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Semantic Completeness of Intuitionistic Predicate Logic in a Fully Constructive Meta-Theory

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SEMANTIC COMPLETENESS OF INTUITIONISTIC PREDICATE LOGIC IN A FULLY CONSTRUCTIVE META-THEORY

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Abstract

SEMANTIC COMPLETENESS OF INTUITIONISTIC

PREDICATE LOGIC

IN A FULLY CONSTRUCTIVE META-THEORY

A constructive proof of the semantic completeness of intuitionistic predicate logic is explored using set-generated complete Heyting Algebra. We work in a constructive set theory that avoids impredicative axioms; for this reason the result is not only intuitionistic but fully constructive. We provide background that makes the thesis accessible to the uninitiated.

Keywords: Semantic Completeness, Intuitionism, Constructivism, Predicativism, Heyting Algebra, Partially Ordered Classes.

Dedicated to my family for encouraging and fostering a curiosity which inevitably lead me to academia. My mother for not only providing me an academic role model but a temperament suited for the journey. My father for his undying love and support. My brother for valuing me beyond my potential. Whitley for her patience, care and the joy she brings to my life.

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Contents

1 Introduction

Intuitionistic and Constructive Mathematics comes in many guises. Unfortunately, there is not universal agreement on the what differentiates intuitionistic and constructive mathematics. We will follow a particular precedent that is gaining popularity, but bear in mind that different sources may follow alternative precedents and use terminology differently. We commence by tracing out some of the history to give some insight. As a school of mathematical thought intuitionism arose around the time of the foundational crisis of mathematics in the early 20th century. This early form of intuitionism is now referred to as Brouwer's intuitionism. Brouwer's intuitionism outright rejects classical principles of mathematics and not surprisingly was met with harsh criticism [3,8]. Later forms of intuitionism were much less contentious. Rather than rejecting classical principles they simply abstained from them. They could be considered economical versions of classical mathematics [8]. The type of intuitionism that this work refers to can be classified as mathematics done with out assuming the Law of Excluded Middle or **LEM** for short. Recall, **LEM** states for any statement ϕ, either ϕ holds or $\neg\phi$ holds. **LEM** seems obvious to the classically trained mathematician, and abstention from it initially "causes some anxiety" [8]. To ease this anxiety it is useful to think of Intuitionistic Mathematics as an attempt at determining what can be achieved with less assumptions and only direct principles. What results is a mathematics that has inherent computational value. Unfortunately, when one gives up **LEM**, one must also give up equivalent principles (e.g. the double negation elimination: $\neg\neg\phi$ implies ϕ) When we give up double negation elimination we lose proof by contradiction and one direction of the contrapositive law: $\neg\psi \longrightarrow \neg\phi$ implies $\phi \longrightarrow \psi$. We also lose stronger principles (e.g. the full Axiom of Choice or **AC** for short). These principles are as familiar to the classical mathematician as **LEM**, so parting with them is equally troubling. I will present Diaconescu's proof (from [8]) that **AC** implies **LEM** in Section 2. Although intuitionistic mathematics dispenses with a number of useful classical tools the resulting mathematics is hardly deficient. Surprisingly, a large part of classical mathematics has been retrieved intuitionistically. This was first seen by Errett Bishop in Constructive Analysis [4] (also see [1], [2] and [3]). Notice we encounter our first clash in terminology but in fact Bishop style mathematics also qualifies as constructive, a term we have yet to discuss. Bishop successfully reformulated the majority of modern analysis with intuitionistic foundations. Certain concepts look drastically different when viewed with these foundations, but the majority of the landscape is familiar. Bishop's monumental leap motivated many to pursue more universal intuitionistic foundations for Bishop Style mathematics. Resultingly, intuintionistic alternatives to Zermelo Fraenkel Set Theory (**ZF** for short) were proposed. Initially, Intuitionistic Set Theroy (**IZF** for short), which has similar axioms to **ZF** with minor changes, was developed. We will see a treatment of formal set theory in Section 5.

Constructivism, at its heart, is about building mathematics from the bottom up rather than relying on platonic existence. Constructivism gained modest popularity during the times of the foundational crisis as well. This period bore another school of mathematical thought that has yet to take its final form: Predicativism (see [3,15,16,17]). Predicativism avoids troublesome self-referential mathematical activity (i.e. definitions and constructions). Unfortunately, there is no universal agreement on what is or is not troublesome. Sorting through the different arguments is messy and quite frankly philosophical (see [17] for more). For now let us discuss impredicativity informally. An impredicative statement invokes (mentions or quantifies over) the set that is being defined or a set that contains the thing being defined [3,15]. A hallmark example of an impredicative definition is the definition of the least upper bound [17]. The least upper bound of a set A is a particular element x of the set of upper bounds of A. The set of upper bounds necessarily includes x. Thus, to define the least upper bound of A we must quantify over a set which contains it. The least upper bound as defined is hypothetical; meaning that we are only describing a property that it has to possess but no existence claim is being made. If one constructs a concrete candidate all we must do is check that it satisfies the necessary property. For this reason impredicative definitions are often viewed as innocuous, from a constructive perspective, as long as the construction of the object is predicative. There is precedence for characterizing constructive mathematics as intuitionistic mathematics that avoids impredicative constructions. We will follow [5,11] — opting to work within foundations that many agree are constructive rather than trying to nail down what constructive mathematics is. A surprisingly large amount of mathematics can be recovered using constructive systems [4,5,6,11,12,14,15], like Martin Löf's Type Theory (**ITT**), Homotopy Type Theory (**HoTT**) or Constructive Zermelo Fraenkel Set Theory (**CZF**), etc. **CZF** has similar axioms to **IZF** but avoids the Power Set Axiom, which is a hallmark example of an impredicative construction (the Power Set Axiom states: $\forall a \exists y \forall x[x \in y \longleftrightarrow x \subseteq a]$). The everyday mathematician does not concern themselves with formalities of set theory, but they are familiar with the language and style of formal set theory. For this reason **CZF** seems a natural choice for this thesis.

For any mathematical theory there is an underlying logic. Associated with classical mathematics is classical logic. A formal language or theory is a precise language that formalizes such a logic. For classical logic there is: 1) Propositional logic which consists of propositional symbols and rules for how other propositions are built out of the familiar logical connectives, and 2) Predicate logic which extends propositional logic by adding predicates and quantifiers (in predicate logic, sentences are called formulae) and additional rules for building formulae. Historically, a logic was called a 'calculus'. For example, Classical Propositional Calculus (or **CPC** for short) and Classical Predicate Calculus (or **CQC**) (You may ask why do we use Q for predicate in the acronym? Good question. Because P was already taken). We tend not to use Calculi in place of Logic when speaking, but the acronyms have stuck around. One proceeds to formalize the notion of provability in a formal language. Proof systems are typically developed via Hilbert Style Systems or Natural Deduction. We will discuss Natural Deduction in Section 4. At this stage a formal language is purely syntactic, meaning it consists only of symbols and rules for manipulating symbols. No meaning or interpretation of the symbols is required. To supply meaning one develops a semantics of a formal language. In (classical) propositional logic, truth tables offer an elementary example of semantics. More generally, one assigns to each propositional variable an element of a Boolean Algebra (with only two elements) and defines how the assignment is extended to all propositions built out of the connectives. Recall, a tautology is a proposition which is true under every possible assignment (true in every column of the truth table). In (classical) predicate logic semantics can be defined similarly (i.e. Tarksi Semantics). In predicate logic we call the analog of a tautology a valid formula. Notice, in a formal language with a proof system and semantics one can now view propositions or formulae themselves as mathematical objects. Then one commences an investigation of the formal language itself. Such an investigation is often called meta-theory. Kurt Gödel, a logician and mathematician, was a prominent meta-theorist. Gödel has many famous meta-theoretic results which is the basis of this work. The Semantic Completeness Theorem (for classical propositional logic) is a very simple statement: If a proposition ϕ is a tautology then it is provable. Notice this meta-theoretic statement connects the semantics of **CPC** with the proof system of **CPC**. There is a similar meta-theoretic statement for **CQC**: If a formula ϕ is valid then it is provable. Gödel provided proofs of the semantic completeness of **CPC** and **CQC** but many others have revisited and improved upon these results in a variety of ways. One thing to note is that proofs in a meta-theory may be informal, but typically one chooses a formal theory that is stronger than the theory of interest. For example to prove statements about **CPC** or **CQC** one may work in **ZF**.

Once intuitionistic/constructive mathematics was established an underlying logic was soon formalized, by Heyting [3] and others. The typical systems are called Intuitionistic Propositional Logic and Intuitionistic Predicate Logic, or **IPC** and **IQC** for short. The nomenclature is a bit unfortunate, but stands for historical reasons. Notice at the level we are working there is no formal notion of set. Therefore, notions of impredicativity do not arise. For this reason there is no distinction between intuitionistic and constructive logic. Intuitionistic logic is weaker than classical logic. More precisely, the theorems of **IPC** form a subset of the theorems of **CPC** and similarly for **IQC** and **CQC**. If **LEM** (or any equivalent statement) is added to either **IPC** or **IQC** the resulting logic is **CPC** or **CQC**, respectively. Symbolically, **IQC** $\subset$ **CQC** and **IQC** $+$ **LEM** $=$ **CQC**. One develops provability and semantics in much the same way as classical logic but with care not to introduce non-intuitionistic principles. In Section 4 we develop a proof system of **IQC** using Natural Deduction. One thing to note is that the semantics of **IPC** and **IQC** can be done in a variety of ways (e.g. Kripke Semantics, topological semantics, algebraic semantics, etc). The semantics chosen for this work is called Heyting Algebra semantics: it assigns propositions (in **IPC**) or formulae (in **IQC**) to elements of a Heyting Algebra, which is analogous to the semantics we discussed for **CPC** and **CQC**. We will explicitly define the semantics of **IQC** in Section 7. Heyting Algebras are a generalization of Boolean Algebras so this is a very natural choice for our semantics. We will see further development of Heyting Algebra in Section 6. Not surprisingly, meta-theoretic questions involving **IPC** and **IQC** were eagerly pursued. There are many semantic completeness results for **IQC** which use classical meta-theories (i.e. they employ **LEM** or an equivalent statement). For example, one commonly proves the following statement: If a proposition ϕ is not provable then it is not valid, which is classically equivalent to semantic completeness. Specifically, this is the contrapostive of semantic completeness, which as you may recall is not, in general, admissible in intuitionistic mathematics. This is a peculiar situation. It seems odd to prove a result about a logic using a principle which the logic itself does not accept. One naturally looks for a completeness proof which is ituitionistic or even constructive.

We now survey the existing work which directly informs this thesis and clearly state what the thesis will achieve. As previously mentioned there are multiple semantics that can be used for **IQC**. Kripke Semantics is used for the classical completeness proof mentioned above. A modified version of Kripke Semantics is used in an intuitionistic completeness proof of **IQC** in [9]. An intuitionistic argument using categorical logic and sheaf semantics is given in [10]. Finally, Troelstra gives a intuitionistic completeness proof of **IQC** (as well as **IPC**) in [1] using Heyting Algebra semantics. The proof by Troelstra is done informally, but can be naturally formalized in **IZF**. Recall **IZF** is the impredicative intuitionistic set theory. The intent of the thesis is to develop all the existing preliminary background information, which includes: intuitionistic and constructive mathematics, formal languages (specifically **IQC**, **IZF**), set theory (specifically **CZF**), theory of partially ordered sets (including Heyting Algebra) and Heyting Algebra semantics. Then we will repackage the proof by Troelstra formally in **IZF**, which will serve as a guide. We then analyze the intuitionistic proof in the interest of finding the deficiencies that lead to its constructive failure. We will uncover a shocking fact about the status of complete lattices in **CZF** that will serve as an obstacle. This leads to the theory of partially ordered classes and the notion of set-generated complete Heyting Algebra. Finally, we define a semantics in this new setting and give a proof of the semantic completeness of **IQC** within **CZF**; which is essentially a constructive extension of Troesltra's proof. It is worth mentioning that the result will not come as a surprise to the audience which is trained in constructive mathematics. It is well known that Heyting Algebras provide a model for intuitionistic logic. In fact, there is a well developed correspondence between Intuitionistic Type Theory, Intuitionistic Predicate Logic and Heyting Algebra (from a Categorical perspective) which may offer a more natural setting for this investigation. The thesis takes the naive approach that was previously laid out because the target audience is likely uninitiated to the subtleties of intuitionistic and constructive mathematics (as well as Type Theory and Category Theory). The goal of the thesis is to bore out the details of the relationship between Heyting Algebra and intuitionistic logic in the most accessible way possible. To the author's knowledge, no cited work has proved the semantic completeness of **IQC** in **CZF** using Heyting Algebra semantics.

2 Intuitionistic and Constructive Mathematics

Intuitionism, as previously mentioned, is a philosophical school of mathematical thought. It holds that mathematics should be based on the concept of knowledge and experienced truth. In its least antagonistic form, intuitionistic and constructive mathematics abstains from many of the indirect methods of classical mathematics: the law of excluded middle (**LEM**), proof by contradiction, the axiom of choice, etc. In its most antagonistic form it outright rejects these classical principles and many others. For this reason intuitionism has been criticised for restricting the mathematician unnecessarily and creating an unfamiliar mathematical landscape. As constructivism subsumes intuitionism these criticisms are transitive. It is a well known fact that many classical results are only provable by indirect methods (e.g. Trichotomy of real numbers, Intermediate value theorem, Well-ordering theorem, etc). Astoundingly, many classical results are recovered constructively with some effort and great care. This has lead to a program, taken up by many, of constructivizing mathematics. Essentially, this program attempts to reformulate results (or entire theories) so that they adhere only to intuitionistic/constructive principles. At times the resulting reformulation looks drastically different than its classical counterpart. Some of these differences have lead to insight that would have never been considered without an interest in intuitionism (e.g. point-free topology, topos theory, etc.) while others have been studied previously but arise naturally when one takes intuitionism and constructivism seriously. Proponents view intuitionistic/constructive mathematics, at the very least, as a useful generalization of classical mathematics. It allows one to investigate what is achievable from a restricted set of assumptions and leads to mathematics with undisputed algorithmic content. This insight is dubbed Curry-Howard correspondence. Its mantra is 'proofs as programs' and says any proof done with only constructive principles can be converted into a computer program. One may explore this correspondence extensively in an intuitionistic type theory (see [22] for more details). The hallmark example is the fact that a constructive existence proof necessarily produces an instance of the object which is claimed to exist. Intuitionistic mathematics has been described by Robert Harper as mathematics as if humans matter. This is because intuitionistic mathematics is proof driven. The **BHK** (Brouwer-Heyting-Kolmogorov) Interpretation makes the philosophical notions of inuitionistism more precise by reinterpreting the logical connectives and quantifiers in this spirit.

Definition 2.1. BHK interpretation

1. A proof of $\perp$ does not exist.
2. A proof of $\phi \wedge \psi$ is a proof of ϕ and a proof of ψ.
3. A proof of $\phi \vee \psi$ is a proof of ϕ or a proof of ψ and a specification of which is proven.
4. A proof of $\phi \longrightarrow \psi$ is a method which converts a proof of ϕ into a proof of ψ.
5. A proof of $\neg\phi$ is a proof of $\phi \longrightarrow \perp$.
6. A proof of $\exists x \in A,\ \phi(x)$ is an object $x \in A$ together with a proof of $\phi(x)$.
7. A proof of $\forall x \in A,\ \phi(x)$ is a method which converts an object $x \in A$ into a proof of $\phi(x)$.

This interpretation describes what constitutes a proof and puts the issue of platonic truth to bed. Something cannot be considered true unless it has been proven from an intuitionistic point of view.

Let us dispense with philosophy and actually look at some examples where intuitionistic mathematics differs from classical mathematics. See [8] for more details. Taking on the **BHK** interpretation it is not hard to see why **LEM** fails. We may not assert the truth of **LEM** without first proving it. To prove **LEM—**under **BHK—** requires us to have a method for proving

ϕ or $\neg\phi$ for an arbitrary formula ϕ. Such a method would allow us a certain amount of omniscience that leads to solutions to many of the toughest mathematical problems and hardly anyone believes such a method exists. These notions are made precise by introducing Omniscience Principles (see [4,8] for more details). This brings us to equivalents of **LEM** that must be avoided when doing Constructive Mathematics. First, I want to discuss proof by contradiction. This technique is very close to the heart of the classical mathematician even if it is viewed as a last resort. It must not be confused with proof of negation which is acceptable intuitionistically. The two are often confused classically because the double-negation elimination makes the distinction invisible. Let us see how they differ. A proof of negation goes like this: assume a proposition ϕ and begin reasoning with available principles. Eventually arriving at a contradiction you must conclude ϕ can not be. So you conclude $\neg\phi$. That is,

$$\begin{array}{c} \phi \\ \vdots \\ \bot \\ \neg\phi \end{array}$$

where $\bot$ represents contradiction. This sounds a lot like proof by contradiction. Why? That is because a proof by contradiction is a proof of negation combined with something called the double negation law, which states: $\neg(\neg\phi) \longleftrightarrow \phi$. The double negation law is a well-known equivalent of **LEM**.

Proposition 2.2. *LEM and the double negation law are equivalent.*

Proof. On the basis of **LEM** we know for any ϕ holds we have ϕ or $\neg\phi$ holds. Thus, $\neg(\neg\phi)$ holds if and only if ϕ holds. Conversely, suppose the double negation law holds. For any formula ϕ, suppose $\neg(\phi \vee \neg\phi)$ holds. Further, suppose ϕ holds. We then conclude $\phi \vee \neg\phi$ holds. This is a contradiction, so by proof of negation $\neg\phi$ holds. But from this we may also conclude $\phi \vee \neg\phi$ holds again. Which also leads to a contradiction. Thus, by proof of negation $\neg\neg(\phi \vee \neg\phi)$ holds. By the double negation law $\phi \vee \neg\phi$ holds. $\square$

It is often mistakenly assumed that Constructive mathematics refutes **LEM**. This is not the case. In fact, under **BHK** we can prove $\neg\neg(\phi \vee \neg\phi)$ by extracting the portion of the proof above.

Proposition 2.3. $\neg\neg(\phi \vee \neg\phi)$

Proof. For any formula ϕ, suppose $\neg(\phi \vee \neg\phi)$ holds. Further, suppose ϕ holds. We then conclude $\phi \vee \neg\phi$ holds. This is a contradiction, so by proof of negation $\neg\phi$ holds. But from this we may also conclude $\phi \vee \neg\phi$ holds again. Which also leads to a contradiction. Thus, by proof of negation $\neg\neg(\phi \vee \neg\phi)$. □

Let us return to proof by contradiction. A proof by contradiction proceeds as follows: we wish to prove ϕ so we assume — for the sake of contradiction — $\neg\phi$ and proceed with available reasoning. We arrive at contradiction and conclude $\neg\phi$ can not be. So using proof of negation we conclude $\neg\neg\phi$ and finally by the double negation law we conclude ϕ. That is,

$$\neg\phi$$

$$\vdots$$

$$\bot$$

$$\neg\neg\phi$$

$$\phi.$$

It is only the last step, which appeals to double negation, that is not available to us intuitionistically/constructively. Let us now discuss contrapositive another classical proof technique that must be dispensed with, although not completely. Using **BHK** we can prove $(\phi \longrightarrow \psi) \longrightarrow (\neg\psi \longrightarrow \neg\phi)$.

Proposition 2.4. $(\phi \longrightarrow \psi) \longrightarrow (\neg\psi \longrightarrow \neg\phi)$

Proof. Suppose $\phi \longrightarrow \psi$ holds. Further suppose $\neg\psi$ holds. Finally, in the interest of proving its negation suppose ϕ holds. Together with the first assumption we conclude ψ holds which is in clear contradiction with the second assumption so by proof of negation we conclude $\neg\phi$ holds. $\square$

The issue, from our perspective, is with the converse direction of the implication. What is provable intuitionistically (by similar reasoning as above) is $(\neg\psi \longrightarrow \neg\phi) \longrightarrow (\phi \longrightarrow \neg\neg\psi))$. But without the double negation law we do not get the general form of the contrapositive. The last principle I want to discuss is the Axiom of Choice (or **AC**). Many mistakenly believe that this controversial axiom is at the heart of non-constructivity. This is in fact not the case. The real issue with **AC** is that in its general form it implies **LEM**. We will now state the Axiom of Choice and prove the above theorem.

Definition 2.5. For any set X, which is a family of sets, if no element of X is empty then there exists a choice function, that is, a function $f : X \to \bigcup X$ such that for every $A \in X$, $f(A) \in A$. Symbolically,

$$\forall X[\emptyset \notin X \longrightarrow \exists f : X \to \bigcup X \; \forall A \in X(f(A) \in A)].$$

Theorem 2.6. *The axiom of choice implies the law of excluded middle.*

Proof. (See [8].) Suppose **AC** holds. Consider an arbitrary proposition P. Consider the sets

$$A = \{x \in \{0,1\} \mid P \vee (x = 0)\}$$

and

$$B = \{x \in \{0,1\} \mid P \vee (x = 1)\}.$$

Clearly, the collection $\{A,B\}$ does not contain the empty set since $0 \in A$ and $1 \in B$. By **AC** there exists a function $f : \{A,B\} \to A \cup B$ such that $f(A) \in A$ and $f(B) \in B$. Now, $A \cup B = \{0,1\}$, so by the definition of the function $f(A) \in \{0,1\}$. Thus, $f(A) = 0$ or $f(A) = 1$. Similarly, $f(B) \in \{0,1\}$ so $f(B) = 0$ or $f(B) = 1$. We may now consider cases:

1. If $f(A) = 1$ (it does not matter which case we consider for $f(B)$ here) then $1 = f(A) \in A$. Thus, $P \vee (1 = 0)$ which is equivalent to P.

2. If $f(B) = 0$ (it does not matter which case we consider for $f(A)$ here) then $0 = f(B) \in B$. Thus, $P \vee (0 = 1)$ which is equivalent to P.

3. If $f(A) = 0$ and $f(B) = 1$. Now we preform a proof of negation (see discussion above). If P holds, then $A = \{0,1\} = B$. By extensionality of functions $0 = f(A) = f(B) = 1$, a contradiction. Thus, $\neg P$ holds.

In each case, we decide whether P or $\neg P$ holds. Thus, **LEM** holds. □

We often refer to this state of affairs by saying **AC** is stronger than **LEM**. Certain schools of Constructive Mathematics accept weaker versions of choice (e.g. Countable Choice and Dependent Choice) that do not imply **LEM**. Strangely, in different foundations for constructive mathematics (i.e. Martin Löf Type Theory) the full version of **AC** is provable, but the subtleties for this anomaly are beyond the scope of this thesis (differences between intensionality and extensionality; see [22] for more details).

Many still find some of the ambiguity of BHK unsatisfactory. Resultingly, a formal theory of intuitionistic logic was developed to lessen some of the ambiguity. Intuitionistic Propositional Calculus (**IPC**) is the formal intuitionstic theory based on the interpretations 1 - 5 from Definition 2.1 while Inuitionstic Predicate Calculus (**IQC**) is the formal intuitionistic theory resulting from adding interpretations 6 and 7 to **IPC**. We will formally define the language **IQC** in Section 3, develop a proof system with in **IQC** in Section 4 and define a semantics for **IQC** in Section 7.

Predicativism was introduced by Russell, Poincare and Weyl [15]. Initially this was a response to the paradoxes of naive set theory, but has taken a life of its own. Informally, we can say that predicative mathematics attempts to avoid circular reasoning. That is, no collection can contains elements definable/constructable ONLY in terms of the collection [15]. Making this principle precise is difficult but following [15] we can say that the comprehension axiom

$$\exists x(y \in x \longleftrightarrow \phi(y))$$

does not hold in general. Instead we can only infer the existence of x if the variables of $\phi(y)$ are restricted to ranging over sets whose existence has been established. This insight rather immediately allows us to see why the power set axiom is viewed, by many as, impredicative. More explicitly, to construct the power set we must quantify over all sets and see how they stand in relation to a certain set. Unfortunately, there is no universal agreement on what constitutes predicativism [15, 16]. So, we will follow the lead of Giovanni Curi in [11] and simply say constructive mathematics (mathematics that is both intuitionistic and predicative) is mathematics done within **CZF**. We will formally define **CZF** in Section 5 and develop some consequences of its axioms.

3 Language of IQC

The formal development of inuitionistic predicate logic (**IQC**) is similar to classical predicate logic. One key difference between classical and intuitionistic logic is that no connective can be defined in terms of another. We first need to define a formal language $\mathscr{L}$, which consists of an alphabet (a finite set of symbols) and a way to determine which strings formed from the alphabet we want in our language. Such strings are typically called well-formed formulae but we will simply call them formulae. For example we wish $\forall x(\phi \wedge \psi)$ to be well-formed but not $x\exists \wedge y\phi$ as it is nonsensical. See [3] for more details about the language of **IQC**.

The two types of expressions that occur are terms and formulae, which are strings of symbols that make up the alphabet of the language. The alphabet is divided into logical and non-logical symbols. Amongst the non-logical symbols are constants, functions, propositions and predicates. Constants, variables and functions operate formally as they do informally and it is not harmful to rely on your intuition. We call these objects terms. An n-place function takes $n \geq 0$ terms as arguments and produces another term. Similarly, a proposition operates formally as you would expect (it is simply a statement). A predicate is a generalization of a proposition: predicates take terms as arguments and produce more complicated formula. An n-place predicate takes $n \geq 0$ terms as arguments. For example, 'Socrates is a man' is a proposition, but 'x is a man' is a 1-place predicate that take arguments. If we let $P(x)$ be the predicate representing 'x is a man' and let c be a constant representing 'Socrates'. Then $P(c)$ represents 'Socrates is a man'.

Definition 3.1. The logical symbols of the language $\mathscr{L}$ consist of the logical connectives ($\wedge$, $\vee$ and $\longrightarrow$), the quantifiers ($\forall$ and $\exists$), punctuation symbols (parenthesis, brackets, commas, etc.), an infinite set of variables $(x, y, z, \ldots)$, and an equality symbol $(=)$. The non-logical symbols of the language $\mathscr{L}$ consist of function symbols and predicate symbols. For any $n \geq 0$ there is a collection of infinitely many n-place function symbols $(f_0^n, f_1^n, f_2^n, \ldots)$ and infinitely many n-place predicate symbols $(P_0^n, P_1^n, P_2^n, \ldots)$. For $n = 0$, the function symbols represent constants of the language and the predicate symbols represent the propositional variables of the language. This defines the alphabet of $\mathscr{L}$.

The set of terms is inductively defined as follows:

1. Any variable is a term.

2. For $n \geq 0$, any n-place function f and terms $t_1, \ldots, t_n$, the expression $f(t_1, \ldots, t_n)$ is a term.

The set of formulae is inductively defined as follows:

1. For $n \geq 0$, any n-place predicate symbol P and terms $t_1, \ldots, t_n$, the expression $P(t_1, \ldots, t_n)$ is a formula.

2. For terms t_1 and t_2, the expression $t_1 = t_2$ is a formula.

3. For formulae ϕ and ψ the expressions $\phi \wedge \psi$, $\phi \vee \psi$ and $\phi \longrightarrow \psi$ are formula.

4. For formula ϕ and variable x the expressions $\forall x \phi$ and $\exists x \phi$ are formulae.

Formulae obtained from rules 1 and 2 are called atomic formulae. We define two special propositional symbols $\perp$ and $\top$ and two notational conventions $\neg\phi$ will represent $\phi \longrightarrow \perp$ and $\phi \longleftrightarrow \psi$ will represent $(\phi \longrightarrow \psi) \wedge (\psi \longrightarrow \phi)$. It is a matter of convention to have an order of operations for the logical constants, but we choose to employ parentheses in this work to avoid confusion. The only variable binding operations we have in **IQC** are $\forall$ and $\exists$. We can define free and bound variables inductively as follows:

1. A variable x is free in an atomic formula ϕ if and only if it occurs in ϕ. No bound variables occur in atomic formula.

2. A variable x is free (bound respectively) in $\phi \wedge \psi$ ($\phi \vee \psi$ or $\phi \longrightarrow \psi$ respectively) if and only if x occurs in either ϕ or ψ.

3. A variable x is free in $\forall y\phi$ ($\exists y\phi$ respectively) if and only if it is free in ϕ and is not y. A variable x is bound in $\forall y\phi$ ($\exists y\phi$ respectively) if and only if it is bound in ϕ or is y.

The formal language just defined has some undesirable restrictions which are not necessarily obvious outright. Tacitly assumed, in any model of a predicate logic, is a domain of discourse. A domain of discourse is a set over which we quantify our variables (e.g. $\forall n \in \mathbb{N}$). The formal theory just described only has models with a single domain of discourse, but for many important areas of study a single domain of discourse is insufficient (e.g. Real Analysis: $\forall \varepsilon \in \mathbb{R}\ \exists N \in \mathbb{N} \ldots$). To overcome this deficiency we introduce the notion of sorts with in a formal language.

Definition 3.2. A formal theory has a countable set of sorts $\mathscr{I}$. Each variable x has an associated sort $i \in \mathscr{I}$. For $n \geq 1$, each function f takes arguments of sort $i_1, \ldots, i_n \in \mathscr{I}$ and has a value of sort $j \in \mathscr{I}$. Note that a constant c (function with $n = 0$) has a sort $i \in \mathscr{I}$. Thus, all terms have a sort. Note that we indicate the sort of a term with a superscript: a term t of sort i is denoted t^i (e.g. $\forall x^i$ indicates an arbitrary variable x of sort i). Now for $n \geq 1$ a predicate R takes arguments of sort $i_1, \ldots, i_n \in \mathscr{I}$. Note that a proposition ϕ (predicates with $n = 0$) has no arguments.

Remark. Note that when a theory has a finite set of sorts one can introduce "sort predicates" and additional axioms which allow us to reduce the theory to a single-sorted logic (see [23] for more). Consider a language with two sorts 1 and 2. Let P_1 and P_2 be predicates with a single argument such that $P_1(x)$ holds if and only if x is of sort 1 and similarly for $P_2(x)$. Now adding

$$\forall x(P_1(x) \vee P_2(x)) \wedge \neg\exists x(P_1(x) \wedge P_2(x))$$

as an axiom to our language reduces it to a single sorted language (see Section 4 for a discussion on axioms and inference rules). If we wish to quantify locally over a sort we simply include it appropriately in the formula (e.g. $\exists x(P_1(x) \wedge \phi(x))$ says there exists a element of sort 1 that satisfies $\phi(x)$). For this reason we will typically work with single sorted theory for simplicity.

At this point we have defined a language for **IQC**, but at present we are only capable of telling which strings of symbols from the alphabet are formulae. We have no means to actually determine which formulae are provable, valid, etc. For example, although $\phi \wedge \neg\phi$ is a formula we do not wish it to be provable. To determine provability we need a proof system and to determine validity we need semantics.

4 Provability with Natural Deduction

We now formalize a proof system for **IQC** using an intuitionistic variant of Natural Deduction (one could equivalently use a Hilbert Style system which uses more axioms and fewer inference rules). See [3,7,18,19] for more details on provability and Natural Deduction. The driving force behind proof is the notion of entailment, that is, when one formula follows some collection of formulae (possibly empty). We now discuss the entailment relation $\vdash$ in a general sense before characterizing entailment in **IQC**. If the assumptions $\phi_1, \ldots, \phi_n$, for $n \geq 0$, entails ϕ we write $\phi_1, \ldots, \phi_n \vdash \phi$. We often condense this notation by writing Γ for the set of assumptions $\phi_1, \cdots, \phi_n$. If $\Gamma \vdash \phi$ we say Γ proves ϕ. It is traditional to call Γ the context. Note if $\Gamma = \emptyset$ we write $\vdash \phi$ and say ϕ is provable. If we wish to add an assumption to a context we often write $\Gamma, \phi \vdash \psi$ instead of $\Gamma \cup \{\phi\} \vdash \psi$. Deductions have the following form: a premise $\Gamma_1 \vdash \phi$ followed by a labeled bar which separates the premise from the conclusion $\Gamma_2 \vdash \psi$, where ϕ and ψ are formulae built up from atomic formulae, logical connectives and quantifiers. For example:

$$\frac{\Gamma_1 \vdash \phi}{\Gamma_2 \vdash \psi} \text{ (Rule Name).}$$

Often the context of the premise and the conclusion are the same (e.g. $\Gamma_1 = \Gamma_2$), but this is not necessarily so. It is our desire for entailment to satisfy the following

$$\frac{}{\phi \vdash \phi} \text{ (Reflexivitity)}$$

$$\frac{\Gamma \vdash \phi \qquad \Gamma, \phi \vdash \psi}{\Gamma \vdash \psi} \text{ (Transitivity)}$$

$$\frac{\Gamma \vdash \phi}{\Gamma, \psi \vdash \phi} \text{ (Weakening)}$$

$$\frac{\Gamma, \phi, \phi \vdash \psi}{\Gamma, \phi \vdash \psi} \text{ (Contraction)}$$

$$\frac{\Gamma \vdash \phi}{\pi\Gamma \vdash \phi} \text{ (Exchange).}$$

Remark. π is a permutation operator of the set Γ. That is, it takes the set Γ and exchanges the order of its elements. One could of course right out the correct form of exchange by expressing Γ as $\phi_1, \ldots, \phi_n$.

It is unproductive to refute reflexivity and transitivity of entailment. Additionally, one might find it difficult to attest to the remaining three properties of entailment but one could imagine an interest in a logic that does not satisfy them. This is not something we plan to explore.

A formal proof in Natural Deduction is a finite tree of entailments where each branch follows from a preceding branch using a rule of inference. What follows are the collection of inference rules that characterize the proof system of **IQC**. The rules come in two flavors: introduction and elimination. These two flavors assist our navigation of formal proofs within the formal system by allowing us to introduce or eliminate a certain connective, quantifier, etc. In some sense the introduction and elimination harmonize to allow for no loss or gain of information. We now characterize the inference rules of **IQC** using deductions.

Definition 4.1. (**IQC** Inference Rules)
For any set of formulae Γ the inference rules are:

$$\frac{}{\Gamma \vdash \top} \text{ ($\top$ Intro)}$$

$$\frac{}{\Gamma, \phi \vdash \phi} \text{ (ϕ Intro)}$$

$$\frac{\Gamma, \phi \vdash \psi}{\Gamma \vdash \phi \longrightarrow \psi} \text{ ($\longrightarrow$ Intro)}$$

$$\frac{\Gamma \vdash \phi \qquad \Gamma \vdash \phi \longrightarrow \psi}{\Gamma \vdash \psi} \text{ ($\longrightarrow$ Elim)}$$

$$\frac{\Gamma \vdash \phi \qquad \Gamma \vdash \psi}{\Gamma \vdash \phi \wedge \psi} \text{ ($\wedge$ Intro)}$$

$$\frac{\Gamma \vdash \phi \wedge \psi}{\Gamma \vdash \phi} \text{ ($\wedge$ Elim$_\mathrm{L}$)}$$

$$\frac{\Gamma \vdash \phi \wedge \psi}{\Gamma \vdash \psi} \text{ ($\wedge$ Elim$_\mathrm{R}$)}$$

$$\frac{\Gamma \vdash \bot}{\Gamma \vdash \phi} \;(\bot \text{ Elim})$$

$$\frac{\Gamma \vdash \phi}{\Gamma \vdash \phi \vee \psi} \;(\vee \text{ Intro}_{\text{L}})$$

$$\frac{\Gamma \vdash \psi}{\Gamma \vdash \phi \vee \psi} \;(\vee \text{ Intro}_{\text{R}})$$

$$\frac{\Gamma \vdash \phi \vee \psi \qquad \Gamma \vdash \phi \longrightarrow \theta \qquad \Gamma \vdash \psi \longrightarrow \theta}{\Gamma \vdash \theta} \;(\vee \text{ Elim})$$

$$\frac{\Gamma \vdash \phi}{\Gamma \vdash \forall x(\phi)} \;(\forall \text{ Intro})$$

$$\frac{\Gamma \vdash \forall x(\phi)}{\Gamma \vdash \phi[t/x]} \;(\forall \text{ Elim})$$

$$\frac{\Gamma \vdash \phi[t/x]}{\Gamma \vdash \exists x(\phi)} \;(\exists \text{ Intro})$$

$$\frac{\Gamma \vdash \exists x(\phi) \qquad \Gamma \vdash \phi \longrightarrow \psi}{\Gamma \vdash \psi} \;(\exists \text{ Elim})$$

$$\frac{\Gamma \vdash t \,=\, t \longrightarrow \phi}{\Gamma \vdash \phi} \;(= \text{Refl})$$

$$\frac{\Gamma \vdash \phi[t/x] \qquad \Gamma \vdash t \,=\, s}{\Gamma \vdash \phi[s/x]} \;(= \text{Repl})$$

Notice there is no elimination rule for $\top$ and not introduction rule for $\bot$. The absence of each maintains the harmony between the rules that was mentioned earlier. Recall, $\neg\phi$ stands for $\phi \longrightarrow \bot$ and $\phi \longleftrightarrow \psi$ stands for $(\phi \longrightarrow \psi) \wedge (\psi \longrightarrow \phi)$. The notation $\phi[t/x]$ is intended to express a formula ϕ where are all free occurrences of the variable x are replaced with the term t. If no such x occurs then there is nothing to replace. This thesis will not address the subtle notions of substitution and avoidance of variable capture (see [3,24] for more details).

We now demonstrate two proofs in Natural Deduction as an illustration. For more examples of Natural Deduction proof trees see [5,18,19]. We first demonstrate the proof tree method by deriving commutativity of $\wedge$.

Theorem 4.2. $\vdash (\phi \wedge \psi) \longrightarrow (\psi \wedge \phi)$.

Proof. We construct a proof tree.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\phi \wedge \psi \vdash \phi \wedge \psi}\,(\phi \wedge \psi \text{ Intro})}{\phi \wedge \psi \vdash \psi}\,(\wedge \text{ Elim}_{\text{R}}) \qquad \dfrac{\dfrac{}{\phi \wedge \psi \vdash \phi \wedge \psi}\,(\phi \wedge \psi \text{ Intro})}{\phi \wedge \psi \vdash \phi}\,(\wedge \text{ Elim}_{\text{L}})}{\phi \wedge \psi \vdash \psi \wedge \phi}\,(\wedge \text{ Intro})}{\vdash (\phi \wedge \psi) \longrightarrow (\psi \wedge \phi)}\,(\longrightarrow \text{ Intro})$$

□

We now wish to prove transitivity of $\rightarrow$ using Natural Deduction inference rules. This will be useful later.

Theorem 4.3. $\vdash ((\phi \longrightarrow \psi) \wedge (\psi \longrightarrow \theta)) \longrightarrow (\phi \longrightarrow \theta)$.

Proof. We construct a proof tree. Let α represent the formula $(\phi \longrightarrow \psi) \wedge (\psi \longrightarrow \theta)$ for the purpose of condensing the proof tree.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\alpha, \phi \vdash \phi}\,(\phi \text{ Intro}) \qquad \dfrac{\dfrac{}{\alpha, \phi \vdash \alpha}\,(\alpha \text{ Intro})}{\alpha, \phi \vdash \phi \longrightarrow \psi}\,(\wedge \text{ Elim}_{\text{L}})}{\alpha, \phi \vdash \psi}\,(\longrightarrow \text{ Elim}) \qquad \dfrac{\dfrac{}{\alpha, \phi \vdash \alpha}\,(\alpha \text{ Intro})}{\alpha, \phi \vdash \psi \longrightarrow \theta}\,(\wedge \text{ Elim}_{\text{R}})}{\alpha, \phi \vdash \theta}\,(\longrightarrow \text{ Elim})}{\alpha \vdash \phi \longrightarrow \theta}\,(\longrightarrow \text{ Intro})}{\vdash ((\phi \longrightarrow \psi) \wedge (\psi \longrightarrow \theta)) \longrightarrow (\phi \longrightarrow \theta)}\,(\longrightarrow \text{ Intro})$$

□

Often times it is enough to simply state which rules one uses to achieve a certain result with out explicitly giving the proof tree.

5 Constructive Set Theory

We now give a brief treatment of Zermelo Fraenkel Set Theory (**ZF** for short) and the intuitionistic and constructive variant (**IZF** and **CZF** for short). This section is very involved. We advise skimming it on a first reading but pay due attention to the axioms of **CZF**, Proposition 5.11, Theorem 5.16 and 5.17. We shall follow the treatment given by Aczel in [5,6] very closely. **ZF** (or **ZFC**, which is **ZF** with the Axiom of Choice) is the formal system that underpins classical mathematics. The axioms of set theory are formal statements that capture intuitive ideas about sets that working mathematician are familiar with. For example the pairing axiom asserts that for any sets A and B there exists a set $C = \{A, B\}$. Some of these axioms formalize notions that are less familiar but under the right reading are still very intuitive. Unfortunately, these axioms are very dense and difficult to read. When focusing on a particular axiom, it is helpful to identify the set which we are asserting to exist. This is typically the set we are 'forming'. Doing this allows you to establish a point of reference and interpret the rest of the axiom more clearly. After the axioms are stated we will discuss them further. See [5,6] (or any standard treatment of **ZF**) for more details.

Definition 5.1. **ZF** is formulated in the language of first-order logic (which we have been calling classical predicate logic or **CQC**) with equality and a single non-logical symbol, the binary connective $\in$. We use $a \subseteq b$ to abbreviate the following formula $\forall u(u \in a \longrightarrow u \in b)$. We use the abbreviation $\forall x \in A\phi(x)$ for $\forall x(x \in A \longrightarrow \phi(x))$ and $\exists x \in A\phi(x)$ for $\exists x(x \in A \wedge \phi(x))$. Such formulae are called bounded because they are not free to quantify over all sets. We use the abbreviation $\exists! x \in X\phi(x)$ for $\exists x \in X \forall y \in X(\phi(y) \longleftrightarrow x = y)$. The set theoretic axioms of **ZF** are as follows:

Extensionality

$$\forall a \forall b[\forall x(x \in a \longleftrightarrow x \in b) \longrightarrow a = b]$$

Pairing

$$\forall a \forall b \exists y \forall x[x \in y \longleftrightarrow (x = a \vee x = b)]$$

Union

$$\forall a \exists y \forall x[x \in y \longleftrightarrow \exists u \in a(x \in u)]$$

Power set

$$\forall a \exists y \forall x[x \in y \longleftrightarrow x \subseteq a]$$

Infinity

$$\exists a[\exists x \in a \wedge (\forall x \in a \exists y \in a(x \in y))]$$

Foundation

$$\forall a[\exists x \in a \longrightarrow \exists x \in a \forall y \in a(y \notin x)]$$

Separation

$$\forall a \exists y \forall x[x \in y \longleftrightarrow x \in a \wedge \phi(x)]$$

for any formula ϕ where y is not free in ϕ.

Replacement

$$\forall a[\forall x \in a \exists! y \phi(x,y) \longrightarrow \exists b \forall y[y \in b \longleftrightarrow \exists x \in a(\phi(x,y))]]$$

for any formula ϕ where b is not free in ϕ.

These axioms are dense and difficult to intuit. Let us discuss them, briefly. The extensionality axiom essentially gives a way of showing equality of sets, while the converse is not required as an axiom because it can be acquired using the substitution property of equality. The pairing axiom assures that we can take two objects and construct a set that consists of only those two objects. The union axiom guarantees the existence of a set y that consists of all the elements of an indicated family of sets a, in agreement with our intuition. The power set axiom guarantees the existence of a collection y of all subsets of a set a. The axiom of infinity guarantees the existence a non-empty set a such that for each element x of a there exists an element y of a that contains x. This gives an infinitely ascending sequence of elements of a and as such guarantees the existence of an infinite set. The foundation axiom assures the set containment does not descend infinitely and as such guarantees the existence of an element that is not a member of any other element. We call a statement an axiom schema if it involves an arbitrary formula. Technically, an axiom schema is not a single axiom but infinitely many axioms, one for each formula. The separation axiom schema guarantees the existence of set y whose elements are elements of a and satisfy $\phi(x)$ for some formula ϕ. This assures us that we can specify subsets of a set. We often represent the subset y as $\{x \in a \mid \phi(x)\}$. The replacement axiom schema is the most complicated axiom of **ZF**. It states that if you are given a function captured by the formula $\phi(x,y)$ then the image of the function is itself a set.

The axioms of **ZF** together with the **AC** form **ZFC**. We will not include the axiom of choice in any intuitionistic or constructive variant because the full axiom of choice implies the law of excluded middle and is deemed non-intuitionistic (see Section 2). Although certain schools of constructive mathematics do allow weaker versions of choice (i.e. countable choice and dependent choice).

We will now define **IZF** (Intuitionistic Set Theory) which is the natural intuitionistic extension of **ZF**. The axioms of **IZF** are identical to those in **ZF** except we will be replacing the Axioms of Foundation and Replacement with Set Induction Schema and Collection, respectively. See [5,6] for more details on **IZF**.

Definition 5.2. **IZF** is formulated using the language of Intuitionistic First-Order Logic (which we have been calling Intuitionistic Predicate Logic or **IQC**), with equality and a single non-logical symbol, the binary connective $\in$. We use similar abbreviations in **IZF** as in **ZF**. The set theoretic axioms of **IZF** are as follows:

Extensionality

$$\forall a \forall b [\forall x (x \in a \longleftrightarrow x \in b) \longrightarrow a = b]$$

Pairing

$$\forall a \forall b \exists y \forall x [x \in y \longleftrightarrow (x = a \vee x = b)]$$

Union

$$\forall a \exists y \forall x [x \in y \longleftrightarrow \exists u \in a (x \in u)]$$

Powerset

$$\forall a \exists y \forall x [x \in y \longleftrightarrow x \subseteq a]$$

Infinity

$$\exists a [\exists x \in a \wedge (\forall x \in a \exists y \in a (x \in y))]$$

Set Induction

$$\forall a [(\forall x \in a \phi(x)) \longrightarrow \phi(a)] \longrightarrow \forall a \phi(a)$$

for any formula ϕ.

Separation

$$\forall a \exists y \forall x [x \in y \longleftrightarrow x \in a \wedge \phi(x)]$$

for any formula ϕ where y is not free in ϕ.

Collection

$$\forall a [\forall x \in a \exists y \phi(x,y) \longrightarrow \exists b \forall x \in a \exists y \in b \phi(x,y)]$$

for any formula ϕ with b not free in ϕ.

We will now briefly discuss the new axioms we have introduced. The axiom of set induction allows us to induct on the membership relation. The axiom of collection is similar to replacement but we do not require y to be unique. Essentially it asserts that we can collect the elements of the right component of a relation into a set.

CZF (Constructive Set Theory) is a constructive set theory similar to **IZF** but has axioms of differing strengths. **CZF** uses bounded seperation instead of normal seperation, it uses strong collection instead of normal collection and finally replaces the power set axiom with the subset collection axiom. We will define **CZF** and state its axioms. Following this we will commence a deep discussion of some of the consequences of **CZF**. See [5,6] for more on **CZF** and related results.

Definition 5.3. **CZF** is formulated in the language of **IQC**, with equality and a single non-logical symbol, the binary connective $\in$. We use similar abbreviations in **CZF** as in **ZF**. The set theoretic axioms of **CZF** are as follows:

Extensionality

$$\forall a \forall b [\forall x (x \in a \longleftrightarrow x \in b) \longrightarrow a = b]$$

Pairing

$$\forall a \forall b \exists y \forall x [x \in y \longleftrightarrow (x = a \vee x = b)]$$

Union

$$\forall a \exists y \forall x [x \in y \longleftrightarrow \exists u \in a (x \in u)]$$

Infinity

$$\exists a [\exists x \in a \wedge (\forall x \in a \exists y \in a (x \in y))]$$

Set Induction

$$\forall a [(\forall x \in a \phi(x)) \longrightarrow \phi(a)] \longrightarrow \forall a \phi(a)$$

Bounded Separation

$$\forall a \exists y \forall x [x \in y \longleftrightarrow x \in a \wedge \phi(x)]$$

for any bounded formula ϕ where y is not free in ϕ. A formula is bounded if all of its quantifiers are bounded (i.e. $\forall x \in y$ or $\exists x \in y$.)

Subset Collection

$$\forall a \forall b \exists c \forall u[\forall x \in a \exists y \in b(\psi(x,y,u))$$

$$\longrightarrow \exists d \in c(\forall x \in a \exists y \in d(\psi(x,y,u)) \wedge \forall y \in d \exists x \in a(\psi(x,y,u)))]$$

for any formula ψ.

Strong Collection

$$\forall a[\forall x \in a \exists y \phi(x,y) \longrightarrow \exists b(\forall x \in a \exists y \in b(\phi(x,y)) \wedge \forall y \in b \exists x \in a(\phi(x,y)))]$$

for any formula ϕ.

Bounded separation is clearly just a weakened version of separation. That is, all formulae that we wish to use in an application of bounded separation must be bounded. This is important because we do not want to form subsets of sets whose formulae allow unbounded quantification (quantification over all sets). Such formulae can lead to impredicative constructions. Strong collection on the other hand strengthens the collection axiom to compensate for the weakening of separation. With strong collection one can prove collection and replacement. This is because strong collection and collection have the same hypotheses but collection has a weaker conclusion and strong collection has a hypothesis weaker than replacement but the conclusion of replacement follows naturally from the conclusion of strong collection. Strong collection is a well-known theorem of **ZF**. From the axioms of infinity, set induction and extensionality we can deduce that there exists a unique set x such that $\forall u[u \in x \longleftrightarrow (u = 0 \vee \exists v \in x(u = v \cup \{v\}))]$ [13]. One often denotes this set ω and thinks of it as the natural numbers without any arithmetic structure. This is essentially the Von Neumann construction of the natural numbers (i.e. $0 \equiv \emptyset, 1 \equiv \{\emptyset\} = \{0\}, 2 \equiv \{\emptyset, \{\emptyset\}\} = \{0, \{0\}\} = \{0,1\}$, etc). Any inductively defined collection can be put into a correspondence with ω and is a set by replacement.

The axioms of **CZF** (or any formal set theory) tell us precisely which collections ARE sets. But for any formula $\phi(x)$ there is no issue with thinking of a collection A such that

$x \in A \longleftrightarrow \phi(x)$, even if A is not provably a set. We call collections that are not provably sets proper classes. We often abbreviate the class A as $\{x \mid \phi(x)\}$. For classes A and B we say $A \subseteq B$ if $\forall x(x \in A \longrightarrow x \in B)$ and $A = B$ if $\forall x(x \in A \longleftrightarrow x \in B)$. One must be cautious working with classes, because they do not formally exist. We must always be able discuss them within the language of our formal system. For example: there is no harm in saying $y \in \{x \mid \phi(x)\}$ because in the formal language we would simple say $\phi(y)$. Similarly, there is no harm in saying $\{x \mid \phi(x)\} = \{y \mid \psi(y)\}$ because formally we would say $\forall z(\phi(z) \longleftrightarrow \psi(z))$.

We now develop some basic set theoretic constructions. Using class notation one can construct all kinds of more complicated classes. Then using the axioms of **CZF** we can prove these constructions are sets under certain circumstances.

From the pairing axiom we can infer for any sets a and b the existence of a set y such that

$$\forall x(x \in y \longleftrightarrow (x = a \vee x = b))$$

and by extensionality the set is unique so we call it $\{a,b\}$. Now we may pair a with itself and form $\{a\} = \{a,a\}$. This is the set with only the element a as a member, the singleton set.

Definition 5.4. We define the ordered pair of sets a and b to be the set

$$\langle a,b\rangle \equiv \{\{a\},\{a,b\}\}.$$

Clearly, the ordered pair is a unique set by the pairing axiom and extensionality.

Proposition 5.5. *If $\langle a,b\rangle = \langle c,d\rangle$ then $a = c$ and $b = d$.*

Proof. A classical proof would use the law of excluded middle. Observe that the cases considered are exhaustive by virtue of the definition of an ordered pair. The proof is very subtle.

Since $\{a\} \in \langle a,b\rangle$ then it must be that $\{a\} \in \langle c,d\rangle$. By definition of the ordered pair either $\{a\} = \{c\}$ or $\{a\} = \{c,d\}$. In either case $a = c$.

Now since $\{a,b\} \in \langle a,b\rangle$ then it must be that $\{a,b\} \in \langle c,d\rangle$. By similar reasoning as before either $\{a,b\} = \{c\}$ or $\{a,b\} = \{c,d\}$. Now either $b = c$ or $b = d$. If $b = c$ then from the previous paragraph we know that $a = c = b$. Which implies $\{c\} = \{c,d\}$. Thus $c = d$ and hence $b = d$. So we conclude $b = d$ in either case. □

Definition 5.6. Let A, B, C be classes and $a, a_1, \ldots, a_n$ be sets. We form the following classes.

1. $\{a_1, \ldots, a_n\} = \{x \mid x = a_1 \vee \ldots \vee x = a_n\}$. When $n = 0$ this is the empty class.
2. $\bigcup A = \{x \mid \exists y \in A(x \in y)\}$
3. $A \cup B = \{x \mid x \in A \vee x \in B\}$
4. $a^+ = a \cup \{a\}$
5. $Pow(A) = \{x \mid x \subseteq A\}$
6. $\{x \in B \mid \phi(x)\} = \{x \mid x \in B \wedge \phi(x)\}$
7. $V = \{x \mid x = x\}$

If A is a set the union axiom asserts the $\bigcup A$ is a set. If A and B are sets then by the pairing axioms $\{A, B\}$ is a set. Thus, $A \cup B = \bigcup\{A, B\}$ is a set. Additionally, if a is a set the pairing axiom asserts $\{a\}$ is a set (in this case we pair a with itself.) Thus $a^+ = a \cup \{a\}$ is a set. Finally, if $n > 0$ and a_i is a set for $1 \leq i \leq n$ then $\{a_i\}$ is a set. Thus, the finite union, $\{a_1, \ldots, a_n\}$ is a set. Why do we not form the two classes $\bigcap A = \{x \mid \forall y \in A(x \in y)\}$ and $A \cap B = \{x \mid x \in A \wedge x \in B\}$? This is due to a subtlety with the empty set.

Definition 5.7. If A is a class and $\theta(x, y)$ a formula then for each $a \in A$ we form the class $B_a = \{y \mid \theta(a, y)\}$ and call $(B_a)_{a \in A}$ a family of classes over A. If $(B_a)_{a \in A}$ is a family of classes over A we form the classes

$$\bigcup_{a \in A} B_a = \{y \mid \exists a \in A(y \in B_a)\},$$

$$\bigcap_{a \in A} B_a = \{y \mid \forall a \in A(y \in B_a)\}.$$

Consider, as an example, the 'class' $\mathbb{R}$ and formula $x \leq y < \infty$. In this case, $B_a = \{y \mid a \leq y < \infty\}$. Thus, $\bigcup_{a \in A} B_a = \mathbb{R}$ and $\bigcap_{a \in A} B_a = \emptyset$.

Remark. One can prove $\mathbb{R}$ is a set in **CZF** [5,6] after choosing a particular representation (i.e. Dedekind reals, Cauchy reals, etc.) but this is not necessary for our work.

Definition 5.8. For classes A and B form the class $A \times B = \{z \mid \exists x \in A \exists y \in B(z = \langle x, y \rangle)\}$. For $r = 1, 2, \ldots$ we can form the r-fold product of A by $A^1 = A$ and $A^{k+1} = A^k \times A$.

We now introduce the union-replacement scheme which is a combination of the union axiom and the replacement scheme. Recall, that strong collection of **CZF** implies Replacement.

Union-Replacement

$$\forall x \in a \exists b \forall y (y \in b \longleftrightarrow \phi(x, y)) \longrightarrow \exists c \forall y (y \in c \longleftrightarrow \exists x \in a \phi(x, y))$$

for any formula ϕ.

Proposition 5.9. *Given extensionality and pairing the union-replacement scheme is equivalent to the conjunction of the union axiom and replacement scheme.*

Proof. Assume union-replacement. Let a be a set. We show replacement by supposing $\forall x \in a \exists ! y \phi(x, y)$. We have shown that singleton classes are sets. So letting $b = \{y\}$ we have

$$\forall x \in a \exists b \forall y (y \in b \longleftrightarrow \phi(x, y))$$

and by union-replacement

$$\exists c \forall y (y \in c \longleftrightarrow \exists x \in a \phi(x, y)).$$

Hence, replacement holds. The union axiom follows from the instance of union-replacement where $\phi(x, y)$ is replaced with $y \in x$. Trivially,

$$\forall x \in a \exists x \forall y (y \in x \longleftrightarrow y \in x).$$

So by union-replacement we have

$$\exists c \forall y (y \in c \longleftrightarrow \exists x \in a (y \in x)).$$

Thus, union holds. Now assume the union axiom and replacement scheme. Let a be a set and suppose $\forall x \in a \exists b \forall y (y \in b \longleftrightarrow \phi(x, y))$. Let b and b' be sets satisfying the previous expression.

By extensionality $b = b'$, so that

$$\forall x \in a \exists ! b \forall y (y \in b \longleftrightarrow \phi(x,y)).$$

By replacement there exists a set d such that for any set z we have

$$z \in d \longleftrightarrow \exists x \in a (\forall y (y \in z \longleftrightarrow \phi(x,y))).$$

By the union axiom we form the union of this set d. That is there exists a set c such that for any set y

$$y \in c \longleftrightarrow \exists z \in d (y \in z).$$

Now by the previous statement we then have

$$y \in c \longleftrightarrow \exists x \in a \phi(x,y).$$

Thus, union-replacement holds. □

We now show certain classes under certain circumstances are sets using union-replacement.

Definition 5.10. Let A be a class and $R \subseteq A \times A$. R is said to be an equivalence relation on A if the following holds for all $a,b,c \in A$:

1. aRa (R is reflexive).

2. if aRb then bRa (R is symmetric).

3. if aRb and bRc then aRc (R is transitive).

For any $a \in A$ we can form the equivalence class

$$[a]_R = \{x \in A \mid xRa\}.$$

Proposition 5.11. *If A is a set and R is a equivalence relation on A (and thus also a set), then* *for each $a \in A$, $[a]_R$ is a set. Moreover, the quotient of A with respect to R,*

$$A/R = \{[a]_R \mid a \in A\},$$

is a set.

Proof. In showing that $[a]_R$ is a set it suffices to show that xRa can be given by a bounded formula, the result is then achieved by applying bounded separation. Observe, xRa if and only if $\exists z \in R z = \langle x, a \rangle$. Recall, $\langle x, a \rangle = \{\{x\}, \{x, a\}\}$, so the formula is bounded and $[a]_R$ is a set.

Now for each $a \in A$ there exists $[a]_R$ such that $\forall y \in [a]_R yRa$, thus by strong collection we can form the quotient set A/R. □

Definition 5.12. For classes A and B if $R \subseteq A \times B$ we call R a class-relation and if $\langle a, b \rangle \in R$ we often write aRb. We also form the classes $dom(R) = \{x \mid \exists y \in B(xRy)\}$ and $ran(R) = \{y \mid \exists x \in A(xRy)\}$. If $R \subseteq A \times B$ is a relation such that $\forall a \in A \exists b \in B(\langle a, b \rangle \in R)$ we call R a multi-valued class-function. If $f \subseteq A \times B$ such that $\forall a \in A \exists! b \in B(\langle a, b \rangle \in f)$ we call f a class-function and write $f(a) = b$. If f is a class function such that $dom(f) = A$ and $ran(f) \subseteq B$ we write $f : A \to B$.

Remark. Formally f does not exist, but by assumption $f \subseteq A \times B$—f is a sub-class of $A \times B$ — so there exists a representative formula $\theta(z)$ of f such that $\forall z(z \in f \longleftrightarrow \theta(z))$. Letting $\phi(x)$ and $\psi(y)$ be the representative formulas of the classes A and B respectively, we would say in the language of set theory f is a class-function if $\forall a \phi(a) \exists! b \psi(b) \theta(\langle a, b \rangle)$. This formally captures that definition of a class-function but makes no mention of any class. This serves as a reminder that all mention of classes CAN be formally removed, but an example of how useful the notion can be.

Proposition 5.13. *If A is a class and $\forall x \in A \exists! y \phi(x, y)$ then there exists a unique class function F with $dom(F) = A$ such that $\forall x \in A \phi(x, F(x))$. Moreover if A is a set so is F.*

Proof. Suppose $\forall x \in A \exists! y \phi(x, y)$. Then

$$\forall x \in A \exists! z \theta(x, z)$$

where $\theta(x,z)$ is $\exists y(z = \langle x,y\rangle \wedge \phi(x,y))$. The required class function is

$$F = \{z \mid \exists x \in A\theta(x,z)\}.$$

Now $z \in F$ if and only if $\exists x \in A\exists y(z = \langle x,y\rangle \wedge \phi(x,y)$. But we know $\forall x \in A\exists! y\phi(x,y)$. So $dom(F) = A$ and by letting $F(x) = y$ we have $\forall x \in A\phi(x,F(x))$. Suppose there exists G that satisfies $dom(G) = A$ and $\forall x \in A\phi(x,G(x))$. By assumption $\forall x \in AF(x) = G(x)$, that is F is unique. Finally, if A is a set then by replacement F is a set. □

Proposition 5.14. *If A is a set and $F : A \to B$ then F is a set (and thus F is simply a function.)*

Proof. Since $\forall x \in A\exists! y\langle x,y\rangle \in F$ Proposition 5.13 implies there is a unique function f with $dom(f) = A$ and $\forall x \in A(\langle x, f(x)\rangle \in F$. Thus, $F = f$ and F is a set. □

Remark. In Proposition 5.14 B may be a class or a function.

The subset collection axiom is among the most intricate axioms in any of the set theories we have encountered. Because of this we are going to introduce the concept of fullness to aid our intuition about subset collection. See [13] for further details.

Definition 5.15. For sets A and B let B^A be the class of all functions with domain A and range contained in B. Let $mv(B^A)$ be the class of all multi-valued functions from A to B. A set C is full in $mv(B^A)$ if $C \subseteq mv(B^A)$ and

$$\forall R \in mv(B^A)\exists S \in C(S \subseteq R).$$

Remark. Notice in Definition 5.12. S can not be empty because there is no empty element in $mv(B^A)$. Recall, the definition of a multi-valued class-function requires us to identify at least one $b \in B$ for every $a \in A$.

We now state some additional axioms and proceed to prove some relationships between them, subset collection and power set.

Fullness

For all sets A and B there exists a set C such that C is full in $mv(B^A)$.

Exponentiation

For all sets A and B the class B^A is a set. More formally,

$$\forall a \forall b \exists c \forall f [f \in c \longleftrightarrow (f : a \to b)].$$

Let $\mathbf{CZF}^-$ be **CZF** without subset collection.

Theorem 5.16. *($\mathbf{CZF}^-$) The following hold:*

1. *Subset collection and fullness are equivalent.*

2. *Fullness implies exponentiation.*

3. *The powerset axiom implies subset collection*

Proof. We follow [5,6].

1. Suppose subset collection holds. Let A and B be sets and $\phi(x,y,u)$ be the formula for $y \in u \wedge \exists z \in B(y = \langle x,z \rangle)$. For all $R \in mv(B^A)$ we have $\forall x \in A \exists z \in B(\langle x,z \rangle \in R)$ which is equivalent to $\forall x \in A \exists y \in A \times B \phi(x,y,R)$. By subset collection there exists C such that the previous line implies $\exists S \in C(\forall y \in S \exists x \in A \phi(x,y,R))$ but recall ϕ says $y \in R$. Thus, we may conclude there exists a C such that $\forall R \in mv(B^A) \exists S \in C(S \subseteq R)$, that is there exists C full in $mv(B^A)$.

Now suppose fullness holds. Let A and B be sets and C be full in $mv(B^A)$. For any set u suppose $\forall x \in A \exists y \in B(\phi(x,y,u))$. Let $\psi(x,z,u)$ be the formula for $\exists y \in B(z = \langle x,y \rangle \wedge \phi(x,y,u))$. Then $\forall x \in A \exists z(\psi(x,z,u))$. By strong collection there exists $v \subseteq A \times B$ such that

$$\forall x \in A \exists z \in v(\psi(x,z,u)) \wedge \forall z \in v \exists x \in A(\psi(x,z,u)).$$

The first conjunct can be expanded to

$$\forall x \in A \exists z \in v \exists y \in B(z = \langle x,y \rangle \wedge \phi(x,y,u))$$

which of course is equivalent to

$$\forall x \in A \exists y \in B \exists z \in v (z = \langle x, y \rangle \wedge \phi(x, y, u)).$$

Now $\exists z \in v(z = \langle x, y \rangle)$ implies $\langle x, y \rangle \in v$ (elimination of the existential quantifier) so we may dispense with mention of z and simply conclude

$$\forall x \in A \exists y \in B (\langle x, y \rangle \in v \wedge \phi(x, y, u)).$$

The second conjunct can be expanded to

$$\forall z \in v \exists x \in A \exists y \in B (z = \langle x, y \rangle \wedge \phi(x, y, u)).$$

Now for any $x \in A$ and $y \in B$ suppose $\langle x, y \rangle \in v$. Then (by universal quantifier elimination) we conclude $\phi(x, y, u)$ that is

$$\forall x \in A \forall y \in B (\langle x, y \rangle \in v \longrightarrow \phi(x, y, u)).$$

As C is full there exists $w \in C$ such that $w \subseteq v$. Define $ran(w) = \{y \mid \exists x(\langle x, y \rangle \in w\} \subseteq B$. Then since $w \in mv(B^A)$ we have for all $x \in A$ there is $y \in B$ such that $\langle x, y \rangle \in w \subseteq v$ and by the above $\phi(x, y, u)$ holds. That is,

$$\forall x \in A \exists y \in ran(w) \phi(x, y, u)$$

and by definition we have for all $y \in ran(w)$ there is $x \in A$ such that $\langle x, y \rangle \in w \subseteq v$ and by the above $\phi(x, y, u)$ holds. That is,

$$\forall y \in ran(w) \exists x \in A \phi(x, y, u).$$

Thus, $D = \{ran(c) \mid c \in C\}$ witnesses the instance of subset collection pertaining to ϕ. Notice that C being a set together with strong collection implies D is a set.

2. Suppose C is full in $mv(B^A)$. Let $f \in B^A \subseteq mv(B^A)$. By Fullness $\exists R \in C(R \subseteq f)$. Now

suppose $\langle x,y\rangle \in f$ and $\langle x,y'\rangle \in R$. Since $R \subseteq f$ we have $\langle x,y'\rangle \in f$. Since f is a function, $y = y'$ and $\langle x,y\rangle \in R$ (and $f \subseteq R$.) So by extensionality $R = f$. Thus, $f \in B^A$ iff $f \in C$ and f is a function or $B^A = \{f \in C \mid f : A \to B\}$. Since C is a set we conclude, by bounded separation, that B^A is a set.

3. It suffices to show that the power set axiom implies fullness. Let A and B be sets. By the power set axiom $Pow(A \times B)$ is a set. Let $\pi_1 : A \times B \to A$ via $\pi_1(\langle a,b\rangle) = a$ for $a \in A$ and $b \in B$. For $U \in Pow(A \times B)$ define $\pi_1(U) = \{x \in A \mid \exists u \in U(\pi_1(u) = x)\}$. Let $C = \{U \in Pow(A \times B) \mid \pi_1(U) = A\}$. By bounded separation C is a set. We claim $C = mv(B^A)$. Suppose $U \in C$. Let $x \in A = \pi_1(U)$ then there exists $u \in U$ such that $\pi_1(u) = x$. But for any such $u \in U$ there exists $y \in B$ such that $u = \langle x,y\rangle$. So, there exists $y \in B$ such that $\pi_1(\langle x,y\rangle) = x$. Thus, for any $x \in A$ there exists $y \in B$ such that $\langle x,y\rangle \in U$ and we conclude $U \in mv(B^A)$. Suppose, conversely, $U \in mv(B^A)$. For any $x \in A$ there exists a $y \in B$ such that $\langle x,y\rangle \in U$ and $\pi_1(\langle x,y\rangle) = x$. So, $\pi_1(U) = A$ and we conclude $U \in C$. Clearly, $mv(B^A)$ is full in itself. $\square$

Theorem 5.17. *The following hold:*

1. *In **CZF**$^-$ the power set axiom is equivalent to the statement* $\forall A \forall B \exists C(C = mv(B^A))$.

2. ***CZF** does not prove the power set axiom.*

3. ***CZF** does not prove* $\forall A \forall B \exists C(C = mv(B^A))$.

Proof. 1. From [13]. In proving 3. from the previous theorem we showed that power set implies that for all sets A and B, $mv(B^A)$ is a set. So it remains to show the converse. Suppose for all sets A and B, $mv(B^A)$ is a set. Let C be any set and $D = mv(\{0,1\}^C)$. By assumption D is a set. To every subset X of C assign $X^* = \{\langle u,0\rangle \mid u \in X\} \cup \{\langle z,1\rangle \mid z \in C\}$. As a result $X^* \in D$. For each $S \in D$ let $pr(S) = \{u \in C \mid \langle u,0\rangle \in S\}$. We then have $X = pr(X^*)$ for every $X \subseteq C$, thus $Pow(C) \subseteq \{pr(S) \mid S \in D\}$. Let $S \in D$. Clearly, $pr(S) \subseteq C$ so $pr(S) \in Pow(C)$. Thus $Pow(C) = \{pr(S) \mid S \in D\}$. Since, for every $S \in D$ there exists $pr(S)$ such that $pr(S) \subseteq C$, by strong collection $\{pr(S) \mid S \in D\}$ is a set. Thus $Pow(C)$ is set.

2. This proof may be beyond the scope of the thesis. But by [6] the strength of **CZF** supplemented with the power set axiom exceeds the proof theoretic strength of second order arithmetic while the strength of **CZF** is only a fraction of second order arithmetic.

3. Follows from 1. and 2. □

Essentially subset collection allows us to prove the collections of functions between sets form a set, i.e. the exponentiation axiom. Subset collection is not strong enough to prove the power set axiom so in that sense **CZF** is deficient to **IZF**, but because of this deficiency **CZF** is a good foundation for constructive (i.e. predicative) mathematics.

Until further notice we will be working within **IZF**. Recall **IZF** does not have the replacement axiom, instead it has the weaker collection axiom. Notice, all the class and set constructions NOT involving replacement can be justified in **IZF** exactly as in **CZF**. For constructions that require replacement one can typically make use of power set, collection and separation (e.g. in **IZF** quotients class are sets via power set and separation.) We will not get into any set constructions for **IZF** here. This activity would be worthwhile but it is sufficient to observe that **CZF** is a strictly weaker theory than **IZF** and as such all theorems of **CZF** are theorems of **IZF**. Thus, from here on we operate under the reasonable assumption that the justification of a collection being a set in **CZF** is enough to justify it is a set in **IZF**. We will use the results of Section 5 freely (and likely with out reference) for the remainder of the thesis.

Using **IZF** we will work toward a semantic completeness proof of **IQC**. This proof will be intuitionistic but since it is done within **IZF** it will be impredicative. In Section 9 we will turn to **CZF** for the remainder of the thesis and work toward a semantic completeness proof that is predicative and hence constructive.

6 Posets, Lattices and Heyting Algebra

We commence this section with the definition of a partial order. A partial order is relation on a set that is reflexive, transitive and anti-symmetric and generalizes the notion of order on the familiar number systems. With the notion of a partially ordered set in hand we may define a lattice. A lattice may be studied dually from an order theoretic or algebraic perspective and we rely on this duality. We then investigate special lattices: bounded, complete, etc. We then give a definition of a Heyting Algebra and complete Heyting Algebra. The notion of completeness is of utmost importance. Notice that in this thesis the term 'completeness' is slightly overloaded. Completeness of a lattice, not to be confused with semantic completeness of a logic, is a property that is very much a generalization of the conditional completeness of the real numbers. In fact, if we supplement the real numbers with positive and negative infinity (i.e. $\mathbb{R} \cup \{-\infty, +\infty\}$) we have a complete lattice as defined in this section.

Definition 6.1. A partial order on a set X is a relation $R \subseteq X \times X$ satisfying

1. $\forall a \in X$, aRa (reflexivity).

2. $\forall a, b, c \in X$, aRb and bRc implies aRc (transitivity).

3. $\forall a, b \in X$, aRb and bRa implies $a = b$ (anti-symmetry).

A partially ordered set (poset) is a set equipped with a partial order (X, R).

Definition 6.2. Given a poset $(X, \leq)$ and $M \subseteq X$ we say $s \in X$ is a supremum (or join) of M if it satisfies

1. $\forall m \in M$, $m \leq s$.

2. $\forall x \in X$ ($\forall m \in M$, $m \leq x$ implies $s \leq x$).

If a supremum exists it is unique. We may define the infimum similarly. We will often use join and supremum interchangeably, similarly with meet and infimum.

Remark. We will favor the meet/join terminology.

Definition 6.3. A lattice is a poset $(X, \leq)$ such that for every $a, b \in X$ the infimum of $\{a, b\}$ denoted $a \wedge b$ and the supremum of $\{a, b\}$ denoted $a \vee b$ both exist. A lattice is bounded if there exist $0, 1 \in X$ such that $\forall x \in X$, $0 \leq x \leq 1$. A lattice is complete if every subset has infimum and supremum denoted $\bigwedge A$ and $\bigvee A$ for $A \subseteq X$.

Remark. Notice that a complete lattice is bounded.

Definition 6.4. A Heyting Algebra is a bounded lattice with a binary operation $\rightarrow$ satisfying $x \leq a \rightarrow b$ iff $x \wedge a \leq b$. We refer to the operation with the above property as a Heyting operation. Define the psuedo-complement of x to be $\neg x \equiv x \rightarrow 0$. Notice this implies $x \wedge \neg x = 0$. A complete Heyting Algebra is a Heyting Algebra that is complete as a lattice.

When the context becomes less clear we subscript the operations and bounds of the Heyting Algebra.

Remark. In a Heyting Algebra H given $a,b \in H$ we have $a \to b = \bigvee\{x \in H \mid x \wedge a \leq b\}$. By definition $x \wedge a \leq b$ is equivalent to $x \leq a \to b$. So, $a \to b$ is an upperbound of $\{x \in H \mid x \wedge a \leq b\}$. Observe, $a \to b \in \{x \in H \mid x \wedge a \leq b\}$ since $a \to b \leq a \to b$ implies $(a \to b) \wedge a \leq b$. So, if there exists a $z \in H$ such that $y \leq z$ for all $y \in \{x \in H \mid x \wedge a \leq b\}$ we have, in particular, $a \to b \leq z$.

Lemma 6.5. *Given a lattice L and $a,b,c \in L$ if $a \leq b$ then $a \wedge c \leq b \wedge c$ and $a \vee c \leq b \vee c$.*

Proof. First, note that $a \wedge c \leq a \leq b$ and $a \wedge c \leq c$. Since $b \wedge c$ is the infimum of b and c we must have $a \wedge c \leq b \wedge c$.

Now, note that $a \leq b \leq b \vee c$ and $c \leq b \vee c$. Since $a \vee c$ is the supremum of a and c we must have $a \vee c \leq b \vee c$. □

Proposition 6.6. *A Heyting Algebra is a poset $(H, \leq)$ together with two distinguished elements 0 and 1 as well as binary operations $\wedge$, $\vee$ and $\to$ which satisfy*

1. $x \leq 1$
2. $x \wedge y \leq x$ *and* $x \wedge y \leq y$
3. *if* $z \leq x$ *and* $z \leq y$ *then* $z \leq x \wedge y$
4. $0 \leq x$
5. $x \leq x \vee y$ *and* $y \leq x \vee y$
6. *if* $x \leq z$ *and* $y \leq z$ *then* $x \vee y \leq z$
7. $x \to x = 1$
8. $x \wedge (x \to y) = x \wedge y$
9. $(x \to y) \wedge y = y$
10. $x \to (y \wedge z) = (x \to y) \wedge (x \to z)$

for all $x,y,z \in H$.

Proof. Since H is bounded lattice properties 1-6 are obvious.

7. We know $x \to x \leq 1$, by 1. By 2. we have $1 \wedge x \leq x$. Using the definition of the Heyting operation we have $1 \leq x \to x$.

8. Clearly, $x \wedge (x \to y) \leq x$. We also have $x \to y \leq x \to y$ and by the definition of the Heyting operation $x \wedge (x \to y) \leq y$. Thus, $x \wedge (x \to y) \leq x \wedge y$ by 3. On the other hand, since $y \wedge x \leq y$ so $y \leq x \to y$. By Lemma 6.6, $x \wedge y \leq x \wedge (x \to y)$.

9. We know $(x \to y) \wedge y \leq y$ by 2. From the above we know $y \leq x \to y$ and $y \leq y$. Thus, $y \leq (x \to y) \wedge y$ by 3.

10. Let $a \in H$. We know $a \leq x \to (y \wedge z)$ is equivalent to $a \wedge x \leq y \wedge z$. Which by 2. and 3. is equivalent to $a \wedge x \leq y$ and $a \wedge x \leq z$. Using the Heyting operation this is equivalent to $a \leq x \to y$ and $a \leq x \to z$. By 2. and 3. this is equivalent to $a \leq (x \to y) \wedge (x \to z)$. The desired result follows.

□

This characterization of a Heyting Algebra looks mechanically similar to the system **IPC**. This can be seen by interpreting the distinguished elements 0 and 1 as absurdity $\perp$ and truth $\top$ the binary operations as the binary connectives, and the order relation as provability. Under this reading, we can combine Property 2. and 8. to arrive at $x \wedge (x \to y) \leq y$, which represents modus ponens! In fact, Heyting Algebra was invented as a means of formalising intuitionistic propositional logic. We can further justify this choice by considering the three element Heyting Algebra $H = \{0, 1/2, 1\}$ ordered by $0 < 1/2 < 1$. We can define the operations of join and meet on H in an obvious way. Let us partially work out the Heyting operation for H. Using Definition 6.4 we can determine that $0 \to 0 = 1$ since $1 \wedge 0 = 0 \leq 0$ implies $1 \leq 0 \to 0$. Next, $1/2 \to 0 = 0$ since $1/2 \to 0 = 1/2$ implies $1/2 = 1/2 \wedge 1/2 \leq 0$ which of course cannot be. Finally, $1 \to 0 = 0$ since $1 \to 0 = 1/2$ implies $1/2 = 1/2 \wedge 1 \leq 0$ which cannot be. Note we can only use indirect reasoning here because with finite sets equality is decidable. Now for $a = 1/2$ we have $\neg a = a \to 0 = 0$ and $a \vee \neg a = 1/2$. This provides a counter model to the law of excluded middle and as such **LEM** is not valid (see Section 7 for more on validity). Within the framework of model theory, Heyting Algebras are called models of **IPC**. Unfortunately, to model **IQC**, a Heyting Algebra is not quite enough. We need to restrict our attention to complete Heyting Algebras. In fact, complete Heyting Algebras are stronger than necessary — we only require a Heyting Algebra with enough arbitrary meets and joins. This is because we only need certain arbitrary meets and joins which allow us to model the quantifiers. But this restriction is somewhat arbitrary so we opt to use complete Heyting Algebras as our models for **IQC**. We now give a useful characterization of complete Heyting Algebras and proceed to show how any Heyting Algebra can be extended to a complete Heyting Algebra. This latter result justifies our the choice of a model for **IQC**.

Proposition 6.7. *A complete lattice L is a complete Heyting Algebra iff the following distributive law*

$$a \wedge \bigvee S = \bigvee \{a \wedge s \mid s \in S\}$$

holds for $a \in L$ and $S \subseteq L$.

Proof. Suppose that we have a complete Heyting Algebra H. Let $a \in H$ and $S \subseteq H$. We now show the distributive law holds. First note that $a \wedge s \leq a \wedge \bigvee S$ for $s \in S$, by Lemma 6.6, since $s \leq \bigvee S$. By definition of supremum we have $\bigvee\{a \wedge s \mid s \in S\} \leq a \wedge \bigvee S$. Now, note that $a \wedge s \leq \bigvee\{a \wedge s \mid s \in S\}$ for all $s \in S$. By definition of the Heyting operation we have $s \leq a \to \bigvee\{a \wedge s \mid s \in S\}$ for all $s \in S$. By definition of supremum we have $\bigvee S \leq a \to \bigvee\{a \wedge s \mid s \in S\}$. Finally, the definition of the Heyting operation yields $a \wedge \bigvee S \leq \bigvee\{a \wedge s \mid s \in S\}$. So the distributive law, $a \wedge \bigvee S = \bigvee\{a \wedge s \mid s \in S\}$, holds in any complete Heyting Algebra.

Now suppose that the distributive law holds for a complete lattice L. We define a Heyting operation as $a \to b \equiv \bigvee\{x \in L \mid x \wedge a \leq b\}$. If $c \wedge a \leq b$ then $c \in \{x \in L \mid x \wedge a \leq b\}$ and thus $c \leq a \to b$. Now, if $c \leq a \to b = \bigvee\{x \in L \mid x \wedge a \leq b\}$ then by Lemma 6.6 we have $c \wedge a \leq \bigvee\{x \in L \mid x \wedge a \leq b\} \wedge a = \bigvee\{x \wedge a \in L \mid x \wedge a \leq b\}$ by the distributive law. Finally, since b is an upper bound of the set we have $\bigvee\{x \wedge a \in L \mid x \wedge a \leq b\} \leq b$ and thus $c \wedge a \leq b$. So, L is a Heyting Algebra. Since it is complete as a lattice then it is in fact a complete Heyting Algebra. □

Remark. If b_i denotes an element of a complete lattice (or Heyting Algebra) for each i in some index set $\mathscr{I}$ then we often use

$$\bigvee_{i\in\mathscr{I}} b_i \text{ and } \bigwedge_{i\in\mathscr{I}} b_i$$

as abbreviations for $\bigvee\{b_i \mid i\in\mathscr{I}\}$ and $\bigwedge\{b_i \mid i\in\mathscr{I}\}$ respectively. Also, note that an arbitrary subset can always be indexed over itself. For example, given a lattice L and $S\subseteq L$ we have $S=\{s \mid s\in S\}$ and so

$$\bigvee S = \bigvee_{s\in S} s.$$

In light of this the distributive law can be similarly stated as

$$a\wedge \bigvee_{s\in S} s = \bigvee_{s\in S} a\wedge s.$$

for $a\in L$ and $S\subseteq L$.

Theorem 6.8. *For any Heyting Algebra H there exists an embedding into a complete Heyting Algebra H' that preserves the bounds, all operations and existing arbitrary meets and joins of H.*

Proof. (see Troelstra [1])

Let H be a Heyting Algebra. A complete ideal (or c-ideal) of H is a subset $I\subseteq H$ that satisfies

1. $0\in I$
2. $b\in I$ and $a\leq b$ implies $a\in I$
3. $X\subseteq I$ and $\bigvee_H X$ exists (in H) implies $\bigvee_H X\in I$.

Now let H' be the set of complete ideals of H. We first show H' is a complete lattice. Let $J=\{I_\alpha\in H' \mid \alpha\in\mathscr{J}\}$ where $\mathscr{J}$ is an arbitrary index. J has an arbitrary meet given by the set theoretic intersection, that is, $\bigwedge_{\alpha\in\mathscr{J}} I_\alpha = \bigcap_{\alpha\in\mathscr{J}} I_\alpha$. J has arbitrary join given by $\bigvee_{\alpha\in\mathscr{J}} I_\alpha = \{\bigvee_H X \mid X\subseteq \bigcup_{\alpha\in\mathscr{J}} I_\alpha \text{ and } \bigvee_H X \text{ exists}\}$.

We show that these objects are complete ideals and are in fact meet and join, respectively. Clearly, $0\in I_\alpha$ for each $\alpha\in\mathscr{J}$, so that $0\in \bigcap_{\alpha\in\mathscr{J}} I_\alpha$. Let $b\in \bigcap_{\alpha\in\mathscr{J}} I_\alpha$ and $a\leq b$, then $b\in I_\alpha$ for each $\alpha\in\mathscr{J}$. Thus $a\in I_\alpha$ for each $\alpha\in\mathscr{J}$, so that $a\in \bigcap_{\alpha\in\mathscr{J}} I_\alpha$. Now let $X\subseteq \bigcap_{\alpha\in\mathscr{J}} I_\alpha$

and suppose $\bigvee_H X$ exists. Then $X \subseteq I_\alpha$ for each $\alpha \in J$. Thus $\bigvee_H X \in I_\alpha$ for each $\alpha \in \mathscr{J}$, so that $\bigvee_H X \in \bigcap_{\alpha\in\mathscr{J}} I_\alpha$. Thus, $\bigcap_{\alpha\in\mathscr{J}} I_\alpha$ is a c-ideal. To see it is the meet we first observe that $\bigcap_{\alpha\in\mathscr{J}} I_\alpha \subseteq I_\beta$ for each $\beta \in \mathscr{J}$. Suppose there exist some other c-ideal M such that $M \subseteq I_\beta$ for each $\beta \in \mathscr{J}$. Then $M \subseteq \bigcap_{\alpha\in\mathscr{J}} I_\alpha$. Thus, $\bigcap_{\alpha\in\mathscr{J}} I_\alpha$ is the desired meet of J.

Now we start by letting $Q = \{\bigvee_H X \mid X \subseteq \bigcup_{\alpha\in\mathscr{J}} I_\alpha$ and $\bigvee_H X$ exists$\}$. Observe that $\emptyset \subseteq \bigcup_{\alpha\in\mathscr{J}} I_\alpha$ and $\bigvee_H \emptyset = 0$ exists in H, so that $0 \in Q$. Now let $b \in Q$ and $a \leq b$. Since $b \in Q$ there exists $X \subseteq \bigcup_{\alpha\in\mathscr{J}} I_\alpha$ such that $\bigvee_H X = b$. So $a \leq \bigvee_H X$ gives us $a = a \wedge \bigvee_H X$. By Lemma 6.6, $a \wedge x \leq a \wedge \bigvee_H X$ for all $x \in X$, so that $a \wedge \bigvee_H X$ is an upper bound for $\{a \wedge x \mid x \in X\}$. Observe for any $y \in H$

$$\forall x \in X, a \wedge x \leq y$$

is equivalent to

$$\forall x \in X, x \leq a \to y$$

by applying the definition of the Heyting operation. This is equivalent to

$$\bigvee_H X \leq a \to y$$

using property of join. This is equivalent to

$$a \wedge \bigvee_H X \leq y.$$

So $a = a \wedge \bigvee_H X = \bigvee_H \{a \wedge x \mid x \in X\}$. Observe that if $x \in I_\alpha$ then since $a \wedge x \leq x$ we have $a \wedge x \in I_\alpha$. Then, we have $\{a \wedge x \mid x \in X\} \subseteq \bigcup_{\alpha\in\mathscr{J}} I_\alpha$ and conclude that $a \in Q$. Now, given $a \in Q$ let $S_a = \{w \in \bigcup_{\alpha\in\mathscr{J}} I_\alpha \mid w \leq a\}$. Clearly, a is an upper bound of S_a. Let m be any other upper bound of S_a. Since $a \in Q$ there exists $X \subseteq \bigcup_{\alpha\in\mathscr{J}} I_\alpha$ such that $a = \bigvee_H X$. Notice $X \subseteq S_a$ so m is also an upper bound for X. Thus $a \leq m$ so that $a = \bigvee_H S_a$. Now let $X \subseteq Q$ such that $\bigvee_H X$ exists. Notice that $S_x \subseteq \bigcup_{\alpha\in\mathscr{J}} I_\alpha$ for every $x \in X$ so $\bigcup_{x\in X} S_x \subseteq \bigcup_{\alpha\in\mathscr{J}} I_\alpha$. Now for any $w \in \bigcup_{x\in X} S_x$ we have for some $x \in X$, $w \leq x \leq \bigvee_H X$. So, $\bigvee_H X$ is an upper bound of $\bigcup_{x\in X} S_x$. Finally observe

$$\forall w \in \bigcup_{x\in X} S_x (w \leq z)$$

is equivalent to

$$\forall x \in X \forall w \in S_x (w \leq z)$$

which is equivalent to

$$\forall x \in X (x \leq z), \text{ since } x = \bigvee_H S_x$$

which is equivalent to

$$\bigvee_H X \leq z.$$

That is, $\bigvee_H \bigcup_{x \in X} S_x = \bigvee_H X$. We conclude that $\bigvee_H X \in Q$. So Q is a c-ideal. Now it remains to show that Q is the join J. If $x \in I_\alpha$ for any $\alpha \in \mathscr{J}$ then $\{x\} \subseteq \bigcup_{\alpha \in \mathscr{J}} I_\alpha$ and $x = \bigvee_H \{x\}$ so $x \in Q$. Thus, $I_\alpha \subseteq Q$ for any $\alpha \in \mathscr{J}$. Let $M \in H'$ be any other upper bound of J. Let $x \in Q$, then $x = \bigvee_H S_x$. Now $S_x \subseteq \bigcup_{\alpha \in \mathscr{J}} I_\alpha \subseteq M$. Realizing that M is a c-ideal we utilize Condition 3 of the definition of c-ideal to obtain $x \in M$ so that $Q \subseteq M$. Thus, Q is the desired join of J. So, H' is a complete lattice.

Now to show H' is a complete Heyting Algebra (see Proposition 6.7) we show that the infinite distributive law holds. Recall that that meet in H' is simply the intersection. Now $x \in \bigvee_{\alpha \in \mathscr{J}} (I \wedge I_\alpha)$ is equivalent to

$$x = \bigvee_H X, X \subseteq \bigcup_{\alpha \in \mathscr{J}} (I \wedge I_\alpha) = \bigcup_{\alpha \in \mathscr{J}} (I \cap I_\alpha) = I \cap \bigcup_{\alpha \in \mathscr{J}} I_\alpha$$

which is equivalent to

$$x = \bigvee_H X, X \subseteq I \text{ and } X \subseteq \bigcup_{\alpha \in \mathscr{J}} I_\alpha$$

which is equivalent to

$$x \in I \text{ and } x \in \bigvee_{\alpha \in \mathscr{J}} I_\alpha$$

which is equivalent to

$$x \in I \cap \bigvee_{\alpha \in \mathscr{J}} I_\alpha = I \wedge \bigvee_{\alpha \in \mathscr{J}} I_\alpha.$$

So we conclude that $I \wedge \bigvee_{\alpha \in \mathscr{J}} I_\alpha = \bigvee_{\alpha \in \mathscr{J}} (I \wedge I_\alpha)$. By Proposition 6.7, H' is a complete Heyting Algebra.

We now define the embedding $i : H \to H'$ via $i(x) = \{y \in H \mid y \leq x\}$ and show that i

preserves $\wedge_H, \vee_H, \rightarrow_H, 0_H, 1_H$, and all existing $\bigvee_H$ and $\bigwedge_H$. It suffices to show that i preserves $\bigvee_H$ and $\bigwedge_H$ since for any $a,b \in H$ we have $a \vee_H b = \bigvee_H \{a,b\}$, $a \wedge_H b = \bigwedge_H \{a,b\}$, $0_H = \bigvee_H \emptyset$, $1_H = \bigwedge_H \emptyset$ and $a \rightarrow_H b = \bigvee_H \{x \in H \mid x \wedge a \leq b\}$ (see Remark following Definition 6.4.) Let $X \subseteq H$ and suppose $\bigwedge_H X$ exists in H. We show that $i(\bigwedge_H X) = \bigwedge_{x \in X} i(x)$. Now $z \in i(\bigwedge_H X)$ is equivalent to

$$z \leq \bigwedge_H X$$

which is equivalent to

$$z \leq x, \forall x \in X$$

which is equivalent to

$$z \in i(x), \forall x \in X$$

which is equivalent to

$$z \in \bigcap_{x \in X} i(x) = \bigwedge_{x \in X} i(x).$$

Thus, $i(\bigwedge_H X) = \bigwedge_{x \in X} i(x)$. Now suppose $\bigvee_H X$ exists in H. We show that $i(\bigvee_H X) = \bigvee_{x \in X} i(x)$. Now

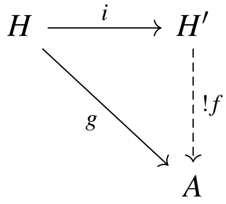
$$z \in i(\bigvee_H X)$$

is equivalent to

$$z \leq \bigvee_H X.$$

Now for any $x \in X$ we have $x \in i(x)$. Thus, $X \subseteq \bigcup_{x \in X} i(x)$ so by the definition of the join in H' we have $\bigvee_H X \in \bigvee_{x \in X} i(x)$. By Property 2 of c-ideals this implies $z \in \bigvee_{x \in X} i(x)$. To show the opposite implication suppose $z \in \bigvee_{x \in X} i(x)$. Thus, $z = \bigvee_H Y$ for some $Y \subseteq \bigcup_{x \in X} i(x)$. For any $y \in Y$ there is an $x \in X$ such that $y \in i(x)$. This yields $y \leq x \leq \bigvee_H X$. That is, $\bigvee_H X$ is an upper bound for Y. Thus, we have $z = \bigvee_H Y \leq \bigvee_H X$. Hence, we conclude $i(\bigvee_H X) = \bigvee_{x \in X} i(x)$. As desired the embedding preserves the existing meets and join and by our observation all operations and bounds of the Heyting Algebra. This concludes the proof. □

The completion of a Heyting Algebra is universal in the categorical sense, that is, for any other complete Heyting Algebra A and map g from H into A there exists a unique map f such that $i \circ f = g$. We can represent this fact by the following commutative diagram.



The previous proof was very involved but absolutely necessary for later work.

7 Validity with H-Valuations

In this section we define validity of formulae using Heyting Algebra semantics. We commence by defining valuations — maps from formulae of **IQC** into a Heyting Algebra/ We can take the idea of truth tables in a classical logic as a starting point. Where tautologies are statements which are true in every column of the truth table. We identify the top element of a complete Heyting Algebra with truth. Then we will consider formulae which map to the top element, 1, of every Heyting Algebra to be valid.

For simplicity we consider a single-sorted theory of **IQC** without equality. We follow [1] closely but make slight alterations for improved clarity.

Definition 7.1. An H-valuation on **IQC** is a map from $\mathbb{IQC}$, the set of formula of **IQC**, to a generic complete Heyting Algebra H, that is $v : \mathbb{IQC} \to H$. To do this we first give an assignment for the terms, functions and predicates. We then extend this assignments to all formulas. To the single-sort i we assign an arbitrary set D. An H-valued relation on D^n is a mapping $D^n \to H$. We assign each n-place relation symbol R with arguments of sort i to an H-valued relation $v(R)$ on D^n. We assign the proposition P (predicate with $n = 0$) to $v(P) \in H$. We assign each n-place function symbol f with arguments and value of sort i to a function $v(f) : D^n \to D$. We assign the constant c (function with $n = 0$) of sort i to $v(c) \in D$. We assign a variable x of sort i to $v(x) \in D$. Thus, we can assign any term t of sort i to $v(t) \in D$. We summarize the valuation for terms:

- $v(x) \in D$ where x is a variable of sort i.

- $v(c) \in D$ where c is a constant of sort i.

- $v(f(t_1,\dots,t_n)) = v(f)(v(t_1),\dots,v(t_n))$ where $v(f) : D^n \to D$ and $t_1,\dots,t_n$ are terms of sort i

and extend the valuation to all formulae:

- $v(\bot) = 0_H$
- $v(\top) = 1_H$
- $v(P) \in H$ where P is a proposition symbol.
- $v(R(t_1,\dots,t_n)) = v(R)(v(t_1),\dots,v(t_n))$ where $v(R) : D^n \to H$ and $t_1,\dots,t_n$ are terms of sort i.
- $v(\phi \wedge \psi) = v(\phi) \wedge_H v(\psi)$
- $v(\phi \vee \psi) = v(\phi) \vee_H v(\psi)$
- $v(\phi \longrightarrow \psi) = v(\phi) \rightarrow_H v(\psi)$
- $v(\forall x(\phi)) = \bigwedge_H \{v(\phi[d/x]) \mid d \in D\}$
- $v(\exists x(\phi)) = \bigvee_H \{v(\phi[d/x]) \mid d \in D\}$

for any formula $\phi, \psi \in \mathbb{IQC}$.

The subscripts are meant to assist the audience in differentiating between logical constants/connectives of **IQC** and elements/operations in the complete Heyting Algebra H. H-valuations on many-sorted languages would be defined similarly, but we one would see a complication of the valuation extension for terms, function and relation symbols and quantifiers. Note that for a (not necessarily complete) Heyting Algebra, H, a partial H-valuation on **IQC** can be defined, but we may not be able to extend the valuation to the quantifiers. This is because in general a Heyting Algebra may not have sufficient arbitrary meets and joins.

Definition 7.2. Given a Heyting Algebra H and a partial H-valuation $v : \mathbb{IQC} \to H$ on **IQC**, we say that the partial H-valuation is quantifier complete if whenever $v(\phi[d/x]) \in H$ for all $d \in D$ we have $\bigwedge_H \{v(\phi[d/x]) \mid d \in D\} \in H$ and $\bigvee_H \{v(\phi[d/x]) \mid d \in D\} \in H$.

Definition 7.3. We say a formula ϕ of **IQC** is H-valid for a complete Heyting Algebra H, denoted $\models_H \phi$, if $v(\phi) = 1_H$ for every H-valuation $v : \mathbb{IQC} \to H$. Further, we say ϕ is valid, denoted $\models \phi$, if $\models_H \phi$ for every complete Heyting Algebra H.

8 Intuitionistic Completeness of IQC

In this section we commence a discussion about semantic completeness. One thing to note, the term completeness is overloaded here, as mentioned in Section 6. This is an unfortunate 'coincidence' of conflicting terminology of the logical theory and lattice theory which we just so happen to be simultaneously employing. We will try to be intentional with the phrasing but fortunately the context will usually keep us out of trouble. A logic is semantically complete if every valid formulae is provable. Soundness is the converse of semantic completeness. Together, soundness and semantic completeness provide a correspondence between validity and provability. We commence by stating, without proof, the soundness theorem.

Theorem 8.1. **IQC** *is sound, that is every formula that is provable is also valid. Symbolically, if* $\vdash \phi$ *then* $\models \phi$.

Proof. See [1] for details. One proceeds with induction on the size of natural deduction proofs and shows that validity is preserved by the inference rules. □

We now prove semantic completeness for single-sorted **IQC** without equality.

Lemma 8.2. *There exists a Heyting Algebra H and quantifier complete H-valuation v such that if* $v(\phi) = 1_H$ *then* $\vdash \phi$.

Proof. Consider the following Heyting Algebra defined on the set $\mathbb{IQC}$ of IQC formula. To construct this Heyting Algebra we define an equivalence relation on $\mathbb{F}$ as such

$$\phi \sim \psi \quad \text{iff} \quad \vdash \phi \longleftrightarrow \psi.$$

Define a Heyting Algebra $H \equiv \mathbb{F}/\sim$ to be the set of equivalence classes $[\phi] = \{\psi \in \mathbb{F} \mid \psi \sim \phi\}$. Order H with

$$[\phi] \leq_H [\psi] \quad \text{iff} \quad \vdash \phi \longrightarrow \psi.$$

Now set $0_H = [\bot]$ and $1_H = [\top]$ and define

- $[\phi] \wedge_H [\psi] = [\phi \wedge \psi]$
- $[\phi] \vee_H [\psi] = [\phi \vee \psi]$
- $[\phi] \to_H [\psi] = [\phi \longrightarrow \psi]$
- if it exists $\bigwedge_H \{[\phi[d/x]] \mid d \in D\} = [\forall x(\phi)]$
- if it exists $\bigvee_H \{[\phi[d/x]] \mid d \in D\} = [\exists x(\phi)]$.

We now define a partial H-valuation $v : \mathbb{IQC} \to H$ via $v(\phi) = [\phi]$. We claim the H-valuation is quantifier complete (see Proposition 8.3). Now, by assumption we have $[\phi] = v(\phi) = 1_H = [\top]$. By properties of equivalence classes (see Definition 5.10) we have $\phi \sim \top$. That is, by our equivalence relation we have $\vdash \phi \longleftrightarrow \top$. Now, using ($\top$ Intro), ($\wedge$ Elim_R) on the previous line and ($\longrightarrow$ Elim) (from Section 4) we conclude $\vdash \phi$. □

Proposition 8.3. *The H-valuation defined in the previous proof is quantifier complete.*

Proof. Let $d \in D$. Note that $\vdash \forall x \phi \longrightarrow \phi[d/x]$ (by $\forall$ Elim and $\longrightarrow$ Intro). Thus, $[\forall x \phi] \leq_H [\phi[d/x]]$ for all $d \in D$. We have established that $[\forall x \phi]$ is a lower bound of $\{[\phi[d/x]] \mid d \in D\}$. Now, suppose $[\psi] \leq_H [\phi[d/x]]$ for all $d \in D$. So we have $\vdash \psi \longrightarrow \phi[d/x]$ for all $d \in D$. Then in particular, for some variable y not occurring free in ψ we have $\vdash \psi \longrightarrow \phi[y/x]$. This is essentially variable renaming so we have $\vdash \psi \longrightarrow \phi$. Thus, $\vdash \psi \longrightarrow \forall x \phi$ (by $\forall$ Intro, Transitivity of $\longrightarrow$, and $\longrightarrow$ Elim from Section 4). Finally, we conclude that $[\psi] \leq_H [\forall x \phi]$. We have established that $[\forall x \phi]$ is the greatest lower bound of $\{[\phi[d/x]] \mid d \in D\}$. That is, $\bigwedge_H \{[\phi[d/x]] \mid d \in D\} = [\forall x(\phi)]$.

Again let $d \in D$. Note that $\vdash \phi[d/x] \longrightarrow \exists x \phi$ (by $\exists$ Intro). Thus, $[\phi[d/x]] \leq_H [\exists x \phi]$ for all $d \in D$. We have established that $[\exists x \phi]$ is a upper bound of $\{[\phi[d/x]] \mid d \in D\}$. Now, suppose $[\phi[d/x]] \leq_H [\psi]$ for all $d \in D$. So we have $\vdash \phi[d/x] \longrightarrow \psi$ for all $d \in D$. Then, for some variable y free in ψ we have $\vdash \phi[y/x] \longrightarrow \psi$. Again, this is just variable renaming so we have $\vdash \phi \longrightarrow \psi$. Finally we have $\vdash \exists x \phi \longrightarrow \psi$ (by $\exists$ Elim and $\longrightarrow$ Intro from Section 4). Finally, we conclude that $[\exists x \phi] \leq_H [\psi]$. We have established that $[\exists x \phi]$ is the least upper bound of $\{[\phi[d/x]] \mid d \in D\}$. That is, $\bigvee_H \{[\phi[d/x]] \mid d \in D\} = [\exists x(\phi)]$. □

Theorem 8.4. **IQC** *is semantically complete, that is every formula that is valid is also provable. Symbolically, if* $\models \phi$ *then* $\vdash \phi$.

Proof. Suppose $\models \phi$. Let H be the free Heyting Algebra on $\mathbb{F}$ and H' the universal completion of H (Theorem 6.8) and consider the H'-valuation $v(\phi) = i([\phi])$, where $[\phi]$ is the quantifier complete valuation on H (see Lemma 8.2) and i is the inclusion map for the universal completion (see Theorem 6.8). Clearly, we have $\models_{H'} \phi$. Thus, $i([\phi]) = 1_{H'}$. Since the universal completion preserves bounds we can infer that $[\phi] = 1_H$. Applying Lemma 8.2 we conclude that $\vdash \phi$. □

By following Troelstra (see [1]) we have established the semantic completeness of **IQC**. As we will see in the next section this proof can only be formalized in **IZF** as it requires a critical use of the power set axiom.

9 Motivation for Partially Ordered Classes

To proceed constructively will first require us to choose a system that is intuitionistic and predicative. The obvious candidate is **CZF** as it is the predicative variant of **IZF**. Using **CZF** as a meta-theory leads to some serious obstacles. Fortunately, quotients can be constructed in **CZF** (see Proposition 5.11). The real issue is that without the power set, axiom performing completions of a lattice (see Proposition 6.8) does not provide a set (see below). A proper class is a collection that is not provably a set. The standard completion requires us to form a certain collection of subsets of a lattice. Typically this collection is provably a set by an appeal to the power set axiom and the axiom of bounded separation. Mathematicians prefer to work with sets. But, the way through this difficulty is to forgo comfort and simply work with the resulting proper classes, albeit carefully. Fortunately, there is a rich theory of partially ordered classes treated in [11] which we have at our disposal. We will follow the standard treatment closely in Section 10, but please refer to [5,6,11] for further details. What follows is an outline of the next few Sections:

1. Develop the theory of partially ordered classes (poclass) and investigate complete lattices and Heyting Algebras in this setting.

2. Identify a special type of complete lattice that arise from a set sized lattice and have more familiar properties.

3. Define an appropriate semantics for **IQC** using this type of complete Heyting Algebras.

4. Prove the semantic completeness of **IQC** with in this framework.

Before we initiate the plan outlined above we will give a proof that no (non-trivial) complete lattice (and thus no complete Heyting Algebra) is provably a set in **CZF**. This will serve as motivation for what follows as it is not at all obvious that the route outlined above is necessary without such a glaring result. This proof, by Giovanni Curi among others, is given below and will follow the slides given in [12]. We have been introduced to the set theories of **CZF**, **IZF** and **ZF**. Reference Section 5 for the subtle distinctions in the axioms of these theories. Recall, the set of theorems of **CZF** form a proper subset of the theorems of **IZF** and the theorems of **IZF** form a proper subset of the theorems of **ZF**. That is, **CZF** $\subset$ **IZF** $\subset$ **ZF**. When the Law of Excluded Middle (**LEM**) is added to either **CZF** or **IZF** the set of theorems of either coincides with the theorems of **ZF**. Additionally if the axioms of full Separation (**Sep**) and power set (**Pow**) are added to CZF then its theorems coincide with **IZF**. Symbolically, **CZF** + **LEM** = **IZF** + **LEM** = **ZF** and **CZF** + **Sep** + **Pow** = **IZF**. We will be working in **CZF** and modest extensions from here on out.

First recall that for any set X, the collection $Pow(X)$ is not provably a set in **CZF**, that is to say it is a proper class. Two principles which have been shown to be consistent with **CZF** (see [20] for details) are:

Troelstra's principle of uniformity

If $(\forall x)(\exists n \in \omega)\phi(x,n)$, then $(\exists n \in \omega)(\forall x)\phi(x,n)$.

Subcountablility of every set

$(\forall x)(\exists U \in Pow(\omega))(\exists f) f : U \twoheadrightarrow x$, where $f : A \twoheadrightarrow B$ indicates that f is onto.

These two principles are in fact inconsistent with classical mathematics. This thesis does not wish to discuss the philosophical nature of these principles. Rather we wish to use them to put a restriction on **CZF** (which as you recall does not have any anti-classical principles). Combing these two principles one arrives at:

Generalized Uniformity Principle (GUP)

For every set a, if $(\forall x)(\exists y \in a)\phi(x,y)$, then $(\exists y \in a)(\forall x)\phi(x,y)$.

Since Troelstra's Principle of Uniformity and Subcountability of every set are consistent with **CZF**, **GUP** is also consistent with **CZF**. In fact it is consistent with various extensions of **CZF** that do not exceed the proof-theoretic strength of **IZF** [12]. Such arbitrary extensions of **CZF** will be denoted **CZF***.

It is now time to give a particularly useful application of **GUP**. We will give a full treatment of partially ordered classes in Section 10. But out of necessity we now define a large join semi-lattice.

Definition 9.1. A (large) $\bigvee$-semilattice is a partially ordered class that has suprema for arbitrary subsets. A $\bigvee$-semilattice is degenerate if it only contains one element.

Remark. We call a collection large to emphasize that it is a proper class rather than a set.

Proposition 9.2. *No non-degenerate $\bigvee$-semilattice L can be proved to have a set of elements in **CZF***.*

Proof. Suppose L is a set. Then for any set y the collection

$$\{x \in L \mid 0 \in y\}$$

is a set by bounded separation. Therefore, since L is a $\bigvee$-semilattice, $(\forall y)(\exists a \in L)a = \bigvee\{x \in L \mid 0 \in y\}$. In **CZF***+**GUP** one may conclude

$$(\exists a \in L)(\forall y)a = \bigvee\{x \in L \mid 0 \in y\}.$$

Since this is true for any set y let us first consider $y = 0$. In this case, $a = \bigvee\emptyset = \bot$. Next consider $y = \{0\}$. In this case, $a = \bigvee L = \top$. So we are forced to conclude $\bot = a = \top$ and thus L is degenerate. Applying the constructively viable form of the contrapositive we conlcude that if L is not degenerate then it is not a set in **CZF*** $+$ **GUP**. Thus, as to remain consistent with **GUP**, if L is not degenerate then L is not provably a set in **CZF***. □

One may ask themselves how can such a result hold? Surely finite sets, such a the Boolean Algebra $\{0,1\}$, are complete and provably sets. For sure, $\{0,1\}$ is provably a set in **CZF** but it is not intuitionistically complete (for the completeness of $\{0,1\}$ implies the weak law of exlcuded middle). In fact no finite set can be complete in an intuitionistic setting (see [21]).

10 Set-Generated Heyting Algebra

We now generalize the concepts of Section 6 to proper classes. Recall that proper classes do not formally exist but it is possible to discuss statements about them within the language of **CZF** (see Section 5). We first define a partially ordered class. In light of the results of Section 9, we quickly move towards more interesting matters such as (large) join complete semilattices, which we denote $\bigvee$-semilattice. Recall we say large to indicate that the collection is not a set but a proper class. We then commence an investigation of set-generated semilattices, which have similar behavior to lattices in the impredicative setting. Finally, we define (large) complete Heyting Algebras and specify when they are set-generated. The section concludes with some results which are of importance going forward. Refer to [11] for further details.

Definition 10.1. A partially ordered class (poclass) $(X,\leq)$ is a class X together with a class-relation $\leq$ that satisfies $\forall x,y,z\in X$

1. $x\leq x$ (Reflexivity)

2. $x\leq y$ and $y\leq x$ implies $x=y$ (Anti-Symmetry)

3. $x\leq y$ and $y\leq z$ implies $x\leq z$ (Transitivity).

Definition 10.2. Given a poclass $(X,\leq)$ and a subset $A\subseteq X$ the join $s\in X$ of A satisfies

1. $\forall a\in A, a\leq s$

2. $\forall x\in X(\forall a\in A, a\leq x$ implies $s\leq x)$.

The infimum (or meet) of A is defined analogously.

Definition 10.3. A (large) $\bigvee$-semilattice is a poclass $(L, \leq)$ where every subset of L has a join. A (large) $\bigwedge$-semilattice is defined similarly. (Note that a $\bigvee$-semilattice is not necessarily a $\bigwedge$-semilattice and vice-versa). If a poclass is both a $\bigvee$-semilattice and a $\bigwedge$-semilattice then we call it a (large) complete lattice.

Remark. Every $\bigvee$-semilattice L has a bottom element. To see this, observe $\emptyset \subseteq L$ so we know $\bigvee \emptyset$ exists. Any $x \in L$ is vacuously an upper bound of $\emptyset$ so we have $\bigvee \emptyset \leq x$. We define $0 \equiv \bigvee \emptyset$.

We typically say that a structure that is provably a set is 'carried' by a set and otherwise it is 'carried' by a proper class. As we are working in **CZF** where no non-degenerate semilattice is carried by a set we will not always specify the largeness of a semilattice.

The **CZF** counterpart to the classical notion of a semilattice (carried by a set) is the concept of a set-generated $\bigvee$-semilattice.

Definition 10.4. A $\bigvee$-semilattice L is set-generated if it has a subset $B \subseteq L$ such that for all $x \in L$,

1. $\downarrow^B x \equiv \{b \in B \mid b \leq x\}$ is a set,

2. $\bigvee \downarrow^B x = x$.

We call such a set B a generating set.

Remark. $\bigvee$-semilattice may have more than one generating set. Notice if B is a generating set of L then $B \cup \{0\}$ is also a generating set of L. For this reason we often assume a generating set contains the bottom element.

The power class $Pow(X)$ for a set X ordered by inclusion is the prototypical example of a set-generated $\bigvee$-semilattice, with generating set $B = \{\{x\} \mid x \in X\}$.

Proposition 10.5. *B as defined above is a generating set for $Pow(X)$.*

Proof. First we need to show B is a set. Clearly a function from X to B can be defined. For example $f : X \to B$ via $f(x) = \{x\}$. Thus, for every $x \in X$ there exists a $y \in B$ such that $f(x) = y$, with $y = \{x\}$. By strong collection with $\phi(x, \{x\})$ being the formula for $f(x) = \{x\}$ and extensionality we conclude B is a set. (More explicitly, by strong collection we have there exists a set B' such that $\forall x \in X \exists \{x\} \in B'(f(x) = \{x\})$ and $\forall \{x\} \in B' \exists x \in X(f(x) = \{x\})$ and by extensionality $B = B'$ thus B is a set.) Let $S \in Pow(X)$ or $S \subseteq X$. Note that $\downarrow^B S \equiv \{b \in B \mid b \subseteq S\} = \{b \in B \mid b = \{x\} \wedge x \in S\}$ is a set by bounded separation. Now clearly, S is an upper bound for $\downarrow^B S$. Let $S' \subseteq X$ such that $b \subseteq S'$ for all $b \in \downarrow^B S$. Let $z \in S$ that is $\{z\} \subseteq S$. This implies $\{z\} \in \downarrow^B S$ which implies $\{z\} \subseteq S'$. Thus $z \in S'$ and we have $S \subseteq S'$. Thus, $\bigvee \downarrow^B S = S$. □

Proposition 10.6. *A set-generated* $\bigvee$*-semilattice is a complete lattice.*

Proof. Let $(L, \leq)$ be a $\bigvee$-semilattice generated by a set B which contains $\perp$ (see remark after Definition 10.4) and $S \subseteq L$. Consider the collection $Y \subseteq B$ of lower bounds of S. That is $l \in Y$ if and only if $l \in B$ and $l \leq s$ for all $s \in S$. Notice Y is non-empty since $\perp \leq s$ for all $s \in S$ (see the remark following Definition 10.4). Clearly, Y is a set by Bounded Separation so by assumption $\bigvee Y$ exists. We claim $\bigvee Y$ is the meet of S. Any $s \in S$ is an upper bound of Y so by definition $\bigvee Y \leq s$ for all $s \in S$. Thus, $\bigvee Y$ is a lower bound of S. Let m be another lower bound S, that is $m \leq s$ for all $s \in S$. Now $\downarrow^B m$ is a set and $\downarrow^B m \subseteq Y$. So we have $m = \bigvee \downarrow^B m \leq \bigvee Y$ and hence conclude $\bigwedge S = \bigvee Y$. □

Definition 10.7. A (large) complete Heyting Algebra is a (large) complete lattice satisfying the infinite distributive law. That is, a complete lattice H is a complete Heyting Algebra if

$$a \wedge \bigvee_{s \in S} s = \bigvee_{s \in S} a \wedge s$$

for $a \in H$ and $S \subseteq H$. If the underlying lattice is set-generated then we say H is a set-generated complete Heyting Algebra.

Remark. Notice that we have foregone the standard definition of a Heyting Algebra here. This choice is in no way essential but rather a matter of convenience.

Note that in **CZF** a (non-complete) Heyting Algebra may or may not provably be a set. Fortunately, the Heyting Algebra that semantic completeness hinges on forms a set in **CZF**, as the next proposition states.

Proposition 10.8. *The Heyting Algebra on* $\mathbb{IQC}$ *(defined in Lemma 8.2) forms a set in* **CZF**.

Proof. $\mathbb{IQC}$ is of course a set, as it is inductively defined (see paragraph after Definition 5.3), so by Proposition 5.11 the collection $\mathbb{IQC}/\sim$ forms a set. □

In **CZF** we can still perform a universal completion of a Heyting Algebra but without the power set axiom the result is a proper class and thus a (large) complete Heyting Algebra. The next theorem will show that this (large) complete Heyting Algebra is in fact set-generated.

Theorem 10.9. *Given a set H that is a Heyting Algebra there exists an embedding into a set-generated complete Heyting Algebra H' that preserves the existing joins and meets of H.*

Proof. A rehashing of Theorem 6.8 will give us an embedding of H into a (large) complete Heyting Algebra H' that preserves existing meets and joins. It remains to show that H' is in fact set-generated. Recall, H' is the collection of c-ideals of H. We claim that $B = \{\downarrow x \mid x \in H\}$ generates H', where $\downarrow x = \{y \in H \mid y \leq x\}$. We first need to show that B is a set. Clearly there exists a function from elements of H to B (explicitly $f : H \to B$ via $f(x) = \downarrow x$.) Similarly to Proposition 10.5 we can apply strong collection and extensionality to show B is a set. Now let $I \in H'$, that is, suppose I is a complete ideal of H. Consider $\downarrow^B I = \{b \in B \mid b \subseteq I\} = \{b \in B \mid b = \downarrow x \wedge \forall y \leq x, y \in I\}$ which is a set by bounded separation. Clearly, I is an upper bound of $\downarrow^B I$. Let $J \in H'$ be any other upper bound. That is, $b \subseteq J$ for all $b \in \downarrow^B I$. Now let $x \in I$. Since I is a complete ideal we have $\downarrow x \subseteq I$ so that $\downarrow x \in \downarrow^B I$. Now, by assumption $\downarrow x \subseteq J$. In particular, $x \in J$ so that $I \subseteq J$. We then conclude $\bigvee \downarrow^B I = I$. □

11 Constructive Completeness of IQC

In this section we define a semantics for **IQC** on the collection of set-generated complete Heyting Algebras. This is a generalization of the concepts of Section 7. Following this we will proceed to prove the constructive version of semantic completeness of **IQC**. In Section 10 we established the necessary behavior of a set-generated complete Heyting Algebra so we may now proceed as we did in the intuitionistic proof of semantic completeness of **IQC** (see Section 8).

Definition 11.1. Given a set-generated complete Heyting Algebra H we define an H-valuation

$v : \mathbb{IQC} \to H$ just as in Section 7. Given a Heyting Algebra H we also define a partial H-valuation and a quantifier complete H-valuation just as before.

Definition 11.2. Given a set-generated complete Heyting Algebra H we say a formula ϕ of **IQC** is H-valid, denoted $\models_H \phi$, if $v(\phi) = 1_H$ for every H-valuation $v : \mathbb{IQC} \to H$. We say ϕ is valid, denoted $\models \phi$, if $\models_H \phi$ for every set-generated complete Heyting Algebra H.

It may seem unnecessary to restrict the semantics to set-generated complete Heyting Algebra, but a 'collection' of (not necessarily set-generated) complete Heyting Algebra is in some sense too large to consider in their entirety in any meaningful way. This is because we have no means of formally discussing collections of proper classes, only collections of sets. Formally, with in the language of **CZF**, we could describe the class of all sets which generate a complete Heyting Algebra with a formula. We would then associate a set-generated Heyting Algebra with its generating set. See Section 5 for more information on how we treat classes in **CZF**.

What follows is a constructive — in the sense that it is done within **CZF** — proof of the semantic completeness of **IQC**.

Theorem 11.3. *The theory* **IQC** *is semantically complete, that is every formula that is valid is also provable. Symbolically, if* $\models \phi$ *then* $\vdash \phi$.

Proof. Suppose $\models \phi$. Let H be the Heyting Algebra on $\mathbb{F}$ and H' the universal set-generated completion of H (see Theorem 10.9) and consider the H'-valuation $v(\phi) = i([\phi])$, where $[\phi]$ is the quantifier complete valuation on H (see Proposition 8.3) and i is the inclusion map for the universal completion (see Theorem 6.8). By Definition 11.2 we have $\models_{H'} \phi$ and thus $i([\phi]) = 1_{H'}$. Since the universal completion preserves bounds we can infer that $[\phi] = 1_H$. We conclude (by Lemma 8.2) that $\vdash \phi$. □

Conclusion

The goal of the thesis, a constructive proof of the semantic completeness of intuitionistic predicate logic (**IQC**), has been achieved. In Section 2, we explored intuitionistic and constructive mathematics and how this perspective differs from classical mathematics. We concluded roughly that intuitionistic mathematics abstains from the law of excluded middle (**LEM**) and constructive mathematics is done with predicative constructions. The latter point rules out potential foundations for constructivism that possess the power set axiom. In Section 3,4 and 7, we discussed formal languages with proof systems and semantics (with a primary focus on intuitionistic predicate logic). In Section 5, we developed some preliminary set theoretic results in Constructive Zermelo Fraenkel set theory (**CZF**) and began a critical discussion on axiomatic strength and proper classes. In Section 6, we developed some order theory with the intention of defining a Heyting Algebra. The notion of Heyting Algebra is quintessential to this thesis and is the crux of the definition of semantics for intuitionistic logic, given in Section 7. In Section 8, we gave an intuitionistic proof of the semantic completeness of intuitionistic predicate logic inspired by Troesltra [1]. This proof fails to be fully constructive because it requires the power set axiom; which is not available in **CZF** or any constructive foundation. In Section 9, we motivated the need for a theory or partially ordered classes. This motivation came from the shocking result that no complete lattice is provably a set in any predicative extension of **CZF**. We then formally introduced the theory of partially ordered classes in Section 10. In Section 11, we redefined a semantics for **IQC** using set-generated complete Heyting Algebra and proved the semantic completeness of **IQC** constructivly.

Potential for future work ranges in difficulty. A natural next step is to extend the semantic completeness result for full **IQC** (i.e. many-sorted **IQC** with equality). The proof would be similar, although some complications arise by adding many-sorts and equality. Alternatively one could explore the possibility that there are quantifier complete Heyting Algebra that are provably sets in **CZF** (of primary interest is the Heyting Algebra of Lemma 8.2). If this were the case one may achieve a constructive result with out an appeal to a theory of partially ordered classes. The semantics would need to be defined on a collection that contains the quantifier complete Heyting Algebra but no complete Heyting Algebra. My suspicion is that this is not going to be the case, but active investigation is underway. Finally one could formalize the proof in a proof assistant for verification (e.g. Lean). Formalizing the proof in a proof assistant is doable but likely will require a timeline of 6+ months. Learning Lean and developing packages that support **CZF** are initial roadblocks. The process of formalization would be its own battle.

An alternative would be to take up the suggestion given in the introduction and approach the problem from the perspective of Type Theory. This setting is more amenable to mechanization and may be fruitful in the process of formalization.

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