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Generalized Multifractional Brownian Motion: Definition and Preliminary Results

Antoine Ayache, Jacques Lévy Véhel

Projet Fractales, INRIA Rocquencourt,
Domaine de Voluceau, Rocquencourt,
BP 105, 78153 Le Chesnay Cedex - FRANCE
ayache@ceremade.dauphine.fr
Jacques.Levy_Vehel@.inria.fr

Abstract. The Multifractional Brownian Motion (MBM) is a generalization of the well known Fractional Brownian Motion. One of the main reasons that makes the MBM interesting for modelization, is that one can prescribe its regularity: given any Hölder function $H(t)$, with values in $]0,1[$, one can construct an MBM admitting at any t_0, a Hölder exponent equal to $H(t_0)$. However, the continuity of the function $H(t)$ is sometimes undesirable, since it restricts the field of application. In this work we define a gaussian process, called the Generalized Multifractional Brownian Motion (GMBM) that extends the MBM. This process will also depend on a functional parameter $H(t)$ that belongs to a set $\mathcal{H}$, but $\mathcal{H}$ will be much more larger than the space of Hölder functions.

1 Introduction

It is classical that there exists a unique gaussian process, self-similar of order H, vanishing at the origin and having stationary increments. This continuous, centered gaussian process is called the Fractional Brownian Motion (FBM). It has been introduced in 1968 by Mandelbrot and Van Ness [6]. This process can be defined as the stochastic integral, for $t \in \mathbb{R}$,

$$B_H(t) = \int_{-\infty}^{+\infty} \frac{e^{it\xi} - 1}{|\xi|^{H+1/2}} dW(\xi), \tag{1}$$

where $dW(\xi)$ is a Brownian measure (a good choice of $dW(\xi)$ leads to real valued FBM's). It is worthwhile noting that $\{B_{1/2}(t)\}$ is the well known Brownian Motion.

One of the main interests of the FBM in modeling is that one can prescribe the regularity of its paths. More precisely, the Hölder exponent of the FBM at any point t_0 is almost surely equal to H; for a random process $\{X(t)\}$ this exponent $\alpha(t_0)$ is defined as follows,

$$\alpha(t_0) = \sup \left\{ \alpha, \ \lim_{h \longrightarrow 0} \frac{X(t_0+h) - X(t_0)}{|h|^{\alpha}} = 0 \right\}. \tag{2}$$

However, the fact that the parameter H does not depend on t_0, is sometimes undesirable since it restricts the field of application. Some phenomena do not admit a constant Hölder exponent : for instance, the use of FBM for synthesizing artificial mountains does not allow to take into account erosion phenomena. The variation of the regularity may even contain an essential part of the signal information. For instance the variation of the Hölder exponent has been used for images segmentation [5].

An extension of the FBM, has been proposed independently by Lévy Véhel and Peltier [8] and by Benassi, Jaffard and Roux [3]. It is called the Multifractional Brownian Motion (MBM). The MBM can be defined by (1), except that the parameter H, is replaced by a Hölder function $H(t)$, with values in $]0,1[$. This process shares many propreties with the FBM; for instance, at any point t_0, the Hölder exponent of the MBM is, almost surely, equal to $H(t_0)$ and the MBM is asymptotically locally self-similar of order $H(t_0)$. In addition the local Haussdorff and Box dimensions of the graph of the MBM at t_0, are both almost surely equal to $2 - H(t_0)$.

However, the continuity of the function $H(t)$ is sometimes undesirable, since it restricts the field of application. In some contexts (for example financial crash or image segmentation), it may seem more realistic that the Hölder exponent be a discontinuous function. Thus, it would be useful to construct a gaussian process extending the MBM that admits a discontinuous Hölder exponent, and this will be the aim of our paper. The method we use is inspired from the study of the Generalized Weierstrass function in [4].

The remainder of this article is organized as follows. In section 2, we define the Generalized Multifractional Brownian Motion (GMBM) of parameters the function $H(t) \in \mathcal{H}$ and the real $\lambda > 1$ where the space $\mathcal{H}$ is much more larger than the space of Hölder functions (see Definitions 1 and 2). In section 3 and in section 4, we will show that the GMBM shares some essential properties with the MBM : at any point t_0, the Hölder exponent of the GMBM is almost surely equal to $H(t_0)$ and the GMBM is asymptotically locally self-similar of order $H(t_0)$.

2 Definition of the GMBM

We have seen that the MBM depends on a funtional parameter that belongs to the space of Hölder functions. The GMBM will also depend on a functional parameter that belongs to a functional space $\mathcal{H}$. Let us first define $\mathcal{H}$.
We will say that a function $h : \mathbb{R} \longrightarrow \mathbb{R}$ is *a* (β, c)*-Hölder function* where $\beta > 0$ and $c > 0$ if and only if for all t_1, t_2 satisfying $|t_1 - t_2| < 1$, we have,

$$|h(t_1) - h(t_2)| \leq c|t_1 - t_2|^{\beta}.$$

Definition 1. *$\mathcal{H}$ will be the set of the functions $H(t)$ defined on $\mathbb{R}$, such that $H(t) = \liminf_{n \longrightarrow \infty} H_n(t)$ where $(H_n(t))_{n \in \mathbb{N}}$ is a sequence of (β, c_n)-Hölder functions with values in $[a, b] \subset]0, 1[$ that satisfy,*

(a) for all $\epsilon > 0$ and t_0, there exist $n_0 = n_0(t_0, \epsilon)$ and $h_0 = h_0(t_0, \epsilon) > 0$ such that, for all $n \geq n_0$ and $|h| \leq h_0$ we have $H_n(t_0 + h) \geq H(t_0) - \epsilon$,
(b) for all t, $H(t) < \beta$ and $c_n = O(n)$.

Definition 2. *Let $H(t) = \liminf_{n \longrightarrow \infty} H_n(t)$ be a function in $\mathcal{H}$ and let $\lambda > 1$. The Generalized Multifractional Brownian Motion (GMBM) of parameters the function $H(t)$ and the real λ is the gaussian process $\{Y_{H,\lambda}(t)\}_{t \in \mathbb{R}}$ such that for all $t \in \mathbb{R}$,*

$$Y_{H,\lambda}(t) = \sum_{n=0}^{\infty} \int_{D_n} \frac{e^{it\xi} - 1}{|\xi|^{H_n(t)+1/2}} dW(\xi), \tag{3}$$

where $D_0 = \{|\xi| < 1\}$ and for all $n \geq 1$, $D_n = \{\lambda^{n-1} \leq |\xi| < \lambda^n\}$.

Note that the process (3) not only depends on the function $(H(t))$ but also on the sequence $(H_n(t))$.

It is clear that the GMBM is an extension of the MBM. Indeed when in (3) all the functions $H_n(t)$ are equal to a Hölder function $H(t)$, then the process $\{Y_{H,\lambda}(t)\}_{t \in \mathbb{R}}$ is the MBM of functional parameter $H(t)$.

It is worthwhile noting that when one merely replaces in (1) the parameter H by a discontinuous function $H(t)$ one obtains a discontinuous process. Since our main aim is to control the Hölder exponent at each point, a definition such as (3) is needed.

One can also remark that the set $\mathcal{H}$ contains for instance all the functions that are both lower-semi-continuous (l.s.c.) and piecewise continuous (see Proposition 1); more interestingly, some very irregular functions, for instance functions of the type $b+(a-b)\chi_F(t)$ where $\chi_F(t)$ is the characteristic function of an arbitrary closed set F, as for example the Cantor set, belong also to $\mathcal{H}$. (see Proposition 2). At last, one can note that all the functions of $\mathcal{H}$ are l.s.c.

The following propositions give some insights on the set $\mathcal{H}$.

Proposition 1. *Let $\{t_l\}_{l \in \mathbb{Z}}$ be an increasing sequence and let $H : \mathbb{R} \longrightarrow [a, b] \subset]0, 1[$ be a function such that for all t,*

$$H(t) = \sum_{l \in \mathbb{Z}} (a_l(t) - a)\chi_{]t_l, t_{l+1}[}(t) + (b_l - a) \sum_{l \in \mathbb{Z}} \chi_{\{t_l\}}(t) + a, \tag{4}$$

where, for all l, $a_l(t)$ is a continuous function on $[t_l, t_{l+1}]$ and $b_l \leq \min\{a_{l-1}(t_l), a_l(t_l)\}$. Then, the function $H(t)$ belongs to the set $\mathcal{H}$.

To prove Proposition 1 we need the following Lemma.

Lemma 1. *Let* $a : [u, v] \longrightarrow [c, d]$ *be a continuous function. For all* $\epsilon > 0$, *there exist a real* $\eta_\epsilon > 0$ *and a* C^1 *function* $a_\epsilon : [u, v] \longrightarrow [c, d]$ *satisfying the following properties:*

(i) for all $t \in [u, u + \eta_\epsilon]$, $a_\epsilon(t) = a(u)$ *and for all* $t \in [v - \eta_\epsilon, v]$, $a_\epsilon(t) = a(v)$,
(ii) for all $t \in [u, v]$, $|a_\epsilon(t) - a(t)| \leq \epsilon$,
(iii) for all t, $|a'_\epsilon(t)| \leq e/\eta_\epsilon$ *(we can take* $e = 12$*).*

Proof (of Lemma 1) Let $\epsilon > 0$, the function $a(t)$ being uniformly continuous on $[u, v]$, there exists a real α, $0 < \alpha < (v-u)/16$, such that for all $t_1, t_2 \in [u, v]$, $|t_1 - t_2| \leq \alpha$, we have $|a(t_1) - a(t_2)| \leq \epsilon/2$.

Let $x_p = u + (p + 1)\alpha/4$, $p_m = \max\{p \in \mathbb{N},\ x_p \leq \frac{u+v}{2}\}$ and let $y_q = v - (q + 1)\alpha/4$, $q_m = \max\{q \in \mathbb{N},\ y_q \geq \frac{u+v}{2}\}$. For all x, $\beta \in [0, 1]$, we set

$$P_\beta(x) = -2\beta x^3 + 3\beta x^2. \tag{5}$$

We have, for all x and for all β, $|P'_\beta(x)| \leq 3/2$.

The function $a_\epsilon(t)$ will be defined as follows.
For all $t \in [u, x_0 - \alpha/8]$

$$a_\epsilon(t) = a(u),$$

for all $t \in [y_0 + \alpha/8, v]$

$$a_\epsilon(t) = a(v),$$

for all $t \in [x_0 - \alpha/8, x_0]$

$$a_\epsilon(t) = \begin{cases} P_{(a(x_0)-a(u))}\left(\frac{8}{\alpha}[t - (x_0 - \frac{\alpha}{8})]\right) + a(u) \text{ if } a(x_0) \geq a(u) \\ \\ P_{(a(u)-a(x_0))}\left(1 - \frac{8}{\alpha}[t - (x_0 - \frac{\alpha}{8})]\right) + a(x_0) \text{ else,} \end{cases}$$

$a_\epsilon(t)$ will be defined similarly on $[y_0, y_0 + \alpha/8]$,
for all $p \in \{0, \ldots, p_m - 2\}$ and for $t \in [x_p, x_{p+1}]$

$$a_\epsilon(t) = \begin{cases} P_{(a(x_{p+1})-a(x_p))}\left(\frac{4}{\alpha}[t - x_p]\right) + a(x_p) \text{ if } a(x_{p+1}) \geq a(x_p) \\ \\ P_{(a(x_p)-a(x_{p+1}))}\left(1 - \frac{4}{\alpha}[t - x_p]\right) + a(x_{p+1}) \text{ else,} \end{cases}$$

$a_\epsilon(t)$ will be defined similarly on $[y_{k+1}, y_k]$ for all $q \in \{0, \ldots, q_m - 2\}$,
for all $t \in [x_{p_m-1}, y_{q_m-1}]$

$$a_\epsilon(t) = \begin{cases} P_{(a(y_{q_m-1})-a(x_{p_m-1}))}\left(\frac{1}{y_{q_m-1}-x_{p_m-1}}[t - x_{p_m-1}]\right) + a(x_{p_m-1}) \\ \text{if } a(x_{p_m-1}) \leq a(y_{q_m-1}) \\ \\ P_{(a(x_{p_m-1})-a(y_{q_m-1}))}\left(1 - \frac{1}{y_{q_m-1}-x_{p_m-1}}[t - x_{p_m-1}]\right) + a(x_{q_m-1}) \\ \text{else.} \end{cases}$$

■

Proof (of Proposition 1). It follows from the previous Lemma, that for all $l \in \mathbb{Z}$ and $m \geq 1$, there exist a real $\eta_{l,m} > 0$ and a C^1 function $a_{l,m} : [t_l, t_{l+1}] \longrightarrow [a,b]$ such that,
- for all $t \in [t_l, t_l + \eta_{l,m}]$, $a_{l,m}(t) = a_l(t_l)$ and for all $t \in [t_{l+1} - \eta_{l,m}, t_{l+1}]$ $a_{l,m}(t) = a_l(t_{l+1})$,
- for all $t \in [t_l, t_{l+1}]$, $|a_{l,m}(t) - a_l(t)| \leq 1/m$,
- for all $t \in [t_l, t_{l+1}]$, $|a'_{l,m}(t)| \leq c/\eta_{l,m}$.

One can suppose that for each l, $(\eta_{l,m})_{m \in \mathbb{N}^*}$ is a decreasing sequence and that $\lim_{m \longrightarrow \infty} \eta_{l,m} = 0$.

For all $k \geq 1$, the function $H_k(t)$ will be defined as follows.

$$H_k(t) = \sum_{l \in I_k} (a_{l,m(l,k)}(t) - a)\chi_{[t_l+1/k, t_{l+1}-1/k]}(t) + \sum_{l \in I_k} (\theta_{l,k}(t) - a)\chi_{[t_l-1/k, t_l+1/k]}(t) + \sum_{l \in J_k} (\gamma_{l,k}(t) - a)\chi_{[t_l, t_l+1/k]}(t) + \sum_{l \in L_k} (\delta_{l,k}(t) - a)\chi_{[t_l-1/k, t_l]}(t) + a, \tag{6}$$

where

$$I_k = \{l \in \mathbb{Z},\ (\exists\, m\ \eta_{l,m} > 8/k) \text{ and } (\exists\, m'\ \eta_{l-1,m'} > 8/k)\},$$

$$J_k = \{l \in \mathbb{Z},\ l \notin I_k \text{ and } l+1 \in I_k\},$$

and $L_k = \{l \in \mathbb{Z},\ l \notin I_k \text{ and } l-1 \in I_k\}$,
note that all these sets are finite;
for all $l \in I_k$, and $t \in [t_l - 1/k, t_l]$

$$\theta_{l,k}(t) = P_{(a_{l-1}(t_l) - b_l)}(1 - k[t - (t_l - 1/k)]) + b_l,$$

and for all $t \in [t_l, t_l + 1/k]$

$$\theta_{l,k}(t) = P_{(a_l(t_l) - b_l)}(k[t - t_l]) + b_l,$$

for all $l \in J_k$

$$\gamma_{l,k}(t) = P_{(a_l(t_l) - a)}(k[t - t_l]) + a,$$

for all $l \in L_k$

$$\delta_{l,k}(t) = P_{(a_l(t_l) - a)}(1 - k[t - (t_l - 1/k)]) + a,$$

and $m(l,k) = \max\{m,\ \eta_{l,m} > 8/k\}$.

∎

Proposition 2. *Let $H : \mathbb{R} \longrightarrow [a,b] \subset]0,1[$ be the function defined by*

$$H(t) = \sum_{l \in \mathbb{N}} (a_l(t) - a)\chi_{]\alpha_l, \beta_l[}(t) + a, \tag{7}$$

where
- *for all* $l,\ l' \in I\!N,\ l \neq l'$ *we have* $]\alpha_l, \beta_l[\cap]\alpha_{l'}, \beta_{l'}[=\emptyset$,
- *for all* l, $a_l(t)$ *is a continuous function on* $[\alpha_l, \beta_l]$.

Since any open set may be written as a countable disjoined union of open intervals, a consequence of the previous proposition is that all the functions of the type $(b-a)\chi_U(t)+a$, where $U \subset I\!R$ is open, belong to $\mathcal{H}$.

The proof of Proposition 2 is similar to the one of Proposition 1.

■

Further properties of $\mathcal{H}$ will be investigated in a forthcoming work.

Theorem 1. *With probability one,* $t \longmapsto Y_{H,\lambda}(t,\omega)$ *is a continuous function*

To prove Theorem 1, we will need the following Lemma.

Lemma 2. *Let* $\{X(t)\}_{t\in K}$ *be a real valued gaussian process defined on a compact interval* K *and suppose there exist* 2 *reals* $c>0$ *and* $\alpha>0$, *such that for all* $t,\ t' \in K$

$$E([X(t)-X(t')]^2) \leq c|t-t'|^{\alpha}. \tag{8}$$

Then with probability one, $t \longmapsto X(t,\omega)$ *is a continuous function.*

Proof (of Lemma 2). Let us show that the process $\{X(t)\}_{t\in K}$ satisfy the following property; there exists 3 reals $\mathcal{C}>0$, $\beta>0$ and $\gamma>0$ such that for all $t,\ t' \in K$

$$E(|X(t)-X(t')|^{\beta}) \leq \mathcal{C}|t-t'|^{1+\gamma}. \tag{$*$}$$

When the inequality $(*)$ is satisfied, it results from Kolmogorov criterium (see [10] page 57) that almost surely

$$\lim_{h\longrightarrow 0} \sup_{\substack{t,\ t' \in K \\ |t-t'|<h}} |X(t)-X(t')| = 0.$$

Thus with probability one, $t \longmapsto X(t,\omega)$ is a continuous function.

To show $(*)$, let us recall that (see [7] page 110), when Z is a centered gaussian random variable, then for every real $r>0$,

$$E(|Z|^r) = \frac{2^{r/2}\Gamma\left(\frac{r+1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}(E(|Z|^2))^{r/2}.$$

Thus taking in the previous equality $Z = X(t)-X(t')$ and $r=\frac{2(1+\gamma)}{\alpha}$ ($\gamma>0$ being arbitrary), then using (8) we get $(*)$.

■

Proof (of Theorem 1) Let us first show that for all $l \in \mathbb{Z}$, with probability one $t \longmapsto Y(t,\omega)$ is a continuous function on $K_l = [\frac{l}{4}, \frac{l}{4} + \frac{1}{2}]$. It follows from Lemma 2, that it is sufficient to establish that: $\forall$, t, $t' \in K_l$

$$E([Y(t) - Y(t')]^2) \leq c|t-t'|^{2a}. \tag{9}$$

We have,

$$E([Y(t) - Y(t')]^2) \leq 2(A+B+G+H),$$

where

$$A = 2\int_0^1 |e^{it\xi} - 1|^2 (|\xi|^{-H_0(t)-1/2} - |\xi|^{-H_0(t')-1/2})^2 \, d\xi,$$

$$B = 8\sum_{n=1}^{\infty} \int_{\lambda^{n-1}}^{\lambda^n} (|\xi|^{-H_n(t)-1/2} - |\xi|^{-H_n(t')-1/2})^2 \, d\xi,$$

$$H = 2\int_0^1 |\xi|^{-2H_0(t')-1} |e^{i(t-t')\xi} - 1|^2 \, d\xi,$$

$$G = 2\sum_{n=1}^{\infty} \int_{\lambda^{n-1}}^{\lambda^n} |\xi|^{-2H_n(t')-1} |e^{i(t-t')\xi} - 1|^2 \, d\xi.$$

Let us give ad-hoc upper bounds of A, B, H and G. The theorem on finite increments entails that for all $\xi \neq 0$, t, t' and n, there exists $\tau \in [a+1/2, b+1/2]$, such that

$$||\xi|^{-H_n(t)-1/2} - |\xi|^{-H_n(t')-1/2}| = |\log|\xi|||\xi|^{-\tau}|H_n(t) - H_n(t')|. \tag{10}$$

Then it follows from (10) and from Definition 1 (b) that,

$$\begin{aligned} A &\leq 2\left(\int_0^1 |e^{it\xi} - 1|^2 \tfrac{(\log\xi)^2}{\xi^{2b+1}} \, d\xi\right) |H_0(t) - H_0(t')|^2 \\ &\leq 2\left(\left(\tfrac{|l|}{4} + \tfrac{1}{2}\right)^2 \int_0^1 \tfrac{(\log\xi)^2}{\xi^{2b-1}} \, d\xi\right) c_0^2 |t-t'|^{2\beta}, \end{aligned} \tag{11}$$

and

$$\begin{aligned} B &\leq 8\textstyle\sum_{n=1}^{\infty} \left(\int_{\lambda^{n-1}}^{\lambda^n} \tfrac{(\log\xi)^2}{\xi^{2a+1}} \, d\xi\right) c_n^2 |t-t'|^{2\beta} \\ &\leq 8\textstyle\sum_{n=1}^{\infty} \lambda^{n-1}(\lambda-1)(n\log\lambda)^2 \lambda^{-(n-1)(2a+1)} c_n^2 |t-t'|^{2\beta} \\ &\leq 8(\lambda-1)(\log\lambda)^2 \left(\textstyle\sum_{n=1}^{\infty} n^2 \lambda^{-2a(n-1)} c_n^2\right) |t-t'|^{2\beta}. \end{aligned} \tag{12}$$

Moreover,

$$H \leq 2\left(\int_0^1 \xi^{-2H_0(t')+1} \, d\xi\right) |t-t'|^2 \leq 2\left(\int_0^1 \xi^{-2b+1} \, d\xi\right) |t-t'|^2. \tag{13}$$

At last we have,

$$G \leq 2\sum_{n=1}^{\infty} \int_{\lambda^{n-1}}^{\lambda^n} \xi^{-2a-1}|e^{i(t-t')\xi} - 1|^2 \, d\xi \leq 2\int_1^{+\infty} \xi^{-2a-1}|e^{i(t-t')\xi} - 1|^2 \, d\xi,$$

$$\leq 2\left(\int_1^{|t-t'|^{-1}} \xi^{-2a+1} \, d\xi\right)|t-t'|^2 + 8\left(\int_{|t-t'|^{-1}}^{+\infty} \xi^{-2a-1} \, d\xi\right), \tag{14}$$

$$\leq c|t-t'|^{2a}.$$

Thus (9) results from (11), (12), (13) et (14).

■

3 Pointwise regularity of the GMBM

The following theorem shows that one can prescribe the regularity of the GMBM.

Theorem 2. *Let $\{Y_{H,\lambda}(t)\}_{t\in\mathbb{R}}$ be the GMBM of parameters the function $H(t)$ of $\mathcal{H}$ and the real $\lambda > 1$. Let t_0 be an arbitrary real and let $\alpha_Y(t_0)$ be the Hölder exponent of the process $\{Y_{H,\lambda}(t)\}_{t\in\mathbb{R}}$ at t_0 (see (2)).*

Then

$$\alpha_Y(t_0) = H(t_0),$$

almost surely .

To prove Theorem 2 we need the following Lemma.

Lemma 3. *Let $\{X(t)\}_{t\in I}$ be a real valued centered gaussian process defined on an interval I and let $\alpha_X(t_0)$ be its Hölder exponent at t_0. $H(t)$ will be a deterministic funtion with values in $[a,b] \subset]0,1[$.*

(i) Suppose that for all $\epsilon > 0$, one can find two reals $c = c(t_0,\epsilon) > 0$ and $h_0 = h_0(t_0,\epsilon) > 0$ such that, for all $|h|$, $|h'| \leq h_0$,

$$E([X(t_0+h) - X(t_0+h')]^2) \leq c|h-h'|^{2(H(t_0)-\epsilon)}. \tag{15}$$

Then, $\alpha_X(t_0) \geq H(t_0)$ almost surely .

(ii) Suppose that for all $\epsilon > 0$, there exists a sequence (h_n) satisfying $\lim_{n\longrightarrow\infty} h_n = 0$ and a real $c = c(t_0,\epsilon) > 0$ such that, for all n,

$$E([X(t_0+h_n) - X(t)]^2) \geq c|h_n|^{2(H(t_0)+\epsilon)}, \tag{16}$$

then $\alpha_X(t_0) \leq H(t_0)$ almost surely .

This lemma follows from general results on the regularity of gaussian processes. However, for sake of completness we prove it here by an elementary method.

Proof Let us fix $t_0 \in I$, $\epsilon > 0$ and suppose that $I =]u, v[$. (15) entails that there exists 2 reals $c > 0$ and $h_0 > 0$ satisfying the following property: for all h, h' such that $|h|$, $|h'| < h_0$ and $t_0 + h$, $t_0 + h' \in I$,

$$E([X(t_0 + h) - X(t_0 + h')]^2) \leq c|h - h'|^{2(H(t_0)-\epsilon/2)}. \tag{17}$$

Let $\eta = \frac{1}{2}\min(h_0, t - u, v - t)$ and $J = [-\eta, \eta]$. The gaussian process $\{Y_{t,\epsilon}(k)\}_{k\in J}$, will be defined on J as follows

$$Y_{t_0,\epsilon}(k) = \begin{cases} \frac{X(t_0+k)-X(t_0)}{|k|^{H(t_0)-\epsilon}}, \text{ if } k \in J - \{0\} \\ \\ 0 \text{ if } k = 0. \end{cases} \tag{18}$$

Suppose for a while that for all h, $h' \in J$

$$E(|Y_{t_0,\epsilon}(h') - Y_{t_0,\epsilon}(h)|^2) \leq C'|h - h'|^{\epsilon}. \tag{19}$$

It follows then from (18) and from Lemma 2 that almost surely $\lim_{k\longrightarrow 0} Y_{t_0,\epsilon}(k) = 0$ which means that almost surely $\alpha_X(t_0) \geq H(t_0) - \epsilon$. Thus from now on our aim will be to show that the inequality (19) holds. It is clear that the inequality (19) is true when $h = 0$ or when $h' = 0$. For all h, $h' \in J - \{0\}$ we have

$$\begin{aligned} &E(|Y_{t_0,\epsilon}(h') - Y_{t_0,\epsilon}(h)|^2) \\ &\quad \leq 2E\left(\frac{|X(t_0 + h') - X(t_0 + h)|^2}{|h'|^{2(H(t_0)-\epsilon)}}\right) \\ &\quad +2\left(\frac{1}{|h'|^{H(t_0)-\epsilon}} - \frac{1}{|h|^{H(t_0)-\epsilon}}\right)^2 E(|X(t_0 + h) - X(t_0)|^2). \end{aligned}$$

It follows then from (17) that,

$$\begin{aligned} E(|Y_{t_0,\epsilon}(h') - Y_{t_0,\epsilon}(h)|^2) &\leq 2c\frac{|h' - h|^{2(H(t_0)-\epsilon/2)}}{|h'|^{2(H(t_0)-\epsilon)}} \\ &+ 2c\left(\frac{1}{|h'|^{H(t_0)-\epsilon}} - \frac{1}{|h|^{H(t_0)-\epsilon}}\right)^2 |h|^{2(H(t_0)-\epsilon/2)}. \end{aligned}$$

From now on we will suppose that $0 < |h| \leq |h'|$,
so we have $\frac{1}{|h'|^{2(H(t_0)-\epsilon)}} \leq \frac{2^{2(H(t_0)-\epsilon)}}{|h'-h|^{2(H(t_0)-\epsilon)}}$, and consequently,

$$E(|Y_{t_0,\epsilon}(h') - Y_{t_0,\epsilon}(h)|^2) \leq c2^{2H(t_0)+1}|h' - h|^{\epsilon} + 2c\left(1 - \left|\frac{h}{h'}\right|^{H(t_0)-\epsilon}\right)^2 |h|^{\epsilon}.$$

Let us study the case where $h' > 0$, the case where $h' < 0$ being similar.

When $h \in [-h', h'/2]$, since $|h' - h| \geq |h|$ the inequality (19) obviously holds. When $h \in]h'/2, h']$, the Theorem on finite increments entails that

$$1 - \left|\frac{h}{h'}\right|^{H(t_0)-\epsilon} \leq 2(H(t_0) - \epsilon)\left(1 - \frac{h}{h'}\right),$$

consequently, we have

$$\left(1-\left|\frac{h}{h'}\right|^{H(t_0)-\epsilon}\right)^2 |h|^\epsilon \leq 4\left(1-\frac{h}{h'}\right)^2 h'^\epsilon,$$
$$\leq 4|h'-h|^\epsilon,$$

and then, we get (19) from this last inequality.

Now, we are going to show (ii). For all $\epsilon > 0$, there exists (h_n) a sequence of non vanishing reals converging to 0 and there exists $c > 0$ such that

$$E[(X(t_0+h_n)-X(t_0))^2] \geq c|h_n|^{2(H(t_0)+\epsilon/2)}.$$

Since $\{X(t)\}$ is a gaussian process we have

$$\frac{X(t_0+h_n)-X(t_0)}{|h_n|^{H(t_0)+\epsilon}} \rightsquigarrow \mathcal{N}(0,\sigma_n^2),$$

and the last inequality implies that $\lim_{n\longrightarrow\infty} \sigma_n = +\infty$. Let us show that the sequence of random variables $\left(\frac{|h_n|^{H(t_0)+\epsilon}}{X(t_0+h_n)-X(t_0)}\right)$ converges in probability to 0. For all real $\eta > 0$, we have

$$P\left(\frac{|h_n|^{H(t_0)+\epsilon}}{|X(t_0+h_n)-X(t_0)|} < \eta\right) = P\left(\frac{|X(t_0+h_n)-X(t_0)|}{|h_n|^{H(t_0)+\epsilon}} > 1/\eta\right)$$
$$= \int_{(|x|>1/\eta)} \frac{1}{\sqrt{2\pi}\sigma_n} \exp(-x^2/2\sigma_n^2)\,dx$$
$$= \frac{1}{\sqrt{2\pi}} \int_{(|x|>1/\eta\sigma_n)} \exp(-x^2/2)\,dx.$$

Thus

$$\lim_{n\longrightarrow\infty} P\left(\frac{|h_n|^{H(t_0)+\epsilon}}{|X(t_0+h_n)-X(t_0)|} < \eta\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp(-x^2/2)\,dx = 1.$$

Therefore, there exists a sub-sequence $k \mapsto n_k$ such that

$$\lim_{k\longrightarrow+\infty} \frac{|X(t_0+h_{n_k})-X(t_0)|}{|h_{n_k}|^{H(t_0)+\epsilon}} = +\infty,$$

almost surely. This means that almost surely, $\alpha_X(t_0) \leq H(t_0)+\epsilon$.

■

Proof (of Theorem 2).

Step 1 : we will establish that for all t_0, $\alpha_Y(t_0) \geq H(t_0)$ almost surely. It follows from Lemma 3 (i) that it is sufficient to show that for all $\epsilon > 0$, there exist 2 reals $c = c(t_0,\epsilon) > 0$ and $h_0 = h_0(t_0,\epsilon) > 0$, such that for all $|h|$, $|h'| \leq h_0$

$$E([Y(t_0+h)-Y(t_0+h')]^2) \leq c|h-h'|^{2(H(t_0)-\epsilon)}. \qquad (20)$$

Let $h_0(t_0,\epsilon) \leq 1/2$ and $n_0 = n_0(t_0,\epsilon)$ be as in Definition 1 (a). For all $|h|$, $|h'| \leq h_0$, we have

$$E[(Y(t_0+h) - Y(t_0+h'))^2] \leq 2(R+S+T+U),$$

where

$$R = 2\int_0^1 |e^{i(t_0+h)\xi} - 1|^2 (|\xi|^{-H_0(t_0+h)-1/2} - |\xi|^{-H_0(t_0+h')-1/2})^2 \, d\xi,$$

$$S = 8\sum_{n=1}^{\infty} \int_{\lambda^{n-1}}^{\lambda^n} (|\xi|^{-H_n(t_0+h)-1/2} - |\xi|^{-H_n(t_0+h')-1/2})^2 d\xi,$$

$$T = 2\int_0^1 |\xi|^{-2H_0(t_0+h')-1} |e^{i(h-h')\xi} - 1|^2 \, d\xi + 2\sum_{n=1}^{n_0-1} \int_{\lambda^{n-1}}^{\lambda^n} |\xi|^{-2H_n(t_0+h')-1} |e^{i(h-h')\xi} - 1|^2 \, d\xi,$$

$$U = 2\sum_{n=n_0}^{\infty} \int_{\lambda^{n-1}}^{\lambda^n} |\xi|^{-2H_n(t_0+h')-1} |e^{i(h-h')\xi} - 1|^2 \, d\xi.$$

Let us give ad-hoc upper bounds of R, S, T, and U. A similar method to that we have used to give upper bounds of A and B, (see the proof of Theorem 1) leads to

$$\begin{aligned} R &\leq 2\left((|t_0| + h_0)^2 \int_0^1 \tfrac{(\log \xi)^2}{\xi^{2b-1}} \, d\xi\right)^2 c_0^2 |h-h'|^{2\beta}, \\ S &\leq 8\left((\log\lambda)^2(\lambda-1)\textstyle\sum_{n=1}^{\infty} n^2 \lambda^{-(n-1)2a} c_n^2\right) |h-h'|^{2\beta}. \end{aligned} \tag{21}$$

Moreover, as for all t and n, $H_n(t) \in [a,b]$, we get

$$T \leq 2\left(\int_0^1 \xi^{-2b+1}\, d\xi\right) |h-h'|^2 + 2\left(\int_1^{\lambda^{n_0}} \xi^{-2a+1}\, d\xi\right) |h-h'|^2. \tag{22}$$

At last, Definition (a) implies that

$$\begin{aligned} U &\leq \textstyle\sum_{n=n_0}^{\infty} \int_{\lambda^{n-1}}^{\lambda^n} \xi^{-2H(t_0)+2\epsilon-1} |e^{i(h-h')\xi} - 1|^2 \, d\xi \\ &\leq 2\textstyle\int_1^{+\infty} \xi^{-2H(t_0)+2\epsilon-1} |e^{i(h-h')\xi} - 1|^2 \, d\xi \\ &\leq 2\left(\textstyle\int_1^{|h-h'|^{-1}} \xi^{-2H(t_0)+2\epsilon+1}\, d\xi\right) |h-h'|^2 + 8\textstyle\int_{|h-h'|^{-1}}^{+\infty} \xi^{-2H(t_0)+2\epsilon-1}\, d\xi \\ &\leq c|h-h'|^{2(H(t_0)-\epsilon)}. \end{aligned} \tag{23}$$

Thus, it follows from (21), (22) et (23) that the process $\{Y(t)\}$ satisfies (20).

Step 2: we will establish that for all t_0, $\alpha_Y(t_0) \geq H(t_0)$ almost surely. It follows from Lemma 3 (ii) that it is sufficient to show that for all $\epsilon > 0$, there exists a sequence (h_n) of non vanishing real numbers converging to 0 and there exists $c = c(t_0, \epsilon) > 0$ such that for all n,

$$E([Y(t_0 + h_n) - Y(t_0)]^2) \geq c|h_n|^{2(H(t_0)+\epsilon)}. \tag{24}$$

As $H(t_0) = \liminf_{n\longrightarrow\infty} H_n(t_0)$, there exists a subsequence $k \mapsto n_k$ such that for all k, $H_{n_k+1}(t_0) \leq H(t_0) + \epsilon$. From now on we will suppose that $n = n_k$. Let $h_n = \lambda^{-n}$, we have

$$\begin{aligned}
&||Y(t_0 + h_n) - Y(t_0)||_{L^2(\Omega)} = \left(E[(Y(t_0 + h_n) - Y(t_0))^2]\right)^{1/2} \\
&\geq \left(\int_{\lambda^n}^{\lambda^{n+1}} |\xi|^{-2H_{n+1}(t_0)-1}|e^{i(t_0+h_n)\xi} - e^{it_0\xi}|^2\, d\xi\right)^{1/2} \\
&\quad - \left(\int_{\lambda^n}^{\lambda^{n+1}} \left(|\xi|^{-H_{n+1}(t_0)-1/2} - |\xi|^{-H_{n+1}(t_0+h_n)-1/2}\right)^2 |e^{i(t_0+h_n)\xi} - 1|^2\, d\xi\right)^{1/2} \\
&\geq \left(\int_{\lambda^n}^{\lambda^{n+1}} |\xi|^{-2H(t_0)-2\epsilon-1}|e^{ih_n\xi} - 1|^2\, d\xi\right)^{1/2} \\
&\quad -2\left(\int_{\lambda^n}^{\lambda^{n+1}} \left(|\xi|^{-H_{n+1}(t_0)-1/2} - |\xi|^{-H_{n+1}(t_0+h_n)-1/2}\right)^2 d\xi\right)^{1/2}.
\end{aligned}$$

Thus setting $u = h_n\xi$, we get

$$\begin{aligned}
&\int_{\lambda^n}^{\lambda^{n+1}} |\xi|^{-2H(t_0)-2\epsilon-1}|e^{ih_n\xi} - 1|^2\, d\xi \\
&\qquad = h_n^{2(H(t_0)+\epsilon)} \int_1^{\lambda} u^{-2H(t_0)-2\epsilon-1}|e^{iu} - 1|^2\, du.
\end{aligned} \tag{25}$$

Moreover, it follows from (10) that

$$\begin{aligned}
&\int_{\lambda^n}^{\lambda^{n+1}} \left(|\xi|^{-H_{n+1}(t_0)-1/2} - |\xi|^{-H_{n+1}(t_0+h_n)-1/2}\right)^2 d\xi \\
&\leq \left(\int_{\lambda^n}^{\lambda^{n+1}} (\log\xi)^2\xi^{-2a-1}\, d\xi\right) c_{n+1}^2 h_n^{2\beta} \\
&\leq \left((n+1)^2\lambda^n(\lambda-1)(\log\lambda)^2\lambda^{-(2a+1)n}\lambda^{-2(\beta-H(t_0)-\epsilon)n}\right) c_{n+1}^2 h_n^{2(H(t_0)+\epsilon)} \\
&= \left((n+1)^2(\log\lambda)^2(\lambda-1)\lambda^{-2(a+\beta-H(t_0)-\epsilon)n}\right) c_{n+1}^2 h_n^{2(H(t_0)+\epsilon)}.
\end{aligned} \tag{26}$$

At last, (25) and (26) entail that

$$\begin{aligned}
||Y(t_0 + h_{n_k}) - Y(t_0)||_{L^2(\Omega)} \geq \Big\{ &\left(\int_1^{\lambda} u^{-2H(t_0)-2\epsilon-1}|e^{iu} - 1|^2\, du\right)^{1/2} \\
&-C_\lambda \lambda^{-(a+\beta-H(t_0)-\epsilon)n_k}(n_k+1)c_{n_k+1}\Big\}\, h_{n_k}^{H(t_0)+\epsilon},
\end{aligned}$$

and then Definition 1 (b), implies that for k, big enough,

$$||Y(t_0 + h_{n_k}) - Y(t_0)||_{L^2(\Omega)} \geq \frac{\sqrt{2}}{2}\left(\int_1^{\lambda} u^{-2H(t_0)-2\epsilon-1}|e^{iu} - 1|^2\, du\right)^{1/2} h_{n_k}^{H(t_0)+\epsilon}.$$

■

4 Local self-similarity of the GMBM

Definition 3. *We will say that a process* $\{X(t)\}$ *is* asymptotically locally self-similar of order $\alpha > 0$ at t_0, *if and only if the process* $\left\{\frac{X(t_0+\rho u)-X(t_0)}{\rho^\alpha}\right\}_u$ *converge in distribution to a non trivial limit as* $\rho \to 0^+$.

Proposition 3. *Let* $\{Y_{H,\lambda}(t)\}_{t\in I\!R}$ *be the GMBM of parameters the function* $H(t) = \liminf_{n\longrightarrow\infty} H_n(t)$ *of* $\mathcal{H}$ *and the real* $\lambda > 1$. *Let* t_0 *be a real satisfying the following properties.*

1. *There exist 2 reals* $c = c(t_0) > 0$ *and* $\delta = \delta(t_0) > 1 - a$ *such that for all* n,
$$|H_n(t_0) - H(t_0)| \leq c\lambda^{-2\delta n};$$
2. *These exist* $n_0 = n_0(t_0)$ *and* $h_0 = h_0(t_0) > 0$ *such that, for all* $n \geq n_0$ *and* $|h| \leq h_0$ *we have* $H_n(t_0+h) \geq H(t_0)$.

Then the process $\{Y_{H,\lambda}(t)\}_{t\in I\!R}$ *is asymptotically locally self-similar of order* $H(t_0)$ *at* t_0. *More precisely,*

$$\lim_{\rho\longrightarrow 0^+} Law\left\{\frac{Y(t_0+\rho u)-Y(t_0)}{\rho^{H(t_0)}}\right\}_{u\in I\!R} = Law\left\{\sigma(t_0)B_{H(t_0)}(u)\right\}_{u\in I\!R},$$

where $\{B_{H(t_0)}(u)\}_{u\in I\!R}$ *is the standard fractional brownian motion of parameter* $H(t_0)$ *and where*

$$\sigma^2(t_0) = 4\int_{-\infty}^{+\infty} \frac{\sin^2 u/2}{|u|^{2H(t_0)+1}}\,du.$$

It is worthwhile noting here that there exist many "interesting" functions in $\mathcal{H}$ that satisfy *1.* and *2.* for all real t_0. Indeed, following the same lines as in the proof of Proposition 1 one can show that:

- functions that are both lower-semi-continuous and piecewise constant;
- functions of the type $b + (a-b)X_F(t)$, where $0 < a < b < 1$ and $X_F(t)$ is the characteristic function of any arbitrary closed set F,

satisfy *1.* and *2.*.

Proposition 3 is a consequence of the following Lemmas.

Lemma 4. *For all* t_0, u, v *and* ρ *small enough, we have*

$$\begin{aligned} &E[(Y(t_0+\rho u)-Y(t_0))(Y(t_0+\rho v)-Y(t_0))] \\ &\quad = \sum_{n=0}^{\infty}\int_{D_n}\frac{(e^{i\rho u\xi}-1)(e^{-i\rho v\xi}-1)}{|\xi|^{2H_n(t_0)+1}}d\xi + 0(\rho^{2H(t_0)}) \end{aligned}$$

Proof Let

$$N_n(\xi,x,y)=\frac{e^{i(t_0+x)\xi}-1}{|\xi|^{H_n(t_0+y)+1/2}}$$

$$\begin{aligned}
&E[(Y(t_0+\rho u)-Y(t_0))(Y(t_0+\rho v)-Y(t_0))]\\
&=\sum_{n=0}^{\infty}\int_{D_n}[N_n(\xi,\rho u,\rho u)-N_n(\xi,0,0)][\overline{N_n(\xi,\rho v,\rho v)}-\overline{N_n(\xi,0,0)}]d\xi\\
&=\sum_{n=0}^{\infty}\Big\{\int_{D_n}[N_n(\xi,\rho u,\rho u)-N_n(\xi,\rho u,0)][\overline{N_n(\xi,\rho v,\rho v)}-\overline{N_n(\xi,\rho v,0)}]d\xi\\
&+\int_{D_n}[N_n(\xi,\rho u,\rho u)-N_n(\xi,\rho u,0)][\overline{N_n(\xi,\rho v,0)}-\overline{N_n(\xi,0,0)}]d\xi\\
&+\int_{D_n}[N_n(\xi,\rho u,0)-N_n(\xi,0,0)][\overline{N_n(\xi,\rho v,\rho v)}-\overline{N_n(\xi,\rho v,0)}]d\xi\\
&+\int_{D_n}[N_n(\xi,\rho u,0)-N_n(\xi,0,0)][\overline{N_n(\xi,\rho v,0)}-\overline{N_n(\xi,0,0)}]d\xi\Big\}.
\end{aligned}$$

At last, by a similar method to that we have used to give upper bounds of A, B, H and G in the proof of Theorem 1, we can show that

$$\begin{aligned}
&E[(Y(t_0+\rho u)-Y(t_0))(Y(t_0+\rho v)-Y(t_0))]\\
&\le \alpha_1\rho^{2\beta}|u|^\beta|v|^\beta+\alpha_2\rho^{\beta+H(t_0)}|u|^\beta|v|^{H(t_0)}+\alpha_3\rho^{\beta+H(t_0)}|u|^{H(t_0)}|v|^\beta\\
&+\textstyle\sum_{n=0}^{\infty}\int_{D_n}\frac{(e^{i\rho u\xi}-1)(e^{-i\rho v\xi}-1)}{|\xi|^{2H_n(t_0)+1}}d\xi,
\end{aligned}$$

where α_1, α_2 and α_3 are 3 constants. ■

Lemma 5. *If the sequence $(H_n(t_0))$ satisfies the condition 1. of Proposition 3, then we have*

$$\lim_{\rho\longrightarrow 0+}\frac{1}{\rho^{2H(t_0)}}\sum_{n=0}^{\infty}\int_{D_n}\frac{(e^{i\rho u\xi}-1)(e^{-i\rho v\xi}-1)}{|\xi|^{2H_n(t_0)+1}}d\xi
=\int_{-\infty}^{+\infty}\frac{(e^{iu\eta}-1)(e^{-iv\eta}-1)}{|\eta|^{2H(t_0)+1}}d\eta$$

Proof

$$\int_{-\infty}^{+\infty}\frac{(e^{iu\eta}-1)(e^{-iv\eta}-1)}{|\eta|^{2H(t_0)+1}}d\eta=\frac{1}{\rho^{2H(t_0)}}\int_{-\infty}^{+\infty}\frac{(e^{i\rho u\xi}-1)(e^{-i\rho v\xi}-1)}{|\xi|^{2H(t_0)+1}}d\xi.$$

Thus, we have

$$\left|\frac{1}{\rho^{2H(t_0)}}\sum_{n=0}^{\infty}\int_{D_n}\frac{(e^{i\rho u\xi}-1)(e^{-i\rho v\xi}-1)}{|\xi|^{2H_n(t_0)+1}}-\int_{-\infty}^{+\infty}\frac{(e^{iu\eta}-1)(e^{-iv\eta}-1)}{|\eta|^{2H(t_0)+1}}d\eta\right|$$
$$\leq \frac{1}{\rho^{2H(t_0)}}\sum_{n=0}^{\infty}\int_{D_n}\left|(e^{i\rho u\xi}-1)(e^{-i\rho v\xi}-1)\left(\frac{1}{|\xi|^{2H_n(t_0)+1}}-\frac{1}{|\xi|^{2H(t_0)+1}}\right)\right|d\xi$$

$$\leq 2\rho^{2(1-H(t_0))}|u||v|\left[\int_0^1\left(\frac{1}{\xi^{2H_0(t_0)-1}}+\frac{1}{\xi^{2H(t_0)-1}}\right)d\xi\right.$$
$$\left.+\sum_{n=1}^{\infty}\lambda^{2n}\int_{\lambda^{n-1}}^{\lambda^n}\left|\frac{1}{\xi^{2H_n(t_0)+1}}-\frac{1}{\xi^{2H(t_0)+1}}\right|d\xi\right].$$

This last series is convergent, since the Theorem on finite increments and *1.* (see Proposition 3) imply that

$$\sum_{n=1}^{\infty}\lambda^{2n}\int_{\lambda^{n-1}}^{\lambda^n}\left|\frac{1}{\xi^{2H_n(t_0)+1}}-\frac{1}{\xi^{2H(t_0)+1}}\right|d\xi,$$

$$\leq 2\lambda^{2a}(\lambda-1)\log\lambda\sum_{n=1}^{\infty}n\lambda^{2(1-a)n}|H_n(t_0)-H(t_0)|,$$

$$\leq c\sum_{n=1}^{\infty}n\lambda^{2(1-a-\delta)n}<\infty.$$

■

Lemma 6. *We have*

$$E[(Y(t_0+\rho u)-Y(t_0))(Y(t_0+\rho v)-Y(t_0))]$$
$$=\left(\int_{-\infty}^{+\infty}\frac{(e^{iu\eta}-1)(e^{-iv\eta}-1)}{|\eta|^{2H(t_0)+1}}d\eta\right)\rho^{2H(t_0)}+O(\rho^{2H(t_0)}).$$

Proof This Lemma is a straightforward consequence of Lemmas 4 and 5. ■

Proof (of Proposition 3) Lemma 6 yields the convergence of the finite dimensional distribution of the process $\left\{\frac{Y_{H,\lambda}(t_0+\rho u)-Y_{H,\lambda}(t_0)}{\rho^{H(t_0)}}\right\}_{u\in\mathbb{R}}$ to those of $\{\sigma(t_0)B_{H(t_0)}(u)\}_{u\in\mathbb{R}}$. To have the convergence in distribution for the topology of the uniform convergence on compact sets a tightness result is required.

Following the same lines as in the proof of Theorem 2 and using *2.* one can show that, for integer $l=2$ and for every compact K there exists a finite constant $c(t_0,l)$ such that for $\rho>0$ small enough,

$$\forall\ u,v\ \in K\quad E\left(\frac{Y(t_0+\rho u)-Y(t_0-\rho v)}{\rho^{H(t_0)}}\right)^2\leq c(t_0,l)|u-v|^{lH(t_0)}.$$

Since the processes are Gaussian this inequality can be extended to l large enough to get $H(t_0).l > 1$. Hence one can classically deduce that the sequence of the laws of $\left(\frac{Y(t_0+\rho u)-Y(t_0)}{\rho^{H(t_0)}}\right)_{\rho>0}$ is relatively compact.

■

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