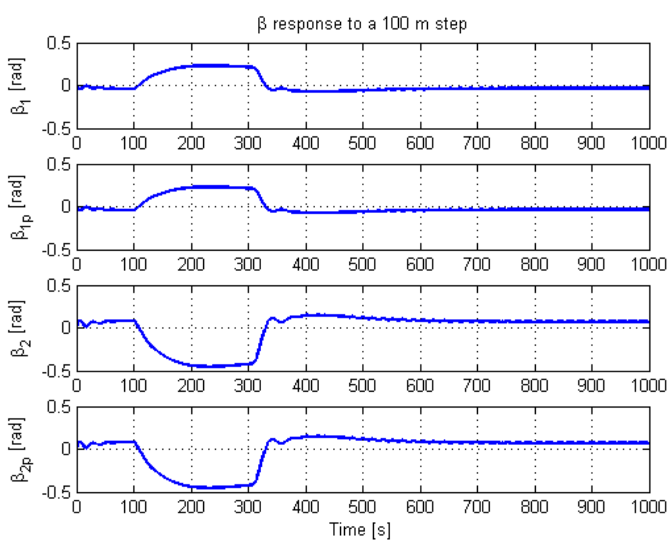
Aubrun, C., Simon, D. and Song, Y.-Q.: 2010, *Co-design approaches for dependable networked control systems*, ISTE-Wiley, chapter 1 : Preliminary notions and state of the art.



(a)



(b)



(c)

Figure 34: Response to a 100m step. (a) : Altitude z, (b) Incidence θ, (c) Fins angle β_i

Benveniste, A. and Berry, G.: 1991, The synchronous approach to reactive and real-time systems, *Proceedings of the IEEE* **79**(9), 1270–1282.

Cervin, A.: 2003, *Integrated Control and Real-Time Scheduling*, PhD thesis, Department of Automatic Control, Lund Institute of Technology, Sweden.

Chen, J., Armstrong, B., Fearing, R. and Burdick, J.: 1988, Satyr and the nymph: Software archetype for real time robotics, *IEEE-ACM Joint Computer Conference*, Dallas, U.S.A.

Chen, T. and Francis, B.: 1995, *Optimal sampled-data control systems*, Springer-Verlag, Berlin, Germany.

Felicioni, F., Jia, N., Simonot-Lion, F. and Song, Y.-Q.: 2010, *Co-design approaches for dependable networked control systems*, ISTE-Wiley, chapter 5 : Overload management through selective data dropping.

Feng, Z. and Allen, R.: 2004, Reduced order $\mathcal{H}_\infty$ control of an autonomous underwater vehicle, *Control Engineering Practice : Guidance and control of underwater vehicles* **12**(12), 1511–1520.

Fossen, T. I.: 1994, *Guidance and Control of Ocean Vehicles*, John Wiley & Sons.

Fridman, E.: 2010, A refined input delay approach to sampled-data control, *To appear in Automatica* .

Fridman, E., Seuret, A. and Richard, J.-P.: 2004, Robust sampled-data stabilization of linear systems: An input delay approach, *Automatica* **40**(8), 1141–1446.

Fujioka, H.: 2009, Stability analysis of systems with aperiodic sample-and-hold devices, *Automatica* **45**(3), 771–775.

Gauthier, C., Sename, O., Dugard, L. and Meissonnier, G.: 2007, An $\mathcal{H}_\infty$ linear parameter-varying (LPV) controller for a diesel engine common rail injection system, *Proceedings of the European Control Conference*, Budapest, Hungary.

Gu, K., Kharitonov, V.-L. and Chen, J.: 2003, *Stability of time-delay systems*, Birkhauser.

Healey, A. and Lienard, D.: 1993, Multivariable sliding mode control for autonomous diving and steering of unmanned underwater vehicles, *Oceanic Engineering* **18**(3), 327–339.

Hespanha, J., Naghshtabrizi, P. and Xu, Y.: 2007, A survey of recent results in networked control systems, *Proceedings of the IEEE* **95**(1), 138–162.

Hetel, L., Daafouz, J. and Iung, C.: 2006, Stabilization of arbitrary switched linear systems with unknown time-varying delays, *Automatic Control, IEEE Transactions on* **51**(10), 1668–1674.

Jalving, B.: 1994, The NDRE-AUV flight control system, *Oceanic Engineering* **19**(4), 497 – 501.

Jalving, B. and Storkersen, N.: 1994, The control system of an autonomous underwater vehicle, *Proceedings of the Third IEEE Conference*, Vol. 2, pp. 851 – 856.

Jiang, Z.-P. and Wang, Y.: 2002, Aconverse Lyapunov theorem for discrete-time systems with disturbances, *Systems & Control Letters* **45**, 49–58.

Kopetz, H. and Bauer, G.: 2003, The time-triggered architecture, *Proceedings of the IEEE* **91**(1), 112–126.

Leith, D. J. and Leithead, W. E.: 2000, Survey of gain-scheduling analysis and design, *International Journal of Control* **73**(11), 1001–1025.

Li, X., Cela, A., Niculescu, S.-I. and Reama, A.: 2009, Some remarks on the stability of networked control systems with periodic scheduling, *Submitted to the European Control Conference*.

Lin, H., Zhai, G. and Antsaklis, P.: 2006, Asymptotic stability and disturbance attenuation properties for a class of networked control systems, *Journal of Control Theory and Application* **4**(1), 76–85.

Liu, K. and Fridman, E.: 2009a, Discontinuous Lyapunov functionals for linear systems with sawtooth delays, *Proceedings of the* 8^{th} *IFAC Workshop on Time-Delay Systems*.

Liu, K. and Fridman, E.: 2009b, Stability analysis of networked control systems: a discontiuous Lyapunov functional approach, *Joint* 48^{th} *IEEE Conference on Decision and Control and* 28^{th} *Chinese Control Conference*, Shanghai, China.

Michiels, W., Niculescu, S.-I. and Moreau, L.: 2004, Using delays and time-varying gains to improve the static output feedback stabilizability of linear systems : A comparison, *IMA Journal of Mathematical Control and Information* **21**(4), 393–418.

Millán, P., Orihuela, L., Vivas, C. and Rubio, F.: 2009, Improved delay-dependent stability criterion for uncertain networked control systems with induced time-varying delays, 1^{rst} *IFAC Workshop on Distributed estimation and Control of Networked control Systems, Necsys'09*, Venise, Italy.

Mirkin, L.: 2007, Some remarks on the use of time-varying delay to model sample-and-hold circuits, *IEEE Trans. on Automatic Control* **52**(6), 1009–1112.

Miyamaoto, S., Aoki, T., Maeda, T., Hirokawa, K., Ichikawa, T., Saitou, T., Kobayashi, H., Kobayashi, E. and Iwasaki, S.: 2001, Maneuvering control system design for autonomous underwater vehicle, *MTS/IEEE Conference and Exhibition* **1**, 482 – 489.

Naghshtabrizi, P., Hespanha, J. P. and Teel, A. R.: 2008a, Stability of delay impulsive systems with application to networked control systems, *Trans. of the Inst. of Measurement and Control,* Special Issue on Hybrid and Switched Systems .

Naghshtabrizi, P., Hespanha, J. and Teel, A.: 2008b, Exponential stability of impulsive systems with application to uncertain sampled-data systems, *Systems and Control Letters* **57**(5), 378–385.

Natale, O., Sename, O. and Canudas de Wit, C.: 2004, Inverted pendulum stabilization through the ethernet network, performance analysis, *American Control Conference ACC'04*, Boston, USA.

Oishi, Y. and Fujioka, H.: 2009, Stability and stabilization of aperiodic sampled-data control systems: An approach using robust linear matrix inequalities, *Joint* 48^{th} *IEEE Conference on Decision and Control and* 28^{th} *Chinese Control Conference*.

Olsson, H., Aström, K.-J., de Wit, C. C., Gäfvert, M. and Lischinsky, P.: 1998, Friction models and friction compensation, *European Journal of Control* **4**(3), 176–195.

Opderbecke, J.: 2009, Description of the scientific mission scenario(s) to be investigated for the marine application, *deliverable D08.01*, FeedNetBack project, `http://feednetback.eu/public-deliverables/public-deliverables-pdf/ploneexfile.2009-12-09.1317343148/`.

Packard, A.: 1994, Gain scheduling via linear fractional transformations, *Systems and Control Letters* **22**(2).

Peet, M., Papachristodoulou, A. and Lall, S.: 2009, Positive forms and stability of linear time-delay systems, *SIAM Journal on Control and Optimization* **47**(6), 3227–3258.

Richard, J.-P.: 2003, Time delay systems: an overview of some recent advances and open problems, *Automatica* **39**, 1667–1694.

Robert, D.: 2007, *Contribution à l'interaction commande/ordonnancement*, PhD thesis (in french), Institut National Polytechnique de Grenoble.

Robert, D., Sename, O. and Simon, D.: 2005, Sampling period dependent RST controller used in control/scheduling co-design, *16th IFAC World Conference*, Prague.

Robert, D., Sename, O. and Simon, D.: 2010, An H_∞ LPV design for sampling varying controllers : experimentation with a T inverted pendulum, *IEEE Transactions on Control Systems Technology* . to appear, DOI 10.1109/TCST.2009.2026179.

Roche, E., Sename, O. and Simon, D.: 2009, LPV / $\mathcal{H}_\infty$ control of an autonomous underwater vehicle (auv), *Proceedings of the European Control Conference*, Budapest, Hungary.

Sala, A.: 2005, Computer control under time-varying sampling period: An lmi gridding approach, *Automatica* **41**(12), 2077–2082.

Sala, A., Cuenca, Á. and Salt, J.: 2009, A retunable pid multi-rate controller for a networked control system, *Information Sciences* **179**(14), 2390–2402.

Salgado-Jimenez, T. and Jouvencel, B.: 2003, Using a high order sliding modes for diving control a torpedo autonomous underwater vehicle, *OCEANS*, Vol. 2, pp. 934 – 939.

Santos, A. S.: 1995, *Contribution à la conception des sous-marins autonomes : architecture des capteurs d'altitude, et commande référencées capteurs*, PhD thesis (in french), Ecole nationale supérieure des Mines de Paris.

Schinkel, M., Chen, W.-H. and Rantzer, A.: 2002, Optimal control for systems with varying sampling rate, *Proceedings of American Control Conference*, Anchorage, USA.

Seuret, A.: 2009, Stability analysis for sampled-data systems with a time-varying period, 48^{th} *IEEE Conference on Decision and Control*.

Seuret, A.: 2010a, Exponential stability and stabilization of sampled-data systems with time-varying period, *Submitted to Proceedings of the* 9^{th} *IFAC Workshop on Time-Delay Systems*.

Seuret, A.: 2010b, A novel stability analysis of sampled-data systems with applications to multi-rate sampling and packet loss, *submitted to Automatica* .

Seuret, A., Edwards, C., Spurgeon, S. and Fridman, E.: 2007, Static output feedback sliding mode control design via an artificial stabilizing delay, *accepted in IEEE Trans on Aut. Control* .

Silvestre, C. and Pascoal, A.: 2004, Control of the INFANTE AUV using gain scheduled static output feedback, *Control Engineering Practice* **12**(12), 1501–1509.

Simon, D., Castillo, E. and Freedman, P.: 1998, Design and analysis of synchronization for real-time closed-loop control in robotics, *IEEE Trans. on Control Systems Technology* **6**(4), 445–461.

Simon, D., Robert, D. and Sename, O.: 2005, Robust control / scheduling co-design: application to robot control, *Proceedings of the 11th IEEE Real-Time and Embedded Technology and Applications Symposium*, San Francisco, USA.

Simon, D., Seuret, A., Hokayem, P., Lygeros, J. and Camacho, E.: 2009, State of the art in control/computing co-design, *deliverable D04.01*, FeedNetBack project, `http://feednetback.eu/public-deliverables/public-deliverables-pdf/ploneexfile.2010-02-15.6643274994/`.

Skogestad, S. and Postlethwaite, I.: 1996, *Multivariable Feedback Control: analysis and design*, John Wiley and Sons.

Sun, X.-M., Liu, G.-P., Rees, D. and Wang, W.: 2008, Stability of systems with controller failure and time-varying delay, *IEEE Trans. on Automatic Control* **53**(10), 2391–2396.

Törngren, M.: 1998, Fundamentals of implementing real-time control applications in distributed computer systems, *Real Time Systems* **14**(3), 219–250.

Wittenmark, B.: 2001, A sample-induced delays in synchronous multirate systems, *European Control Conference*, Porto, Portugal, pp. 3276–3281.

Yue, D., Han, Q.-L. and Lam, J.: 2008, Stability and stabilization of nonuniform sampling systems, *Automatica* **44**(12), 3222–3226.

Zampieri, S.: 2008, A survey of recent results in Networked Control Systems, *Proc.* 17^{th} *IFAC World Congress*, Seoul, Korea.

Zhang, W.-A. and Yu, L.: 2010, Stabilization of sampled-data control systems with control inputs missing, *IEEE trans. on Automatic Control* **55**(2), 447–452.

Zhang, W. and Branicky, M.: 2001, Stability of networked control systems with time-varying transmission period, *Allerton Conf. Communication, Control, and Computing*.

Zhang, W., Branicky, M. and Phillips, S.: 2001, Stability of networked control systems, *IEEE Control Systems Magazine* **21**.