GENERATING SERIES: A COMBINATORIAL COMPUTATION

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GENERATING SERIES : A COMBINATORIAL COMPUTATION

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Abstract. The purpose of this paper is to apply combinatorial techniques for computing coefficients of rational formal series (G_k) in two noncommuting variables and their differences at orders k and k-1.
This in turn may help one to study the reliability and the quality of a model for non-linear black-box identification.
We investigate the quality of the model throughout the criteria of a measure of convergence.
We provide, by a symbolic computation, a valuation relating to the convergence of the family (B_k). This computation is a sum of differential monomials in the input functions and behavior system. We identify each differential monomial with its colored multiplicity and analyse our computation in the light of the free differential calculus.
We propose also a combinatorial interpretation of coefficients of (G_k). These coefficients are powers of an operator Θ which is in the monoid generated by two linear differential operators Δ and Γ. More than a symbolic validation, these computing tools are parameterized by the input and the system's behavior.

Keywords. nonlinear systems, combinatorics, generating series, symbolic computation, model validation.

1 Introduction

Our topic is at the junction of several areas, area of identification and validation of non-linear systems and area of generating series in non commutative variables.

- The model validation is a crucial problem in system identification [9]. It measures confidence in the model to reproduce the behavior of a dynamic system, under some hypothesis.
  In a discrete-time approach, the model validation is really an invalidation since it determines whether a discret sample input-output is inconsistent with the model[10].
  In [11], the authors develop new methods for validation of continuous-time non-linear systems. They use Barrier certificates whose existence prove inconsistency of a model, with experimental data.
  To validate a continuous-time model of an unknown dynamic system [1], we propose an exact symbolic computation of coefficients of rational power series. We use a deterministic model (versus probabilistic one) by considering that data noises are bounded.

- Several methods may be used for determining the input-output behavior of a dynamical system : transfer functions, functional expansions (Volterra series) [14], and generating power series [8]. For single input systems, the transfert function can be used to find a linear approximation by means of Padé approximants [3].
  The formal power series in several noncommutative variables are efficient tools for dealing with functional expansions. The behavior of causal functionals [8, 7]is uniquely described by two noncommutative power series : the generating series and the Chen series. The Chen series measures the input contribution and is independent of the system.
  The generating series (i.e the Fliess'series) is the geometric contribution and it is independent of the input. This formal series in noncommutative variables is defined on the 'encoding alphabet' $Z = \{z_0, z_1, \cdots z_n\}$ corresponding to the system input. Then the output is obtained by interpreting words over Z like iterated integrals of inputs. Finally, we see that noncommutative algebra of formal power series is useful in differential geometry, to study the behavior and the structure of analytic dynamical systems.

Generating series in non commutative variables have been recently studied by several authors [4, 5, 6]. In [2], the authors consider the problem of computing the coefficients of rational series in two commutative variables. It is possibly a more challenging task to deal with the non commutative case.
So, the purpose of this paper is to apply combinatorial techniques for getting a close formula of coefficients of a rational formal series (G_k) in two noncommutative variables and their differences at orders k and k-1.
This in turn may help one to study validation of a family (B_k) of bilinear systems, described by the series (G_k) and global modeling of an unknown dynamical system (Σ).
Computing and bounding these differences, we propose an estimation of the error due to approximations by (B_k). This error computation is a sum of differential monomials in the input functions and behavior system. We identify

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each differential monomial with its colored multiplicity and analyse our computation in the light of the free differential calculus.
This error computation allows one to better measure the impact of noisy inputs on the convergence of (B_k). Indeed, one can determine the contribution of the inputs and of the system in the error computation.

2 Problem statement and preliminaries

2.1 A local modeling of the unknown system

The problem consists in modeling an unknown dynamic system (Σ) for $t \in [0,T] = \bigcup_{i\in I}[t_i, t_i+d]$, when knowing some correlated sets of input/output.
We construct a behavioral model, based on the identification of its input/output functional (the generating series), in a neighborhood of every t_i, up to a given order k [1, 13]. At once a local modeling by a bilinear system $(B_i)_k$ around every t_i is provided. Then a family $((B_i)_{i\in I})_k$, global modeling of the unknown system is produced, such that the outputs of (Σ) and $((B_i)_{i\in I})_k$ coincide up to order k.

2.2 The bilinear system

We consider a certain class $(\mathcal{G}P)$ including the electric equation

$$y^{(1)}(t) = f(y(t)) + u(t) \tag{1}$$

where $u(t)$ is the input function
Σ, the unknown system is an affine system.
In this case, equation (1) can be written

$$(\Sigma) \qquad \left\{ \begin{array}{rcl} \dot{x} & = & A_0(x) + A_1(x)u(t) \\ y(t) & = & x(t) \end{array} \right.$$

- u(t) is the real input
- x(t) is the current state
- $A_0 = a^{(0)}\frac{\partial}{\partial x}$ where $\quad a^{(0)} = f(x)|_{x(0)}$
- $A_1 = \frac{\partial}{\partial x}$

The class $(\mathcal{G}P)$ encloses the nonlinear differential equation relating the current excitation i(t) and the voltage v(t) across a capacitor [14]

$$v^{(1)} + k_1 v + k_2 v^2 = i(t)$$

Let $a^{(i)} = f^{(i)}(x)|_{x(0)}$
We notice that the fundamental formula [14] provides the following bilinear system (B_k), approximating at order k

$$\left\{ \begin{array}{rcl} \dot{x}_k(t) & = & (M_0 + M_1 u(t))x_k(t) \\ \overline{y}_k(t) & = & \lambda x_k(t) \end{array} \right.$$

where $\lambda = (x(0) \quad 1 \quad 0\cdots 0)$

$$x_k(0) = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$M_0 = (C_{z_0 z_1^k})$ (resp $M_1 = (C_{z_1^{k+1}})$) expressed in basis $(C_{z_1^k})$.

$$M_0 = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ a^{(0)} & a^{(1)} & a^{(2)} & \cdots & a^{(k)} \\ 0 & a^{(0)} & 2a^{(1)} & \cdots & 0 \\ 0 & 0 & a^{(0)} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

$$M_1 = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

We have, if x_{ik} is the ith component of the state vector $x_k, \quad \forall i \leq k$

$$x_{1k}^{(1)}(t) = 0 \tag{2}$$

$$x_{2k}^{(1)}(t) = (a^{(0)} + u(t)) + \sum_{l=1}^{k-1} a^{(l)} x_{(l+1)k}(t) \tag{3}$$

$$x_{jk}^{(1)}(t) = (a^{(0)}+u(t))x_{(j-1)k}(t) + \sum_{l=1}^{k-j} \binom{j+l-2}{l} a^{(l)} x_{(j+l-1)k}(t) \tag{4}$$

for $\quad 3 \leq j \leq k-1$

$$x_{kk}(t) = (a^{(0)} + u(t))x_{(k-1)k}(t) \tag{5}$$

So, at order k, we obtain the ith derivative of the state vector x as a function of the previous ones. Our solution consists of two steps : to compute $x_{(k-n)k}^{(k-n+i)}(0)$ and to compute the difference of the ith derivative $x_{2k}^{(i)}(0) - x_{2(k-1)}^{(i)}(0)$.

2.3 Colored partitions and multiplicities

A number partition or multiplicity is a sequence $\mu = (\mu_1, \mu_2, \mu_3, \cdots)$ (often written as $1^{\mu_1}2^{\mu_2}3^{\mu_3}\cdots$) of nonnegative integers [15, 16]. On a single letter a, the differential monomials become :

$$a^{\mu} = (a^{(i_1)})^{e_1}(a^{(i_2)})^{e_2}\cdots(a^{(i_q)})^{e_q},$$

$$1 \leq i_1 < i_2 < \ldots i_q$$

Such a monomial is indexed by the following partition [15, 16] :

$$\mu = (i_1^{\mu_{i_1}} i_2^{\mu_{i_2}} \cdots i_q^{\mu_{i_q}})$$

Let $C = \{a, u\}$ be a set of two colors. We call colored partition on C an element of the free monoid generated by the cartesian product $N^+ \times N^+$ i.e. any finite sequence of couples of nonnegative integers

$$\mu = ((\mu_1^a, \mu_1^u), (\mu_2^a, \mu_2^u), \cdots)$$

So, a colored partition μ will denote the differential monomial

$$a^\mu = (a^{(i_1)})^{e_1} \cdots (a^{(i_p)})^{e_p} (u^{(j_1)})^{f_1} \cdots (u^{(j_q)})^{f_q}$$

$$1 \leq i_1 < i_2 < \dots i_p, \quad 1 \leq j_1 < i_2 < \dots j_q$$

where e_l (resp f_l) $= \mu_{i_l}^a$ (resp $\mu_{j_l}^u$). The weight and the size of μ are defined as follows :

$$wgt(\mu) = \sum_c \sum_k k.\mu_k^c$$

$$size(\mu) = \sum_c \sum_k \mu_k^c$$

The empty partition is noted ϵ.
If L is the set of colored partitions, we define a partial order $\ll$ on L :

$$\nu = \{(\nu_i^a, \nu_i^u)\} \ll \mu = \{(\mu_i^a, \mu_i^u)\}$$

if

$$\nu_i^a \leq \mu_i^a \quad and \quad \nu_i^u \leq \mu_i^u \quad \forall i$$

L, with this partial ordering forms a Young lattice. [17]
We consider now B_i a subset of L defined by :

$$\{\mu / wgt(\mu) = i\}$$

and we note $I(\mu_{max})$ the order ideal generated by μ_{max}, if

$$\mu_{max} = max(\mu / \mu \in B_i)$$

2.4 Forest of increasing trees

A forest of increasing trees on $\{1, \cdots, n\}$, according to [18], is a set of rooted increasing trees, the set of vertices of which is exactly [n] and such each vertex is smaller than all its successors. Reutenauer shows that the n-power of a linear differential operator is equal to the sum of the labels of all forests of increasing trees on $\{1, \cdots n\}$.
We introduce in the next section, the notion of forest of increasing colored trees, to take into account the multiplicity of operators.

3 Main results

3.1 First step : Computation of $x_{(k-n)k}^{(k-n+i)}(0)$

By derivating and term's regrouping, we can show that :

$$x_{(k-n)k}^{(k-n+i)}(0)$$

$$=$$

$$\sum_{m=1}^{min(i+1,k-1)} a^{(m)} \sum_{l=1}^{k-n-1} \binom{k-n-l+m-1}{m}$$

$$(a^{(0)} + u^{(0)})^{l-1} x_{(k-n-l+m)k}^{(k-n-l+m+i-m)}(0)$$

$$+$$

$$\sum_{m=1}^{i+1} u^{(m)} \sum_{l=1}^{k-n-2} \binom{k-n-l+i}{m}$$

$$(a^{(0)} + u^{(0)})^{l-1} x_{(k-n-l)k}^{(k-n-l+i-m)}(0) + 1$$

$$(if \quad m = i+1)$$

We analyze now these equations in the light of the free differential calculus. Considering the derivative $a^{(i)}$ and $u^{(i)}$ specialized at time t=0 as differential letters, it is clear that our computation is a sum of differential monomials in a and u.

3.1.1 Combinatorial analysis of our computation

Let us now interpret combinatorially our computation by identifying each differential monomial with its colored multiplicity. The recursive relation is captured by the operation :

$$\mu_{max} \odot c = \sum_{\substack{\nu \in I(\mu_{max}) \\ wgt(\nu) = j \leq i}} c^{(i-j+1)}.\nu$$

By factorizing according to the colored partitions, we get :

$$x_{(k-n)k}^{(k-n+i)} = \sum_c \sum_{\substack{\nu \in I(\mu_{max}) \\ wgt(\nu) = j \leq i}} c^{(i-j+1)}.\nu.g^1_{(c^{(i-j+1)}\nu)}$$

where :

$$g^l_{a^{(m)}\nu} = (a^{(0)} + u^{(0)})^{m+1} \sum_{p=m}^{n_l+m} \binom{l}{m} g_v^p$$

and

$$g^l_{u^{(m)}\nu} = (a^{(0)} + u^{(0)})^{m-1} \sum_{p=1}^{n_l} \binom{l+i+1}{m} g_v^p$$

with $n_1 = k - n - 1, \quad n_l = l \quad \forall l > 1$

$$g_\epsilon = 1$$

3.1.2 Computation of $x_{(k-n)k}^{(k-n+i)}(0)$

We consider now permutations of a colored partition μ on an alphabet $X = \bigcup_{c\in C} X_c$. A permutation [17] of μ is a word in which each letter belongs to X and for each $x_i \in X$, the total number of appearances of x_i in the word is μ_i^c, for some $c \in C$
Let us note $\pi = \xi_1\xi_2\cdots\xi_{size(\mu)}$ a permutation of μ and σ_μ the set of permutations of μ.
Since, our alphabet

$$X_a = \{a^{(p)}|p = 1, min(k-1, i+1)\}$$

and

$$X_u = \{u^{(p)}|p = 1, i+1)\}$$
$$\xi_j = c^{(i_j)}$$

, for some (c, i_j).
$x_{(k-n)k}^{(k-n+i)}$ is a linear combination of monomial $y_1^{\lambda_1}\cdots y_n^{\lambda_n}(y_i \in X_a \bigcup X_u)$ and all distinct monomials obtained from it by a permutation of variables.
We get finally , if $s = (\sum_j j| \quad \mu_j^u \neq 0)$ and r =size(μ)

$$x_{(k-n)k}^{(k-n+i)} = \sum_{wgt(\mu)=i+1} \mu.(a^{(0)} + u^{(0)})^{k-n+i-r-s} g_\mu^n$$

$$g_\mu^n = \sum_{\pi\in\sigma_\mu} A_1 \prod_{j=2}^{r} A_j + b$$

where:

$$A_j = \begin{cases} \sum_{m_j=i_j}^{m_{j-1}+i_j} \binom{m_j}{i_j} & if \quad \xi_j = a^{(i_j)} \\ \sum_{m_j=1}^{m_{j-1}} \binom{m_j + i - j + 2}{i_j} & if \quad \xi_j = u^{(i_j)} \end{cases}$$

$$A_1 = \begin{cases} \sum_{m_1=m}^{k-n-2+m} \binom{m_1}{i_1} & if \quad \xi_1 = a^{(i_1)} \\ \sum_{m_j=1}^{k-n-2} \binom{m_1 + i + 1}{i_1} & if \quad \xi_1 = u^{(i_1)} \end{cases}$$

and b = 1 if $\xi_1 = u^{(i+1)}$, 0 otherwise.

Remark : $x_{(k-n)k}^{(k-n+i)}$ is not a symmetric polynomial even if its structure is the similar, because input and system contributions are different.
Example : i=1

$$x_{(k-n)k}^{(k-n+1)} = \sum_{l=1}^{k-n-2} \binom{l}{1} \sum_{m=1}^{l} \binom{m}{1} (a^{(0)} + u^{(0)})^{k-n-1} a^{(1)} a^{(1)}$$

$+$

$$\sum_{l=2}^{k-n} \binom{l}{2} (a^{(0)} + u^{(0)})^{k-n} a^{(2)}$$

$+$

$$\sum_{l=1}^{k-n-2} \binom{l+2}{2} (a^{(0)} + u^{(0)})^{k-n-2} u^{(2)}$$

$+$

$$(\sum_{l=1}^{k-n-1} \binom{l}{1} \sum_{m=1}^{l} \binom{m+1}{1}$$

$+$

$$\sum_{l=1}^{k-n-2} \binom{l+2}{1} \sum_{m=1}^{l} \binom{m}{1})(a^{(0)} + u^{(0)})^{k-n-2} a^{(1)} u^{(1)}$$

$+$

$$\sum_{l=1}^{k-n-2} \binom{l+2}{1} \sum_{m=1}^{l} \binom{m+1}{1} (a^{(0)} + u^{(0)})^{k-n-3} u^{(1)} u^{(1)}$$

We give a proof of this assertion in the annex.

3.2 Second step: Computation of $x_{2k}^{(k+i)}(0) - x_{2(k-1)}^{(k+i)}(0)$

The first derivative coincide up to order k-2, but at order k-1, we have
$x_{2k}^{(k-1)} - x_{2(k-1)}^{(k-1)} = 0$ and $x_{jk}^{(k-1)} - x_{j(k-1)}^{(k-1)} \neq 0$.
Let M (resp P) the set of partitions on the single letter a (resp u),
We define :

- W_i a subset of M defined by

$$\{\nu|1 \leq size(\nu) \leq i+2\}$$

- V_i a subset of P defined by

$$\{\lambda|size(\lambda) = \lfloor\frac{i}{2}\rfloor,$$
$$wgt(\lambda) \leq i-2,$$
$$or \quad \lambda = u^{(i-2)},$$
$$or \quad \lambda = u^{(i-1)}\}$$

- S_l a subset of L defined by

$$\{\mu|wgt(\mu) = l\}$$

We define now an operation $\nabla : M \times P \times L \mapsto L$

$$\nabla(\nu, \lambda, \mu) = ((\nu_i + \mu_i^a, \lambda_i + \mu_i^c))_i$$

and a subset P_t of $L \quad \forall 0 \leq t \leq i$

$$P_t = \{\tau = \nabla(\nu, \lambda, \mu)| \quad \mu \in S_t, \lambda \in V_i, \nu \in W_i, wgt(\tau) = k+i-1\}$$

We obtain, by a straightforward computation :

$$x_{2k}^{(k+i)} - x_{2(k-1)}^{(k+i)} =$$

$$\sum_{\substack{\nabla(\nu,\lambda,\mu)\in P_t \\ 0\leq t\leq i}} \nabla(\nu, \lambda, \mu) \quad h_\nu.f_\lambda.g_\mu^1.(a^{(0)} + u^{(0)})^{k+i-2-r_1-s}$$

where

$$f_\lambda = \sum_{\pi\in\sigma_\lambda} \prod_{l=1}^{size(\lambda)} \binom{k+i-2l}{k+i-2l-i_j}$$

$$h_\nu = \begin{cases} \sum_{\pi\in\sigma_\nu^1} \prod_{j=1}^{r-2} \binom{i_j+i_{j+1}-1}{i_{j+1}} \binom{k-2}{i_r} & size(\nu) \neq 1 \\ 1 & size(\nu) = 1 \end{cases}$$

with

$$r = size(\nu)$$

$$r_1 = r + size(\mu)$$

$$s = (\sum_j j | \mu_j^u \neq 0)$$

$$\pi = \xi_1 \xi_2 \cdots \xi_r$$

$$\xi_j = c^{(i_j)}$$

g_μ^1 defined previously.

$$\sigma_\nu^1 = \{\pi \in \sigma_\nu | \pi \neq \nu_1.\mu, size(\nu_1) < size(\nu)$$
$$and \quad \pi \neq (a^{(1)})^{r-1}.\xi_r\}$$

- (i) We remove permutations π which could be equal to a partition $\tau = \nu_1.\mu$, for some τ
- (ii) We remove also permutations which begin by the first derivative of letter a.

Example: i=1, k=6

$$x_{26}^{(7)} - x_{25}^{(7)}$$

=

$$a^{(1)}(x_{26}^{(6)} - x_{25}^{(6)}) + a^{(2)}(x_{36}^{(6)} - x_{35}^{(6)})$$

+

$$a^{(3)}(x_{46}^{(6)} - x_{45}^{(6)}) + a^{(4)}(x_{56}^{(6)} - x_{55}^{(6)}) + a^{(5)}x_{66}^{(6)}$$

At order 6, we have :

- $$x_{26}^{(6)} - x_{25}^{(6)} =$$
  $$(\binom{4}{3} + \binom{4}{2})(a^{(0)} + u^{(0)})^4.a^{(2)}.a^{(3)}$$
  +
  $$\binom{4}{1}(a^{(0)} + u^{(0)})^4.a^{(4)}.a^{(1)}$$
  +
  $$(a^{(0)} + u^{(0)})^5.a^{(5)}$$
- $$x_{36}^{(6)} - x_{35}^{(6)} =$$
  $$(\binom{2}{1}\binom{4}{3})(a^{(0)} + u^{(0)})^4.a^{(1)}.a^{(3)}$$
  +
  $$(\binom{3}{2}\binom{4}{2}(a^{(0)} + u^{(0)})^4.a^{(2)}.a^{(2)}$$
  +
  $$\binom{4}{3}a^{(3)}(\sum_{l=1}^{l=4}\binom{l}{1})(a^{(0)}+u^{(0)})^3(a^{(1)}(a^{(0)}+u^{(0)})+u^{(1)})$$
- $$x_{46}^{(6)} - x_{45}^{(6)} =$$
  $$(\binom{4}{3})(a^{(0)}+u^{(0)})^5.a^{(3)}+(\binom{3}{1}\binom{4}{2}(a^{(0)}+u^{(0)})^4.a^{(1)}.a^{(2)}$$
  +
  $$\binom{4}{2}a^{(2)}(\sum_{l=1}^{l=4}\binom{l}{1})(a^{(0)}+u^{(0)})^3(a^{(1)}(a^{(0)}+u^{(0)})+u^{(1)})$$
- $$x_{56}^{(6)} - x_{55}^{(6)} =$$
  $$(\binom{4}{2})(a^{(0)} + u^{(0)})^5.a^{(2)}$$
  +
  $$\binom{4}{1}a^{(1)}(\sum_{l=1}^{4}\binom{l}{1})(a^{(0)}+u^{(0)})^3(a^{(1)}(a^{(0)}+u^{(0)})+u^{(1)})$$
- $$x_{66}^{(6)}$$
  =
  $$(a^{(0)} + u^{(0)})^5 a^{(1)}(\sum_{l=1}^{4}\binom{l}{1})$$
  +
  $$(a^{(0)} + u^{(0)})^4 u^{(1)}(\sum_{l=1}^{5}\binom{l}{1})$$

We notice in this example that :
- Coefficients of $a^{(2)}a^{(3)}a^{(1)}$ and $a^{(3)}a^{(2)}a^{(1)}$ are different from those of $a^{(1)}a^{(2)}a^{(3)}$, $a^{(2)}a^{(1)}a^{(3)}$(condition (i))
- Coefficient of $a^{(1)}a^{(1)}a^{(4)}$ is nil (condition(ii))

3.3 Coefficients of the generating series

We give, in this section, a combinatorial interpretation of coefficients of generating series.

3.3.1 Forest of colored increasing trees

From the definition of a forest of increasing trees, we introduce the concept of a forest of colored increasing trees, to take into account the multiplicity of operators. We use a notion of a "colored partition" ([17]). For each vertex i, we color i any one of c_i colors. Let C the set of colors. We define colored increasing trees on cartesian product $\{1, \cdots, n\} \times C$.

3.3.2 Combinatorial interpretation

The label of a forest on $\{1, \cdots, n\} \times C$ is a noncommutative monomial and is defined as :

$$\Pi_{(i,c)\in\{1,\cdots,n\}\times C} P^{(\alpha(i,c))} \frac{\partial}{\partial q}^{k}$$

where
$P(q) = 1$ or $P(q) = a^{(0)}$

$\alpha(i,c)$ is the number of sons of the node (i,c)
k is the number of trees of the forest. We extend this concept to multiple operators by introducing colored increasing trees.

3.3.3 Application

We consider the class $(\mathcal{G}P)$ given in the previous section. According to ([8]), the coefficients of the generating series are :

$$\langle G \mid z_{i_1} z_{i_2} \cdots z_{i_k} \rangle = [A_{i_1} \circ A_{i_2} \circ \cdots \circ A_{i_k} \circ h(q)]_0$$

where :

$$A_{i_j} = a^{(0)} \frac{\partial}{\partial q}$$

or

$$A_{i_j} = \frac{\partial}{\partial q}$$

Let us define two differential operators

$$\Delta = a^{(0)} \frac{\partial}{\partial q}$$

$$\Gamma = \frac{\partial}{\partial q}$$

These coefficients are powers of an operator Θ which is in the monoid generated by the two linear differential operators Δ and Γ . $C = \{c_1, c_2\}$

The 2-power of operator Θ is :

$$\Theta^2 = \langle G \mid z_1 z_0 \rangle + \langle G \mid z_0 z_1 \rangle + \langle G \mid z_0 z_0 \rangle + \langle G \mid z_1 z_1 \rangle$$

The colored increasing trees are :

The labels of these trees are monomials $P^{(0)} P^{(1)} \frac{\partial}{\partial q}$, $P^{(0)^2} \frac{\partial}{\partial q}^2$
Each colored vertex is associated to $P(q) = 1$ or $P(q) = a^{(0)}$
We note that, since the observation function h(q) is the identity function, all the powers of $\frac{\partial}{\partial q}^n$, $n \geq 2$ are zero.

3.4 Computation of $(\mathbf{\overline{y}_k(t) - \overline{y}_{k-1}(t)})$

The (k-1) first derivative of the two outputs coincide at point t=0. So, using Taylor's development, we can write

$$\overline{y}_k(t) - \overline{y}_{k-1}(t) = \sum_{i \geq k} (\overline{y}_k^{(i)}(0) - \overline{y}_{k-1}^{(i)}(0)) . \frac{t^i}{i!}$$

Taking into account that $\overline{y}_k^{(i)}(0) = x_{2k}^{(i)}(0)$, we obtain a right computation of the output's difference at order k and k-1.

4 Application: Numerical examples

Our examples are based on the computation of the two consecutive outputs at order 2 and 3 (we note $E3$ this error), in the case of equations like electric equation. Taylor expansion is at order 5. So we get :

$$\overline{y}_3(t) - \overline{y}_2(t)$$

$$=$$

$$\frac{t^3 . a_2 . (a_0 + u(0))^2}{3!} +$$

$$\frac{t^4 (q_{33}^{(4)}(0) - q_{23}^{(4)}(0))}{4!} +$$

$$\frac{t^5 (q_{33}^{(5)}(0) - q_{23}^{(5)}(0))}{5!} + \epsilon(t)$$

where

$$\epsilon(t) = O(t^6)$$

and

$$(q_{33}^{(4)}(0) - q_{23}^{(4)}(0) =$$

$$2a_1 a_2 (a_0 + u(0))^2 + 3a_2 (a_0 + u(0)) u^{(1)}(0)$$

and

$$(q_{33}^{(5)}(0) - q_{23}^{(5)}(0) = 3a_1 a_2 (a_0 + u(0))^2 +$$

$$7a_1 a_2 (a_0 + u(0)) u^{(1)}(0) +$$

$$a_2^2 (a_0 + u(0))^3 +$$

$$4a_2 (a_0 + u(0)) u^{(2)}(0) +$$

$$3a_2 (u^{(1)})^2(0)$$

4.1 First examples

We try in this example to measure the impact of the system. We show in [19] that the system is bounded input bounded output (BIBO) for $a^{(1)} \leq 0$ and not BIBO for $a^{(1)} > 0$.
So, we took $k_1 = k_2 = 1$ in the previous example ($a^{(1)} \leq 0$ for these values of parameters).
In the array below, the parameters are $k_1 = k_2 = -1$ ($a^{(1)} > 0$). The input function is the same (sin(t)).
We see that the error is more important when the system is not stable (not BIBO).

t	E3
0.1	0.0016
0.5	0.41
1.0	6.65
1.5	38.54
2.	140.8

Table 1: First Table .

t	E3
0.1	- 0.0201
0.5	- 1.7
1.0	-3.4
1.5	19.47
2.	211.2

Table 2: Second Table .

4.2 Second examples

We measure here the impact of input functions. The parameters of the system are $k_1 = k_2 = 1$(Case of stability of the system).

We take $i(t) = 100 * t^6 + 10$.
For an input function $i(t) = 100 * t^6 + 1000$, the error is about 10^9.
We point up three facts, through these examples

- The error is too important beyond some interval and our model is not "acceptable"
- The error depends on system behavior and its stability
- The error is different from smooth inputs to rough inputs

5 Conclusion

The validation which is presented in this paper is not statistical.
It consists in evaluating the convergence of a family (B_k) of bilinear systems on the unknown system (Σ) by an effective symbolic computation. It allows to determine intervals where our model is "'acceptable"', as shown with numerical examples.
It displays the respective contributions of the input and of the system itself.
A symbolic overestimation of the model's output, for a bounded input, displays too the contributions of the input and of the system.
More than a symbolic validation, these computing tools are parameterized by the input and the system's behavior. They can particularly provide a valuation process for rough and oscillating inputs as well as for smooth inputs.

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