On the complexity of cycle enumeration using Zeons

René Schott, Stacey Staples

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René Schott[*], G. Stacey Staples[†]

Abstract

Nilpotent adjacency matrix methods are employed to enumerate k-cycles in simple graphs on n vertices for any $k \leq n$. The worst-case time complexity of counting k-cycles in an n-vertex simple graph is shown to be $\mathcal{O}(n^{\alpha+1}2^n)$, where $\alpha \leq 3$ is the exponent representing the complexity of matrix multiplication. The average case time complexity of counting k-cycles in an n-vertex simple random graph with equiprobable edges of probability p is found to be $\mathcal{O}(n^{\alpha+1}(1+p)^n)$. When k is fixed, the enumeration of all k-cycles in an n-vertex graph is of time complexity $\mathcal{O}(n^{\alpha+k-1})$. The storage complexity of our approach is $\mathcal{O}(n^2 2^n)$. Experimental results detailing computation times (in seconds) are compared with algorithms based on the approaches of Bax and Tarjan.

AMS subject classification: 68Q25, 60B99, 05C38, 05C85
Keywords: cycles, enumeration, complexity, zeons

1 Introduction

In earlier theoretical work, the current authors have shown that the complexity of a number of NP-class problems from graph theory require only a polynomial number of operations in a 2^n-dimensional commutative algebra denoted by $\mathcal{C}\ell_n{}^{\rm nil}$, and referred to herein as a "zeon algebra" [6]. In particular, the problem of enumerating k-cycles in any graph on n vertices requires $\mathcal{O}(n^\alpha \log k)$ $\mathcal{C}\ell_n{}^{\rm nil}$ operations, or "$\mathcal{C}\ell$ops," where $\alpha \leq 3$ denotes

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the exponent associated with matrix multiplication. The authors have applied nilpotent adjacency methods to the study of random graphs [4] and explored connections between nilpotent adjacency matrices and quantum random variables [5].

In the current work, computational complexity is studied in greater detail by counting algebraic operations at the basis blade level. While counting the number of basis blade operations in ${\mathcal{C}\ell_n}^{\rm nil}$ performed by an algorithm may be a natural measure of complexity if one assumes the existence of a computer architecture capable of naturally dealing algebraic elements, it is not natural in the context of classical computing.

The zeon algebra does however lend itself to convenient symbolic computations. The current work illustrates these symbolic computations and considers some practical advantages for doing so. In particular, experimental comparisons between the zeon approach and other classical algorithms are presented.

Examples generated with Mathematica were computed on a 2.4 GHz MacBook Pro with 4 GB of 667 MHz DDR2 SDRAM running Mathematica 6 for MAC OS X with the *Combinatorica* package. Cycle enumeration is accomplished using the nilpotent adjacency matrix approach, Bax's approach, and the HamiltonianCycle procedure found in the Mathematica package *Combinatorica*. Time plots comparing the three approaches are included. Mathematica code used to generate examples can be found online through the second-named author's web page, `http://www.siue.edu/~sstaple`.

2 Theoretical Considerations

A *graph* $G = (V, E)$ is a collection of vertices V and a set E of unordered pairs of vertices called *edges*. Two vertices $v_i, v_j \in V$ are said to be *adjacent* if there exists an edge $e_{ij} = \{v_i, v_j\} \in E$. In this case, the vertices v_i and v_j are said to be *incident* with e_{ij}.

A *k-walk* $\{v_0, \ldots, v_k\}$ in a graph G is a sequence of vertices in G with *initial vertex* v_0 and *terminal vertex* v_k such that there exists an edge $(v_j, v_{j+1}) \in E$ for each $0 \leq j \leq k-1$. A k-walk contains k edges. A *k-path* is a k-walk in which no vertex appears more than once. A *closed k-walk* is a k-walk whose initial vertex is also its terminal vertex. A *k-cycle* is a closed k-path with $v_0 = v_k$. It is well-known that the problem of enumerating a graph's cycles is known to be NP-complete [3].

Bax's approach to cycle enumeration uses powers of a graph's adjacency matrix with the principle of inclusion-exclusion to count all Hamiltonian cycles in $\mathcal{O}(2^n\text{poly}(n))$ time $\mathcal{O}(\text{poly}(n))$ storage. [1]. Enumerating only those cycles of length k is accomplished by applying Bax's algorithm to all k-vertex subgraphs. Consequently, the complexity for counting k-cycles is $\mathcal{O}\left(\binom{n}{k}2^k\,\text{poly}(k)\right)$. For fixed k, this is $\mathcal{O}(\text{poly}(n))$, since $\binom{n}{k} \leq n^k$ for all $n \geq k$. For k increasing with n however, this is $\mathcal{O}\left(\binom{n}{n/2}2^{n/2}\text{poly}(n)\right) = \mathcal{O}\left(2^{3n/2}\text{poly}(n)\right)$.

Tarjan's algorithm enumerates all cycles in a graph on n vertices with time complexity $\mathcal{O}((n+|E|)(C+1))$ when applied to a graph with C cycles [8]. The storage complexity is $\mathcal{O}(n+|E|+S)$, where S is the sum of the lengths of all cycles. Note that the number of cycles on a k-vertex subgraph is potentially of order $k!$, while the number of such subgraphs is of order $\binom{n}{k}$.

A convenient and practical Tarjan-type implementation is the HamiltonianCycle procedure found in the Mathematica package *Combinatorica*. The algorithm uses backtracking and look-ahead to enumerate all Hamiltonian cycles in a graph on n vertices. The implementation utilized for the examples in this paper enumerates cycles of length k in an n-vertex graph G by applying HamiltonianCycle to all k-vertex subgraphs of G. Implementations of this Tarjan-like approach are referred to henceforth as "CombiTarjan."

2.1 Nilpotent adjacency matrices

Definition 2.1. The *n-particle zeon algebra*, denoted by ${\mathcal{C}\ell_n}^{\text{nil}}$, is defined as the real abelian algebra generated by the collection $\{\zeta_i\}$ $(1 \leq i \leq n)$ along with the scalar $1 = \zeta_0$ subject to the following multiplication rules:

$$\zeta_i\,\zeta_j = \zeta_j\,\zeta_i \quad \text{for } i \neq j, \text{ and} \tag{2.1}$$

$$\zeta_i{}^2 = 0 \quad \text{for } 1 \leq i \leq n. \tag{2.2}$$

It is evident that a general element $u \in {\mathcal{C}\ell_n}^{\text{nil}}$ can be expanded as

$$u = \sum_{I \in 2^{[n]}} u_I\,\zeta_I\,, \tag{2.3}$$

where $I \in 2^{[n]}$ is a subset of $[n] = \{1, 2, \ldots, n\}$ used as a multi-index, $u_I \in \mathbb{R}$, and $\zeta_I = \prod_{\iota \in I} \zeta_\iota$.

Remark 2.2. The zeon algebra ${\mathcal{C}\ell_n}^{\text{nil}}$ can be realized as a commutative subalgebra of the Grassmann algebra $\bigwedge V$ over a $2n$-dimensional vector space V with orthonormal basis $\{\gamma_i\}$ by defining $\zeta_i = \gamma_i\gamma_{n+i}$ for each $1 \leq i \leq n$.

A canonical basis element ζ_I is referred to as a *blade*. The number of elements in the multi-index I is referred to as the *grade* of the blade ζ_I.

The *scalar sum* evaluation of an element $u \in \mathcal{C}\ell_n{}^{\text{nil}}$ is defined by

$$\left\langle\left\langle \sum_{I \in 2^{[n]}} u_I \,\zeta_I \right\rangle\right\rangle = \sum_{I \in 2^{[n]}} u_I. \tag{2.4}$$

Definition 2.3. A *blade operation* in $\mathcal{C}\ell_n{}^{\text{nil}}$ is defined as computing the sum or product of two basis blades. In particular, for multi-indices I and J, each of the following computations is regarded as a blade operation:

$$(a\zeta_I)(b\zeta_J) = \begin{cases} 0 & \text{if } I \cap J \neq \emptyset, \\ (ab)\,\zeta_{I\cup J} & \text{otherwise;} \end{cases} \tag{2.5}$$

$$a\zeta_I + b\zeta_J = \begin{cases} (a+b)\zeta_I & \text{if } I = J, \\ a\zeta_I + b\zeta_J & \text{otherwise.} \end{cases} \tag{2.6}$$

Recalling the correlation between subsets of $[n]$ and bit strings of length n, each basis blade ζ_I is uniquely associated with a binary string $\underline{I}$. Letting $\mathcal{S}_n$ denote the set of all length-n bit strings with bitwise logical operators and defining

$$\underline{I} \ominus \underline{J} := \begin{cases} 0 & \text{if } \underline{I} \,\text{AND}\, \underline{J} \neq \underline{\emptyset}, \\ \underline{I} \,\text{OR}\, \underline{J} & \text{otherwise,} \end{cases} \tag{2.7}$$

the pair $(\mathcal{S}_n, \ominus)$ is seen to be an Abelian semigroup. The group algebra $\mathbb{R}\mathcal{S}_n$ is then isomorphic to $\mathcal{C}\ell_n{}^{\text{nil}}$.

Note that blade addition in $\mathbb{R}\mathcal{S}_n$ is made explicit by

$$a\underline{I} + b\underline{J} = \begin{cases} (a+b)\underline{I} & \text{if } \underline{I} \,\text{XOR}\, \underline{J} = \underline{\emptyset}, \\ a\underline{I} + b\underline{J} & \text{otherwise.} \end{cases} \tag{2.8}$$

The cost of a basis blade multiplication in $\mathcal{C}\ell_n{}^{\text{nil}}$ is then equal to that of computing first the bitwise AND and then the bitwise OR of two n-bit words, which is known to be $\mathcal{O}(n)$. Summing a pair of basis blades is similarly $\mathcal{O}(n)$.

Given arbitrary elements $u, v \in \mathcal{C}\ell_n{}^{\text{nil}}$, let ν_u and ν_v denote the respective numbers of nonzero coefficients in the canonical zeon expansions of u and v. The number of blade products involved when computing uv is then $\mathcal{O}(\nu_u\nu_b)$, and the number of blade sums is similarly $\mathcal{O}(\nu_u\nu_v)$. Taking the costs of blade operations into consideration, the complexity of expanding the product uv is seen to be $\mathcal{O}(n\,\nu_u\nu_v)$.

This complexity is implicit in proofs throughout the remainder of the paper.

Remark 2.4. The Mathematica implementation of ${\mathcal{C}\ell_n}^{\text{nil}}$ used in the examples contained herein is based on subset operations rather than binary representations of subsets and bit operations. The additional overhead is offset by the relatively low dimensions of the examples.

Definition 2.5. Let G be a graph on n vertices, either simple or directed with no multiple edges, and let $\{\zeta_i\}$, $1 \leq i \leq n$ denote the nilpotent generators of ${\mathcal{C}\ell_n}^{\text{nil}}$. Define the *nilpotent adjacency matrix* associated with G by

$$\mathcal{A}_{ij} = \begin{cases} \zeta_j \text{ if } (v_i, v_j) \in E(G) \\ 0 \text{ otherwise.} \end{cases} \tag{2.9}$$

Letting the vertices $V = \{v_1, \ldots, v_n\}$ be associated with the standard basis of $\mathbb{R}^n$ and recalling Dirac notation, the i^{th} row of $\mathcal{A}$ is conveniently denoted by $\langle v_i|\,\mathcal{A}$, while the j^{th} column is denoted by $\mathcal{A}\,|v_j\rangle$.

Theorem 2.6. *Let $\mathcal{A}$ be the nilpotent adjacency matrix of an n-vertex graph G. For any $k > 1$ and $1 \leq i, j \leq n$,*

$$\left\langle v_i|\mathcal{A}^k|v_j\right\rangle = \sum_{\substack{(w_1,\ldots,w_k)\in V^k \\ (w_k=v_j)\wedge(m\neq\ell\Rightarrow w_m\neq w_\ell)}} \zeta_{\{w_1,\ldots,w_k\}} = \sum_{\substack{I\subseteq V \\ |I|=k}} \omega_I\zeta_I, \tag{2.10}$$

where ω_I denotes the number of k-step walks from v_i to v_j visiting each vertex in I exactly once when initial vertex $v_i \notin I$, and revisiting v_i exactly once when $v_i \in I$. In particular, for any $k \geq 3$ and $1 \leq i \leq n$,

$$\left\langle v_i|\mathcal{A}^k|v_i\right\rangle = \sum_{\substack{I\subseteq V \\ |I|=k}} \omega_I\zeta_I, \tag{2.11}$$

where ω_I denotes the number of k-cycles on vertex set I based at $v_i \in I$.

Proof. Because the generators of ${\mathcal{C}\ell_n}^{\text{nil}}$ square to zero, a straightforward inductive argument shows that the nonzero terms of $\left\langle v_i|\mathcal{A}^k|v_j\right\rangle$ are multivectors corresponding to two types of k-walks from v_i to v_j: self-avoiding walks (i.e., walks with no repeated vertices) and walks in which v_i is repeated exactly once at some step but are otherwise self-avoiding. Walks of the second type are zeroed in the k^{th} step when the walk is closed. Hence, terms of $\left\langle v_i|\mathcal{A}^k|v_i\right\rangle$ represent the collection of k-cycles based at v_i.

□

In light of this theorem, the name "nilpotent adjacency matrix" is justified by the following corollary.

Corollary 2.7. *Let $\mathcal{A}$ be the nilpotent adjacency matrix of a simple graph on n vertices. For any positive integer $k \leq n$, the entries of $\mathcal{A}^k$ are homogeneous elements of grade k in ${\mathcal{C}\ell_n}^{\text{nil}}$. Moreover, $\mathcal{A}^k = \mathbf{0}$ for all $k > n$.*

Another immediate corollary is that

$$\left\langle\left\langle \operatorname{tr}\left(\mathcal{A}^k\right)\right\rangle\right\rangle = k\ \left|\{k\text{-cycles in } G\}\right|, \tag{2.12}$$

since each k-cycle appears with k choices of base point along the main diagonal of $\mathcal{A}^k$.

Note that the complexity of computing $\mathcal{A}^k$ may vary depending on various methods of computing powers. The *iterated method* requires $k-1$ matrix products to compute

$$\mathcal{A}^k := \begin{cases} \mathcal{A} & \text{if } k = 1, \\ \mathcal{A}^{k-1}\mathcal{A} & \text{otherwise.} \end{cases} \tag{2.13}$$

Given the binary representation of positive integer k, the *successive squares method* requires $\lfloor \log_2 k \rfloor$ matrix products and matrix sums to compute. In particular, letting $\underline{k}$ be a set of nonnegative integers such that $k = \sum_{\ell \in \underline{k}} 2^\ell$, then

$$\mathcal{A}^k = \sum_{\ell \in \underline{k}} \mathcal{A}^{2^\ell} \tag{2.14}$$

While the successive squares method is generally more efficient than the iterated method, the application to nilpotent adjacency matrices is not straightforward. The next result is based on the iterated method.

First, define the following useful notation for positive integers n and $k \leq n$:

$$\tau_k^n := \sum_{\ell=1}^{k} \binom{n}{k}. \tag{2.15}$$

Theorem 2.8. *The average-case complexity for enumerating cycles of arbitrary length in a homogeneous random graph on n vertices with edge probability p using the nilpotent adjacency matrix method is $\mathcal{O}\left(n^{\alpha+1}(1+p)^n\right)$. Moreover, for $k \leq n$ the average-case complexity of enumerating k-cycles is $\Omega\left(n^{\alpha+1}\tau_{k-1}^{n-1}\right)$.*

Proof. In light of Theorem 2.6, for any $k \leq n$, computing $\mathcal{A}^{k+1} = (\mathcal{A}^k)A$ requires computing

$$\left\langle v_i|\mathcal{A}^{k+1}|v_j\right\rangle = \sum_{\ell=1}^{n} \left\langle v_i|\mathcal{A}^k|v_\ell\right\rangle \langle v_\ell|\mathcal{A}|v_j\rangle \tag{2.16}$$

for all $1 \leq i, j \leq n$. Entries of $\mathcal{A}^k$ are homogeneous grade-k elements of $\mathcal{Z}_n$. Thus, the average number of blade products computed is the product of the expected numbers of nonzero coefficients in the canonical expansions of $\left\langle v_i|\mathcal{A}^k|v_\ell\right\rangle$ and $\langle v_\ell|\mathcal{A}|v_j\rangle$.

Claim. Let $n \geq 3$ and $2 < k \leq n$. For any $1 \leq i, j \leq n$, the expected number of nonzero coefficients in the canonical expansion of $\left\langle v_i|\mathcal{A}^k|v_j\right\rangle$ satisfies the following inequality:

$$p^k \binom{n-1}{k-1} \leq \mathbb{E}(\sharp \{\text{nonzero coefficients}\}) \leq p^{k-1}\binom{n-1}{k-1}. \tag{2.17}$$

Moreover, in the case $k = 2 < n$,

$$p^2(n-2) \leq \mathbb{E}(\sharp \{\text{nonzero coefficients}\}) \leq p(n-1). \tag{2.18}$$

Proof of claim. By Theorem 2.6, the expected number of nonzero coefficients in the canonical zeon expansion of $\left\langle v_i|\mathcal{A}^k|v_j\right\rangle$ is equal to the expected number of k-vertex subsets $I \subseteq V$ such that there exists a k-step walk from v_i to $v_j \in I$ visiting each vertex of I exactly once when $v_i \notin I$ and revisiting v_i exactly once when $v_i \in I$.

The expected number of vertex sets I on which k-walks $v_i \to v_j$ exist is determined by partitioning the collection of walks into three classes: walks with no repeated vertices, walks that repeat v_i on the second step–and therefore repeat an edge, and walks that revisit v_i on some step other than the second.

Class I: When a collection of k-walks $v_i \to v_j$ exists on k independent equiprobable edges with no revisited vertices,

$$\begin{aligned}\mathbb{E}(\sharp\{I : \exists k\text{-walk on } I\}) &= \frac{\mathbb{E}(k\text{-walks } v_i \to v_j \text{ on } I)}{(k-1)!} \\ &= \frac{p^k(n-2)!/(n-k-1)!}{(k-1)!} = p^k\binom{n-2}{k-1},\end{aligned} \tag{2.19}$$

since v_i is excluded and the walk must terminate at v_j.

Class II: Given a collection of k-walks $v_i \to v_j$ on $k-1$ independent equiprobable edges, revisiting only v_i exactly once in the second step,

$$\mathbb{E}(\sharp\{I : \exists k\text{-walk on } I\}) = \frac{\mathbb{E}(k\text{-walks } v_i \to v_j \text{ on } I)}{(k-2)!}$$
$$= \frac{p^{k-1}(n-2)!/((n-2)-(k-2))!}{(k-2)!} = p^{k-1}\binom{n-2}{k-2}. \quad (2.20)$$

Class III: Given a collection of k-walks $v_i \to v_j$ on k independent equiprobable edges, revisiting v_i exactly once in some step other than the second,

$$\mathbb{E}(\sharp\{I : \exists k\text{-walk on } I\}) = \frac{\mathbb{E}(k\text{-walks } v_i \to v_j \text{ on } I)}{(k-1)!}$$
$$= \frac{p^{k}(n-2)!/((n-2)-(k-1))!}{(k-1)!} = p^{k}\binom{n-2}{k-1}. \quad (2.21)$$

While the walks themselves are partitioned into these classes, the vertex sets corresponding to the walks are partitioned into sets V_1 containing v_i and V_2 not containing v_i. Note that a single vertex set $I \in V_1$ may correspond to walks revisiting v_i on the second step as well as walks revisiting v_i on different steps.

The lower bound on the expected number of nonzero coefficients is obtained by summing expected numbers of vertex sets corresponding to walks of Classes I and III. Using Pascal's Identity, the lower bound on the expected number of nonzero coefficients is then given by

$$p^k\binom{n-2}{k-1} + p^k\binom{n-2}{k-2} = p^k\binom{n-1}{k-1}.$$

The upper bound is similarly found by summing expected numbers of vertex sets corresponding to walks of classes I and II:

$$p^k\binom{n-2}{k-1} + p^{k-1}\binom{n-2}{k-2} = p^{k-1}\left(p\binom{n-2}{k-1} + \binom{n-2}{k-2}\right)$$
$$\leq p^{k-1}\left(\binom{n-2}{k-1} + \binom{n-2}{k-2}\right) = p^{k-1}\binom{n-1}{k-1}.$$

In the special case $k=2$, it becomes evident that the expected number of nonzero coefficients in the canonical expansion of $\langle v_i|\mathcal{A}^2|v_j\rangle$ is equal to the expected degree of v_i when $i=j$, and equal to the expected number of two step walks on distinct vertices $v_i \to v_\ell \to v_j$ when $i \neq j$; i.e.,

$$\mathbb{E}(\sharp\{\text{nonzero coefficients}\}) = \begin{cases} p(n-1) & i=j, \\ p^2(n-2) & \text{otherwise.} \end{cases} \quad (2.22)$$

Hence, when $k = 2$, the lower bound on the expected number of nonzero coefficients is of the form $p^k \binom{n-1}{k-1} - p^k$, while the upper bound is the same as in the more general case. This completes the proof of the claim.

For $1 \leq k \leq n$, the expected number of blade multiplications is bounded above by

$$\sum_{\ell=1}^{n} p^{k-2} \binom{n-1}{k-2} p = np^{k-1} \binom{n-1}{k-2}. \tag{2.23}$$

Hence, the expected number of blade multiplications in the matrix product $\mathcal{A}^{k-1}\mathcal{A}$ is bounded above by $n^{\alpha+1} p^{k-1} \begin{pmatrix} n-1 \\ k-2 \end{pmatrix}$. Applying this result recursively, the average number of blade multiplications required to compute $\mathcal{A}^k$ is found to be bounded above by

$$n^{\alpha+1} \sum_{\ell=2}^{k} p^{\ell-1} \binom{n-1}{\ell-2}. \tag{2.24}$$

Observing that

$$\begin{aligned} \sum_{\ell=2}^{k} p^{\ell-1} \binom{n-1}{\ell-2} &= \sum_{\ell=0}^{k-2} p^{\ell+1} \binom{n-1}{\ell} = p \sum_{\ell=0}^{k-2} p^{\ell} \binom{n-1}{\ell} \\ &\leq p \sum_{\ell=0}^{n-1} p^{\ell} \binom{n-1}{\ell} = p(1+p)^{n-1}, \end{aligned} \tag{2.25}$$

cycle enumeration is of average-case complexity $\mathcal{O}(n^{\alpha+1}(1+p)^n)$.

When $k > 3$, the expected number of blade multiplications performed when computing $\langle v_i | \mathcal{A}^{k-1}\mathcal{A} | v_j \rangle$ is bounded below by

$$\sum_{\ell=1}^{n} p^{k-1} \binom{n-1}{k-2} p = np^{k} \binom{n-1}{k-2}. \tag{2.26}$$

When $k = 3$, the lower bound is

$$\sum_{\ell=1}^{n} p^2 (n-2) p = np^3 (n-2), \tag{2.27}$$

while $k = 2$ gives lower bound

$$\sum_{\ell=1}^{n} p^2 = np^2. \tag{2.28}$$

Combining (2.26), (2.27), and (2.28), a lower bound on the expected number of blade multiplications performed in computing $\mathcal{A}^k$ for $k \geq 2$ by the iterative method is

$$
\begin{aligned}
n^{\alpha} &\left(np^2 + np^3(n-2) + \sum_{\ell=4}^{k-1} np^{\ell} \binom{n-1}{\ell-2} \right) \\
&= n^{\alpha+1} \left(p^2 + p^3(n-2) + \sum_{\ell=4}^{k-1} p^{\ell} \binom{n-1}{\ell-2} \right) \\
&\geq n^{\alpha+1} p^{k-1} \left(n-1 + \sum_{\ell=4}^{k-1} \binom{n-1}{\ell-2} \right) \\
&= n^{\alpha+1} p^{k-1} \left(n-1 + \sum_{\ell=2}^{k-3} \binom{n-1}{\ell} \right) \\
&= n^{\alpha+1} p^{k-1} \sum_{\ell=1}^{k-1} \binom{n-1}{\ell} = n^{\alpha+1} p^{k-1} \tau_{k-1}^{n-1}. \qquad (2.29)
\end{aligned}
$$

I.e., the average-case complexity of enumerating k-cycles is $\Omega\left(n^{\alpha+1}p^{k-1}\tau_{k-1}^{n-1}\right)$. □

Example 2.9. Computation times of enumerating $\lfloor n/2 \rfloor$-cycles in random graphs are depicted in Figures 1, 2, and 3.

The worst-case complexity of cycle enumeration is established by setting $p = 1$ in the statement of Theorem 2.8.

Corollary 2.10. *The worst-case time complexity for enumerating k-cycles in a graph on n vertices using the nilpotent adjacency matrix method is $\mathcal{O}\left(n^{\alpha+1}2^n\right)$.*

Example 2.11. The average-case complexity of enumerating 3-cycles in a homogeneous random graph on n vertices with edge probability p using the iterated nilpotent adjacency matrix method is $\Theta(n^{\alpha+2})$.

As the next theorem shows, the fixed cycle length case is very well-behaved in terms of complexity.

Theorem 2.12. *For fixed $k \in \mathbb{N}$, the complexity of enumerating k-cycles in an n-vertex graph is $\mathcal{O}(n^{\alpha+k-1})$.*

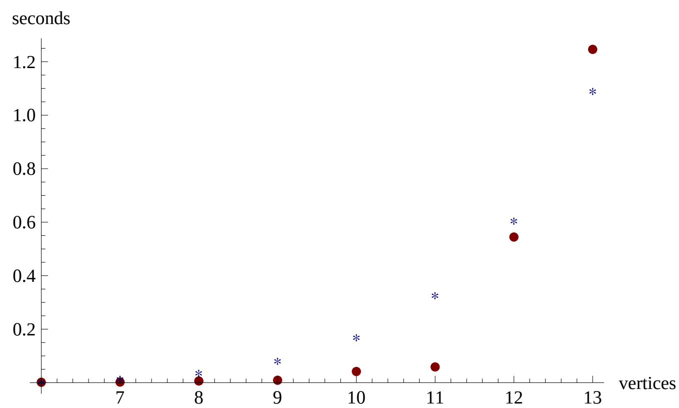


Figure 1: Mean run times of zeon method over 100 trials of counting $\lfloor n/2 \rfloor$-cycles in n-vertex graphs with edge probability $p = 0.3$. Plotted with asterisks are values $cn^4(1+p)^n$ where $c = 1.26707 * 10^{-6}$, obtained by least squares method.

Proof. The case $k = 3$ is clear from the special case in the proof of Theorem 2.8. When $k > 3$, the maximum number of nonzero coefficients in the canonical zeon expansion of $\langle v_i | \mathcal{A}^{k-1} | v_j \rangle$ is $\binom{n-1}{k-2}$. Asymptotically, $\binom{n-1}{k-2} \approx \frac{(n-1)^{k-2}}{(k-2)!} = \mathcal{O}(n^k)$. Hence, computing $\mathcal{A}^k$ requires computing at most

$$n^\alpha \sum_{\ell=0}^{k-2} \binom{n-1}{\ell} = \mathcal{O}(n^\alpha n^{k-2}) = \mathcal{O}(n^{k+(\alpha-2)}). \tag{2.30}$$

blade products. □

Example 2.13. Computation times of enumerating 5-cycles in random graphs appear in Figures 4, 5, and 6.

2.2 Lower bounds

We turn now to considerations of lower bounds on complexity using the iterated method. In particular, a lower bound on complexity of counting k_n-cycles where k_n increases with n.

Proposition 2.14. *Let (k_n) be a sequence in $\mathbb{N}$. If $\exists M \in \mathbb{N}$ such that $k_n > \lceil n/2 \rceil$ for all $n \geq M$, then the average-case complexity of enumerating k_n-cycles in a homogeneous random graph on n vertices with edge probability p using the iterated nilpotent adjacency matrix method is $\Theta\left(n^{\alpha+1}(1+p)^n\right)$.*

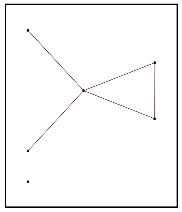
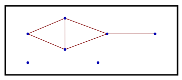
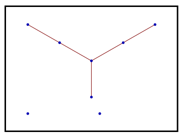
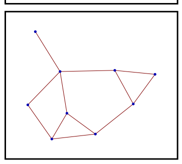
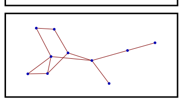
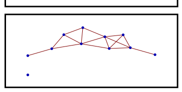
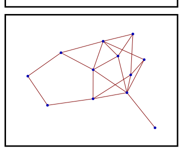
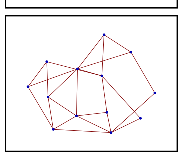
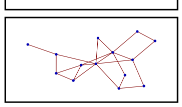
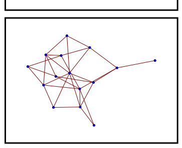
n	p	Zeon Time	Graph	Bax Time	CombiTarjan time	cycle size	♯{k-cycles}
6	0.3	0.041951		0.014575	0.009360	3	2
7	0.3	0.005449		0.017005	0.017279	3	4
8	0.3	0.002638		0.071668	0.028735	4	0
9	0.3	0.006998		0.128964	0.101792	4	2
10	0.3	0.017523		0.695656	0.271784	5	6
11	0.3	0.047338		1.294477	0.736030	5	12
12	0.3	0.304311		6.504564	2.635960	6	80
13	0.3	0.378919		12.118591	3.452875	6	98
14	0.3	1.086447		69.379095	7.004031	7	22
15	0.3	3.681130		130.205070	24.613884	7	404

Figure 2: Times (in secs) required to enumerate $\lfloor n/2 \rfloor$-cycles in randomly generated n-vertex graphs having equiprobable edges ($p = 0.3$).

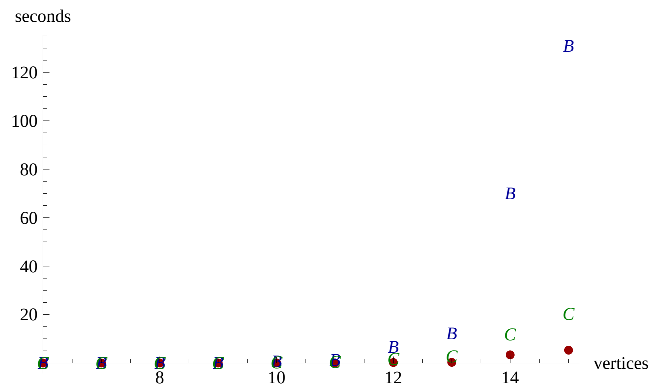


Figure 3: Mean run times over twenty trials of counting $\lfloor n/2 \rfloor$-cycles in n-vertex graphs with edge probability $p = 0.25$. Plotmarkers: B–Bax, C–Combi-Tarjan, *–Zeon.

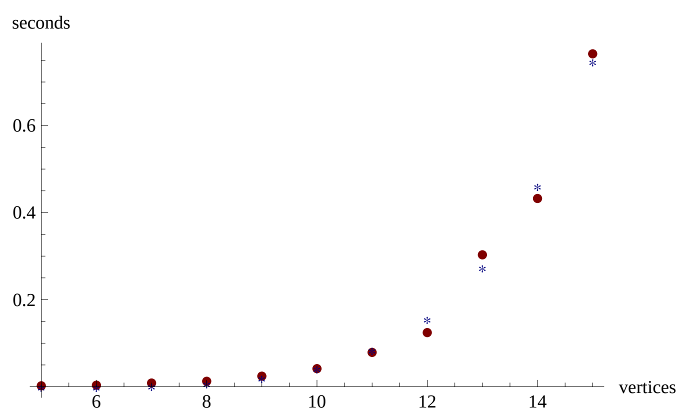


Figure 4: Mean run times of zeon method over 100 trials of counting 5-cycles in n-vertex graphs with edge probability $p = 0.3$. Plotted with asterisks are values $c\,n^7$ where $c = 4.38386 * 10^{-9}$, obtained by least squares method.

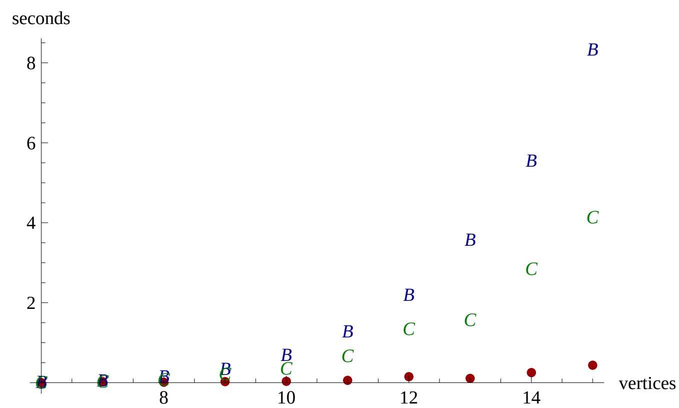


Figure 5: Average run times over twenty trials of counting 5-cycles in n-vertex graphs with edge probability $p = 0.3$. Plotmarkers: B–Bax, C–Combi-Tarjan, *–Zeon.

Proof. By Theorem 2.8, the average time complexity of enumerating k_n-cycles is $\mathcal{O}(n^{\alpha+1}(1+p)^n)$. It remains to establish that (in the iterated method) the average complexity is also $\Omega(n^{\alpha+1}(1+p)^n)$.

Observe that the proof of Theorem 2.8 implies that the expected number of nonzero coefficients in the canonical zeon expansion of $\langle v_i|\mathcal{A}^k|v_j\rangle$ is bounded below by $p^k\binom{n-2}{k-1}$ for $1 \leq k \leq n-1$, even in the special case $k = 2$.

Suppose (k_n) and M satisfy the conditions stated in the proposition. Given $0 \leq p \leq 1$, it is clear from symmetry of binomial coefficients that $p^\ell\binom{n-1}{\ell} \geq p^{(n-1)-\ell}\binom{n-1}{n-\ell}$ for $0 \leq \ell \leq k_n$. An immediate consequence is that in the case $k_n \geq n/2+1$ for even $n \geq M$,

$$\sum_{\ell=0}^{k_n-2} p^\ell\binom{n-2}{\ell} \geq \sum_{\ell=0}^{n/2-1} p^\ell\binom{n-2}{\ell} \geq \frac{1}{2}\sum_{\ell=0}^{n-2} p^\ell\binom{n-2}{\ell} = \frac{1}{2}(1+p)^{n-2}. \tag{2.31}$$

For odd $n \geq M$, $k_n \geq \frac{n+1}{2}+1$ implies

$$\sum_{\ell=0}^{k_n-2} p^\ell\binom{n-2}{\ell} \geq \sum_{\ell=0}^{(n-1)/2} p^\ell\binom{n-2}{\ell} > \frac{1}{2}\sum_{\ell=0}^{n-2} p^\ell\binom{n-2}{\ell} = \frac{1}{2}(1+p)^{n-2}. \tag{2.32}$$

□

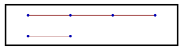
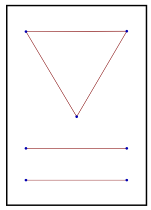
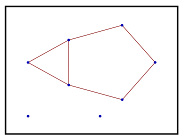
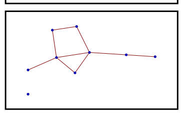
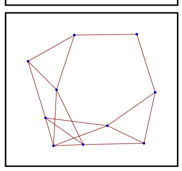
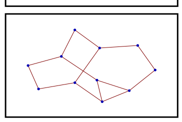
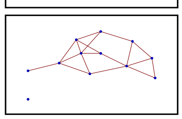
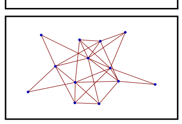
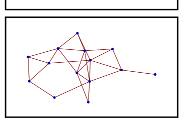
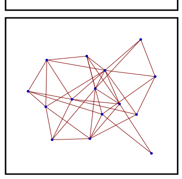
n	p	Zeon Time	Graph	Bax Time	Combinatorica time	cycle size	♯{k-cycles}
6	0.3	0.001911		0.017523	0.005827	5	0
7	0.3	0.003465		0.061099	0.013855	5	0
8	0.3	0.007210		0.157496	0.070600	5	2
9	0.3	0.011970		0.353054	0.139510	5	2
10	0.3	0.051886		0.702653	0.448507	5	22
11	0.3	0.016244		1.290107	0.367940	5	0
12	0.3	0.066979		2.209488	0.917483	5	16
13	0.3	0.732873		3.596704	3.470682	5	254
14	0.3	0.179088		5.584722	2.569243	5	60
15	0.3	1.305130		8.399761	5.819182	5	298

Figure 6: Times (in secs) required to enumerate 5-cycles in randomly generated n-vertex graphs.

2.3 Computing powers by successive-squares

While the successive squares method is more efficient than the iterated method for ordinary matrix multiplication; i.e., $\mathcal{O}(n^\alpha \log n)$ vs. $\mathcal{O}(n^{\alpha+1})$, such is not necessarily the case for nilpotent adjacency matrices.

Lemma 2.15. *The average-case complexity for enumerating k-cycles in a homogeneous random graph on n vertices with edge probability p using the successive-squares nilpotent adjacency matrix method is*

$$\mathcal{O}\left(n^{\alpha+1}\sum_{\ell=0}^{\log_2 k} p^{2^\ell}\binom{n-1}{2^\ell-1}^2\right).$$

Proof. As in the proof of Theorem 2.8, the expected number of nonzero terms in the canonical zeon expansion of $\left\langle v_i|\mathcal{A}^k|v_j\right\rangle$ is bounded above by $p^{k-1}\binom{n-1}{k-1}$. Hence, the expected number of blade multiplications performed in computing an entry of the squared matrix $\left\langle v_i|\mathcal{A}^{2k}|v_j\right\rangle$ is bounded above by $\frac{n}{p^2}p^{2k}\binom{n-1}{k-1}^2$. Summing over successive squares then gives the result. □

The next proposition reveals inefficiencies in the successive-squares method of computing powers of nilpotent adjacency matrices.

Proposition 2.16. *Let $\mathcal{A}$ denote the nilpotent adjacency matrix of a homogeneous random graph on n vertices with equiprobable edges of probability p, and let $3 \le k \le n/2$. Let $\mathcal{N}\left((\mathcal{A}^k)^2\right)$ denote the number of blade multiplications performed in squaring $\mathcal{A}^k$ resulting in zero, i.e., the number of* null blade multiplications *performed. Then,*

$$\mathbb{E}\left(\mathcal{N}\left((\mathcal{A}^k)^2\right)\right) = \Theta\left(n^3p^{2k}\left[\binom{n-1}{k-1}^2 - \binom{n-2}{k-1}\binom{n-k}{k-1}\right]\right). \tag{2.33}$$

Proof. Given a fixed k-blade ζ_I, the number of k-blades indexed by sets nontrivially intersecting I is given by $\binom{n}{k} - \binom{n-k}{k}$. In other words, for fixed multi-index I,

$$\sharp\{\zeta_J \in \mathcal{C}\ell_n{}^{\text{nil}} : \zeta_I\zeta_J = 0\} = \binom{n}{k} - \binom{n-k}{k}. \tag{2.34}$$

Considering a random graph on n vertices with equiprobable edges of probability p, the expected number of nonzero coefficients in the canonical zeon expansion of $\left\langle v_i|\mathcal{A}^k|v_j\right\rangle$ is between $p^k\binom{n-1}{k-1}$ and $p^{k-1}\binom{n-1}{k-1}$. We consider now the expected number of null products occurring when computing $\left\langle v_i|\mathcal{A}^{2k}|v_j\right\rangle$ by squaring $\mathcal{A}^k$.

Recall that

$$\left\langle v_i|\mathcal{A}^{2k}|v_j\right\rangle = \sum_{\ell=1}^{n} \left\langle v_i|\mathcal{A}^k|v_\ell\right\rangle \left\langle v_\ell|\mathcal{A}^k|v_j\right\rangle. \tag{2.35}$$

Because each term in the canonical zeon expansion of $\left\langle v_i|\mathcal{A}^k|v_\ell\right\rangle$ is of the form $\alpha_I\zeta_{I\setminus\{\ell\}}\zeta_\ell$,

$$(\ell = j) \Rightarrow \left\langle v_i|\mathcal{A}^k|v_\ell\right\rangle \left\langle v_\ell|\mathcal{A}^k|v_j\right\rangle = 0. \tag{2.36}$$

The expected number of null products computed when $\ell = j$ is thus the product of the expected numbers of nonzero coefficients in the respective canonical zeon expansions. Hence, the lower and upper bounds on expected numbers of null blade products are squares of the lower and upper bounds on the expected numbers of nonzero coefficients, i.e., when $k \neq 2$,

$$p^{2k}\binom{n-1}{k-1}^2 \leq \mathbb{E}\left(\sharp\{\text{null blade products when}\,\ell = j\}\right) \leq p^{2(k-1)}\binom{n-1}{k-1}^2. \tag{2.37}$$

On the other hand, when $\ell \neq j$, we consider blades indexed by sets containing j and sets not containing j. Let X_j denote the number of nonzero coefficients indexed by sets containing v_j in $\langle v_i|\mathcal{A}^k|v_\ell\rangle$. Employing reasoning from the proof of Theorem 2.8, the expected value of X_j satisfies

$$p^k\binom{n-2}{k-2} \leq \mathbb{E}(X_j) \leq p^{k-1}\binom{n-2}{k-2},$$

since v_j and v_ℓ must be included in any k-walks represented.

Letting $\tilde{X}_j$ denote the number of nonzero coefficients indexed by sets *not* containing v_j in $\langle v_i|\mathcal{A}^k|v_\ell\rangle$, similar reasoning gives

$$p^k\binom{n-2}{k-1} \leq \mathbb{E}(\tilde{X}_j) \leq p^{k-1}\binom{n-2}{k-1}.$$

For arbitrary basis blade indexed by a set I not containing v_j, let $\tilde{Y}_j$ denote the number of nonzero coefficients in the expansion of $\left\langle v_\ell|\mathcal{A}^k|v_j\right\rangle$ whose index sets nontrivially intersect I. Then,

$$p^k\left(\binom{n-1}{k-1} - \binom{n-k}{k-1}\right) \leq \mathbb{E}(\tilde{Y}_j) \leq p^{k-1}\left(\binom{n-1}{k-1} - \binom{n-k}{k-1}\right). \tag{2.38}$$

When $v_j \in I$, letting Y_j denote the number of nonzero coefficients in the expansion of $\langle v_\ell|\mathcal{A}^k|v_j\rangle$ whose index sets nontrivially intersect I gives

$$p^k\binom{n-1}{k-1} \le \mathbb{E}(Y_j) \le p^{k-1}\binom{n-1}{k-1}. \tag{2.39}$$

The expected number of null blade products occurring in the multiplication $\langle v_i|\mathcal{A}^k|v_\ell\rangle \langle v_\ell|\mathcal{A}^k|v_j\rangle$ when $\ell \neq j$ is then given by

$$\mathbb{E}\left(\sharp\{\text{null blade products when}\,\ell \neq j\}\right) = \mathbb{E}(\tilde{X}_j\tilde{Y}_j + X_jY_j).$$

With all these considerations in mind,

$$\begin{aligned} p^k\binom{n-2}{k-1}p^k\left(\binom{n-1}{k-1}-\binom{n-k}{k-1}\right) + p^k\binom{n-2}{k-2}p^k\binom{n-1}{k-1} \\ = p^{2k}\left[\binom{n-1}{k-1}^2 - \binom{n-2}{k-1}\binom{n-k}{k-1}\right] \\ \le \mathbb{E}\left(\sharp\{\text{null blade products when}\,\ell \neq j\}\right) \\ \le p^{2(k-1)}\left[\binom{n-1}{k-1}^2 - \binom{n-2}{k-1}\binom{n-k}{k-1}\right] \\ = p^{k-1}\binom{n-2}{k-1}p^{k-1}\left(\binom{n-1}{k-1}-\binom{n-k}{k-1}\right) + p^{k-1}\binom{n-2}{k-2}p^{k-1}\binom{n-1}{k-1}. \end{aligned} \tag{2.40}$$

Returning to the sum (2.35), the expected number of null products computed in calculating $\langle v_i|\mathcal{A}^{2k}|v_j\rangle$ by squaring $\mathcal{A}^k$ is seen to satisfy

$$\begin{aligned} p^{2k}\binom{n-1}{k-1}^2 + (n-1)p^{2k}\left[\binom{n-1}{k-1}^2 - \binom{n-2}{k-1}\binom{n-k}{k-1}\right] \\ \le \mathbb{E}\left(\sharp\{\text{null blade products}\}\right) \\ \le p^{2(k-1)}\binom{n-1}{k-1}^2 + (n-1)p^{2(k-1)}\left[\binom{n-1}{k-1}^2 - \binom{n-2}{k-1}\binom{n-k}{k-1}\right]. \end{aligned} \tag{2.41}$$

The expected number of null blade products computed in squaring $\mathcal{A}^k$

is then obtained by summing over all matrix indices $1 \leq i, j \leq n$. Hence,

$$n^2 p^{2k} \binom{n-1}{k-1}^2 + (n^3 - n^2) p^{2k} \left[\binom{n-1}{k-1}^2 - \binom{n-2}{k-1}\binom{n-k}{k-1} \right]$$
$$\leq \mathbb{E}\left(\mathcal{N}((\mathcal{A}^k)^2)\right)$$
$$\leq n^2 p^{2(k-1)} \binom{n-1}{k-1}^2 + (n^3 - n^2) p^{2(k-1)} \left[\binom{n-1}{k-1}^2 - \binom{n-2}{k-1}\binom{n-k}{k-1} \right]. \tag{2.42}$$

□

The expected number of null blade multiplications is seen to be dependent on p. The inefficiency of squaring varies with graph density, as illustrated in Example 2.17.

Example 2.17. Let $\mathcal{A}$ be the nilpotent adjacency matrix of a randomly-generated graph on 10 vertices with equiprobable edges of probability $p = 0.5$. The expected number of null blade products computed in calculating $\mathcal{A}^{10}$ by squaring $\mathcal{A}^5$ is between 15196.3 and 60785.2. On the other hand, when $p = 0.1$, the expected number of null blade products in the same computation is between 0.0015561 and 0.15561.

2.4 Remarks on space complexity

The algorithms presented by Bax have space complexity $\mathcal{O}(\text{poly}(n))$. On the other hand, Tarjan's algorithm actually lists cycles, which can result in $\mathcal{O}(n!)$ space complexity.

By storing only vertex sets on which cycles exist rather than the cycles themselves, the space complexity of the nilpotent adjacency matrix method is less than that of Tarjan's method.

Lemma 2.18. *Enumerating cycles in a simple graph on n vertices using nilpotent adjacency matrix methods has storage complexity $\mathcal{O}(n^2 2^n)$.*

Proof. The nilpotent matrix method requires construction of $n \times n$ matrices whose entries are elements of a 2^n-dimensional algebra; i.e., in the worst case, $\mathcal{O}(2^n)$ coefficients must be associated with each matrix entry. Consequently, the space complexity is $\mathcal{O}(n^2 2^n)$. □

3 Conclusion

Given a computing architecture in which one blade multiplication is done in $\mathcal{O}(n)$ time, the average case complexity of enumerating k-cycles in a homogeneous random graph on n vertices with equiprobable edges of probability p is $\mathcal{O}(n^{\alpha+1}(1+p)^n)$. This is more efficient than Bax's algorithm, which enumerates all cycles in $\mathcal{O}(2^n \text{poly}(n))$ time. In sparse graphs, Tarjan's algorithm offers advantages in time complexity, but the zeon approach offers advantages in space complexity.

In relatively low-dimensional cases, the nilpotent matrix approach to cycle enumeration offers practical advantages, even when implemented on a classical computer using Mathematica. To summarize the experimental results, the nilpotent adjacency matrix method offers practical advantages over Bax and CombiTarjan when enumerating k-cycles in relatively sparse n-vertex graphs with $k < n$. The advantage is most striking in the case $k \approx n/2$, since this case maximizes the number of subgraphs being considered in both the Bax and CombiTarjan methods.

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