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Abstract In this paper, we use duality arguments "à la Michel Pierre" to establish global existence of classic solutions for a class of parabolic reaction-diffusion systems modeling, for instance, the evolution of reversible chemical reactions.

1 Introduction

This paper is motivated by the general question of global existence in time of solutions to the following reaction-diffusion system

$$(\mathcal{S})\left\{\begin{array}{lcll} u_t - d_1\Delta u & = & w^\gamma - u^\alpha v^\beta & (0,+\infty)\times\Omega, \quad (E_1)\\ v_t - d_2\Delta v & = & w^\gamma - u^\alpha v^\beta & (0,+\infty)\times\Omega, \quad (E_2)\\ w_t - d_3\Delta w & = & -w^\gamma + u^\alpha v^\beta & (0,+\infty)\times\Omega, \quad (E_3)\\ \dfrac{\partial u}{\partial n}(t,x) = \dfrac{\partial v}{\partial n}(t,x) = \dfrac{\partial w}{\partial n}(t,x) & = & 0 & (0,+\infty)\times\partial\Omega,\\ u(0,x) & = & u_0(x)\geq 0 & x\in\Omega,\\ v(0,x) & = & v_0(x)\geq 0 & x\in\Omega,\\ w(0,x) & = & w_0(x)\geq 0 & x\in\Omega, \end{array}\right.$$

where Ω is a bounded regular open subset of $\mathbb{R}^N$, $(d_1,d_2,d_3,\alpha,\beta,\gamma)\in(0,+\infty)^3\times[1,+\infty)^3$.

Note that the system $(\mathcal{S})$ satisfies two main properties, namely :
(P) the nonnegativity of solutions of $(\mathcal{S})$ is preserved for all time ;
(M) the total mass of the components u, v, w is a priori bounded on all finite intervals $(0,t)$.

If α,β and γ are positive integers, system $(\mathcal{S})$ is intended to describe for example the evolution of a reversible chemical reaction of type

$$\alpha U + \beta V \rightleftharpoons \gamma W$$

where u, v, w stand for the density of U, V and W respectively.
This chemical reaction is typical of general reversible reactions and contains the major difficulties encountered in a large class of similar problems as regards global existence of solutions.

Let us make precise what we mean by solution.

By *classical solution* to $(\mathcal{S})$ on $Q_T = (0,T)\times\Omega$, we mean that, at least
(i) $(u,v,w)\in\mathcal{C}([0,T);L^1(\Omega)^3)\cap L^\infty([0,\tau]\times\Omega)^3, \forall\tau\in(0,T)$;
(ii) $\forall k,\ell = 1\ldots N$, $\forall p\in(1,+\infty)$

$$\partial_t u, \partial_t v, \partial_t w, \partial_{x_k}u, \partial_{x_k}v, \partial_{x_k}w, \partial_{x_k x_\ell}u, \partial_{x_k x_\ell}v, \partial_{x_k x_\ell}w, u, v, w \in L^p((0,T)\times\Omega)\,;$$

(iii) equations in $(\mathcal{S})$ are satisfied a.e (almost everywhere).

By *weak solution* to $(\mathcal{S})$ on $Q_T = (0,T)\times\Omega$, we essentially mean solution in the sense of distributions or, equivalently here, solution in the sens of the variation of constants formula with the corresponding semigroups. More precisely

$$\begin{aligned} u(t) &= S_{d_1}(t)u_0 + \int_0 S_{d_1}(t-s)(w^\gamma(s) - u^\alpha(s)v^\beta(s))\,ds \\ v(t) &= S_{d_2}(t)v_0 + \int_0 S_{d_2}(t-s)(w^\gamma(s) - u^\alpha(s)v^\beta(s))\,ds \\ w(t) &= S_{d_3}(t)u_0 + \int_0 S_{d_3}(t-s)(-w^\gamma(s) + u^\alpha(s)v^\beta(s))\,ds \end{aligned}$$

where $S_{d_i}(.)$ is the semigroup generated in $L^1(\Omega)$ by $-d_i\Delta$ with homogeneous Neumann boundary condition, $1\le i\le 3$.

By just integrating the sum $(E_1)+(E_2)+2(E_3)$ in space and time, and taking into account the boundary conditions $\left(\int_\Omega \Delta(d_1u + d_2v + d_3w) = 0\right)$, we obtain

$$\int_\Omega u(t) + v(t) + 2w(t) = \int_\Omega u_0 + v_0 + 2w_0 \qquad t\ge 0. \tag{1}$$

Together with the nonnegativity of u, v and w, estimate (1) implies that

$$\forall t\ge 0\ ,\ \|u(t)\|_{L^1(\Omega)}, \|v(t)\|_{L^1(\Omega)}, \|w(t)\|_{L^1(\Omega)} \le \|u_0 + v_0 + 2w_0\|_{L^1(\Omega)}. \tag{2}$$

In other words, the total mass of three components does not blow up ; $u(t)$, $v(t)$ and $w(t)$ rest bounded in $L^1(\Omega)$ uniformly in time.

Although one has uniform L^1-bound in time, classical solutions may not globally exist for diffusion coefficients d_1, d_2, d_3 which are not equal (global existence obviously holds if $d_1 = d_2 = d_3$). As surprisingly proved in [12] and [16], it may indeed happen that, under assumptions (P) and (M), solutions blow up in finite time in L^∞ ! In particular, classical bounded solutions do not exist globally in time.

If $u_0, v_0, w_0 \in L^\infty(\Omega)$, local existence and uniqueness of nonnegative and uniformly bounded solution to $(\mathcal{S})$ are known (see e.g. [13]). More precisely, there exists $T>0$ and a unique classical solution (u,v,w) of $(\mathcal{S})$ on $[0,T)$. If $T_{\max}$ denotes the greatest of these T's, then

$$\big(T_{\max} < +\infty\big) \Longrightarrow \lim_{t\nearrow T_{\max}} \big(\|u(t)\|_{L^\infty(\Omega)} + \|v(t)\|_{L^\infty(\Omega)} + \|w(t)\|_{L^\infty(\Omega)}\big) = +\infty. \tag{3}$$

To prove global existence (*i.e.* $T_{\max} = +\infty$), it is sufficient to obtain an a priori estimate of the form

$$\forall t\in[0,T_{\max}), \qquad \|u(t)\|_{L^\infty(\Omega)} + \|v(t)\|_{L^\infty(\Omega)} + \|w(t)\|_{L^\infty(\Omega)} \le H(t), \tag{4}$$

where $H : [0, +\infty) \to [0, +\infty)$ is a nondecreasing and continuous function.

This type of estimates is far of being obvious for our system except the case where diffusion coefficients d_1, d_2, d_3 are equal *i.e* $d_1 = d_2 = d_3 = d$. Indeed, $Z = u + v + 2w$ satisfies

$$(E) \begin{cases} Z_t - d\Delta Z = 0 & (0, +\infty) \times \Omega, \\ \dfrac{\partial Z}{\partial n} = 0 & (0, +\infty) \times \partial\Omega, \\ Z(0, x) = Z_0(x) & x \in \Omega, \end{cases}$$

where $Z_0(x) = u_0(x) + v_0(x) + 2w_0(x)$.
In particular, we deduce by maximum principle that

$$\|u(t) + v(t) + 2w(t)\|_{L^\infty(\Omega)} \leq \|u_0 + v_0 + 2w_0\|_{L^\infty(\Omega)}, \qquad t \geq 0.$$

Together with nonnegativity, this implies

$$\|u(t)\|_{L^\infty(\Omega)} + \|v(t)\|_{L^\infty(\Omega)} + \|w(t)\|_{L^\infty(\Omega)} \leq \|u_0 + v_0 + 2w_0\|_{L^\infty(\Omega)}, \qquad t \geq 0.$$

In other words, $u(t)$, $v(t)$ and $w(t)$ stay uniformly bounded in $L^\infty(\Omega)$ and therefore $T_{\max} = +\infty$.

In the case where the diffusion coefficients are different from each other, global existence is considerably more complicated. It has been studied by several authors in the following cases.
First case $\alpha = \beta = \gamma = 1$.

In this case, global existence of classical solutions has been obtained by Rothe [13] for dimension $N \leq 5$. Later, it has first been proved by Pierre [10] for all dimensions N and then by Morgan [9].

The exponentiel decay towards equilibrium has been studied by Desvillettes-Fellner [2] in the case of one space dimension.

The global existence of weak solutions has been proved by Laamri [7] for initial data u_0, v_0 and w_0 only in $L^1(\Omega)$.
Second case $\gamma = 1$ regardless of α and β.

In this case, global existence of classical solutions has been obtained by Feng [4] in all dimensions N and more general boundary conditions.
Third case $\alpha + \beta \leq 2$ or $\gamma \leq 2$.

In this case, Pierre [11] has proved global existence of weak solutions for initial data u_0, v_0 and w_0 only in $L^2(\Omega)$.

Our paper mainly completes the investigations of [[4], [9], [10], [13]] and [[7], [11]]. As far as we know, our results are new either when $\alpha+\beta < \gamma$, or when $1 < \gamma < \dfrac{N+6}{N+2}$ regardless of α and β. For the sake of clarity, we decided to focus in this work on the question of global existence in time of solutions in the case of homogeneous Neumann boundary conditions. So, we shall prove global existence of classical solutions to system $(\mathcal{S})$ in the following cases :
* $\alpha + \beta < \gamma$;
* ($d_1 = d_3$ or $d_2 = d_3$) and for any (α, β, γ) ;
* $d_1 = d_2$ and for any (α, β, γ) such that $\alpha + \beta \neq \gamma$;
* $1 < \gamma < \dfrac{N+6}{N+2}$ and for any (α, β).

For the sake of completeness and for the reader's convenience, we shall also give a direct proof different from that of Feng [4] in the special case $\gamma = 1$.

Notation : Throughout this study, we denote by C_i's various positive numbers depending only on the data and for $p \in [1, +\infty[$

$$\|u(t)\|_p = \left(\int_\Omega |u(t,x)|^p \, dx \right)^{1/p}, \quad \|u\|_{L^p(Q_T)} = \left(\int_0^T \int_\Omega |u(t,x)|^p \, dtdx \right)^{1/p},$$

$$\|u(t)\|_\infty = \text{esse } \sup_{x\in\Omega} |u(t,x)|, \quad \|u\|_{L^\infty(Q_T)} = \text{esse } \sup_{(t,x)\in Q_T} |u(t,x)|.$$

2 The main results

One of the main ingredients for the proof of our results is the following lemma which is based on the regularizing effects of the heat equation. This lemma has been introduced by Hollis-Martin-Pierre in [5].

Lemma 1 *Let $T > 0$ and (ϕ, ψ) the classical solution of*

$$\begin{cases} \phi_t - d_1 \Delta \phi &= f(\phi, \psi) \quad (t,x) \in (0,T) \times \Omega \\ \psi_t - d_2 \Delta \psi &= g(\phi, \psi) \quad (t,x) \in (0,T) \times \Omega \\ \dfrac{\partial \phi}{\partial n}(t,x) &= 0 \quad (t,x) \in (0,T) \times \partial\Omega \\ \dfrac{\partial \psi}{\partial n}(t,x) &= 0 \quad (t,x) \in (0,T) \times \partial\Omega \\ \phi(0,x) &= \phi_0(x) \quad x \in \Omega \\ \psi(0,x) &= \psi_0(x) \quad x \in \Omega. \end{cases}$$

Assume that $f + g = 0$, then for each $p \in (1, +\infty)$, there exists C such that for all $t \in (0,T)$

$$\|\psi\|_{L^p(Q_t)} \le C \left[\|\phi\|_{L^p(Q_t)} + 1 \right]. \tag{5}$$

A more general version of this lemma can be founded in [11, lemma 3.4]. □

2.1 The case $\alpha + \beta < \gamma$

Theorem 1 *Assume that $0 \le u_0, v_0, w_0 \le M$ where M is a positive real.*
If $\alpha + \beta < \gamma$, then the system $(\mathcal{S})$ admits a global classical solution.

Proof :
• Let $T \in (0, T_{\max})$ and let $t \in (0,T]$. Thanks to the nonnegativity of u, v and w, we deduce from the equation (E_1) that u is bounded from above by the solution U of

$$(P_1) \begin{cases} U_t - d_1 \Delta U &= w^\gamma \quad (t,x) \in (0,T) \times \Omega \\ \dfrac{\partial U}{\partial n}(t,x) &= 0 \quad (t,x) \in (0,T) \times \partial\Omega \\ U(0,x) &= u_0(x) \quad x \in \Omega, \end{cases}$$

and we deduce from the equation (E_2) that v is bounded from above by the solution V of

$$(P_2) \begin{cases} V_t - d_2 \Delta V &= w^\gamma \quad (t,x) \in (0,T) \times \Omega \\ \dfrac{\partial V}{\partial n}(t,x) &= 0 \quad (t,x) \in (0,T) \times \partial\Omega \\ V(0,x) &= v_0(x) \quad x \in \Omega. \end{cases}$$

Therefore it is sufficient to show that $w \in L^p(Q_T)$ for p large enough.

• Let $q > 1$. Multiplying the equation (E_3) by w^q and integrating over Q_T, we get

$$\frac{1}{q+1}\int_\Omega w^{q+1}(T) + qd_3 \int\int_{Q_T} |\nabla w|^2 w^{q-1} + \int\int_{Q_T} w^{q+\gamma} = \int\int_{Q_T} u^\alpha v^\beta w^q + K_0 \qquad (6)$$

where

$$K_0 = \frac{1}{q+1}\int_\Omega w_0^{q+1}.$$

Thanks to Hölder's inequality, we have

$$\int\int_{Q_T} u^\alpha v^\beta w^q \le \|u\|^\alpha_{L^{\alpha r}(Q_T)} \|v\|^\beta_{L^{\beta s}(Q_T)} \|w\|^q_{L^{\gamma+q}(Q_T)} \qquad (7)$$

where

$$\frac{1}{r} + \frac{1}{s} + \frac{q}{q+\gamma} = 1.$$

Since $\alpha + \beta < \gamma$, we can choose r such that $r\alpha \le q+\gamma$ and s such that $s\beta \le q+\gamma$. To convince oneself, it is enough to draw the straight line with cartesian equation $x + y = \dfrac{\gamma}{q+\gamma}$ and to identify the points with coordinates $(\dfrac{\alpha}{q+\gamma}, 0)$ and $(0, \dfrac{\beta}{q+\gamma})$.

Then $L^{q+\gamma}(Q_T) \subset L^{\alpha r}(Q_T)$ and $L^{q+\gamma}(Q_T) \subset L^{\beta s}(Q_T)$. Consequently, there exists C_1 such that

$$\int\int_{Q_T} u^\alpha v^\beta w^q \le C_1 \|u\|^\alpha_{L^{\gamma+q}(Q_T)} \|v\|^\beta_{L^{\gamma+q}(Q_T)} \|w\|^q_{L^{\gamma+q}(Q_T)}. \qquad (8)$$

By virtue of lemma 1, there exists C_2 such that

$$\|u\|_{L^{\gamma+q}(Q_T)} \le C_2(1 + \|w\|_{L^{\gamma+q}(Q_T)}) \qquad (9)$$

and there exists C_3 such that

$$\|v\|_{L^{\gamma+q}(Q_T)} \le C_3(1 + \|w\|_{L^{\gamma+q}(Q_T)}). \qquad (10)$$

Thanks to (9) and (10), estimate (8) can be written

$$\int\int_{Q_T} u^\alpha v^\beta w^q \le C_4 \left(1 + \|w\|_{L^{\gamma+q}(Q_T)}\right)^\alpha \left(1 + \|w\|_{L^{\gamma+q}(Q_T)}\right)^\beta \left(1 + \|w\|_{L^{\gamma+q}(Q_T)}\right)^q. \qquad (11)$$

If $\|w\|_{L^{\gamma+q}(Q_T)} \le 1$ then the proof ends up. Otherwise, there exists C_5 such that

$$\int\int_{Q_T} u^\alpha v^\beta w^q \le C_5 \|w\|^{q+\alpha+\beta}_{L^{\gamma+q}(Q_T)}. \qquad (12)$$

So we deduce from (6)

$$\int\int_{Q_T} w^{q+\gamma} \le C_5 \|w\|^{q+\alpha+\beta}_{L^{\gamma+q}(Q_T)} + K_0. \qquad (13)$$

With the notation $R := \int\int_{Q_T} w^{q+\gamma}$, estimate (13) can be written

$$R \le C_5 R^{\frac{q+\alpha+\beta}{q+\gamma}} + K_0. \qquad (14)$$

Since $q+\alpha+\beta<q+\gamma$, by applying Young's inequality to (14), we obtain

$$(1-\varepsilon)R \leq K_0 + C_6. \tag{15}$$

Then, for $\varepsilon \in (0,1)$, we have the desired estimate

$$\|w\|_{L^{q+\gamma}(Q_T)} \leq C_7. \tag{16}$$

Going back to (P_1) and (P_2), we have, by choosing q such that $\dfrac{q+\gamma}{\gamma} > \dfrac{N+2}{2}$ and thanks to the L^p-regularity theory for the heat operator (see [6]),

$$\|u\|_{L^\infty(Q_T)} \leq C_8 \tag{17}$$

$$\|v\|_{L^\infty(Q_T)} \leq C_9. \tag{18}$$

Now going back to (E_3), we deduce from (17) and (18) that there exists C_{10} such that

$$\|w\|_{L^\infty(Q_T)} \leq C_{10}. \tag{19}$$

This implies that $T_{\max} = +\infty$. □

Remark This method seems to be specific to the case $\alpha+\beta<\gamma$. It fails when $\alpha+\beta \geq \gamma$ since some restrictions on the parameters α, β, γ and on the diffusion coefficients will appear.

2.2 Case where $d_1 = d_3$ or $d_2 = d_3$ or $d_1 = d_2$.

Theorem 2 *Assume that* $0 \leq u_0, v_0, w_0 \leq M$.
(i) If $d_1 = d_3$ *or* $d_2 = d_3$*, then system* $(\mathcal{S})$ *admits a global classical solution for any* (α,β,γ).
(ii) If $d_1 = d_2$*, then the system* $(\mathcal{S})$ *admits a global classical solution for any* (α,β,γ) *such that* $\alpha+\beta \neq \gamma$.

Proof :
(i) Assume that $d_1 = d_3 = d$, we have

$$(u+w)_t - d\Delta(u+w) = 0 \; ; \; \frac{\partial(u+w)}{\partial n} = 0 \; ; \; (u+w)(0,x) = u_0(x) + w_0(x).$$

We deduce by maximum principle

$$\|u(t)+w(t)\|_\infty \leq \|u_0+w_0\|_\infty. \tag{20}$$

Together with the nonnegativity of u et w, this implies that $u(t)$ and $w(t)$ are uniformly bounded in $L^\infty(\Omega)$.
By going back to (E_2) and thanks to the L^p-regularity theory for the heat operator (see [6]), we conclude that $\|v(t)\|_\infty$ is uniformly bounded in $L^\infty(\Omega)$ on all interval $[0,T]$ so that $T_{\max} = +\infty$.
(ii) Assume that $d_1 = d_2 = d$. The case $\alpha+\beta<\gamma$ was already handled in the theorem 1, so it remains only to tackle the case $\gamma < \alpha+\beta$. Moreover, one can assume that $u_0 \neq v_0$ since if $u_0 = v_0$ the result is obvious.
Since $d_1 = d_2 = d$, we have

$$(u-v)_t - d\Delta(u-v) = 0 \; ; \; \frac{\partial(u-v)}{\partial n} = 0 \; ; \; (u-v)(0,x) = u_0(x) - v_0(x).$$

The maximum principle then implies $\|u(t)-v(t)\|_\infty \le \|u_0 - v_0\|_\infty = C$. Hence we have

$$\begin{aligned} u^{\alpha+\beta} &= u^\alpha v^\beta + u^\alpha(u^\beta - v^\beta) \\ &= u^\alpha v^\beta + u^\alpha \beta(\theta u + (1-\theta)v)^{\beta-1}(u-v) \text{ where } \theta \in]0,1[\\ &\le u^\alpha v^\beta + u^\alpha \beta 2^{\beta-1} C (u^{\beta-1} + v^{\beta-1}). \end{aligned}$$

Thanks to Young's inequality, there exists $C_{11} > 0$ and $C_{12} > 0$ such that

$$C_{11} u^{\alpha+\beta} \le u^\alpha v^\beta + C_{12}. \tag{21}$$

By virtue of (21), equation (E_1) implies that

$$u_t - d_1 \Delta u + C_{11} u^{\alpha+\beta} \le w^\gamma + C_{12}. \tag{22}$$

Let $q > 1$. Multiplying (22) by u^q and integrating over Q_T, we obtain

$$\frac{1}{q+1}\int_\Omega u^{q+1}(T) + q d_2 \int\int_{Q_T} |\nabla u|^2 u^{q-1} + C_{11} \int\int_{Q_T} u^{q+\alpha+\beta} \le \int\int_{Q_T} w^\gamma u^q + C_{12} \int\int_{Q_T} u^q + K_1 \tag{23}$$

where

$$K_1 = \frac{1}{q+1}\int_\Omega u_0^{q+1}.$$

Thanks to Hölder's inequality, we have

$$\int\int_{Q_T} w^\gamma u^q \le \left(\int\int_{Q_T} w^{\gamma r}\right)^{1/r} \left(\int\int_{Q_T} u^{qs}\right)^{1/s} \tag{24}$$

where $r = \dfrac{\alpha+\beta+q}{\gamma}$ and $s = \dfrac{\alpha+\beta+q}{q+\alpha+\beta-\gamma}$.
Lemma 1 implies that there exists C_{13} such that

$$\left(\int\int_{Q_T} w^{\gamma r}\right)^{1/r} = \|w\|^\gamma_{L^{q+\alpha+\beta}(Q_T)} \le C_{13}^\gamma \left(1 + \|u\|_{L^{q+\alpha+\beta}(Q_T)}\right)^\gamma.$$

If $\|u\|_{L^{q+\alpha+\beta}(Q_T)} \le 1$ then the proof ends up. Otherwise, there exists C_{14} such that

$$\left(\int\int_{Q_T} w^{\gamma r}\right)^{1/r} \le C_{14} \|u\|^\gamma_{L^{q+\alpha+\beta}(Q_T)}. \tag{25}$$

Since $qs < q + \alpha + \beta$, we have $L^{q+\alpha+\beta}(Q_T) \subset L^{qs}(Q_T)$, then there exists C_{15} such that

$$\left(\int\int_{Q_T} u^{qs}\right)^{1/s} \le C_{15} \|u\|^q_{L^{q+\alpha+\beta}(Q_T)}. \tag{26}$$

Denote $S := \int\int_{Q_T} u^{q+\alpha+\beta}$. Estimates (25) and (26) imply that

$$\int\int_{Q_T} w^\gamma u^q \le C_{16} S^{\frac{q+\gamma}{q+\alpha+\beta}}. \tag{27}$$

Moreover, since $L^{q+\alpha+\beta}(Q_T) \subset L^q(Q_T)$, there exists C_{17} such that

$$C_{12} \int\int_{Q_T} u^q \leq C_{17} S^{\frac{q}{q+\alpha+\beta}}. \tag{28}$$

Since $\gamma < \alpha + \beta$, by applying Young's inequality, there exists C_{18} such that

$$C_{16} S^{\frac{q+\gamma}{q+\alpha+\beta}} \leq \frac{\varepsilon}{2} S + C_{18}. \tag{29}$$

Applying again Young's inequality, there exists C_{19} such that

$$C_{17} S^{\frac{q}{q+\alpha+\beta}} \leq \frac{\varepsilon}{2} S + C_{19}. \tag{30}$$

Consequently, estimate (23) implies

$$(C_{11} - \varepsilon) S \leq C_{18} + C_{19} + K_1. \tag{31}$$

By choosing $\varepsilon < C_{11}$ in (31), there exists C_{20} such that

$$\|u\|_{L^{q+\alpha+\beta}(Q_T)} \leq C_{20}. \tag{32}$$

Thanks to lemma 1 and estimate (32) there exists C_{21} such that

$$\|w\|_{L^{q+\alpha+\beta}(Q_T)} \leq C_{21}. \tag{33}$$

By going back to (P_1) and (P_2), we have by choosing q such that $\dfrac{q+\alpha+\beta}{\gamma} > \dfrac{N+2}{2}$ and thanks to the L^p-regularity theory for the heat operator (see [6])

$$\|u\|_{L^\infty(Q_T)} \leq C_{22} \tag{34}$$

$$\|v\|_{L^\infty(Q_T)} \leq C_{23}. \tag{35}$$

Now let's go back to (E_3), we deduce from (34) and (35) that there exists C_{24} such that

$$\|w\|_{L^\infty(Q_T)} \leq C_{24}. \tag{36}$$

This implies that $T_{\max} = +\infty$. □
Remark Even in the last case *i.e* $d_1 = d_2$, global existence or blow-up in the limit case $\alpha+\beta = \gamma$ remain an open problem. □

2.3 Case $1 \leq \gamma < \dfrac{N+6}{N+2}$ regardless of α and β.

Theorem 3 *Assume that* $0 \leq u_0, v_0, w_0 \leq M$ *where* $M > 0$. *If* $1 \leq \gamma < \dfrac{N+6}{N+2}$, *then the system* $(\mathcal{S})$ *admits a global classical solution for any* $(\alpha, \beta) \in [1, +\infty)^2$.

Proof :
Let $T \in (0, T_{\max})$ and let $t \in (0, T]$. Thanks to the nonnegativity of u, v and w, we deduce from the equation (E_1) that u is bounded from above by the solution U of

$$(P_1)\begin{cases} U_t - d_1\Delta U = w^\gamma & (t,x) \in (0,T)\times\Omega \\ \dfrac{\partial U}{\partial n}(t,x) = 0 & (t,x) \in (0,T)\times\partial\Omega \\ U(0,x) = u_0(x) & x \in \Omega. \end{cases}$$

Therefore it is sufficient to show that $w \in L^p(Q_T)$ for p large enough. For this we have to distinguish the case $\gamma = 1$ and the case $\gamma > 1$.

• **Case** $\gamma = 1$ **and** $\alpha, \beta \geq 1$**.**

Let us recall that global existence of classical solutions for $(\mathcal{S})$ when $\alpha = \beta = \gamma = 1$ has been studied by several authors. It has been obtained by Rothe [13] for dimension $N \leq 5$. Later, it has first been proved by Pierre [10] for all dimensions N and then by Morgan [9].
Independantly, Feng [4] has proved global existence in the case $\gamma = 1$ regardless of α and β and more general boundary conditions.
For the sake of completeness and for the reader's convenience, we give here a simple and direct proof in the last case ($\gamma = 1$ regardless of α and β). In our proof, we use an idea introduced by Pierre in [10] and applied in [8].

For any $p \geq 1$, we deduce from (P_1) and the semigroup property

$$\|u(t)\|_p \leq \|u_0\|_p + \int_0^t \|w(s)\|_p \, ds. \tag{37}$$

By applying Hölder's inequality for $p > 1$ and thanks to (5), we obtain

$$\int_0^t \|w(s)\|_p \, ds \leq t^{1/p'} \left(\int_0^t \int_\Omega w^p \, dsdx \right)^{1/p} \leq t^{1/p'} C_{25} \left[1 + \left(\int_0^t \int_\Omega u^p \, dsdx \right)^{1/p} \right] \tag{38}$$

where $p' = \dfrac{p}{p-1}$.
For $t \in (0, T]$, let us set $h(t) := \int_\Omega |u(t,x)|^p \, dx$. Inequality (37) can be written

$$h(t)^{1/p} \leq C_{26} + C_{27} \left(\int_0^t h(s) \, ds \right)^{1/p}. \tag{39}$$

Taking the p^{th} power of (39) we obtain

$$h(t) \leq 2^{p-1} C_{26}^p + 2^{p-1} C_{27}^p \int_0^t h(s) \, ds. \tag{40}$$

But, inequality (40) is a linear Gronwall's inequality, then

$$\|u\|_{L^p(Q_T)} \leq C_{28}. \tag{41}$$

Repeating the method above with v instead of u, we obtain

$$\|v\|_{L^p(Q_T)} \leq C_{29}. \tag{42}$$

Estimates (41) and (42) imply that for some $q > \dfrac{N+2}{2}$

$$\|u^\alpha v^\beta\|_{L^q(Q_T)} \leq C_{30}. \tag{43}$$

Going back to equation (E_3) we have, thanks to the L^q-regularity theory for the heat operator,

$$\|w\|_{L^\infty(Q_T)} \leq C_{31}. \tag{44}$$

This conclude the proof for the case $\gamma = 1$ regardless of α and β. □

• **Case** $1 < \gamma < \dfrac{N+6}{N+2}$**.**
The proof in this case is based on lemma 1 and these two following lemmas.

Lemma 2 (Michel Pierre) *Let $T > 0$ and let Z the solution of*

$$\begin{cases} Z_t - \Delta(A(t,x)Z) & \leq & 0 & (t,x) \in (0,T) \times \Omega, \\ \dfrac{\partial Z}{\partial n}(t,x) & = & 0 & (t,x) \in (0,T) \times \partial\Omega, \\ Z(0,x) & = & Z_0(x) & x \in \Omega. \end{cases}$$

Assume that $0 < d < A(t,x) < D$ where $(d,D) \in (0,+\infty)^2$. Then, there exists $C = C(T,d,D,\Omega)$ such that

$$\|Z\|_{L^2(Q_T)} \leq C\|Z_0\|_{L^2(\Omega)}.$$

For a general version of this lemma, see [11, proposition 6.1] or [3, theorem 3.1]. □

Lemma 3 *Let (p,q) such that $1 \leq p \leq q \leq +\infty$, $d > 0$ and $S_d(t)$ the semigroup generated in $L^p(\Omega)$ by $-d\Delta$ with homogeneous Neumann boundary condition. Then*

$$\|S_d(t)Y\|_q \leq (C(\Omega)m(t))^{\frac{-N}{2}(\frac{1}{p}-\frac{1}{q})}\|Y\|_p, \text{ for all } Y \in L^p(\Omega), \ \ t > 0 \tag{45}$$

where $m(t) = \min(1,t)$.

For a proof of this lemma see for instance [13, Lemma 3, p. 25] or [1, Theorem 3.2.9, p. 90]. □

We now go back to the proof of theorem 3.

By applying lemma 2 to the system $(\mathcal{S})$ where $Z = u + v + 2w$ and $A = \dfrac{d_1 u + d_2 v + 2d_3 w}{u + v + 2w}$, we have $u,\ v,\ w\ \in L^2(Q_T)$. More precisely, there exists C_{32} such that

$$\|u\|_{L^2(Q_T)}, \|v\|_{L^2(Q_T)}, \|w\|_{L^2(Q_T)} \leq C_{32}. \tag{46}$$

Now, we have thanks to the estimate (45) with $p > 1$ and $q = +\infty$

$$\|u(t)\|_\infty \ \ \leq \ \ \|u_0\|_\infty + C_{33}\int_0^t (t-s)^{\frac{-N}{2p}}\|w^\gamma(s)\|_p\,ds. \tag{47}$$

By applying Hölder's inequality, we obtain

$$\int_0^t (t-s)^{\frac{-N}{2p}}\|w(s)^\gamma\|_p\,ds \leq \left(\int_0^t (t-s)^{\frac{-Np'}{2p}}\,ds\right)^{1/p'}\left(\int_0^t \|w^\gamma(s)\|_p^p\,ds\right)^{1/p}. \tag{48}$$

We first remark that the integral $\int_0^t (t-s)^{\frac{-N}{2(p-1)}} ds$ converges when $p > \dfrac{N+2}{2}$ and we have

$$\begin{aligned}\int_0^t (t-s)^{\frac{-Np'}{2p}}\, ds &= t^{1-N/(2(p-1))} \int_0^1 (1-y)^{\frac{-N}{2(p-1)}}\, dy \\ &\leq C(T)^{p/(p-1)} = T^{1-N/(2(p-1))} \int_0^1 (1-y)^{\frac{-N}{2(p-1)}}\, dy.\end{aligned}$$

On the other hand, lemma 1 implies that

$$\left(\int_0^t \|w^\gamma(s)\|_p^p ds\right)^{1/p} = \|w\|^\gamma_{L^{p\gamma}(Q_t)} \leq C_{34}^\gamma \left(1 + \|u\|_{L^{p\gamma}(Q_t)}\right)^\gamma. \tag{49}$$

If $\|u\|_{L^{p\gamma}(Q_t)} \leq 1$ then the proof ends up. Otherwise there exists C_{35} such that

$$\left(\int_0^t \|w^\gamma(s)\|_p^p ds\right)^{1/p} \leq C_{35}\|u\|^\gamma_{L^{p\gamma}(Q_T)}. \tag{50}$$

Since

$$\begin{aligned}\|u\|^\gamma_{L^{p\gamma}(Q_T)} = \left(\int\int_{Q_T} u^{p\gamma}\right)^{1/p} &= \left(\int\int_{Q_T} u^{p\gamma-p+\varepsilon+p-\varepsilon}\right)^{1/p} \\ &\leq \|u\|^{1-\varepsilon/p}_{L^\infty(Q_T)} \left(\int\int_{Q_T} u^{p\gamma-p+\varepsilon}\right)^{1/p},\end{aligned}$$

it follows that (47) can be written

$$\|u(t)\|_\infty \leq \|u_0\|_\infty + C_{36}\|u\|^{1-\varepsilon/p}_{L^\infty(Q_T)} \left(\int\int_{Q_T} u^{p\gamma-p+\varepsilon}\right)^{1/p}. \tag{51}$$

If $p(\gamma-1) < 2$, by choosing $\varepsilon \in (0, \min(p, 2-p(\gamma-1))$, we deduce from (46) and (51) that there exists C_{37} such that

$$\|u\|_{L^\infty(Q_T)} \leq C_{37}. \tag{52}$$

Note that the above condition $p(\gamma-1) < 2$ holds if $\gamma < 1 + \dfrac{2}{p} < 1 + \dfrac{4}{N+2} = \dfrac{N+6}{N+2}$.
We establish in the same way that there exists C_{38} such that

$$\|v\|_{L^\infty(Q_T)} \leq C_{38}. \tag{53}$$

Finally, for (E_3), we deduce from (52) and (53) that there exists C_{39} such that

$$\|w\|_{L^\infty(Q_T)} \leq C_{39}. \tag{54}$$

This concludes the proof in the case $1 < \gamma < \dfrac{N+6}{N+2}$. □

Remark : Our conjecture is that $\gamma^* = \dfrac{N+6}{N+2}$ is not optimal. In fact, when $N = 1$ one can prove that the result of theorem 3 still holds for $\gamma^* = 7/2$. □

3 Conclusion

• All our results are still true if we replace homogeneous Neumann boundary conditions by homogeneous Dirichlet boundary conditions, it suffices to replace lemma 3 by the following one.

Lemma 4 *Let* (p, q) *such that* $1 \leq p \leq q \leq +\infty$*,* $d > 0$ *and* $S_d(t)$ *the semigroup generated in* $L^p(\Omega)$ *by* $-d\Delta$ *with homogeneous Dirichlet boundary. Then*

$$\|S_d(t)Y\|_q \leq (4\pi t)^{\frac{-N}{2}(\frac{1}{p}-\frac{1}{q})} \|Y\|_p, \text{ for all } Y \in L^p(\Omega), \quad t > 0. \tag{55}$$

For a proof of this lemma, see for instance [14, Proposition 48.4, p. 441]. □

• In the case where the diffusion coefficients are not equal (i.e. $d_i \neq d_j$ for all $1 \leq i \neq j \leq 3$), global existence of classical solutions for $(\mathcal{S})$ or blow-up is still an open question when

$$\frac{N+6}{N+2} \leq \gamma \leq \alpha + \beta.$$

Our guess is that system $(\mathcal{S})$ admits a classical global solution for all $\frac{N+6}{N+2} \leq \gamma < \alpha + \beta$ and that there is a finite time blow-up when $\gamma = \alpha + \beta$ and the dimension N is large. ■

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