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AN OBSERVER FOR A NONLINEAR AGE-STRUCTURED MODEL OF A HARVESTED FISH POPULATION

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Abstract. We consider an age-structured model of a harvested population. This model is a discrete-time system that includes a nonlinear stock-recruitment relationship. Our purpose is to estimate the stock state. To achieve this goal, we built an observer, which is an auxiliary system that uses the total number of fish caught over each season and gives a dynamical estimation of the number of fish by age class. We analyse the convergence of the observer and we show that the error estimation tends to zero with exponential speed if a condition on the fishing effort is satisfied. Moreover the constructed observer (dynamical estimator) does not depend on the poorly understood stock-recruitment relationship. This study shows how some tools from nonlinear control theory can help to deal with the state estimation problem in the field of renewable resource management.

1. **Introduction.** The problem of natural stock management has received great attention during the last decades. Developers of management policies in the exploitation of renewable resource stocks need to have a good estimate of the available resource. Current mathematical models together with computer simulations are useful in describing the evolution of complex systems. One of the important problems in control theory is to reconcile the available data with the used mathematical model. This problem is known as the observability problem, and it is

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related to the construction of "observers"(called some times software sensors) for dynamical systems. In this paper, we show how to apply this theory to address the stock estimation problem for an exploited fish population. The biological model we use is the standard fisheries age-structured model (see for instance [8, 7, 16]):

$$\left\{\begin{array}{rcl} x_1(k+1) & = & f\Big(\sum_{i=1}^{n} b_i x_i(k)\Big) \\ x_2(k+1) & = & x_1(k)e^{-M_1-q_1\tau E(k)} \\ \vdots & & \vdots \\ x_{n-1}(k+1) & = & x_{n-2}(k)e^{-M_{n-2}-q_{n-2}\tau E(k)} \\ x_n(k+1) & = & x_{n-1}(k)e^{-M_{n-1}-q_{n-1}\tau E(k)} + x_n(k)e^{-M_n-q_n\tau E(k)} \end{array}\right. \qquad (1)$$

where

- n is the number of age-classes,
- $x_i(k)$ is the number of individuals in the ith class at time k,
- b_i is the fecundity rate of class i,
- M_i is the natural mortality rate of class i,
- q_i is the catchability coefficient of individuals of the ith class,
- $E(k)$ is the fishing effort at time k,
- τ is the length of harvesting season,
- f is the stock-recruitment function.

We suppose that the number of fish harvested over each period $[k, k+1)$ is available for measurement. This number can be expressed as follows (see [7] pp. 146-148, for instance):

$$y(k) = \sum_{i=1}^{n} \frac{q_i\tau E(k)}{q_i\tau E(k) + M_i}\Big(1 - e^{-M_i-q_i\tau E(k)}\Big)x_i(k). \qquad (2)$$

The goal of this paper is to give a simple tool that would allow us to give a dynamical estimation of the state $(x_1(k), \ldots, x_n(k))$ of the stock using the available information which is the value of the captures. To achieve this goal we shall build an observer for system (1).

To fix the ideas let us take a peculiar three-dimensional numerical example of model (1) with the depensatory stock-recruitment function $f(x_0) = \dfrac{x_0^2}{1+\beta x_0^2}$:

$$\left\{\begin{array}{rcl} x_1(k+1) & = & f\Big(\sum_{i=1}^{3} b_i x_i(k)\Big) = \dfrac{(\sum_{i=1}^3 b_i x_i(k))^2}{1+\beta(\sum_{i=1}^3 b_i x_i(k))^2}, \\ x_2(k+1) & = & x_1(k)e^{-M_1-q_1\tau E(k)}, \\ x_3(k+1) & = & x_2(k)e^{-M_2-q_2\tau E(k)} + x_3(k)e^{-M_3-q_3\tau E(k)} \end{array}\right. \qquad (3)$$

with the following parameters:

Depensation recruitment function parameter	$\beta = 0.6$
Fecundity parameters	$b = [2\ \ 3\ \ 3]$,
Catchability coefficients	$q = [0.12\ \ 0.24\ \ 1]$,
Natural mortality rates	$M = [0.8\ \ 0.8\ \ 0.8]$,
Length of harvesting season	$\tau = 1$,
Fishing effort	$E(k) = 10$.

The variables $x_i(k)$ give the number in millions of individuals of the class i at time k. If we can measure the state of (3) at some time k_0, then the equation (3) allow us to compute the values of $x_1(k)$, $x_2(k)$ and $x_3(k)$ for all time $k \geq k_0$. Suppose, for instance, that the real state of the stock at time $k = 0$ is known and it is given by $x_1(0) = 0.181$, $x_2(0) = 0.021$ and $x_3(0) = 0.015$ (in millions of individuals); then by (3), the state of the stock will be (for instance) at time $k = 19$:

$$x_1(19) = 27.1 \times 10^{-3},\ x_2(19) = 8.37 \times 10^{-3},\ \text{and}\ x_3(19) = 0.55 \times 10^{-3}.$$

It can moreover be shown that the above initial condition leads to the extinction of the considered population; that is, $x_i(k)$ tends toward zero as time k tends toward infinity. In practice, the stock will vanish as soon as k becomes larger than 20. However, in practice we do not have access to the values of $x_1(0)$, $x_2(0)$ and $x_3(0)$. All we can measure is the output of the system. Here it is the value of the captures defined by equation (2). For (3) associated to the above parameters values, its expression is given by:

$$y(k) = 0.518799\, x_1(k) + 0.719428\, x_2(k) + 0.925907\, x_3(k).$$

Therefore, at time $k = 0$ we know only that $y(0) = 0.122899$. This value of the output corresponds to the real unknown initial condition, but it also corresponds to the following possible values: $\bar{x}_1(0) = 0.16$, $\bar{x}_2(0) = 0.05$, $\bar{x}_3(0) = 0.0042336$, since $0.518799 \times 0.16 + 0.719428 \times 0.05 + 0.925907 \times 0.0042336 = 0.122899$. Hence, one can take $(\bar{x}_1(0), \bar{x}_2(0), \bar{x}_3(0))$ as an initial condition for system (3), and in this case the state of the stock will be at time $k = 19$:

$$\bar{x}_1(19) = 1474 \times 10^{-3},\ \bar{x}_2(19) = 199 \times 10^{-3},\ \text{and}\ \bar{x}_3(19) = 8.13 \times 10^{-3}.$$

These values are different from the values obtained for the real initial condition. Moreover, the simulations show that the solution of (3) corresponding to the initial condition $(\bar{x}_1(0), \bar{x}_2(0), \bar{x}_3(0))$ will converge to a positive steady state whose coordinates are $(1.474, 0.199, 0.008)$, while the real state will tend toward $(0, 0, 0)$. To summarize, the dynamical model (3) does not suffice to compute the value of the state at a given time k nor to predict the behavior of the system, because one needs to know the value of the real initial condition, which is unavailable for measurement. To overcome this difficulty we shall use a tool from control theory called an observer. That is, we shall construct another dynamical system whose state $\hat{\mathbf{x}}$ will provide an estimate of the real unmeasured state of the considered model, and this will be true regardless of the observer's initial condition: we need not care about the choice of the initial condition of the observer. For (3), the state of the observer will converge rapidly to the real state of the system (issued from the supposed real initial condition $\mathbf{x}(\mathbf{0}) = (0.181, 0.021, 0.015)$), even if we take as an initial condition for the observer the false one $\bar{\mathbf{x}}(\mathbf{0}) = (0.16, 0.05, 0.0042336)$. This is shown in Table 1, which compares the values obtained at different times by system (3) initialized respectively at the true initial condition $\mathbf{x}(\mathbf{0})$ and at the false one $\bar{\mathbf{x}}(\mathbf{0})$, as well as the values obtained by the observer also initialized at the false initial condition $\bar{\mathbf{x}}(\mathbf{0})$. For lack of space we only give the values of the first component, but the same observations are valid for the second and the third component of the state. It can be seen that the values provided by the observer are practically equal to the values of the real state as soon as time k becomes larger than 3. The simulations summarized in the table below have been done with SCILAB.

TABLE 1. Simulation values for system (3).

k	$x_1(k) \times 10^3$	$\bar{x}_1(k) \times 10^3$	$\hat{x}_1(k) \times 10^3$
0	181.0000	160.0000	160.0000
1	195.0483	204.4220	196.8801
2	192.2349	202.3581	192.0978
3	192.6023	210.1368	192.6010
4	192.4746	221.8007	192.4745
10	189.0404	990.8104	189.0404
15	152.3006	1471.1720	152.3006
17	99.0640	1473.7021	99.0640
19	27.1224	1473.9699	27.1224
20	6.5337	1473.9912	6.5337
21	0.6299	1473.9981	0.6299
24	0.0000	1474.0014	0.0000

Now we briefly recall the definition of an *observer* in control theory. Suppose that the dynamical evolution of some phenomena is modelled by the following system:

$$\begin{cases} \mathbf{x(k+1)} = \mathbf{G}(\mathbf{x(k)}, \mathbf{u(k)}) \\ \mathbf{y(k)} = \mathcal{O}(\mathbf{x(k)}, \mathbf{u(k)}) \end{cases} \tag{4}$$

where $\mathbf{x(k)} \in \mathbb{R}^n$ is the state of the system at time k, and $\mathbf{u(k)} \in U \subset \mathbb{R}^m$ is the input or the control. We usually do not have access to the whole state: we can observe or measure only a part (or some function) of the actual state of the system. Therefore we introduce another variable, $\mathbf{y(k)} \in \mathbb{R}^q$, which is called the measurable output of the system. For instance, the state of the fishery model (1) is given by the number of fish in each class, the control is the fishing effort E or the fishing mortality $q_i E$, and the measurable output corresponds to the captures. The expression of the function $\mathcal{O}$ is given by (2).

An observer for the the system (4) is a dynamical system whose inputs are the inputs and outputs of the system (4), which produces an estimate $\hat{\mathbf{x}}(\mathbf{k})$ of the state $\mathbf{x(k)}$ such that the estimation error $\mathbf{x(k)} - \hat{\mathbf{x}}(\mathbf{k})$ tends to zero as time k goes to infinity and must remain small if it starts small (see [23],[9]). The observer will be said to be an exponential observer if there exists $\rho < 1$ such that, for all $k \geq 0$ and if for all initial conditions $(\mathbf{x(0)}, \hat{\mathbf{x}}(\mathbf{0}))$, one has

$$| \hat{\mathbf{x}}(\mathbf{k}) - \mathbf{x(k)} | \leq \rho^k \, | \hat{\mathbf{x}}(\mathbf{0}) - \mathbf{x(0)} | .$$

For linear systems

$$\begin{cases} \mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k), \\ \mathbf{y(k)} = \mathbf{C}\mathbf{x}(k), \\ \mathbf{x(k)} \in \mathbb{R}^n, \mathbf{u(k)} \in \mathbb{R}^m, \mathbf{y(k)} \in \mathbb{R}^p, \\ \mathbf{A} \text{ is a } n \times n \text{ square matrix }, \mathbf{B} \text{ is a } m \times n \text{ matrix and } \mathbf{C} \text{ is a } p \times n \text{ matrix}, \end{cases} \tag{5}$$

the observer design has been completely solved by the Luenberger observer [15] which is simply given by

$$\hat{\mathbf{x}}(\mathbf{k+1}) = \mathbf{A}\hat{\mathbf{x}}(\mathbf{k}) + \mathbf{B}\mathbf{u}(\mathbf{k}) + \mathbf{K}(\mathbf{y}(\mathbf{k}) - \mathbf{C}\hat{\mathbf{x}}(\mathbf{k})).$$

The Luenberger observer converges; that is, $| \hat{\mathbf{x}}(\mathbf{k}) - \mathbf{x(k)} |$ tends to zero exponentially if it is possible to find a matrix $\mathbf{K}$ in such a way that the eigenvalues of the matrix $\mathbf{A}$–$\mathbf{KC}$ are all with modulus less than one. It has been proved that such a matrix $\mathbf{K}$ exists if the pair $(\mathbf{C},\mathbf{A})$ is observable or at least detectable. The system (5) (or the pair $(\mathbf{C},\mathbf{A})$) is observable if any two distinct initial conditions produce two distinct outputs; that is, $\mathbf{x(0)} \neq \bar{\mathbf{x}}(\mathbf{0}) \Longrightarrow \mathbf{Cx(k)} \not\equiv \mathbf{C\bar{x}(k)}$, where $\mathbf{x(k)}$ (respectively $\bar{\mathbf{x}}(\mathbf{k})$) is the solution of system (5) emanating from the initial condition $\mathbf{x(0)}$ (respectively from $\bar{\mathbf{x}}(\mathbf{0})$). This is equivalent to the following: $\mathbf{Cx(k)} \equiv 0 \Longrightarrow \mathbf{x(k)} \equiv \mathbf{0}$. The pair $(\mathbf{C},\mathbf{A})$ is detectable if the following holds : $\mathbf{Cx(k)} \equiv \mathbf{0} \Longrightarrow \lim_{k\to+\infty} \mathbf{x(k)} = \mathbf{0}$. A simple algebraic criterion allows one to check whether a pair of matrices $(\mathbf{C},\mathbf{A})$ is observable. The pair $(\mathbf{C},\mathbf{A})$ is observable if and only if the matrix

$$\mathbf{O}_{(\mathbf{C},\mathbf{A})} = \begin{pmatrix} \mathbf{C} \\ \mathbf{CA} \\ \mathbf{CA^2} \\ \vdots \\ \mathbf{CA^{n-1}} \end{pmatrix}$$

is of rank n. In this case, we say that the system (5) or the pair $(\mathbf{C},\mathbf{A})$ satisfies the Kalman rank condition for observability [23]. When the pair $(\mathbf{C},\mathbf{A})$ (or the linear system (5)) is not observable (i.e., rank $\mathbf{O}_{(\mathbf{C},\mathbf{A})} = r < n$), then there exists an invertible matrix $\mathbf{P}$ such that $\mathbf{PAP^{-1}} = \begin{pmatrix} \mathbf{A_{11}} & \mathbf{0} \\ \mathbf{A_{21}} & \mathbf{A_{22}} \end{pmatrix}$, and $\mathbf{CP^{-1}} = \begin{pmatrix} \mathbf{C_1} & \mathbf{0} \end{pmatrix}$, where $\mathbf{A_{11}}$ is a $r \times r$ matrix, $\mathbf{C_1}$ is a $p \times r$ matrix, and the pair $(\mathbf{C_1},\mathbf{A_{11}})$ is observable. The pair $(\mathbf{C},\mathbf{A})$ is detectable if all the eigenvalues λ_i of the matrix $\mathbf{A_{22}}$ satisfy $|\lambda_i| < 1$.

For nonlinear systems, there is unfortunately no "universal"solution. The observer design problem for nonlinear systems is still a very active research area in control theory. Several methods have been developed for some classes of systems, especially for continuous-time systems [13, 14, 18, 12, 25, 6]. This is not an exhaustive list, because the literature on the subject is extensive (339 references in MathScinet). Some applications of nonlinear observers to continuous biological models have been done (see, for instance, [1, 6, 5]). However there are fewer results concerning nonlinear discrete-time systems. Among them, one can mention [3, 22, 17, 11, 24, 4]. Most of the available results are local (the observer converges only for small initial error) or involve the solvability of some nonlinear functional equations. Moreover these results assume that the function $\mathbf{G}$ modeling the dynamics of the system is completely and precisely known. Here we are interested in the design of a global observer for the fishery system (1), which exhibits an additional difficulty because the recruitment function f is poorly known. Many mathematical expressions have been proposed for the stock-recruitment relationship in the literature. The widely-used recruitment functions [7, 2, 16, 20, 21] are (α, β and γ are positive parameters) as follows:

$$\begin{array}{ll} Beverton\ and\ Holt & f(x_0) = \alpha x_0/(1+\beta x_0)\ ; \\ Ricker & f(x_0) = \alpha x_0 e^{-\beta x_0}\ ; \\ Powerfunction & f(x_0) = \alpha x_0^{1-\beta}\ ; \\ Shepherd & f(x_0) = \alpha x_0/(1+\beta x_0^c)\ ,\ (c>0). \\ Deriso-Schnute & f(x_0) = \alpha x_0\,(1-\beta x_0)^{1/\gamma}; \\ Saila-Lorda & f(x_0) = \alpha x_0^{\gamma} e^{-\beta x_0}. \end{array}$$

Here $x_0 = \sum_{i=1}^{n} b_i x_i$ represents the number of newborns.

Therefore, to use model (1) to estimate the stock for a given population or to use it for fisheries management, one must choose the appropriate recruitment function. This is not an easy task because criteria for making the "good"choice are not generally available. Here we shall build an estimator (observer) which is independent of the recruitment function f. The observer we built will actually work even in the case where the stock-recruitment relationship is stochastical. More precisely, the observer will give a dynamical estimate $\hat{\mathbf{x}}(\mathbf{k})$ of the state $\mathbf{x}(\mathbf{k})$ of the model (1) without using the recruitment function f. The convergence of the observer will be guaranteed if the minimal value of the fishing effort is larger than some positive constant. Previous tentatives to solve this problem have been done in [10] for $n = 3$ (three age classes) and for n age classes in [19]. However the constructions made in [10] and [19] were done with the following output $y(k) = \sum_{i=1}^{n} x_i(k)\, e^{-M_i}(1 - e^{-q_i \tau E(k)})$, which was assumed to be the number of harvested fish. Unfortunately, this is not correct. Moreover, our sufficient condition for the convergence of the observer is weaker than that of [19].

2. **Stock estimation with an observer.** Our aim is to propose an observer for system (1) considered with the output given by equation (2). To this end we introduce the following notations:

$$E_{min} \le E(k) \le E_{max} \quad \forall k \ge 0,$$

$$q_{min} \le q_i \le q_{max} \ \text{ for } i = 1...n,$$

$$m \le M_i \le M \ \text{ for } i = 1...n,$$

and we assume that $q_1 \neq 0$.

Now let us consider the following candidate observer:

$$\begin{cases} \hat{x}_1(k+1) = & \dfrac{M_1 + q_1\tau E(k+1)}{q_1\tau E(k+1)\Big(1 - v_1(k+1)\Big)}\; y(k+1) \\ & -\displaystyle\sum_{i=1}^{n-1} \dfrac{q_{i+1}\Big(M_1 + q_1\tau E(k+1)\Big)\Big(1 - v_{i+1}(k+1)\Big)}{q_1\Big(M_{i+1} + q_{i+1}\tau E(k+1)\Big)\Big(1 - v_1(k+1)\Big)}\, v_i(k)\hat{x}_i(k) \\ & -\dfrac{q_n\Big(M_1 + q_1\tau E(k+1)\Big)\Big(1 - v_n(k+1)\Big)}{q_1\Big(M_n + q_n\tau E(k+1)\Big)\Big(1 - v_1(k+1)\Big)}\, v_n(k)\hat{x}_n(k), \\ \hat{x}_2(k+1) = & v_1(k)\hat{x}_1(k), \\ \vdots & \vdots \\ \hat{x}_{n-1}(k+1) = & v_{n-2}(k)\hat{x}_{n-2}(k), \\ \hat{x}_n(k+1) = & v_{n-1}(k)\hat{x}_{n-1}(k) + v_n(k)\hat{x}_n(k), \end{cases} \tag{6}$$

where $v_i(k) = e^{-M_i - q_i\tau E(k)}$. This system can be written in a condensed form as

$$\mathbf{\hat{x}(k+1)} = \mathbf{A(k)\hat{x}(k)} + y(k+1)\mathbf{F(k)}, \tag{7}$$

where the matrix $A(k)$ is given in the proof of Proposition 1 and

$$\mathbf{F(k)} = \begin{pmatrix} \frac{M_1 + q_1\tau E(k+1)}{q_1\tau E(k+1)(1 - v_1(k+1))} \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

The main result can then be stated as follows

Proposition 1. *There exists $\eta > 0$ such that if $E_{min} > \eta$, then the system (6) is a global exponential observer for system (1).*

Proof. We have

$$\begin{aligned} y(k+1) = & \sum_{i=1}^{n} \frac{q_i\tau E(k+1)}{M_i + q_i\tau E(k+1)}\Big(1 - v_i(k+1)\Big)x_i(k+1) \\ = & \frac{q_1\tau E(k+1)}{M_1 + q_1\tau E(k+1)}\Big(1 - v_1(k+1)\Big)x_1(k+1) \\ & + \sum_{i=2}^{n} \frac{q_i\tau E(k+1)}{M_i + q_i\tau E(k+1)}\Big(1 - v_i(k+1)\Big)x_i(k+1) \\ = & \frac{q_1\tau E(k+1)}{M_1 + q_1\tau E(k+1)}\Big(1 - v_1(k+1)\Big)x_1(k+1) \\ & + \sum_{i=1}^{n-1} \frac{q_{i+1}\tau E(k+1)}{M_{i+1} + q_{i+1}\tau E(k+1)}\Big(1 - v_{i+1}(k+1)\Big)x_{i+1}(k+1)\,. \end{aligned}$$

Thanks to (1), we have

$$\begin{cases} x_{i+1}(k+1) = v_i(k)x_i(k), \text{ for } i = 1, ..., n-2, \\ \text{and } x_n(k+1) = v_{n-1}(k)x_{n-1}(k) + v_n(k)x_n(k). \end{cases}$$

Therefore,

$$\begin{aligned} y(k+1) = & \frac{q_1 \tau E(k+1)}{M_1 + q_1 \tau E(k+1)} \Big(1 - v_1(k+1)\Big) x_1(k+1) \\ & + \sum_{i=1}^{n-1} \frac{q_{i+1} \tau E(k+1)}{M_{i+1} + q_{i+1} \tau E(k+1)} \Big(1 - v_{i+1}(k+1)\Big) v_i(k) x_i(k) \\ & + \frac{q_n \tau E(k+1)}{M_n + q_n \tau E(k+1)} \Big(1 - v_n(k+1)\Big) v_n(k) x_n(k) \,. \end{aligned}$$

Let $\mathbf{e(k)} = \mathbf{x(k)} - \mathbf{\hat{x}(k)}$ be the error. Then, taking into account the dynamical equations of the system (1) and of the observer (6), we can write

$$\begin{aligned} e_1(k+1) = & \; x_1(k+1) - \hat{x}_1(k+1) \\ = & \; -\sum_{i=1}^{n-1} \frac{q_{i+1}\Big(M_1 + q_1 \tau E(k+1)\Big)\Big(1 - v_{i+1}(k+1)\Big)}{q_1\Big(M_{i+1} + q_{i+1}\tau E(k+1)\Big)\Big(1 - v_1(k+1)\Big)} v_i(k) e_i(k) \\ & - \frac{q_n\Big(M_1 + q_1 \tau E(k+1)\Big)\Big(1 - v_n(k+1)\Big)}{q_1\Big(M_n + q_n \tau E(k+1)\Big)\Big(1 - v_1(k+1)\Big)} v_n(k) e_n(k), \end{aligned}$$

and

$$e_i(k+1) = v_{i-1}(k)x_{i-1}(k) - v_{i-1}(k)\hat{x}_{i-1}(k) = v_{i-1}(k)e_{i-1}(k) \text{ for } i = 2...n-1,$$

and

$$e_n(k+1) = v_{n-1}(k)e_{n-1}(k) + v_n(k)e_n(k).$$

We denote by α_i the following functions:

$$\alpha_i(k) = -\frac{q_{i+1}\Big(M_1 + q_1 \tau E(k+1)\Big)\Big(1 - v_{i+1}(k+1)\Big)}{q_1\Big(M_{i+1} + q_{i+1}\tau E(k+1)\Big)\Big(1 - v_1(k+1)\Big)} v_i(k), \quad \text{for } i = 1 \dots n-1$$

and

$$\alpha_n(k) = -\frac{q_n\Big(M_1 + q_1 \tau E(k+1)\Big)\Big(1 - v_n(k+1)\Big)}{q_1\Big(M_n + q_n \tau E(k+1)\Big)\Big(1 - v_1(k+1)\Big)} v_n(k).$$

With these notations we can write

$$\mathbf{e(k+1)} = \mathbf{A(k)e(k)}, \tag{8}$$

where the time-varying matrix $\mathbf{A(k)}$ is defined as follows:

$$\mathbf{A(k)} = \begin{pmatrix} \alpha_1(k) & \alpha_2(k) & \alpha_3(k) & \dots & \alpha_n(k) \\ v_1(k) & 0 & 0 & \dots & 0 \\ 0 & v_2(k) & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & v_{n-1}(k) & v_n(k) \end{pmatrix}.$$

We shall use the following matrix norm: $\|\mathbf{A}(\mathbf{k})\|_1 = max_j \sum_{i=1}^{n} |a_{ij}(k)|$. To prove that system (6) is a global exponential observer for system (1), it is sufficient to prove that $\|\mathbf{A}(\mathbf{k})\|_1 \leq \delta < 1$.

For $i = 1...n$, let: $s_i(k) = \left| \dfrac{\alpha_i(k)}{v_i(k)} \right|$. We have

$$\text{for } i = 1...n-1,\ s_i(k) = \frac{M_1 q_{i+1}\Big(1 - v_{i+1}(k+1)\Big)}{q_1\Big(M_{i+1} + q_{i+1}\tau E(k+1)\Big)\Big(1 - v_1(k+1)\Big)} + \frac{q_{i+1} q_1 \tau E(k+1)\Big(1 - v_{i+1}(k+1)\Big)}{q_1\Big(M_{i+1} + q_{i+1}\tau E(k+1)\Big)\Big(1 - v_1(k+1)\Big)}.$$

Since $\dfrac{q_{i+1} q_1 \tau E(k+1)}{q_1\Big(M_{i+1} + q_{i+1}\tau E(k+1)\Big)} < 1$, we can write

$$s_i(k) \leq \frac{M_1 q_{i+1}\Big(1 - v_{i+1}(k+1)\Big)}{q_1\Big(M_{i+1} + q_{i+1}\tau E(k+1)\Big)\Big(1 - v_1(k+1)\Big)} + \frac{1 - v_{i+1}(k+1)}{1 - v_1(k+1)}.$$

The map $x \mapsto \dfrac{1 - e^{-x}}{x}$ is decreasing; hence

$$\frac{1 - v_{i+1}(k+1)}{M_{i+1} + q_{i+1}\tau E(k+1)} = \frac{1 - e^{-M_{i+1} - q_{i+1}\tau E(k+1)}}{M_{i+1} + q_{i+1}\tau E(k+1)} \leq \frac{1 - e^{-m - q_{min}\tau E_{min}}}{m + q_{min}\tau E_{min}}.$$

So we obtain

$$s_i(k) \leq \frac{M q_{max}}{q_1} \frac{(1 - e^{-m - q_{min}\tau E_{min}})}{m + q_{min}\tau E_{min}} \frac{1}{1 - v_1(k+1)} + \frac{1 - v_{i+1}(k+1)}{1 - v_1(k+1)}.$$

Since $0 < v_{i+1}(k+1) = e^{-M_{i+1} - q_{i+1}\tau E(k+1)} \leq 1$, we have $1 - v_{i+1}(k+1) \leq 1$. Therefore, we have for $i = 1 \dots n-1$,

$$s_i(k) \leq \frac{M q_{max}}{q_1} \frac{(1 - e^{-m - q_{min}\tau E_{min}})}{m + q_{min}\tau E_{min}} \frac{1}{1 - v_1(k+1)} + \frac{1}{1 - v_1(k+1)}.$$

The same thing can be done for $s_n(k)$, which gives

$$s_n(k) \leq \frac{M q_{max}}{q_1} \frac{(1 - e^{-m - q_{min}\tau E_{min}})}{m + q_{min}\tau E_{min}} \frac{1}{1 - v_1(k+1)} + \frac{1}{1 - v_1(k+1)}.$$

The map $x \mapsto \dfrac{1}{1 - e^{-x}}$ is decreasing, it follows that

$$\frac{1}{1 - v_1(k+1)} = \frac{1}{1 - e^{-M_1 - q_1\tau E(k+1)}} \leq \frac{1}{1 - e^{-m - q_{min}\tau E_{min}}}.$$

Therefore,

$$s_i(k) \leq \frac{M q_{max}}{q_1} \frac{1}{m + q_{min}\tau E_{min}} + \frac{1}{1 - e^{-m - q_{min}\tau E_{min}}},\ \forall\, i = 1, ..., n.$$

And hence we have for all $i = 1...n$,

$$|\alpha_i(k)| + |v_i(k)| = s_i(k) v_i(k) + v_i(k) = (s_i(k) + 1) v_i(k)$$
$$\leq \left(\frac{M q_{max}}{q_1} \frac{1}{m + q_{min}\tau E_{min}} + \frac{1}{1 - e^{-m - q_{min}\tau E_{min}}} + 1 \right) e^{-m - q_{min}\tau E_{min}}.$$

Therefore $\|\mathbf{A}(\mathbf{k})\|_1 \leq \delta(E_{min})$, where

$$\delta(E_{min}) = \left(\frac{Mq_{max}}{q_1} \frac{1}{m + q_{min}\tau E_{min}} + \frac{1}{1 - e^{-m-q_{min}\tau E_{min}}} + 1 \right) e^{-m-q_{min}\tau E_{min}}.$$

We will show now that there exists $\eta > 0$ in such a way that $E_{min} > \eta$ implies $\delta(E_{min}) < 1$.

Let $X = e^{-m-q_{min}\tau E_{min}}$ and consider $P(X) = -\dfrac{Mq_{max}\, X}{q_1\, Log(X)} + \dfrac{X}{1-X} + X - 1$. One can remark that $P(X) = \delta(E_{min}) - 1$.

We have $lim_{X\to 0}P(X) = -1$ and $lim_{X\to 1^-}P(X) = +\infty$. So, there exists $x_1 \in]0, 1[$ such that $P(x_1) = 0$. Let $x^* = inf\{x \in]0, 1[/P(x) = 0\}$; then, we have $P(X) < 0$ for all X satisfying $0 < X < x^*$. Now, $0 < X < x^*$ is equivalent to $E_{min} > \dfrac{-Log(x^*) - m}{q_{min}\tau}$ since $X = e^{-m-q_{min}\tau E_{min}}$. It is then sufficient to choose $\eta = \dfrac{-Log(x^*) - m}{q_{min}\tau}$, and this completes the proof of Proposition 1. □

Remark 1. When the natural mortality rates are all equal, it is possible to give a weaker condition on the minimal value of the fishing effort that ensures the convergence of the observer.

Proposition 2. *Assume that $q_1 \leq q_i$ $\forall i = 2, ..., n$, and moreover that the natural mortality coefficient is the same for all stages, that is, $M_1 = M_2 = \ldots = M_n = m$. Then, the system (6) is a global exponential observer for the system (1) if*

$$E_{min} > \frac{1}{q_1\tau}\left(Log\left[\frac{q_{max}}{q_1} + 1\right] - m \right).$$

Proof. We have

$$\begin{aligned}
&|\alpha_i(k)| + |v_i(k)| = \\
&\left(\frac{q_{i+1}\Big(M_1 + q_1\tau E(k+1)\Big)\Big(1 - e^{-M_{i+1}-q_{i+1}\tau E(k+1)}\Big)}{q_1\Big(M_{i+1} + q_{i+1}\tau E(k+1)\Big)\Big(1 - e^{-M_1-q_1\tau E(t+1)}\Big)} + 1 \right) e^{-M_i-q_i\tau E(k)} \\
&= \left(\frac{q_{i+1}\Big(m + q_1\tau E(k+1)\Big)\Big(1 - e^{-m-q_{i+1}\tau E(k+1)}\Big)}{q_1\Big(m + q_{i+1}\tau E(k+1)\Big)\Big(1 - e^{-m-q_1\tau E(t+1)}\Big)} + 1 \right) e^{-m-q_i\tau E(k)} \\
&= \left(\frac{q_{i+1}}{q_1} \frac{\dfrac{1 - e^{-m-q_{i+1}\tau E(k+1)}}{m + q_{i+1}\tau E(k+1)}}{\dfrac{1 - e^{-m-q_1\tau E(t+1)}}{m + q_1\tau E(k+1)}} + 1 \right) e^{-m-q_i\tau E(k)}.
\end{aligned}$$

Since $q_1 \leq q_i$, we have

$$m + q_1\tau E(k+1) \leq m + q_{i+1}\tau E(k+1),$$

and using the fact that the map $x \mapsto \dfrac{1}{1-e^{-x}}$ is decreasing, we get

$$\frac{1 - e^{-m-q_{i+1}\tau E(k+1)}}{m + q_{i+1}\tau E(k+1)} \leq \frac{1 - e^{-m-q_1\tau E(t+1)}}{m + q_1\tau E(k+1)}.$$

Thus, $|\alpha_i(k)| + |v_i(k)| \leq \left(\frac{q_{i+1}}{q_1} + 1\right) e^{-m-q_i\tau E(k)} \leq \left(\frac{q_{i+1}}{q_1} + 1\right) e^{-m-q_1\tau E(k)}$. It follows that

$$\|\mathbf{A}(\mathbf{k})\|_1 \leq \left(\frac{q_{i+1}}{q_1} + 1\right) e^{-m-q_1\tau E(k)}.$$

And hence, $\|\mathbf{A}(\mathbf{k})\|_1 < 1$ if $E_{min} > \frac{1}{q_1\tau}\left(Log\left[\frac{q_{max}}{q_1} + 1\right] - m\right)$. □

Remark 2. When it is possible to have the number of harvested individuals from the first class ($i = 1$) (i.e., when the output

$$y_1(k) = \frac{q_1\tau E(k)}{q_1\tau E(k) + M_1}\left(1 - e^{-M_1-q_1\tau E(k)}\right) x_1(k)$$

is available for measurement), then the construction of the observer is simpler and does not involve any condition on the fishing effort except that it does not vanish during the harvesting season. The dynamical equation of the observer in this case is given by the following:

$$\left\{\begin{array}{lcl} \hat{x}_1(k+1) & = & \dfrac{q_1\tau E(k+1) + M_1}{q_1\tau E(k+1)\Big(1 - v_1(k+1)\Big)} y_1(k+1) \\ \hat{x}_2(k+1) & = & v_1(k)\hat{x}_1(k) \\ \vdots & & \vdots \\ \hat{x}_n(k+1) & = & v_{n-1}(k)\hat{x}_{n-1}(k) + v_n(k)\hat{x}_n(k) \end{array}\right. \tag{9}$$

Remark 3. In general, the fishery's literature provides the fishing mortality rates instead of the catchability parameters. On the other hand the available catch data usually gives the total weight of the fish caught during a season. Therefore, we will consider that the measurable output is the seasonal biomass yield (i.e., the total weight of harvested fishes over each period $[k, k+1)$) instead of their number. If we denote by w_i the mean weight of individuals of class i, then the seasonal biomass yield is

$$Y(k) = \sum_{i=1}^{n} \frac{q_i\tau E(k)}{M_i + q_i\tau E(k)}(1 - e^{-M_i-q_i\tau E(k)})w_i x_i(k). \tag{10}$$

Let $\varphi_i(k) = q_i\tau E(k)$ be the fishing mortality rate of class i, then we can write:

$$Y(k) = \sum_{i=1}^{n} \frac{\varphi_i(k)}{M_i + \varphi_i(k)}(1 - e^{-M_i-\varphi_i(k)})w_i x_i(k).$$

With these notations and defining $v_i(k) = e^{-M_i-\varphi_i(k)}$, an observer for system (1) whose output is the seasonal biomass yield $Y(k)$ can be written as

$$\left\{\begin{aligned}
&\hat{x}_1(k+1) = \frac{Y(k+1)\Big(M_1+\varphi_1(k+1)\Big)}{\varphi_1(k+1)\Big(1-v_1(k+1)\Big)w_1}\\
&-\sum_{i=1}^{n-1}\frac{\varphi_{i+1}(k+1)\Big(M_1+\varphi_1(k+1)\Big)\Big(1-v_{i+1}(k+1)\Big)v_i(k)\,w_i\,\hat{x}_i(k)}{\varphi_1(k+1)\Big(M_{i+1}+\varphi_{i+1}(k+1)\Big)\Big(1-v_1(k+1)\Big)w_1}\\
&-\frac{\varphi_n(k+1)\Big(M_1+\varphi_1(k+1)\Big)\Big(1-v_n(k+1)\Big)v_n(k)\,w_n\,\hat{x}_n(k)}{\varphi_1(k+1)\Big(M_n+\varphi_n(k+1)\Big)\Big(1-v_1(k+1)\Big)w_1},\\
&\hat{x}_2(k+1) = v_1(k)\,\hat{x}_1(k),\\
&\vdots\\
&\hat{x}_{n-1}(k+1) = v_{n-2}(k)\,\hat{x}_{n-2}(k),\\
&\hat{x}_n(k+1) = v_{n-1}(k)\,\hat{x}_{n-1}(k) + v_n(k)\,\hat{x}_n(k).
\end{aligned}\right. \tag{11}$$

3. **Numerical examples.**

3.1. **An oscillating system.** To illustrate the efficiency of the observer, we give a simulation that shows the observer works well even if the system (1) does not have a stable steady state. To this end we consider a three-age-class system with a Ricker stock-recruitment function $f(x_0) = x_0\,e^{-\beta x_0}$, with $x_0 = \sum_{i=1}^{3} b_i x_i$. We use the following parameters:

Ricker function parameters	$\alpha = 1,\ \beta = 0.003$,
Fecundity parameters	$b = [15\ \ 20\ \ 20]$,
Catchability coefficients	$q = [0.24\ \ 0.36\ \ 0.42]$,
Natural mortality rates	$M = [0.2\ \ 0.2\ \ 0.2]$,
Length of harvesting season	$\tau = 2/3$,
Fishing effort	$E(k) = 8.33 + e^{-t}$.

With these parameters the model exhibits oscillations. The simulations have been done with Scilab. The time evolutions of the state variables x_1, x_2 and x_3, as well as their estimates $\hat{x}_1$, $\hat{x}_2$ and $\hat{x}_3$, are drawn in Figures 1, 2, and 3. It can be seen that the convergence of the estimate variable $\hat{x}_i$ to the real state x_i is quite fast.

3.2. **An example with a stable equilibrium.** We now consider the system (1) with the Beverton-Holt stock-recruitment function $\alpha x_0/(1+\beta x_0)$ associated to the following parameters:

Beverton-Holt function parameters	$\alpha = 1,\ \beta = 0.0002$,
Fecundity parameters	$b = [8\ \ 10\ \ 10]$,
Catchability coefficients	$q = [0.24\ \ 0.36\ \ 0.42]$,
Natural mortality rates	$M = [0.2\ \ 0.2\ \ 0.2]$,
Length of harvesting season	$\tau = 2/3$,
Fishing effort	$E = 8$.

The corresponding three-dimensional dynamical system has a globally asymptotically stable steady state whose coordinates are $(4527, 1030, 135)$. Once again the estimates $\hat{x}_1$, $\hat{x}_2$ and $\hat{x}_3$ delivered by the observer converge rapidly to the real states x_1, x_2 and x_3, as seen in Figures 4, 5, and 6.

4. **Conclusion.** An observer for a standard age-structured model has been presented and explicitly constructed. This observer is simple to use and to implement. It allows one to obtain a dynamical estimate of the state of the stock by using as the only available data the values of the captures. This means that if one can measure the output $y(k)$ (here, it is the total catch), then the observer will give an estimation of the number of individuals by age classes $x_1(k), \dots x_n(k)$ that are not measurable in practice or are at least difficult or expensive (acoustic methods for example) to measure. A good estimate of the stock is important, at least for establishing management policies. The observer's convergence is quite fast. It has the advantage of not using the stock-recruitment function.

The estimator developed in this paper can not be used directly for prediction when we do not know the analytical expression of the recruitment function$f$, because it uses the output at time $k+1$; that is, to calculate the estimate $\hat{x}(k+1)$, the observer needs the value of the output at the same time $k+1$. However, the estimator can indirectly help in prediction when the expression of the recruitment function and the map $k \mapsto E(k)$ are available in the following way. When the observer allows recovery of a good estimate $\hat{\mathbf{x}}(\mathbf{k_0})$ (with the desired precision) of the real unknown state $\mathbf{x}(\mathbf{k_0})$ and then model (1) can be used for prediction for $k \geq k_0$ by taking $\hat{\mathbf{x}}(\mathbf{k_0})$ as an initial condition (for instance in example (3.2)), one can take $k_0 = 4$, since for this value of time the values of $\mathbf{x}(\mathbf{k_0})$ and $\hat{\mathbf{x}}(\mathbf{k_0})$ are practically equal. In practice we do not have the values of $\mathbf{x}(\mathbf{k})$, and so we can not compare with the values delivered by the observer in order to determine the value k_0 of time for which we have $|\mathbf{x}(\mathbf{k_0}) - \hat{\mathbf{x}}(\mathbf{k_0})| < \epsilon$, where ϵ is the desired precision. To determine k_0, it is sufficient to simulate the observer with different initial conditions, and then k_0 is the first time for which the different curves coincide. We have done this for example (3.2). Figure 7 gives the time evolution of the third coordinate $\hat{x}_3$ corresponding to three different initial conditions for the observer dynamical system (6). This shows that one can take $k_0 = 4$, and the same conclusion can be derived from the curves corresponding to the two other components $\hat{x}_1$ and $\hat{x}_2$.

The condition on the fishing effort that allows the convergence of the observer can be weakened and the convergence can be made faster if we add a corrective term to the observer dynamics (6) as follows: the output (2) can be written in a matrix form $y(k) = \mathbf{C}(\mathbf{k})\mathbf{x}(\mathbf{k})$ where the time varying matrix $\mathbf{C}(\mathbf{k})$ depends on $E(k)$; the new candidate observer is then

$$\hat{\mathbf{x}}(\mathbf{k+1}) = \mathbf{A}(\mathbf{k})\hat{\mathbf{x}}(\mathbf{k}) + y(k+1)\mathbf{F}(\mathbf{k}) + \mathbf{L}(\mathbf{k})\Big(y(k) - \mathbf{C}(\mathbf{k})\hat{\mathbf{x}}(\mathbf{k})\Big),$$

where the matrix $\mathbf{L}(\mathbf{k})$ has to be computed. A work in this direction is in progress.

Optimal control theory has been widely used in renewable resource management. The present work indicates that the estimation problem can be investigated from the point of view of control engineering.

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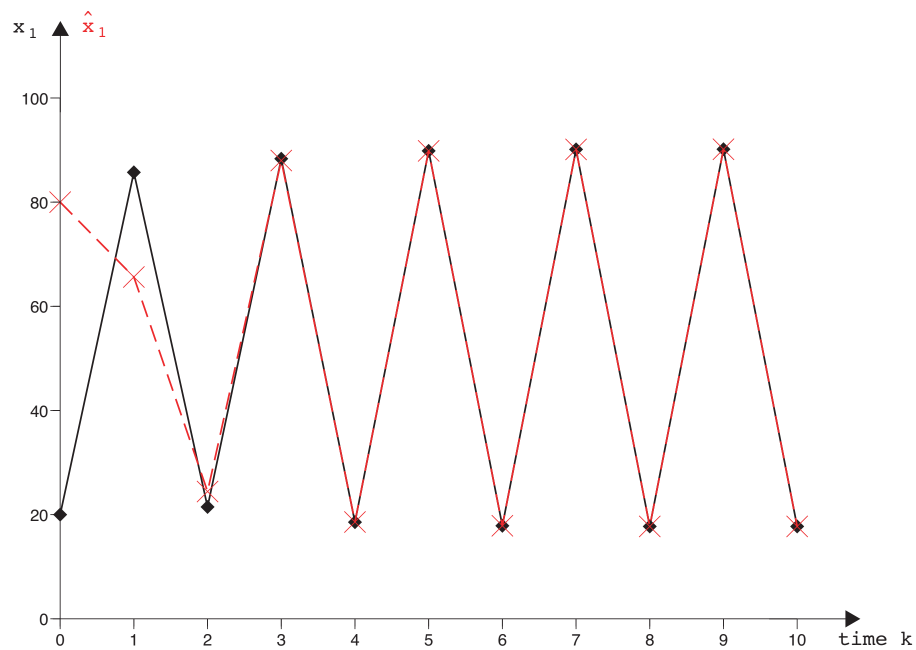

FIGURE 1. x_1 (solid line) and its estimate $\hat{x}_1$ (dashed line)

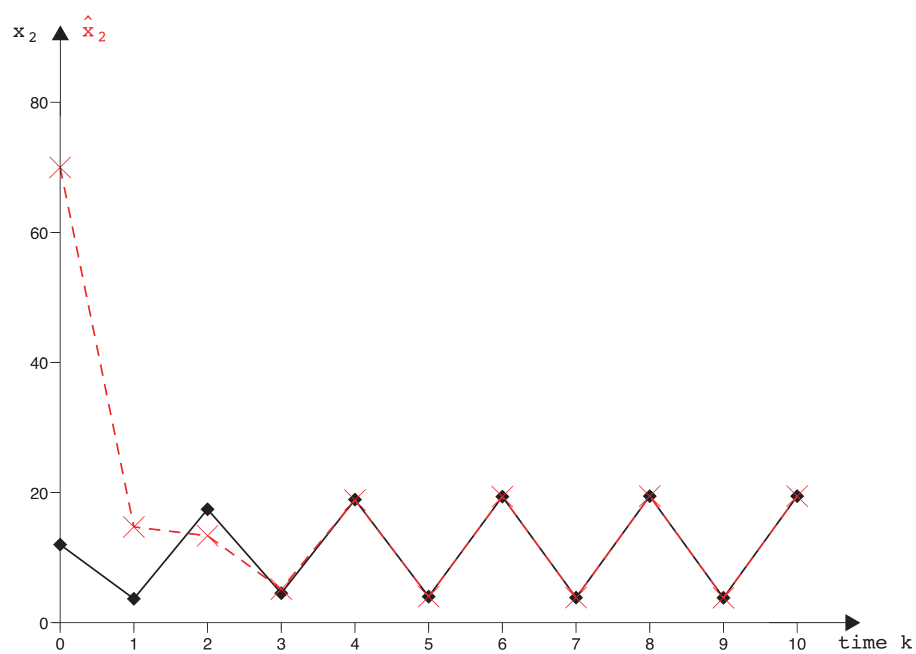

FIGURE 2. x_2 (solid line) and its estimate $\hat{x}_2$ (dashed line)

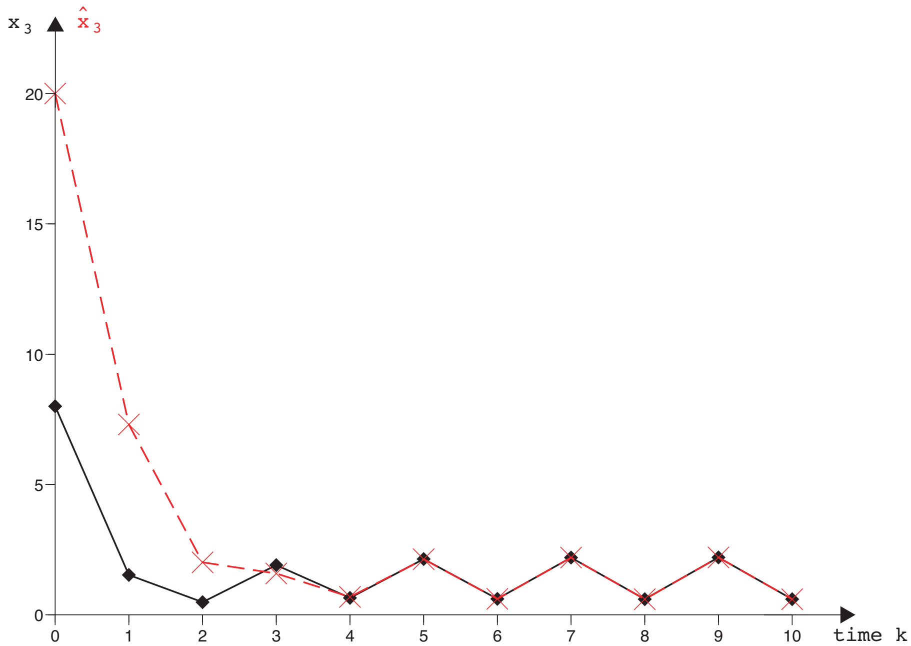

FIGURE 3. x_3 (solid line) and its estimate $\hat{x}_3$ (dashed line)

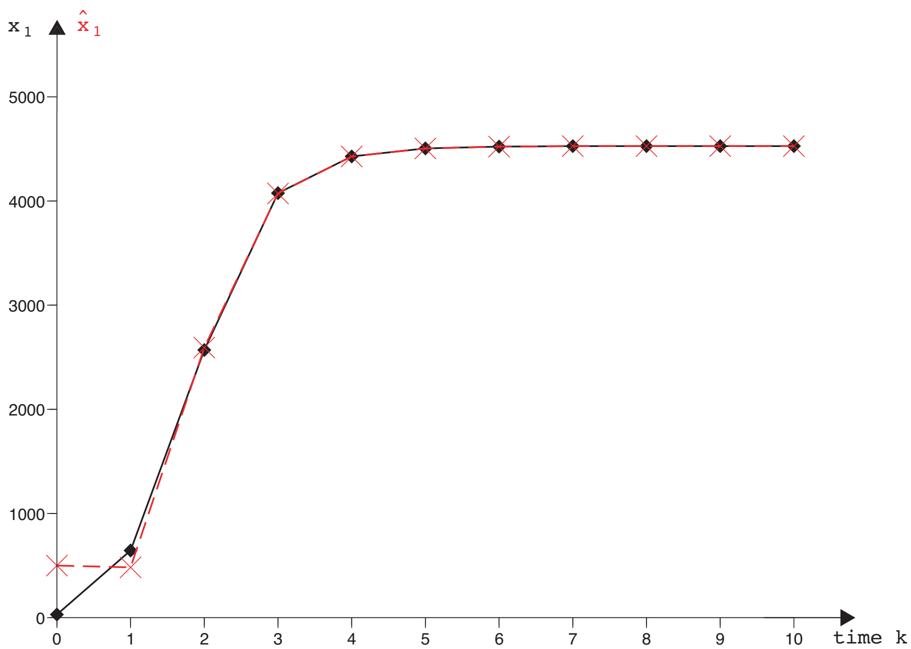

FIGURE 4. x_1 (solid line) and its estimate $\hat{x}_1$ (dashed line)

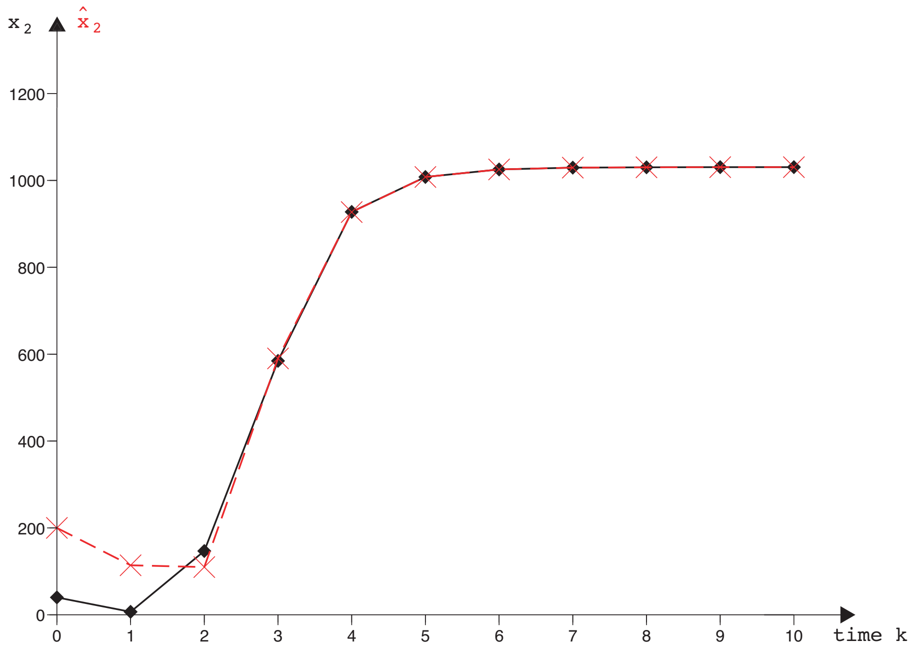

FIGURE 5. x_2 (solid line) and its estimate $\hat{x}_2$ (dashed line)

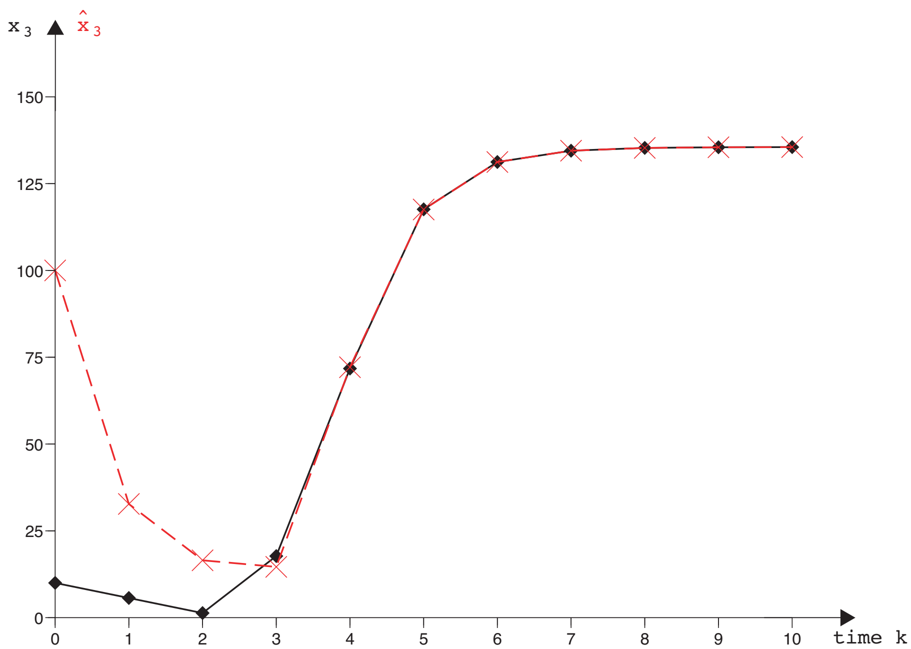

FIGURE 6. x_3 (solid line) and its estimate $\hat{x}_3$ (dashed line)

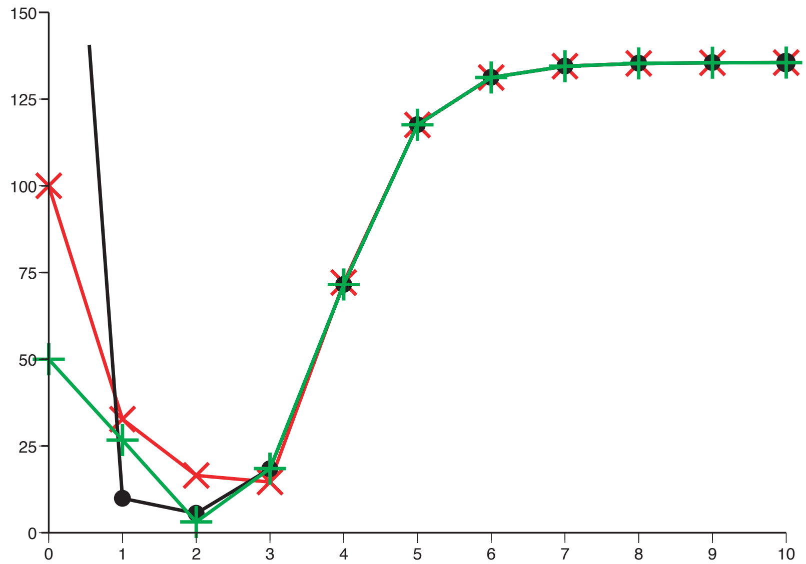

FIGURE 7. Time evolution of $\hat{x}_3$ corresponding to different initial conditions for the observer.

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