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Global Carleman estimate on a network for the wave equation and application to an inverse problem.

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Abstract

We are interested in an inverse problem for the wave equation with potential on a star-shaped network. We prove the Lipschitz stability of the inverse problem consisting in the determination of the potential on each string of the network with Neumann boundary measurements at all but one external vertices. Our main tool, proved in this article, is a global Carleman estimate for the network.

Keywords: networks, wave equation, inverse problem, Carleman estimate.

AMS subject classifications: 35R30, 93C20, (34B45)

1 Introduction and main result

In this paper we consider a star-shaped network $\mathcal{R}$ of $n+1$ edges e_j, of length $l_j > 0$, $j \in \{0, .., n\}$, connected at one vertex that we assume to be the origin 0 of all the edges (see Figure 1). For any function $f : \mathcal{R} \to \mathbb{R}$ we set

$$f_j = f \big|_{e_j} \quad \text{the restriction of } f \text{ to the edge } e_j,$$

$$[f]_0 = \sum_{j=0}^{n} f_j(0) \text{ the transmission bracket at the vertex 0.}$$

More precisely we consider on this plane 1-D network a wave equation with a different potential on each string, given by the following system

$$\begin{cases} u_{j,tt}(x,t) - u_{j,xx}(x,t) + p_j(x)u_j(x,t) = g_j(x,t), & \forall j \in [\![0,n]\!], (x,t) \in (0,l_j) \times (0,T), \\ u_j(l_j,t) = 0, & \forall j \in [\![0,n]\!], t \in (0,T), \\ u(x,0) = u^0(x), u_t(x,0) = u^1(x), & x \in \mathcal{R}, \end{cases}$$

$$(1)$$

under the assumptions of continuity and of Kirschoff law at the vertex 0, given by

$$u_j(0,t) = u_i(0,t) =: u(0,t), \quad \forall i,j \in \{0,...,n\}, \, 0 < t < T, \tag{2}$$

$$[u_x(t)]_0 := \sum_{j=0}^{n} u_{j,x}(0,t) = 0, \quad 0 < t < T. \tag{3}$$

In the sequel, we shall use the following notations:

$$L^2(\mathcal{R}) = \left\{ f : \mathcal{R} \to \mathbb{R}, \, f_j \in L^2(0,l_j), \, \forall j \in \{0,...,n\} \right\},$$

$$H_0^1(\mathcal{R}) = \left\{ f : \mathcal{R} \to \mathbb{R}, \, f_j \in H^1(0,l_j), \, f_j(l_j) = 0, \, f_j(0) = f_i(0), \forall i,j \in \{0,...,n\} \right\}.$$

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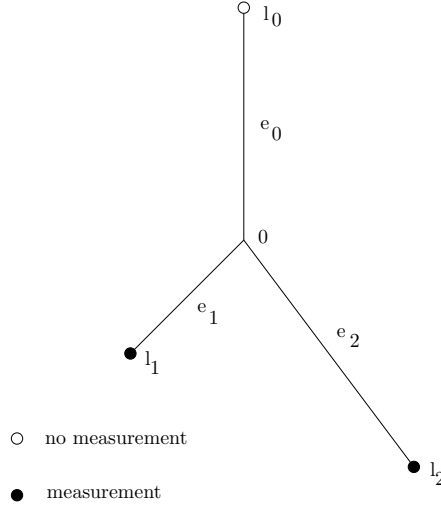
Figure 1: A star-shaped network with 3 strings.

For shortness, for $f \in L^1(\mathcal{R}) = \left\{ f : \mathcal{R} \to \mathbb{R}, f_j \in L^1(0, l_j), \forall j \in \{0, ..., n\} \right\}$ we often write,

$$\int_{\mathcal{R}} f \, dx = \sum_{j=0}^{n} \int_{0}^{l_j} f_j(x) dx.$$

Then the norms of the Hilbert spaces $L^2(\mathcal{R})$ and $H_0^1(\mathcal{R})$ are defined by

$$\|f\|_{L^2(\mathcal{R})}^2 = \int_{\mathcal{R}} |f|^2 \, dx \text{ and } \|f\|_{H_0^1(\mathcal{R})}^2 = \int_{\mathcal{R}} |f_x|^2 \, dx.$$

First of all, assuming that $u^0 \in H_0^1(\mathcal{R})$, $u^1 \in L^2(\mathcal{R})$, $p \in L^\infty(\mathcal{R})$ and $g \in L^1(0, T; L^2(\mathcal{R}))$ are known, the Cauchy problem is well-posed and one can also prove that

$$u \in C([0, T], H_0^1(\mathcal{R})) \cap C^1([0, T], L^2(\mathcal{R})).$$

This result can be deduced from [21] for instance.

We are interested in the inverse problem of the determination of the potential p on each of the $n + 1$ strings of the network from only n boundary measurements: $(u_{i,x}(l_i, \cdot))_{i=1..n}$ on $(0, T)$. We will prove the well-posedness of this problem, giving the appropriate stability estimate. The proof will mainly rely on a global Carleman estimate for this network of wave equations, result given in section 2 also interesting by itself.

Statement of the inverse problem: Is it possible to retrieve the potential $p = p(x)$, for $x \in \mathcal{R}$ from the n measurements $(u_{i,x}(l_i, \cdot))_{i=1..n}$ on $(0, T)$ where u is the solution to (1) ?

We will actually give local answer to this question. If we denote by $u[p]$ the weak solution of (1), assuming that $p \in L^\infty(\mathcal{R})$ is a given potential, we are concerned with the stability around p. That is to say p and $u[p]$ are known while q is unknown and we prove the following local lipschitz stability result.

To precisely state the results we will prove in this article, we needs to introduce, for $m \geq 0$, the set

$$L_{\leq m}^\infty(\mathcal{R}) = \left\{ q : \mathcal{R} \to \mathbb{R}, q_j \in L^\infty(0, l_j), \forall j \in \{0, ..., n\}, \ s.t. \ \|q\|_{L^\infty(\mathcal{R})} \leq m \right\}.$$

Theorem 1 *There exists $T_0 > 0$ such that for all $T \geq T_0$, $p \in L^\infty(\mathcal{R})$, $u^0 \in H_0^1(\mathcal{R})$, $u^1 \in L^2(\mathcal{R})$ and $g \in L^1(0, T; L^2(\mathcal{R}))$, if we assume*

$$u[p] \in H^1(0, T; L^\infty(\mathcal{R}))$$

$$|u^0(x)| \geq r > 0, \ ae \ in \ \mathcal{R},$$

then there exists a constant $C = C(T, p, u^0, u^1, m, l_0, ..., l_n) > 0$ such that $\forall q \in L_{\leq m}^\infty$:

$$||q - p||_{L^2(\mathcal{R})} \leq C \sum_{j=1}^n ||u_{j,x}[p](l_j) - u_{j,x}[q](l_j)||_{H^1(0,T)}.$$

Remark 1 *One should notice that we can guarantee $u[p] \in H^1(0, T; L^\infty(\mathcal{R}))$ with more constraints on the hypothesis on the data u^0, u^1, g in equation (1). For exemple, $u^0 \in H^2(\mathcal{R}) \cap H_0^1(\mathcal{R})$, $u^1 \in H_0^1(\mathcal{R})$ and $g \in W^{1,1}(0, T; L^2(\mathcal{R}))$ works in that case.*

In the field of inverse problems in partial differential equations (pde), where the question is to determine some parameter(s) from measurement(s) of the solution of the pde, the main concern is the well-posedness of the problem. The notion of well-posed problem, introduced by Hadamard, relies on the existence, uniqueness and stability of the solution of the problem. Concerning pde's, the main issue is often that the inverse problem is ill-posed.

In order to give a non exhaustive state of the art concerning the problem we consider in this paper, one should expect some information about inverse problems for hyperbolic equations and about 1-D networks of strings.

On the one hand, the book of V. Isakov [14] adresses some techniques linked to the study of inverse problem for several pde's. Historically, the first answers concerned the uniqueness of solution for inverse problems and on this topic, the article of L. Bukhgeim and M. V. Klibanov [8] is the first to give a method using Carleman estimates to prove the uniqueness of a one measurement inverse problem for hyperbolic equations. Uniqueness results for some inverse source problem have been proved by M. V. Klibanov in [15] and a stability result from M. Yamamoto for the wave equation, deriving from it, can be read in [28]. It is indeed possible to obtain local Lipschitz stability around a single known solution, provided that this solution is regular enough and contains enough information, as it can also be read in [7], [16].

Many other related inverse results for hyperbolic equations use the same strategy. A complete list is too long to be given here. To cite some of them see the articles of J.-P. Puel, M. Yamamoto and O. Yu. Imanuvilov with for instance, [24] and [28] where the pde is given with Dirichlet boundary data and the inverse problem is studied from Neumann measurements and [13] for the reverse case. One should also quote for instance [29] for a case of two unknowns to recover and [3] that concerns the determination of potential but for an hyperbolic equation where the principal part of the operator has a discontinuity on an interface. These references are all based upon the use of local or global Carleman estimates. Related to this, there are also general pointwise Carleman estimates that can be useful in similar inverse problems [17].

Global Carleman estimates and applications to one-measurement inverse problems were also obtained in the case of variable but still regular coefficients, as in [12], [19] and [5] for instance.

On the other hand, the control, observation and stabilization problems of networks have been the object of intensive research (see [11, 18, 31] and the references therein). These works use results from several domains: non-harmonic Fourier series, Diophantine approximations, graph theory, wave propagation techniques.

In [11] and in [9, 10], controllability results for the wave equation on networks are proved by showing observability inequalities and under assumptions about the irrationality properties of the ratios of the lengths of the string. One should mention that in these works, the control is only applied at one single end of the network.

Stabilization results for the wave equation on networks have been considered by several authors in some particular situations. We refer, for instance, to [1], where explicit decay rates are obtained for networks with some special structures. We also refer to [22] where the

problem is considered in the presence of delay terms in the feedback law. In [26] the aim is to develop a systematic method for the stabilization of the general network by means of a damping term located on one single exterior node and to give a general result allowing to transform a weak observability result for the corresponding conservative system (see [11]) into a stabilization one for the damped one. The authors obtain the explicit decay estimates (in general weaker than exponential) of the energy for smooth initial data which depend on the geometric and topological properties of the network.

Inverse problems for waves on networks are as usual intimately related to the control problem. We refer to the paper [2] for the analysis of tree-like networks through the so-called boundary control approach [4], where the determination of constant coefficient, length and angles of the network is considered but the question of stability is not adressed. Moroever in the recent paper [27] the authors consider some inverse scattering problems on star-shaped graphs. More precisely they first prove the identifiability of the geometry of this graph (the number of edges and their lengths) by studying the asymptotic behavior of only one reflection coefficient in the high frequency limit. Second they show that under some assumptions on the graph, the measurement of two reflection coefficients associated to the two different sets of boundary conditions at the external vertices of the tree determines uniquely the potential. Finally in [23], the authors consider the inverse problem of determining point wave sources in tree-shaped networks and show that the Neumann boundary observation on a part of the lateral boundary determines uniquely the point sources.

As far as we know, global Carleman estimates for networks were not written and proved before this article. The main interest of this tool is the direct use of it to obtain (as we do in Section 3) the stability of the inverse problem we describe here (determination of potential). Moreover, many results of controllability of string networks concern only the wave equation without lower order terms (see [18, 25] for instance), that is not the case here. And as one may know, an observability estimate can be derived from a global Carleman estimate quite easily (see [30] for instance) and therefore lead to exact controllability results.

This paper is organized as follows. A global Carleman estimate for a wave equation with potential on a star-shaped 1-D network is proved in Section 2. We use the Bukhgeim-Klibanov method in Section 3 to prove the Lipschitz stability of the inverse problem stated in Theorem 1.

2   A global Carleman estimate on a network

In this section, we will establish a Carleman estimate for the wave operator $L = \partial_{tt} - \partial_{xx} + p$ defined on a star-shaped network $\mathcal{R}$ and applied to a function $v : \mathcal{R} \times (-T, T) \to \mathbb{R}$ satisfying $v_j(l_j, t) = 0$ for all $j \in [\![0, n]\!]$ and $t \in (-T, T)$ and $v(x, \pm T) = 0$ for all $x \in \mathcal{R}$.

Let $\lambda > 0$. We define the weight function $\varphi = \varphi(x, t)$ for all $(x, t) \in \mathcal{R} \times (-T, T)$ by

$$\varphi(x, t) \quad = \quad e^{\lambda \phi(x, t)}, \tag{4}$$

where ϕ satisfies the following lemma.

Lemma 1 *We define the weight function ϕ on each edge e_j of the network as follows:*
For all $j \in [\![0, n]\!]$, $x \in (0, l_j)$ and $t \in \mathbb{R}$,

$$\phi\mid_{e_j} (x, t) := \phi_j(x, t) = (x - x_j)^2 - \beta t^2 + M_j. \tag{5}$$

There exist $(x_0, x_1, ..., x_n) \in \mathbb{R}^+ \times (\mathbb{R}^-)^n$, $\beta \in (0, 1)$ and $T > 0$ such that $\beta T \leq \dfrac{\min_{i \in [\![1, n]\!]} |x_i|}{2}$ and there exists $M_0, M_1, ... M_n > 0$, such that for all $i, j \in [\![0, n]\!]$, $t \in (-T, T)$, $x \in (0, l_j)$,

$$\phi_i(0, t) = \phi_j(0, t), \tag{6}$$

$$\phi_j(x, t) \geq 1 \tag{7}$$

and such that the following $(n + 1) \times (n + 1)$ symetric matrix is definite positive, thus,

$$\exists \alpha > 0, \, \forall t \in [-T, T], \, \forall \xi \in \mathbb{R}^{n+1}, \, (A_\phi(t)\xi, \xi) \geq \alpha |\xi|^2, \tag{8}$$

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with $A_\phi(t) :=$

$$
\begin{pmatrix}
\phi_{0,x}(0) + \phi_{1,x}(0) & \phi_{0,x}(0) & \cdots & \phi_{0,x}(0) & -\phi_{0,x}(0)[\phi_x]_0 \\
& \ddots & \ddots & \vdots & -\phi_{0,x}(0)[\phi_x]_0 \\
& & \ddots & \phi_{0,x}(0) & \vdots \\
& & & \phi_{0,x}(0) + \phi_{n,x}(0) & -\phi_{0,x}(0)[\phi_x]_0 \\
& & & & \phi_{0,x}(0)[\phi_x]_0^2 + [\phi_x^3]_0 - \phi_t^2(0,t)[\phi_x]_0
\end{pmatrix}.
$$

Moreover, one has directly that for all $j \in [\![0, n]\!]$ and for all $x \in (0, l_j)$,

$$\phi_j(x, 0) > M_j \tag{9}$$

and if we denote $\rho := \max\limits_{i \in [\![0,n]\!], x \in [0,l_i]} |x - x_i|$, then if $T > \dfrac{2\rho^2}{\min\limits_{i \in [\![1,n]\!]} |x_i|}$, one can choose $\beta > \dfrac{\rho^2}{T^2}$

such that

$$\phi_j(x, \pm T) < M_j, \tag{10}$$

and there exists $\eta > 0$ so that for all $t \in [-T, -T + \eta] \cup [T - \eta, T]$,

$$\phi_j(x, t) \leq M_j. \tag{11}$$

Remark 2 *The use of a more natural weight function satisfying continuity and Kirchoff laws at the central node does not allow to satisfy the required inequalities for the Carleman estimate in view of a n measurements inverse problem. Thus, we propose in this article this type of weight function ϕ with a matrix A_ϕ coming directly from the proof of the Carleman estimates below. This kind of weight function has been introduced first by Benabdallah, Dermenjian and Le Rousseau in [6] for the heat equation with a discontinuous coefficient.*

Proof. The matrix A_ϕ is definite positive if and only if all of its leading principal minors are positive. Using the definition (5), we get $\phi_{i,x}(0, t) = -2x_i$. We choose $x_0 > l_0 > 0$ and $x_i = x_1 < 0$, $\forall i \geq 1$ thus the symmetric matrix $A_\phi(t)$ becomes

$$
A_\phi(t) = -2
\begin{pmatrix}
x_0 + x_1 & x_0 & \cdots & x_0 & 2x_0(x_0 + nx_1) \\
& \ddots & \ddots & \vdots & 2x_0(x_0 + nx_1) \\
& & \ddots & x_0 & \vdots \\
& & & x_0 + x_1 & 2x_0(x_0 + nx_1) \\
& & & & a_{n+1,n+1}
\end{pmatrix}
$$

where $a_{n+1,n+1} = 4x_0(x_0 + nx_1)^2 + 4(x_0^3 + nx_1^3) - 4\beta^2 t^2(x_0 + nx_1)$, and we actually need

$$(-2)(-2x_1)^{i-1}(ix_0 + x_1) > 0, \forall i \in [\![1, n]\!] \tag{12}$$

$$\mathrm{Det}(A_\phi(t)) > 0, \forall t \in [-T, T]. \tag{13}$$

On the one hand, choosing $x_1 < -nx_0$ satisfies (12). On the other hand, the function $\mathrm{Det}(A_\phi(t)) = P(x_0, x_1, \beta^2 t^2)$ is a polynomial in x_1 of degree $n + 3$. More precisely, since $a_{n+1,n+1}$ is of maximum order in x_1 in the matrix $A_\phi(t)$, then the leading order term in x_1 of $\mathrm{Det}(A_\phi(t))$ comes from the product of the diagonal terms. Therefore, taking $\beta t = \frac{|x_1|}{2}$, the polynomial $P(x_0, x_1, x_1^2/4)$ has a leading order term in x_1 given by

$$(-2x_1)^{n+3} n(1 - 1/4)$$

and choosing x_1 sufficiently large negative gives $P(x_0, x_1, x_1^2/4) > 0$ and is compatible with (12). Hence, let $\beta > 0$ and $T > 0$ be such that $\beta T \leq \frac{|x_1|}{2}$, and then, for all $t \in [-T, T]$,

$$\mathrm{Det}(A_\phi(t)) = P(x_0, x_1, \beta^2 t^2) \geq P(x_0, x_1, x_1^2/4) > 0$$

which proves (13).

In order to have the continuity (6) at the central node 0 and the positiveness (7) of the weights ϕ_j, it suffices to choose $M_j > 0$ large enough for all j and such that

$$x_i^2 + M_i = x_j^2 + M_j, \forall i, j \in [\![0, n]\!].$$

5

Finally let T such that $T > \dfrac{2\rho^2}{\min\limits_{i\in[\![1,n]\!]}|x_i|}$. Then $\dfrac{\rho^2}{T^2} < \dfrac{\min_{i\in[\![1,n]\!]}|x_i|}{2T}$ and we can choose

$\beta \in (0,1)$ such that $\dfrac{\rho^2}{T^2} < \beta \leq \dfrac{\min_{i\in[\![1,n]\!]}|x_i|}{2T}$. Then the proof of (10)-(11) relies on the fact

that with $\beta > \dfrac{\rho^2}{T^2}$

$$\phi_j(x, \pm T) - M_j = (x - x_j)^2 - \beta T^2 < 0, \ \forall j \in [\![0, n]\!], \ \forall x \in (0, l_j).$$

$\square$

Remark 3 *This lemma gathers all the properties on the weight function ϕ defined by (5) we will need in this article. But we want to underline that they are of two kinds. On the one hand, (6), (7) and (8) are needed for the proof of the Carleman estimate and this means that there is no constraint on the time T. On the other hand, (9), (10) and (11) are strongly needed for the proof of the stability of the inverse problem and therefore, the observation time T has to be large enough. The proof of Lemma 1 was not meant to give an optimal minimal time T.*

The main result of this section is the following global Carleman inequality.

Theorem 2 *Let $p \in L^\infty(\mathcal{R})$ such that $\|p\|_{L^\infty(\mathcal{R})} \leq m$. Assume that L is the partial differential operator defined by $L = \partial_{tt} - \partial_{xx} + p$ on the star-shaped network $\mathcal{R}$ as in equation (1) and $\varphi = e^{\lambda\phi}$ with ϕ satisfying Lemma 1. There exists $s_0 > 0$, $\lambda_0 > 0$ and a constant $M = M(s_0, \lambda_0, \mathcal{R}, T, m, \beta, x_0, ..., x_n) > 0$ such that:*

$$s\lambda \int_{-T}^{T} \int_{\mathcal{R}} \varphi \left(v_t^2 + v_x^2\right) e^{2s\varphi} \, dxdt + s^3\lambda^3 \int_{-T}^{T} \int_{\mathcal{R}} \varphi^3 v^2 e^{2s\varphi} \, dxdt$$

$$\leq M \int_{-T}^{T} \int_{\mathcal{R}} |Lv|^2 e^{2s\varphi} \, dxdt + Ms\lambda \sum_{j=1}^{n} \int_{-T}^{T} v_{j,x}^2(l_j) e^{2s\varphi(l_j)} \, dt, \tag{14}$$

for all $s > s_0$, $\lambda > \lambda_0$ and v satisfying the internal node conditions (2) and (3) and

$$\begin{cases} Lv \in L^2(\mathcal{R} \times (-T, T)), \\ v \in H^1(-T, T; H_0^1(\mathcal{R})), \\ v(\pm T) = \partial_t v(\pm T) = 0 \text{ in } \mathcal{R}. \end{cases}$$

Proof. Let $s > 0$, $\lambda > 0$ and the weigth function φ be defined by $\varphi(x, t) = e^{\lambda\phi(x,t)}$ where ϕ is given by (5) and satisfies (6), (7) and (8). We set $w = e^{s\varphi}v$ and $Pw = e^{s\varphi}L(e^{-s\varphi}w)$. After computing Pw we split the terms as follows:

$$Pw = P_1w + P_2w + Rw$$

where, for a constant $C > 0$ to be chosen later,

$$P_1w = w_{tt} - w_{xx} + s^2\lambda^2\varphi^2(\phi_t^2 - \phi_x^2)w, \tag{15}$$

$$P_2w = (C-1)s\lambda\varphi(\phi_{tt} - \phi_{xx})w - s\lambda^2\varphi(\phi_t^2 - \phi_x^2)w - 2s\lambda\varphi(\phi_t w_t - \phi_x w_x), \tag{16}$$

$$Rw = pw - Cs\lambda\varphi(\phi_{tt} - \phi_{xx})w. \tag{17}$$

Therefore, $P_1w + P_2w = Pw - Rw$ and

$$\int_{-T}^{T} \int_{\mathcal{R}} |Pw - Rw|^2 \, dxdt = \int_{-T}^{T} \int_{\mathcal{R}} |P_1w|^2 \, dxdt + \int_{-T}^{T} \int_{\mathcal{R}} |P_2w|^2 \, dxdt$$

$$+ \ 2\int_{-T}^{T} \int_{\mathcal{R}} P_1w P_2w \, dxdt.$$

The main goal of the proof will be to minimize the cross-term in $P_1w P_2w$ by positive and dominant terms looking similar to the one of the left hand side of (14) and negative boundary terms that will be moved to the right hand side of the estimate. In the sake of clarity, we will devide the proof in several steps.

6

Step 1. Main calculations

We set $\langle P_1 w, P_2 w \rangle_{L^2(\mathcal{R} \times (-T,T))} = \sum_{i,k=1}^{3} I_{i,k}$ where $I_{i,k}$ is the integral of the product of the ith-term in $P_1 w$ and the kth-term in $P_2 w$. Therefore after some integrations by parts,

$$
\begin{aligned}
I_{1,1} &= (C-1)s\lambda \int_{-T}^{T} \int_{\mathcal{R}} \varphi w_{tt}(\phi_{tt} - \phi_{xx})w \, dx dt \\
&= (1-C)s\lambda \int_{-T}^{T} \int_{\mathcal{R}} \varphi w_t^2 (\phi_{tt} - \phi_{xx}) \, dx dt \\
&\quad + \frac{C-1}{2}s\lambda^3 \int_{-T}^{T} \int_{\mathcal{R}} \varphi w^2 \phi_t^2 (\phi_{tt} - \phi_{xx}) \, dx dt \\
&\quad + \frac{C-1}{2}s\lambda^2 \int_{-T}^{T} \int_{\mathcal{R}} \varphi w^2 \phi_{tt}(\phi_{tt} - \phi_{xx}) \, dx dt \ ;
\end{aligned}
$$

$$
\begin{aligned}
I_{1,2} &= -s\lambda^2 \int_{-T}^{T} \int_{\mathcal{R}} \varphi w_{tt}(\phi_t^2 - \phi_x^2)w \, dx dt \\
&= s\lambda^2 \int_{-T}^{T} \int_{\mathcal{R}} \varphi w_t^2 (\phi_t^2 - \phi_x^2) \, dx dt \\
&\quad - s\lambda^2 \int_{-T}^{T} \int_{\mathcal{R}} \varphi w^2 \phi_{tt}^2 \, dx dt - \frac{s\lambda^4}{2} \int_{-T}^{T} \int_{\mathcal{R}} \varphi w^2 \phi_t^2 (\phi_t^2 - \phi_x^2) \, dx dt \\
&\quad - \frac{5s\lambda^3}{2} \int_{-T}^{T} \int_{\mathcal{R}} \varphi w^2 \phi_{tt}\phi_t^2 \, dx dt + \frac{s\lambda^3}{2} \int_{-T}^{T} \int_{\mathcal{R}} \varphi w^2 \phi_{tt}\phi_x^2 \, dx dt \ ;
\end{aligned}
$$

$$
\begin{aligned}
I_{1,3} &= -2s\lambda \int_{-T}^{T} \int_{\mathcal{R}} \varphi w_{tt}(\phi_t w_t - \phi_x w_x) \, dx dt \\
&= s\lambda^2 \int_{-T}^{T} \int_{\mathcal{R}} \varphi w_t^2 \phi_t^2 \, dx dt - 2s\lambda^2 \int_{-T}^{T} \int_{\mathcal{R}} \varphi w_t \phi_t w_x \phi_x \, dx dt \\
&\quad + s\lambda \int_{-T}^{T} \int_{\mathcal{R}} \varphi w_t^2 \phi_{tt} \, dx dt + s\lambda^2 \int_{-T}^{T} \int_{\mathcal{R}} \varphi w_t^2 \phi_x^2 \, dx dt \\
&\quad + s\lambda \int_{-T}^{T} \int_{\mathcal{R}} \varphi w_t^2 \phi_{xx} \, dx dt + s\lambda \int_{-T}^{T} \varphi(0,t)w_t^2(0,t) \left[\phi_x\right]_0 dt
\end{aligned}
$$

since $\varphi_j(0,t) = \varphi_i(0,t) =: \varphi(0,t)$ and $w_{j,t}(0,t) = w_{i,t}(0,t) = w_t(0,t)$ for all $i,j \in \{0,..,n\}$. Using the same tricks, we have

$$
\begin{aligned}
I_{2,1} &= (1-C)s\lambda \int_{-T}^{T} \int_{\mathcal{R}} \varphi w w_{xx}(\phi_{tt} - \phi_{xx}) \, dx dt \\
&= (C-1)s\lambda \int_{-T}^{T} \int_{\mathcal{R}} \varphi w_x^2 (\phi_{tt} - \phi_{xx}) \, dx dt \\
&\quad + \frac{1-C}{2}s\lambda^3 \int_{-T}^{T} \int_{\mathcal{R}} \varphi w^2 \phi_x^2 (\phi_{tt} - \phi_{xx}) \, dx dt \\
&\quad + \frac{1-C}{2}s\lambda^2 \int_{-T}^{T} \int_{\mathcal{R}} \varphi w^2 \phi_{xx}(\phi_{tt} - \phi_{xx}) \, dx dt \\
&\quad + \frac{1-C}{2}s\lambda^2 \int_{-T}^{T} \varphi(0,t)w^2(0,t) \left[\phi_x(\phi_{tt} - \phi_{xx})\right]_0 dt \\
&\quad + (C-1)s\lambda \int_{-T}^{T} \varphi(0,t)w(0,t) \left[w_x(\phi_{tt} - \phi_{xx})\right]_0 dt \ ;
\end{aligned}
$$

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$$I_{2,2} \;=\; s\lambda^2 \int_{-T}^{T}\!\!\int_{\mathcal{R}} \varphi w w_{xx}(\phi_t^2 - \phi_x^2)\,dxdt$$

$$\;=\; -s\lambda^2 \int_{-T}^{T}\!\!\int_{\mathcal{R}} \varphi w_x^2(\phi_t^2 - \phi_x^2)\,dxdt$$

$$+\frac{s\lambda^4}{2}\int_{-T}^{T}\!\!\int_{\mathcal{R}} \varphi w^2 \phi_x^2(\phi_t^2 - \phi_x^2)\,dxdt + \frac{s\lambda^3}{2}\int_{-T}^{T}\!\!\int_{\mathcal{R}} \varphi w^2 \phi_{xx}(\phi_t^2 - \phi_x^2)\,dxdt$$

$$-2s\lambda^3 \int_{-T}^{T}\!\!\int_{\mathcal{R}} \varphi w^2 \phi_{xx}\phi_x^2\,dxdt - s\lambda^2 \int_{-T}^{T}\!\!\int_{\mathcal{R}} \varphi w^2 \phi_{xx}^2\,dxdt$$

$$+\frac{s\lambda^3}{2}\int_{-T}^{T} \varphi(0,t)w^2(0,t)\left[\phi_x(\phi_t^2 - \phi_x^2)\right]_0 dt$$

$$-s\lambda^2 \int_{-T}^{T} \varphi(0,t)w^2(0,t)\left[\phi_x\phi_{xx}\right]_0 dt$$

$$-s\lambda^2 \int_{-T}^{T} \varphi(0,t)w(0,t)\left[w_x(\phi_t^2 - \phi_x^2)\right]_0 dt \;;$$

$$I_{2,3} \;=\; 2s\lambda \int_{-T}^{T}\!\!\int_{\mathcal{R}} \varphi w_{xx}(\phi_t w_t - \phi_x w_x)\,dxdt$$

$$\;=\; -2s\lambda^2 \int_{-T}^{T}\!\!\int_{\mathcal{R}} \varphi w_t w_x \phi_t \phi_x\,dxdt$$

$$+s\lambda^2 \int_{-T}^{T}\!\!\int_{\mathcal{R}} \varphi w_x^2 \phi_x^2\,dxdt + s\lambda^2 \int_{-T}^{T}\!\!\int_{\mathcal{R}} \varphi w_x^2 \phi_t^2\,dxdt$$

$$+s\lambda \int_{-T}^{T}\!\!\int_{\mathcal{R}} \varphi w_x^2 \phi_{tt}\,dxdt + s\lambda \int_{-T}^{T}\!\!\int_{\mathcal{R}} \varphi w_x^2 \phi_{xx}\,dxdt$$

$$-s\lambda \sum_{j=0}^{n} \int_{-T}^{T} \varphi(l_j,t)w_{j,x}^2(l_j,t)\phi_{j,x}(l_j,t)dt$$

$$+s\lambda \int_{-T}^{T} \varphi(0,t)\left[\phi_x w_x^2\right]_0 dt$$

$$-2s\lambda \int_{-T}^{T} \varphi(0,t)w_t(0,t)\phi_t(0,t)\left[w_x\right]_0 dt \;;$$

$$I_{3,1} = (C-1)s^3\lambda^3 \int_{-T}^{T}\!\!\int_{\mathcal{R}} \varphi^3 w^2(\phi_{tt} - \phi_{xx})(\phi_t^2 - \phi_x^2)\,dxdt \;;$$

$$I_{3,2} = -s^3\lambda^4 \int_{-T}^{T}\!\!\int_{\mathcal{R}} \varphi^3 w^2(\phi_t^2 - \phi_x^2)^2\,dxdt \;;$$

$$I_{3,3} \;=\; -2s^3\lambda^3 \int_{-T}^{T}\!\!\int_{\mathcal{R}} \varphi^3 w(\phi_t w_t - \phi_x w_x)(\phi_t^2 - \phi_x^2)\,dxdt$$

$$\;=\; s^3\lambda^3 \int_{-T}^{T}\!\!\int_{\mathcal{R}} \varphi^3 w^2(\phi_{tt} - \phi_{xx})(\phi_t^2 - \phi_x^2)\,dxdt$$

$$+3s^3\lambda^4 \int_{-T}^{T}\!\!\int_{\mathcal{R}} \varphi^3 w^2(\phi_t^2 - \phi_x^2)^2\,dxdt$$

$$+2s^3\lambda^3 \int_{-T}^{T}\!\!\int_{\mathcal{R}} \varphi^3 w^2\left(\phi_t^2\phi_{tt} + \phi_{xx}\phi_x^2\right)\,dxdt$$

$$-s^3\lambda^3 \int_{-T}^{T} \varphi^3(0,t)w^2(0,t)\left[\phi_x(\phi_t^2 - \phi_x^2)\right]_0 dt.$$

We obtain thus,

$$
\begin{aligned}
\langle P_1(w), P_2(w)\rangle_{L^2(\mathcal{R}\times(-T,T))} =& 2s\lambda \int_{-T}^{T}\int_{\mathcal{R}} w_t^2 \varphi \phi_{tt}\, dxdt \\
& - Cs\lambda \int_{-T}^{T}\int_{\mathcal{R}} \varphi w_t^2(\phi_{tt} - \phi_{xx})\, dxdt \\
& + 2s\lambda^2 \int_{-T}^{T}\int_{\mathcal{R}} \varphi(w_t^2\phi_t^2 - 2w_t w_x \phi_t \phi_x + w_x^2\phi_x^2)\, dxdt \\
& + 2s\lambda \int_{-T}^{T}\int_{\mathcal{R}} \varphi w_x^2 \phi_{xx}\, dxdt \\
& + Cs\lambda \int_{-T}^{T}\int_{\mathcal{R}} \varphi w_x^2(\phi_{tt} - \phi_{xx})\, dxdt \qquad (18)\\
& + 2s^3\lambda^4 \int_{-T}^{T}\int_{\mathcal{R}} \varphi^3 w^2(\phi_t^2 - \phi_x^2)^2\, dxdt \\
& + 2s^3\lambda^3 \int_{-T}^{T}\int_{\mathcal{R}} \varphi^3 w^2(\phi_t^2\phi_{tt} + \phi_x^2\phi_{xx})\, dxdt \\
& + Cs^3\lambda^3 \int_{-T}^{T}\int_{\mathcal{R}} \varphi^3 w^2(\phi_{tt} - \phi_{xx})(\phi_t^2 - \phi_x^2)\, dxdt \\
& - s\lambda \sum_{j=0}^{n}\int_{-T}^{T} \varphi(l_j,t)w_{j,x}^2(l_j,t)\phi_{j,x}(l_j,t)\, dt + X + B
\end{aligned}
$$

where B is the sum of the trace term at the central node and X is the sum of the remaining interior terms, in such a way that

$$
|X| \le Ms\lambda^3 \int_{-T}^{T}\int_{\mathcal{R}} w^2\varphi^3\, dxdt
$$

for some suitable constant $M > 0$ since $\phi \ge 1$ gives $\lambda \le \varphi = e^{\lambda\phi}$. Let us denote A_k, $k = 1..8$ the first eight interior integrals in the product of $P_1 w$ by $P_2 w$. Thus we have,

$$
\langle P_1 w, P_2 w\rangle_{L^2(\mathcal{R}\times(-T,T))} = \sum_{k=1}^{8} A_k - s\lambda \sum_{j=0}^{n}\int_{-T}^{T} w_{j,x}^2(l_j,t)\varphi(l_j,t)\phi_{j,x}(l_j,t)dt + X + B. \quad (19)
$$

From now on, $M > 0$ will be a generic constant depending only on the network $\mathcal{R}$, the time T and on $m, \beta, x_0, ..., x_n$ but independent of s and λ.

Step 2. Boundary terms at the central node.

Lemma 2 *Under the hypothesis of Lemma 1 for the weight function, the sum of the trace terms B at the central node 0 in* (18) *is positive for s and λ sufficiently large.*

Proof. Gathering all the terms at the central node 0, we get

$$
B = B_{1,3} + B_{2,1} + B_{2,2} + B_{2,3} + B_{3,3}
$$

where

$$
B_{1,3} = s\lambda \int_{-T}^{T} \varphi(0,t)w_t^2(0,t)[\phi_x]_0\, dt,
$$

$$
\begin{aligned}
B_{2,1} =\ & (C-1)s\lambda \int_{-T}^{T} \varphi(0,t)w(0,t)[(\phi_{tt} - \phi_{xx})w_x]_0\, dt \\
& + \frac{(1-C)}{2}s\lambda^2 \int_{-T}^{T} \varphi(0,t)w^2(0,t)[\phi_x(\phi_{tt} - \phi_{xx})]_0\, dt,
\end{aligned}
$$

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$$B_{2,2} = -s\lambda^2 \int_{-T}^{T} \varphi(0,t)w(0,t)[w_x(\phi_t^2 - \phi_x^2)]_0 \, dt$$
$$+ \frac{s\lambda^3}{2} \int_{-T}^{T} \varphi(0,t)w^2(0,t)[\phi_x(\phi_t^2 - \phi_x^2)]_0 \, dt$$
$$- s\lambda^2 \int_{-T}^{T} \varphi(0,t)w^2(0,t)[\phi_x\phi_{xx}]_0 \, dt,$$

$$B_{2,3} = -2s\lambda \int_{-T}^{T} \varphi(0,t)w_t(0,t)\phi_t(0,t)[w_x]_0 dt + s\lambda \int_{-T}^{T} \varphi(0,t)[\phi_x w_x^2]_0 \, dt,$$

$$B_{3,3} = -s^3\lambda^3 \int_{-T}^{T} \varphi^3(0,t)w^2(0,t)\phi_t^2(0,t)[\phi_x]_0 \, dt + s^3\lambda^3 \int_{-T}^{T} \varphi^3(0,t)w^2(0,t)[\phi_x^3]_0 \, dt.$$

We denote by μ the sum of $B_{1,3}$, $B_{3,3}$, and of the second term in $B_{2,3}$ i.e.

$$\mu := s^3\lambda^3 \int_{-T}^{T} \varphi^3(0,t)w^2(0,t) \left([\phi_x^3]_0 - \phi_t^2(0,t)[\phi_x]_0\right) dt$$
$$+s\lambda \int_{-T}^{T} \varphi(0,t)[\phi_x w_x^2]_0 \, dt + s\lambda \int_{-T}^{T} \varphi(0,t)w_t^2(0,t)[\phi_x]_0 \, dt.$$

Since $w = e^{s\varphi}v$ gives $w_x = s\lambda\varphi\phi_x w + e^{s\varphi}v_x$, then $[w_x]_0 = \sum_{j=0}^{n} w_{j,x}(0) = s\lambda\varphi(0)w(0)[\phi_x]_0$

and we can write $w_{0,x}(0) = s\lambda\varphi(0)w(0)[\phi_x]_0 - \sum_{j=1}^{n} w_{j,x}(0)$. Using this expression, we thus

obtain

$$[\phi_x w_x^2]_0 = \phi_{0,x}(0) \left(s\lambda\varphi(0)w(0)[\phi_x]_0 - \sum_{j=1}^{n} w_{j,x}(0)\right)^2 + \sum_{j=1}^{n} \phi_{j,x}(0)w_{j,x}^2(0)$$

$$= s^2\lambda^2\varphi^2(0)w^2(0)\phi_{0,x}(0)[\phi_x]_0^2 - 2s\lambda\varphi(0)w(0)\phi_{0,x}(0)[\phi_x]_0 \sum_{j=1}^{n} w_{j,x}(0)$$

$$+\phi_{0,x}(0) \left(\sum_{j=1}^{n} w_{j,x}(0)\right)^2 + \sum_{j=1}^{n} \phi_{j,x}(0)w_{j,x}^2(0)$$

$$= s^2\lambda^2\varphi^2(0)w^2(0)\phi_{0,x}(0)[\phi_x]_0^2 - 2s\lambda\varphi(0)w(0)\phi_{0,x}(0)[\phi_x]_0 \sum_{j=1}^{n} w_{j,x}(0)$$

$$+\sum_{j=1}^{n}(\phi_{0,x}(0) + \phi_{j,x}(0))w_{j,x}^2(0) + 2\phi_{0,x}(0) \sum_{i,j=1..n, i\neq j} w_{i,x}(0)w_{j,x}(0).$$

Therefore,

$$\mu = s\lambda \int_{-T}^{T} \varphi(0,t)\left(A_\phi(t)W(t), W(t)\right) dt + s\lambda \int_{-T}^{T} \varphi(0,t)w_t^2(0,t)[\phi_x]_0 \, dt,$$

with $W(t) = \left((w_{j,x}(0,t))_{j=1..n}, s\lambda\varphi(0,t)w(0,t)\right)$. Using Lemma 1, we get

$$\mu \geq \alpha s\lambda \int_{-T}^{T} \varphi(0,t) \sum_{j=1}^{n} w_{j,x}^2(0,t)dt + \alpha s^3\lambda^3 \int_{-T}^{T} \varphi^3(0,t)w^2(0,t)dt$$

$$+ s\lambda[\phi_x]_0 \int_{-T}^{T} \varphi(0,t)w_t^2(0,t) \, dt, \tag{20}$$

with $[\phi_x]_0 = -2\sum_{j=0}^{n} x_j > 0$, because from Lemma 1, $(x_1,\ldots,x_n) \in \left(\mathbb{R}^-\right)^n$ and A_ϕ is definite positive, which gives $x_0 + x_1 < 0$ (from the first leading principal minor).

Moreover, since $\phi_{tt} = -2\beta$, $\phi_{xx} = 2$ and $[w_x]_0 = s\lambda\varphi(0,t)w(0,t)[\phi_x]_0$, we have

$$
\begin{aligned}
|B_{2,1}| &= \left| 2(C-1)(1+\beta)s^2\lambda^2[\phi_x]_0 \int_{-T}^{T} \varphi^2(0,t)w^2(0,t)dt \right. \\
&\quad \left. + (1-C)(1+\beta)s\lambda^2[\phi_x]_0 \int_{-T}^{T} \varphi(0,t)w^2(0,t)dt \right| \\
&\leq M(s^2\lambda^2 + s\lambda^2) \int_{-T}^{T} \varphi^3(0,t)|w(0,t)|^2 dt
\end{aligned}
\tag{21}
$$

since $\varphi(0,t) \geq 1$.

As we have $\phi_t^2 = 4\beta^2 t^2$, $[\phi_x^3]_0 = -8\sum_{j=0}^{n} x_i^3 > 0$ and

$$
\begin{aligned}
\left[w_x\phi_x^2\right]_0 &= 4\left(x_0^2 [w_x]_0 + \sum_{j=1}^{n}(x_j^2 - x_0^2)w_{j,x}(0,t) \right) \\
&= 4\left(s\lambda[\phi_x]_0 x_0^2 \varphi(0,t)w(0,t) + \sum_{j=1}^{n}(x_j^2 - x_0^2)w_{j,x}(0,t) \right),
\end{aligned}
$$

we obtain

$$
\begin{aligned}
|B_{2,2}| &= \left| -2s\lambda^2[\phi_x]_0 \int_{-T}^{T} \varphi(0,t)w^2(0,t)dt - 4\beta^2 s^2\lambda^3[\phi_x]_0 \int_{-T}^{T} \varphi^2(0,t)w^2(0,t)t^2 dt \right. \\
&\quad + 4s\lambda^2 \int_{-T}^{T} w(0,t)\varphi(0,t)\left(s\lambda[\phi_x]_0 x_0^2 \varphi(0,t)w(0,t) + \sum_{j=1}^{n}(x_j^2 - x_0^2)w_{j,x}(0,t) \right) dt \\
&\quad \left. + 2\beta^2 s\lambda^3[\phi_x]_0 \int_{-T}^{T} w^2(0,t)\varphi(0,t)t^2 dt - \frac{s\lambda^3}{2}\int_{-T}^{T} w^2(0,t)\varphi(0,t)[\phi_x^3]_0 dt \right| \\
&\leq M(s\lambda^2 + s\lambda^3 + s^2\lambda^3) \int_{-T}^{T} \varphi^3(0,t)|w(0,t)|^2 dt \\
&\quad + Ms\lambda^2 \sum_{j=1}^{n} \int_{-T}^{T} \varphi(0,t)|w(0,t)||w_{j,x}(0,t)|dt \\
&\leq M(s\lambda^2 + s\lambda^3 + s^2\lambda^3) \int_{-T}^{T} \varphi^3(0,t)|w(0,t)|^2 dt \\
&\quad + M\left(s^{3/2}\lambda^3 \int_{-T}^{T} \varphi(0,t)|w(0,t)|^2 dt + s^{1/2}\lambda \sum_{j=1}^{n} \int_{-T}^{T} \varphi(0,t)|w_{j,x}(0,t)|^2 dt \right)
\end{aligned}
$$

what gives

$$
\begin{aligned}
|B_{2,2}| &\leq M(s\lambda^2 + s\lambda^3 + s^2\lambda^3 + s^{3/2}\lambda^3) \int_{-T}^{T} \varphi^3(0,t)|w(0,t)|^2 dt \\
&\quad + Ms^{1/2}\lambda \sum_{j=1}^{n} \int_{-T}^{T} \varphi(0,t)|w_{j,x}(0,t)|^2 dt.
\end{aligned}
\tag{22}
$$

Finally with an integration by parts we have

$$
\begin{aligned}
|B_{2,3}(1)| &= \left| 4\beta s^2\lambda^2 \int_{-T}^{T} \varphi^2(0,t)tw(0,t)w_t(0,t)[\phi_x]_0 \right| \\
&= \left| -2\beta s^2\lambda^2[\phi_x]_0 \int_{-T}^{T} \varphi^2(0,t)w^2(0,t)dt + 8\beta^2 s^2\lambda^3[\phi_x]_0 \int_{-T}^{T} t^2\varphi^2(0,t)w^2(0,t)dt \right| \\
&\leq M(s^2\lambda^2 + s^2\lambda^3) \int_{-T}^{T} \varphi^3(0,t)|w(0,t)|^2 dt.
\end{aligned}
\tag{23}
$$

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As $\mu = B_{1,3} + B_{3,3} + B_{2,3}(2) = B - (B_{2,1} + B_{2,2} + B_{2,3}(1))$ and gathering (20), (21), (22) and (23), we then obtain for $s \geq 1$, $\lambda \geq 1$

$$\alpha s \lambda \sum_{j=1}^{n} \int_{-T}^{T} \varphi(0,t) |w_{j,x}(0,t)|^2 dt + \alpha s^3 \lambda^3 \int_{-T}^{T} \varphi^3(0,t) |w(0,t)|^2 dt$$

$$+ s\lambda [\phi_x]_0 \int_{-T}^{T} \varphi(0,t) |w_t(0,t)|^2 dt$$

$$\leq B + M s^2 \lambda^3 \int_{-T}^{T} \varphi^3(0,t) |w(0,t)|^2 dt + M s^{1/2} \lambda \sum_{j=1}^{n} \int_{-T}^{T} \varphi(0,t) |w_{j,x}(0,t)|^2 dt.$$

Consequently by taking s and λ sufficiently large we get $B \geq 0$. $\qquad \square$

Step 3. Interior terms.

Lemma 3 *There exists $s_0 > 0$, $\lambda_0 > 0$ and a constant $M(s_0, \lambda_0, \mathcal{R}, T, m, \beta, x_0, ..., x_n) > 0$ such that for all $s > s_0$, $\lambda > \lambda_0$*

$$M \sum_{k=1}^{8} A_k \geq s\lambda \int_{-T}^{T} \int_{\mathcal{R}} \varphi \left(w_t^2 + w_x^2 \right) dxdt + s^3 \lambda^3 \int_{-T}^{T} \int_{\mathcal{R}} \varphi^3 w^2 dxdt.$$

Proof. First of all, let $C \in (0,1)$ be such that

$$2(-2\beta + C(1+\beta)) = \delta_1 > 0$$
$$2(2 - C(1+\beta)) = \delta_2 > 0.$$

This means $\dfrac{2\beta}{\beta + 1} < C < \dfrac{2}{\beta + 1}$ and explains why β is chosen in $(0,1)$.

Therefore, one can observe that since $\phi_{tt} = -2\beta$ and $\phi_{xx} = 2$,

$$A_1 + A_2 = 2s\lambda \int_{-T}^{T} \int_{\mathcal{R}} \varphi w_t^2 (-2\beta + C(1+\beta)) \, dxdt = \delta_1 s\lambda \int_{-T}^{T} \int_{\mathcal{R}} \varphi w_t^2 \, dxdt$$

$$A_3 = 2s\lambda^2 \int_{-T}^{T} \int_{\mathcal{R}} \varphi (w_t \phi_t - w_x \phi_x)^2 \, dxdt \geq 0,$$

$$A_4 + A_5 = 2s\lambda \int_{-T}^{T} \int_{\mathcal{R}} \varphi w_x^2 (2 - C(1+\beta)) \, dxdt = \delta_2 s\lambda \int_{-T}^{T} \int_{\mathcal{R}} \varphi w_x^2 \, dxdt$$

and

$$A_6 + A_7 + A_8 = 2s^3 \lambda^3 \int_{-T}^{T} \int_{\mathcal{R}} \varphi^3 w^2 F_\lambda(\phi) \, dxdt$$

where

$$F_\lambda(\phi_j) = \lambda(\phi_{j,t}^2 - \phi_{j,x}^2)^2 + (\phi_{j,t}^2 \phi_{j,tt} + \phi_{j,x}^2 \phi_{j,xx}) + \frac{C}{2}(\phi_{j,tt} - \phi_{j,xx})(\phi_{j,t}^2 - \phi_{j,x}^2)$$

$$= 16\lambda \left((x - x_j)^2 - \beta^2 t^2 \right)^2 + 4 \left(C(\beta + 1) + 2\beta \right) \left((x - x_j)^2 - \beta^2 t^2 \right)$$
$$+ 8(1 - \beta)(x - x_j)^2.$$

As $8(1 - \beta)(x - x_j)^2 \geq K > 0$ for all $j \in [\![0, n]\!]$, then for λ sufficiently large, one has $F_\lambda(\phi_j) \geq \kappa > 0$ which ends the proof of Lemma 3.

$\qquad \square$

Combining equation (19), Lemma 2 and Lemma 3, we get

$$
\begin{aligned}
\langle P_1(w), P_2(w)\rangle_{L^2(\mathcal{R}\times(-T,T))} \geq &-s\lambda \sum_{j=0}^{n} \int_{-T}^{T} \varphi(l_j,t) w_{j,x}^2(l_j,t)\phi_{j,x}(l_j,t)dt \\
&+ Ms\lambda \int_{-T}^{T}\int_{\mathcal{R}} \varphi\left(w_t^2+w_x^2\right)\,dxdt \\
&+ Ms^3\lambda^3 \int_{-T}^{T}\int_{\mathcal{R}} \varphi^3 w^2\,dxdt \\
\geq &-2s\lambda \sum_{j=1}^{n} \int_{-T}^{T} \varphi(l_j,t) w_{j,x}^2(l_j,t)(l_j-x_j)dt \\
&+ Ms\lambda \int_{-T}^{T}\int_{\mathcal{R}} \varphi\left(w_t^2+w_x^2\right)\,dxdt \\
&+ Ms^3\lambda^3 \int_{-T}^{T}\int_{\mathcal{R}} \varphi^3 w^2\,dxdt
\end{aligned}
\tag{24}
$$

since $x_0 > l_0$ implies $\phi_{0,x}(l_0,t) = 2(l_0-x_0) < 0$.

Moreover

$$
\int_{-T}^{T}\int_{\mathcal{R}} |Pw-Rw|^2\,dxdt \leq 2\int_{-T}^{T}\int_{\mathcal{R}} |P(w)|^2\,dxdt + 2\int_{-T}^{T}\int_{\mathcal{R}} |Rw|^2\,dxdt
$$

$$
\leq \quad 2\int_{-T}^{T}\int_{\mathcal{R}} |Pw|^2\,dxdt + 4m\int_{-T}^{T}\int_{\mathcal{R}} |w|^2\,dxdt + Ms^2\lambda^2\int_{-T}^{T}\int_{\mathcal{R}} \varphi^2|w|^2\,dxdt.
$$

Thus, for s and λ sufficiently large, the two last terms of the right hand side can be absorbed by the dominant term $s^3\lambda^3\int_{-T}^{T}\int_{\mathcal{R}} \varphi^3 w^2\,dxdt$ of (24) and, we have proved the following result:

Theorem 3 *Assume that P is the conjugate operator defined by $Pw = e^{s\varphi}L(e^{-s\varphi}w)$ where the partial differential operator $L = \partial_{tt} - \partial_{xx} + p$ is defined on the star-shaped network $\mathcal{R}$ as in equation (1) (meaning that we assume homogeneous Dirichlet boundary conditions at external nodes and continuity and Kirschoff laws at the internal node), and $\varphi = e^{\lambda\phi}$ with ϕ satisfying Lemma 1 with $s, \lambda > 0$. Assume also that*

$$
\begin{aligned}
P_1w &= w_{tt} - w_{xx} + s^2\lambda^2\varphi^2(\phi_t^2-\phi_x^2)w, \\
P_2w &= Ks\lambda\varphi(\phi_{tt}-\phi_{xx})w - s\lambda^2\varphi(\phi_t^2-\phi_x^2)w - 2s\lambda\varphi(\phi_t w_t - \phi_x w_x),
\end{aligned}
$$

where $\dfrac{\beta-1}{\beta+1} < K < \dfrac{1-\beta}{\beta+1}$.

There exists $s_0 > 0$, $\lambda_0 > 0$ and a constant $M = M(s_0,\lambda_0,\mathcal{R},T,m,\beta,x_0,...,x_n) > 0$ such that:

$$
s\lambda \int_{-T}^{T}\int_{\mathcal{R}} \varphi\left(w_t^2+w_x^2\right)\,dxdt + s^3\lambda^3\int_{-T}^{T}\int_{\mathcal{R}} \varphi^3 w^2\,dxdt + \int_{-T}^{T}\int_{\mathcal{R}} |P_1w|^2 + \int_{-T}^{T}\int_{\mathcal{R}} |P_2w|^2\,dxdt
$$

$$
\leq M\int_{-T}^{T}\int_{\mathcal{R}} |Pw|^2\,dxdt + Ms\lambda \sum_{j=1}^{n} \int_{-T}^{T} w_{j,x}^2(l_j)\,dt,
\tag{25}
$$

for all $s > s_0$, $\lambda > \lambda_0$ and w satisfying

$$
\begin{cases}
Pw \in L^2(\mathcal{R}\times(-T,T)), \\
w \in H^1(-T,T;H_0^1(\mathcal{R})), \\
w(\pm T) = \partial_t w(\pm T) = 0 \text{ in } \mathcal{R}.
\end{cases}
$$

Step 4. Return to the variable v.

Using $\varphi = e^{\lambda\phi} \geq 1$ and the fact that $w = ve^{s\varphi}$ gives for all $x \in \mathcal{R}$ and $t \in (-T, T)$

$$e^{2s\varphi}v_t^2 \leq 2w_t^2 + 2s^2\lambda^2\phi_t^2\varphi^2 w^2,$$
$$e^{2s\varphi}v_x^2 \leq 2w_x^2 + 2s^2\lambda^2\phi_x^2\varphi^2 w^2,$$
$$e^{2s\varphi(l_j)}v_{j,x}^2(l_j) = w_{j,x}^2(l_j) \quad \forall j \in [\![0, n]\!],$$

we easily obtain (14) and the proof of Theorem 2 is complete.

Remark 4 *We can extend this result to the case of tree-shaped networks with Neumann boundary measurements at all but one external vertices (that we can assume to be the root of the tree). Indeed, in this case, the integrations by parts give the "internal" boundary terms of each edge, and then B defined above is the sum at each interior node of the sum of the trace term. More precisely, we obtain instead of one matrix A_ϕ defined in Lemma 1, several matrices of the same kind, corresponding to each interior node. By choosing appropriate coefficients x_j for the weights ϕ_j (which is possible), we prove that these matrices are definite positive. The proof of the Carleman estimate does not change except for these keypoints. The difficulty here is just to use clear and understandable notations for vertices, edges, to orient the graph...*

However for general networks which may contain closed circuits, the choice of appropriate weights ϕ_j (in particular appropriate x_j) is not clear and some difficulties appear. It is still an open problem regarding the carleman estimate and one could also know that the exact controllability of such general networks is also an open question.

3 Stability of the inverse problem

Before giving the proof of Theorem 1, we will begin this section by a stability theorem for the following inverse source problem. Let y be the solution of

$$\begin{cases} y_{tt} - y_{xx} + p(x)y = f(x)R(x,t), & \text{in } \mathcal{R} \times (0, T), \\ y_j(l_j, t) = 0, & \text{in } (0, T), \forall j \in [\![0, n]\!], \\ y_j(0, t) = y_i(0, t), & \text{in } (0, T), \forall i, j \in [\![0, n]\!], \\ [y_x(t)]_0 := \sum_{j=0..n} y_{j,x}(0, t) = 0, & \text{in } (0, T), \\ y(0) = 0, \quad y_t(0) = 0, & \text{in } \mathcal{R}. \end{cases} \qquad (26)$$

Statement of the source inverse problem: Is it possible to retrieve the time independent source $f = f(x)$, for $x \in \mathcal{R}$ from the n measurements $y_{j,x}(l_j, t), j \in [\![1, n]\!]$ on $(0, T)$ where y is the solution to (26) ?

The following answer is obtained using the global Carleman estimate given in Section 2.

Theorem 4 *Assume that $f \in L^2(\mathcal{R})$, $R \in H^1(0, T; L^\infty(\mathcal{R}))$ with*

$$||R_t||_{L^2(0,T;L^\infty(\mathcal{R}))} \leq r$$

and

$$|R(x, 0)| \geq r_0 \geq 0, \ ae \ \text{in } \overline{\mathcal{R}}. \qquad (27)$$

Then there exists a constant $C = C(T, \mathcal{R}, m, r, ...) > 0$ such that $\forall p \in L^\infty_{\leq m}$:

$$||f||_{L^2(\mathcal{R})} \leq C \sum_{j=1}^{n} ||y_{j,x}(l_j, t)||_{H^1(0,T)}.$$

Proof. We will apply the Carleman estimate given in Theorem 3 by (25) to $w = \chi \partial_t y$ where χ is a cutoff function to be detailed later. We divide the proof in several steps.

Step 1. Let us first work on the equation satisfied by $z = \partial_t y$:

$$\begin{cases} z_{tt} - z_{xx} + p(x)z = f(x)R_t(x,t), & \text{in } \mathcal{R} \times (0,T), \\ z_j(l_j, t) = 0, & \text{in } (0,T), \forall j \in [\![0,n]\!], \\ z_j(0,t) = z_i(0,t), & \text{in } (0,T), \forall i,j \in [\![0,n]\!], \\ [z_x(t)]_0 = 0, & \text{in } (0,T), \\ z(0) = 0, \quad z_t(0) = f(x)R(x,0), & \text{in } \mathcal{R}. \end{cases} \qquad (28)$$

We will need some energy estimates for the solution of this equation, that motivates the recalling of the following classical result.

Lemma 4 *Let $\mathcal{R}$ be a general network and assume that $p \in L^\infty(\mathcal{R})$ with $\|p\|_{L^\infty(\mathcal{R})} \leq m$, $g \in L^1(0,T; L^2(\mathcal{R}))$, $u^0 \in H_0^1(\mathcal{R})$ and $u^1 \in L^2(\mathcal{R})$. We consider the wave equation*

$$\begin{cases} u_{tt} - u_{xx} + p(x)u = g(x,t), & \text{in } \mathcal{R} \times (0,T), \\ u_j(l_j, t) = 0, & \text{in } (0,T), \forall j \in [\![0,n]\!], \\ u_j(0,t) = u_i(0,t), & \text{in } (0,T), \forall i,j \in [\![0,n]\!], \\ [u_x(t)]_0 = 0, & \text{in } (0,T), \\ u(0) = u^0, \quad u_t(0) = u^1, & \text{in } \mathcal{R}. \end{cases} \qquad (29)$$

Therefore, the Cauchy problem is well-posed and equation (29) admits a unique weak solution $u \in C([0,T], H_0^1(\mathcal{R})) \cap C^1([0,T], L^2(\mathcal{R}))$ and there exists a constant $C = C(\mathcal{R}, T, m) > 0$ such that for all $t \in (0,T)$, the energy $E_u(t) = \|u_t(t)\|_{L^2(\mathcal{R})}^2 + \|u_x(t)\|_{L^2(\mathcal{R})}^2$ of the system satisfy

$$E_u(t) \leq C \left(\|u^0\|_{H_0^1(\mathcal{R})}^2 + \|u^1\|_{L^2(\mathcal{R})}^2 + \|g\|_{L^1(0,T,L^2(\mathcal{R}))}^2 \right) \qquad (30)$$

and we also have the following trace estimate

$$\sum_{j=0}^{n} \|u_{j,x}(l_j, t)\|_{L^2(0,T)}^2 \leq C \left(\|u^0\|_{H_0^1(\mathcal{R})}^2 + \|u^1\|_{L^2(\mathcal{R})}^2 + \|g\|_{L^1(0,T,L^2(\mathcal{R}))}^2 \right). \qquad (31)$$

We refer to [21] (Chapter 3) for the proof of the existence and uniqueness of solution to equation (29). Estimate (30) can be also deduced from reference [21]. It is a classical result which can be formally obtained by multiplying (29) by $u_{j,t}$, summing up for $j \in \{0, ..., n\}$ the integral of this equality on $(0,T) \times (0, l_j)$ and using some integrations by parts. Concerning estimate (31), we refer to [20] (Chapter 1). This estimate is a hidden regularity result which can be obtained by multipliers technique. Formally, it comes from the multiplication of (29) by $m(x)u_{j,x}$, where $m \in C^1(\bar{\mathcal{R}})$ with $m(0) = 0$ and $m_j(l_j) = 1$, summing up for $j \in \{0, ..., n\}$ the integral of this equality on $(0,T) \times (0, l_j)$ and using some integrations by parts.

We can apply this result to equation (28) since $f \in L^2(\mathcal{R})$ and $R \in H^1(0,T; L^\infty(\mathcal{R}))$ and denoting the corresponding energy by

$$E_z(t) = \|z(t)\|_{H_0^1(\mathcal{R})}^2 + \|z_t(t)\|_{L^2(\mathcal{R})}^2$$

we get for all $t \in (0,T)$:

$$\begin{aligned} E_z(t) &\leq CE_z(0) + C\|fR_t\|_{L^1(0,T;L^2(\mathcal{R}))}^2 \\ &\leq C\|f\|_{L^2(\mathcal{R})}^2 \left(\|R(0)\|_{L^\infty(\mathcal{R})}^2 + \|R\|_{H^1(0,T;L^\infty(\mathcal{R}))}^2 \right), \end{aligned} \qquad (32)$$

that will be usefull later, and

$$\sum_{j=0}^{n} \|z_{j,x}(l_j, t)\|_{L^2(0,T)}^2 \leq C\|f\|_{L^2(\mathcal{R})}^2 \left(\|R(0)\|_{L^\infty(\mathcal{R})}^2 + \|R\|_{H^1(0,T;L^\infty(\mathcal{R}))}^2 \right),$$

which implies that for all $j \in [\![0,n]\!]$, $z_{j,x}(l_j, t) \in L^2(0,T)$.

Remark 5 *One should notice that this last estimate gives finally $y_{j,x}(l_j, t) \in H^1(0, T)$ for all $j \in [\![0, n]\!]$, giving at the same time a meaning to the measurement we make in our inverse problem. At the end of the proof of Theorem 4, we could indeed write the two-sided estimate*

$$C^{-1}||f||_{L^2(\mathcal{R})} \leq C \sum_{j=1}^{n} ||y_{j,x}(l_j, t)||_{H^1(0,T)} \leq C||f||_{L^2(\mathcal{R})}.$$

Step 2. Let us now extend the problem (28) on $(-T, T)$, setting $z(x, t) = z(x, -t)$ for all $(x, t) \in \mathcal{R} \times (-T, 0)$. We also extend R_t on an even way and keep the same notations for the new problem. Therefore, we have

$$z \in C([-T, T], H_0^1(\mathcal{R})) \cap C^1([-T, T], L^2(\mathcal{R})) \text{ and } R_t \in L^2(-T, T; L^\infty(\mathcal{R})).$$

Moreover, from the properties of the weight function ϕ that were given in Lemma 1, and using the parameter η introduced there, we define the cut-off function $\chi \in C^\infty(\mathbb{R}; [0, 1])$ such that

$$\begin{cases} \chi(\pm T) = \chi'(\pm T) = 0 \\ \chi(t) = 1 \qquad \qquad \forall t \in [-T + \eta, T - \eta]. \end{cases} \tag{33}$$

Therefore, we set $v = \chi z$ that satisfies the following equation:

$$\begin{cases} v_{tt} - v_{xx} + p(x)v = \chi f R_t + \chi'' z + 2\chi' z_t, & \text{in } \mathcal{R} \times (-T, T), \\ v_j(l_j, t) = 0, & \text{in } (-T, T), \forall j \in [\![0, n]\!], \\ v_j(0, t) = v_i(0, t), & \text{in } (-T, T), \forall i, j \in [\![0, n]\!], \\ [v_x(t)]_0 = 0, & \text{in } (-T, T), \\ v(0) = 0, \quad v_t(0) = f(x)R(x, 0), & \text{in } \mathcal{R}, \\ v(\pm T) = 0, \quad v_t(\pm T) = 0, & \text{in } \mathcal{R}. \end{cases} \tag{34}$$

Henceforth, $M > 0$ will correspond to a generic constant depending on $s_0, \varepsilon, \lambda, T, \mathcal{R}, x_0, ..., x_n, \beta, \chi, r, r_0,$ and η but independent of $s > s_0$.

Step 3. We use now the same notations as in Section 2 for the proof of the Carleman estimate. For v solution of (34), we set

$$w = e^{s\varphi}v, \quad P_1 w = w_{tt} - w_{xx} + s^2 \lambda^2 \varphi^2 (\phi_t^2 - \phi_x^2)w,$$

and we then have for all $x \in \mathcal{R}$, $w(x, \pm T) = \partial_t w(x, \pm T) = 0$ and $w(x, 0) = 0$.
Inspired from an idea of O. Yu. Imanuvilov and M. Yamamoto [13], we consider the integral

$$\int_{-T}^{0} \int_{\mathcal{R}} P_1 w(x, t).w_t(x, t)\, dxdt.$$

On the one hand, using the properties of w, we can make the following calculation:

$$\int_{-T}^{0} \int_{\mathcal{R}} P_1 w.w_t\, dxdt$$

$$= \int_{-T}^{0} \int_{\mathcal{R}} \left(w_{tt} - w_{xx} + s^2\lambda^2\varphi^2(\phi_t^2 - \phi_x^2)w\right).w_t\, dxdt$$

$$= \frac{1}{2}\int_{\mathcal{R}} |w_t(0)|^2\, dx - \frac{s^2\lambda^2}{2}\int_{-T}^{0}\int_{\mathcal{R}} w^2 \left(\varphi^2(\phi_t^2 - \phi_x^2)\right)_t\, dxdt$$

$$= \frac{1}{2}\int_{\mathcal{R}} |w_t(0)|^2\, dx - 2s^2\lambda^2 \sum_{j=0}^{n}\int_{-T}^{0}\int_{0}^{l_j} w_j^2 \left(\varphi_j^2(\beta^2 t^2 - |x - x_j|^2)\right)_t\, dxdt$$

$$= \frac{1}{2}\int_{\mathcal{R}} e^{2s\varphi(0)}|fR(0)|^2\, dx + 8s^2\lambda^3 \sum_{j=0}^{n}\int_{-T}^{0}\int_{0}^{l_j} w_j^2\varphi_j^2\beta t(\beta^2 t^2 - |x - x_j|^2)\, dxdt$$

$$- 4s^2\lambda^2 \int_{-T}^{0}\int_{\mathcal{R}} w^2\varphi^2\beta^2 t\, dxdt.$$

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On the other hand, from this equality and a Cauchy-Schwarz estimate,

$$\int_{\mathcal{R}} e^{2s\varphi(0)} |fR(0)|^2 \, dx$$

$$= \; 2 \int_{-T}^{0} \int_{\mathcal{R}} P_1 w . w_t \, dxdt + 8s^2\lambda^2 \int_{-T}^{0} \int_{\mathcal{R}} w^2 \varphi^2 \beta^2 t \, dxdt$$

$$-16s^2\lambda^3 \sum_{j=0}^{n} \int_{-T}^{0} \int_{0}^{l_j} w_j^2 \varphi_j^2 \beta t (\beta^2 t^2 - |x - x_j|^2) \, dxdt$$

$$\leq \; 2 \sum_{j=0}^{n} \left(\int_{-T}^{0} \int_{0}^{l_j} \left| P_1^j w_j \right|^2 \, dxdt \right)^{\frac{1}{2}} \left(\int_{-T}^{0} \int_{0}^{l_j} |w_{j,t}|^2 \, dxdt \right)^{\frac{1}{2}}$$

$$+ M s^2 \lambda^3 \sum_{j=0}^{n} \int_{-T}^{0} \int_{0}^{l_j} w_j^2 \varphi_j^2 \left| \beta t (\beta^2 t^2 - |x - x_j|^2) \right| \, dxdt. \tag{35}$$

Using now the "intermediate" Carleman estimate (25) of Theorem 3 for a fixed $\lambda > \lambda_0$ that we omit, choosing s large enough to absorb the last term in the right hand side (35), we obtain

$$\int_{\mathcal{R}} e^{2s\varphi(0)} |fR(0)|^2 \, dx$$

$$\leq \; \frac{2}{\sqrt{s}} \sum_{j=0}^{n} \left(\int_{-T}^{T} \int_{0}^{l_j} \left| P_1^j w_j \right|^2 \, dxdt \right)^{\frac{1}{2}} \left(s \int_{-T}^{T} \int_{0}^{l_j} w_{j,t}^2 \, dxdt \right)^{\frac{1}{2}}$$

$$+ M s^2 \lambda^3 \sum_{j=0}^{n} \int_{-T}^{T} \int_{0}^{l_j} w_j^2 \varphi_j^2 \, dxdt$$

$$\leq \; \frac{M}{\sqrt{s}} \int_{-T}^{T} \int_{\mathcal{R}} |Pw|^2 \, dxdt + M\sqrt{s} \sum_{j=1}^{n} \int_{-T}^{T} w_{j,x}^2(l_j) \, dt$$

$$\leq \; \frac{M}{\sqrt{s}} \int_{-T}^{T} \int_{\mathcal{R}} e^{2s\varphi} |Lv|^2 \, dxdt + M\sqrt{s} \sum_{j=1}^{n} \int_{-T}^{T} v_{j,x}^2(l_j) \, dt, \tag{36}$$

the φ_j's being bounded from above and below.

One should now notice that from (9) and (11) of Lemma 1, we can deduce that

$$\varphi(x,t) \leq e^{\lambda C_0} < e^{\lambda \phi(x,0)} = \varphi(x,0), \quad \forall (x,t) \in \mathcal{R} \times [-T, -T+\eta] \cup [T-\eta, T], \tag{37}$$

$$\varphi(x,t) \leq \varphi(x,0), \quad \forall (x,t) \in \mathcal{R} \times [-T, T]. \tag{38}$$

From equation (34) and the properties (33) of the cut-off function χ and that of the weight φ given in (37) and (38), one gets that

$$\int_{-T}^{T} \int_{\mathcal{R}} e^{2s\varphi} |Lv|^2 \, dxdt$$

$$\leq \; M \int_{-T}^{T} \int_{\mathcal{R}} e^{2s\varphi} |\chi f R_t|^2 \, dxdt + M \int_{-T}^{T} \int_{\mathcal{R}} e^{2s\varphi} \left(|\chi' z_t|^2 + |\chi'' z|^2 \right) \, dxdt$$

$$\leq \; M \int_{-T}^{T} \int_{\mathcal{R}} e^{2s\varphi} |f|^2 |R_t|^2 \, dxdt + M \left(\int_{-T}^{-T+\eta} + \int_{T-\eta}^{T} \right) \int_{\mathcal{R}} e^{2s\varphi} \left(z_t^2 + z^2 \right) \, dxdt$$

$$\leq \; M \int_{-T}^{T} \int_{\mathcal{R}} e^{2s\varphi(0)} |f|^2 |R_t|^2 \, dxdt + M \left(\int_{-T}^{-T+\eta} + \int_{T-\eta}^{T} \right) \int_{\mathcal{R}} e^{2se^{\lambda C_0}} \left(z_t^2 + z^2 \right) \, dxdt$$

$$\leq \; M \|R_t\|_{L^2(-T,T;L^\infty(\mathcal{R}))} \int_{\mathcal{R}} e^{2s\varphi(0)} |f|^2 \, dx$$

$$+ M e^{2se^{\lambda C_0}} \left(\int_{-T}^{-T+\eta} + \int_{T-\eta}^{T} \right) \int_{\mathcal{R}} \left(z_t^2 + z^2 \right) \, dxdt.$$

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Using the energy estimate given in (32), and again the property (37) of the weight φ, one gets

$$
\begin{aligned}
\int_{-T}^{T} \int_{\mathcal{R}} e^{2s\varphi} |Lv|^2 \, dxdt \quad &\leq \quad M \int_{\mathcal{R}} e^{2s\varphi(0)} |f|^2 \, dx + M e^{2se^{\lambda C_0}} \left(\int_{-T}^{-T+\eta} + \int_{T-\eta}^{T} \right) E_z(t) dt \\
&\leq \quad M \int_{\mathcal{R}} e^{2s\varphi(0)} |f|^2 \, dx \\
&\quad + M\eta \left(||R(0)||_{L^\infty(\mathcal{R})}^2 + ||R||_{H^1(0,T;L^\infty(\mathcal{R}))}^2 \right) e^{2se^{\lambda C_0}} \int_{\mathcal{R}} |f|^2 \, dx \\
&\leq \quad M \int_{\mathcal{R}} e^{2s\varphi(0)} |f|^2 \, dx.
\end{aligned}
$$

Gathering this last estimate with (36), we have proved

$$
\int_{\mathcal{R}} e^{2s\varphi(0)} |fR(0)|^2 \, dx \leq \frac{M}{\sqrt{s}} \int_{\mathcal{R}} e^{2s\varphi(0)} |f|^2 \, dx + M\sqrt{s} \sum_{j=1}^{n} \int_{-T}^{T} v_{j,x}^2(l_j) \, dt.
$$

Therefore, the assumption (27) made on R allow to obtain

$$
\int_{\mathcal{R}} e^{2s\varphi(0)} |f|^2 \, dx \leq \frac{M}{\sqrt{s}} \int_{\mathcal{R}} e^{2s\varphi(0)} |f|^2 \, dx + M\sqrt{s} \sum_{j=1}^{n} \int_{-T}^{T} v_{j,x}^2(l_j) \, dt,
$$

and the choice of s large enough gives

$$
\begin{aligned}
\int_{\mathcal{R}} e^{2s\varphi(0)} |f|^2 \, dx \quad &\leq \quad M\sqrt{s} \sum_{j=1}^{n} \int_{-T}^{T} v_{j,x}^2(l_j) \, dt \\
&\leq \quad M\sqrt{s} \sum_{j=1}^{n} \int_{-T}^{T} y_{j,tx}^2(l_j) \, dt.
\end{aligned}
$$

The proof of Theorem 4 is then complete. $\qquad \square$

We will end this section by the proof of Theorem 1 which is a direct consequence of Theorem 4. Indeed, if we set $\tilde{y} = u[q] - u[p]$, where $u[p]$ is solution of (1), $f = p - q$ and $R = u[p]$, then $\tilde{y}$ is the solution of

$$
\begin{cases}
\tilde{y}_{tt} - \tilde{y}_{xx} + q(x)\tilde{y} = f(x)R(x,t), & \text{in } \mathcal{R} \times (0,T), \\
\tilde{y}_j(l_j, t) = 0, & \text{in } (0,T), \, \forall j \in [\![0, n]\!], \\
\tilde{y}_j(0, t) = \tilde{y}_i(0, t), & \text{in } (0,T), \, \forall i, j \in [\![0, n]\!], \\
[\tilde{y}_x(t)]_0 = 0, & \text{in } (0,T), \\
\tilde{y}(0) = 0, \quad \tilde{y}_t(0) = 0, & \text{in } \mathcal{R}.
\end{cases} \tag{39}
$$

where $q = f + p \in L_{\leq m}^\infty(\mathcal{R})$. The key point is that in the proof of Theorem 4, all the constants M depend on the bound m of the L^∞-norm of the potential as stated in Theorem 2. Thus, with $q \in L_{\leq m}^\infty(\mathcal{R})$, we are actually, with equation (39), in a situation similar to the linear inverse problem related to equation (26) and we then obtain the desired result.

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