The Schwarzian derivative on symmetric spaces of Cayley type

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THE SCHWARZIAN DERIVATIVE ON SYMMETRIC SPACES OF CAYLEY TYPE

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Abstract. Let M be a symmetric space of Cayley type and f a conformal diffeomorphism of M. We study a relationship between the conformal factor of f and a generalized Shwarzian derivatine of f.

1. Introduction

Let $M = G/H$ be a Riemannian symmetric space and let $\underline{\mathbf{g}}$ be a $G-$invariant metric on M. Let f be a conformal transformation of $(M, \underline{\mathbf{g}})$, *i.e.*

$$f^*\underline{\mathbf{g}} = \mathbf{c}_f\underline{\mathbf{g}}$$

where $\mathbf{c}_f$ is the conformal factor of f.

When $M = \mathcal{H}$ is the single sheeted hyperboloid (with the unique Lorentz metric), then it is well known that $\mathcal{H}$ is conformally equivalent to $S^1 \times S^1 \setminus \Delta_{S^1}$ where S^1 is the unit cirle and Δ_{S^1} the null space. The group of (orientation preserving) conformal diffeomorphisms of $\mathcal{H}$ is $\mathrm{Diff}(S^1)$, the group of diffeomorphisms of the circle S^1. In [KS] Kostant and Sternberg, pointed out an interesting relationship between the Shwarzian derivative of a transformation $f \in \mathrm{Diff}(S^1)$ and the corresponding conformal factor $\mathbf{c}_f$ (which is a singular function on the null space). More precisely, they prove that $\mathbf{c}_f$ tends to 1 on $\mathcal{H}$ as we approach infinity, and that the Hessain of the (extended) $\mathbf{c}_f$ is the Schwarzian derivative of f.

The single sheeted hyperboloid is the simplest example of a large class of para-hermitian symmetric spaces, the Cayley type spaces. Those spaces are characterized as an open orbit in $S \times S$ where S is the Shilov boundary of a bounded symmetric domain of tube type [K_1]. In this paper, we will use this characterization to extend the results of Kostant and Sternberg to symmetric spaces of Cayley type.

Key words and phrases. Symmetrc space of Cayley type, Schwarzian derivative
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2. Causal symmetric spaces, Symmetric spaces of Cayley type

Let M be a smooth $n-$manifold. A causal structure on M is a cone field $\mathcal{C} = (C_p)_{p\in M}$ where

$$C_p \subset T_pM$$

is a causal cone *ie.* non-zero, closed convex cone which is pointed ($C_p \cap -C_p = \{0\}$), generating ($C_p - C_p = T_pM$) and such that C_p depends smoothly on p.

If $M = G/H$ is a homogeneous space, where G is a Lie group and $H \subset G$ a closed subgroup, then the causal structure is said to be $G-$invariant if for any $g \in G$

$$C_{g\cdot x} = Dg(x)(C_x), \quad \text{for } x \in M,$$

where $Dg(x)$ is the derivative of g at x.

Let $M = G/H$ be a symmetric space, *ie.* there exists an involution σ of G such that $(G^\sigma)^\circ \subset H \subset G^\sigma$ where $(G^\sigma)^\circ$ is the identity component of G^σ.

Let $\mathfrak{g} = \mathrm{Lie}(G)$ be the Lie algebra of G and put

$$\mathfrak{h} = \mathfrak{g}(+1,\sigma), \quad \mathfrak{q} = \mathfrak{g}(-1,\sigma)$$

the eigenspaces of σ. Then $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ and the tangent space $T_{x_0}M$ at $x_0 = 1H$ can be identified with $\mathfrak{q}$. In this identification, the derivative $Dg(x_0)$, $g \in H$ corresponds to $\mathrm{Ad}(g)$. Therefore an invariant causal structure on M is determined by a causal cone C in $\mathfrak{q}$ which is $\mathrm{Ad}(H)-$invariant.

Suppose that G is be semi-simple with a finite centre and that the pair $(\mathfrak{g},\mathfrak{h})$ is irreducible (*ie.* there is no non-trivial ideal in $\mathfrak{g}$ which is invariant under σ). Then, there exists a Cartan involution θ commuting with the given involution σ.

Let K be the corresponding maximal compact subgroup of G. Let $\mathfrak{k} = \mathfrak{g}(+1,\theta)$, $\mathfrak{p} = \mathfrak{g}(-1,\theta)$. Then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is the corresponding Cartan decomposition of $\mathfrak{g}$.

Let $\mathrm{Cone}_H(\mathfrak{q})$ be the set of $\mathrm{Ad}(H)-$ invariant causal cones in $\mathfrak{q}$. Then following Ólafsson (see [H-Ó]) the irreducible symmetric space M is called

(CC): Compactly Causal space if there exists a $C \in \mathrm{Cone}_H(\mathfrak{q})$ such that $C^\circ \cap \mathfrak{k} \neq \emptyset$.

(NCC): Non-Compactly Causal space if there exists a $C \in \mathrm{Cone}_H(\mathfrak{q})$ such that $C^\circ \cap \mathfrak{p} \neq \emptyset$.

(CT): Cayley Type space if it is (CC) and (NCC).

The simplest example of Cayley type symmetric spaces is the space $M = SO_0(2,1)/SO_0(1,1)$ realized as the one-sheeted hyperboloid

$$\{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 - x_3^2 = 1\}.$$

3. Causal compactification of symmetric spaces of Cayley type

Let $D = G/K \subset V_{\mathbb{C}} = V + iV \simeq \mathbb{C}^n$ be a bounded symmetric domain in $\mathbb{C}-$vector space. The group G is the identity component of the group of holomorphic automorphisms of D, and $K = \{g \in G \mid g \cdot o = o\}$ the stabilizer of the base point $o = 1K \in D$, which is a maximal compact subgroup of G.

Suppose that D is of tube type,

$$D \simeq T_\Omega = V + i\Omega$$

where Ω is the symmetric cone of the Euclidean Jordan algebra V.

Let $z \mapsto \bar{z}$ the complex conjugation in $V_{\mathbb{C}}$ with respect to V and e the unit element of V. The set

$$S = \{z \in V_{\mathbb{C}} \mid \bar{z}z = e\}$$

is a connected submanifold of $V_{\mathbb{C}}$ which is a Riemannian symmetric space of compact type

$$S \simeq U/U_e,$$

where U is the identity component of the group of linear transformations $g \in GL(V_{\mathbb{C}})$ such that $gS = S$, and U_e is the stabilizer subgroup of $e \in S$.

There exists on $V_{\mathbb{C}}$ a $U-$invariant spectral norm $z \mapsto |z|$, and one can prove (see [F-K]) that D is the unit disc

$$D = \{z \in V_{\mathbb{C}} \mid |z| < 1\}$$

and S is its Shilov boundary.

Let

$$G(\Omega) = \{g \in \mathrm{GL}(V) \mid g\Omega = \Omega\}.$$

It is a reductive Lie group which acts transitively on Ω. Let $G_0 = G(\Omega)^\circ$ be the identity component of $G(\Omega)$.

Let G^c be the identity component of the group of holomorphic automorphisms of T_Ω. Then G_0 is a Lie subgroup of G^c. The subgroups G_0 and $N^+ = \{t_v : z \mapsto z + v,\ v \in V\}$, together with the inversion $j : z \mapsto -z^{-1}$, generate the group G^c.

For any $x \in V$ we define the quadratic representation $P(x)$ of Jordan algebra V given by $P(x) = 2L(x)^2 - L(x^2)$, where $L(x)$ is the multiplication by x.
The Lie algebra $\mathfrak{g}^c$ of G^c is the set of vector fields on V of the form

$$X(z) = u + Tz + P(z)v \simeq (u, T, v),$$

where T is linear and $u, v \in V$.

Consider on G^c the involutions

$$\begin{aligned} \sigma^c(g) &= \nu \circ g \circ \nu \\ \theta^c(g) &= (-\nu) \circ g \circ (-\nu) \end{aligned}$$

where $\nu : z \mapsto \bar{z}^{-1}$. We use the same letters for the corresponding involutions on the Lie algebra $\mathfrak{g}^c$.
If $X = (u, T, v) \in \mathfrak{g}^c$, then

$$\sigma^c(X) = (v, -T^*, u) \text{ and } \theta^c(X) = (-v, -T^*, -u),$$

Therefore,

$$\begin{aligned} \mathfrak{h}^c &:= \mathfrak{g}^c(\sigma^c, +1) = \{(u, T, u) \mid u \in V,\ T \in \mathfrak{k}_0\} \\ \mathfrak{q}^c &:= \mathfrak{g}^c(\sigma^c, -1) = \{(u, L(v), -u) \mid u, v \in V\},\ (L(v)x = vx) \\ \mathfrak{k}^c &:= \mathfrak{g}^c(\theta^c, +1) = \{(u, T, -u) \mid u \in V,\ T \in \mathfrak{k}_0\} \\ \mathfrak{p}^c &:= \mathfrak{g}^c(\theta^c, -1) = \{(u, L(v), u) \mid u \in V\}. \end{aligned}$$

Consider the convex cones in $\mathfrak{q}^c$

$$\begin{aligned} C_1 &= \{(u, L(v), -u) \mid u + v \in -\bar{\Omega}, u - v \in \bar{\Omega}\}, \\ C_2 &= \{(u, L(v), -u) \mid u + v \in \bar{\Omega}, u - v \in \bar{\Omega}\}. \end{aligned}$$

Then C_1 and C_2 are $\mathrm{Ad}(H^c)-$invariant causal cones and

$$C_1 \cap \mathfrak{p}^c \neq \emptyset,\ C_2 \cap \mathfrak{k}^c \neq \emptyset.$$

Let $c : z \mapsto i(e + z)(e - z)^{-1}$ be the Cayley transform corredponding to the bounded symmetric domain D. Then we have

Theorem 3.1 ([K$_1$, K$_2$]). (1) *$H := c^{-1} \circ G_0 \circ c = H^c := G \cap G^c$.*
(2) *$M = G/H \simeq G^c/H^c$ is a symmetric space of Cayley type, and every Cayley type space is given in this way.*

Let

$$\Delta_S = \{(z, w) \in S \times S \mid \Delta(z - w) = 0\}$$

be the null space of $S \times S$, where Δ is the determinant function of V, extended to $V_{\mathbb{C}}$.

The group G acts diagonally on $S \times S$. Furthermore,

Theorem 3.2 ([K$_1$, K$_2$]). *G acts transitively on $S \times S \setminus \Delta_S$ and the stabilizer of the base point $(e, -e) \in S \times S \setminus \Delta_S$ is the subgroup H. Therefore, $M = G/H \simeq S \times S \setminus \Delta_S$ and $S \times S$ is the (causal) compactification of M.*

For example,

$$\begin{array}{ll} D & = SU(n,n)/S(U(n) \times U(n)) \\ & = \{z \in \mathrm{Mat}(n, \mathbb{C}) \mid I_n - z^* z \gg 0\} \\ S & = U(n) \\ M & = SU(n,n)/GL(n, \mathbb{C}) \times \mathbb{R}^+ \\ & \simeq \{(z, w) \in U(n) \times U(n) \mid \mathrm{Det}(z - w) \neq 0\}, \end{array} \quad \text{and}$$

$$\begin{array}{ll} D & = Sp(n, \mathbb{R})/U(n) \\ & = \{z \in \mathrm{Sym}(n, \mathbb{C}) \mid I_n - z^* z \gg 0\} \\ S & = U(n)/O(n) \\ & = \{z \in U(n) \mid z^t = z\} \\ M & = Sp(n, \mathbb{R})/GL(n, \mathbb{R}) \times \mathbb{R}^+ \\ & \simeq \{(z, w) \in U(n) \times U(n) \mid z^t = z, w^t = w,\ \mathrm{Det}(z - w) \neq 0\}. \end{array}$$

4. The Schwarzian derivative on the one-sheeted hyperboloid

Recall the classical cross-ratio of four points in the complex plane

$$[z_1, z_2, z_3, z_4] = \frac{z_1 - z_3}{z_2 - z_3} : \frac{z_1 - z_4}{z_2 - z_4}$$

Then the Élie Cartan formula for the cross-ratio is given by

Theorem 4.1 (Élie Cartan). *Consider $f : S^1 \to S^1$ and four points $x_1, x_2, x_3, x_4 \in S^1$ tending to $x \in S^1$. Then*

$$\begin{aligned} \tfrac{[f(x_1), f(x_2), f(x_3), f(x_4)]}{[x_1, x_2, x_3, x_4]} - 1 & = \tfrac{1}{6} S(f)(x)(x_1 - x_2)(x_3 - x_4) \\ & \qquad + [\text{ higher order terms }] \end{aligned}$$

where $S(f)$ denote the Schwarzian derivative of f,

$$S(f) = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2.$$

Consider now the the single sheeted hyperboloid

$$\begin{aligned}\mathcal{H} &= SL(2,\mathbb{R})/\mathbb{R}_+^* \\ &\simeq SU(1,1)/SO(1,1) \\ &\simeq S^1\times S^1\setminus \Delta_{S^1} \\ &= \{(e^{i\theta_1},e^{i\theta_2})\ :\ \theta_1\not\equiv\theta_2\}.\end{aligned}$$

$\mathcal{H}$ carries a (unique up to multiplicative constant) Lorentz metric, $\underline{\mathbf{g}}$, invariant under $SL(2,\mathbb{R})$,

$$\underline{\mathbf{g}} = \frac{d\theta_1 d\theta_2}{|e^{i\theta_1}-e^{i\theta_2}|^2}.$$

Let $f : S^1 \to S^1$ be a diffeomorphism viewed as a conformal transformation of $(\mathcal{H},\underline{\mathbf{g}})$, $f^*\underline{\mathbf{g}} = \mathbf{c}_f\underline{\mathbf{g}}$. Applying the Cartan formula, when $\theta_1,\theta_2\to\theta$, we get

$$\begin{aligned}\mathbf{c}_f(\theta_1,\theta_2)-1 &= \tfrac{f^*\underline{g}(\theta_1,\theta_2)}{\underline{g}(\theta_1,\theta_2)}-1 \\ &= \tfrac{1}{6}S(f)(e^{i\theta})(e^{i\theta_1}-e^{i\theta_2})^2+\dots\end{aligned}$$

Then we have

Theorem 4.2 (Kostant-Sternberg [KS])**.** *The conformal factor $\boldsymbol{c}_f\to 1$ as $(\theta_1,\theta_2)\to\Delta_{S^1}$. In the other word $\boldsymbol{c}_f$ tends to 1 on $\mathcal{H}$ as we approach the infinity.*
So let us extend $\boldsymbol{c}_f$ to be defined on $S^1\times S^1$ by setting it equal to 1 on Δ_{S^1}. Then $\boldsymbol{c}_f$ is twice differentiable on $S^1\times S^1$, and has Δ_{S^1} as critical manifold and the Hessian Hess$(\boldsymbol{c}_f)$ is equal to $S(f)$.

5. The Schwarzian derivative on symmetric spaces of Cayley type

The Kantor [Kan] cross-ratio for z_1,z_2,z_3,z_4 in $V_{\mathbb{C}}$, is the rational function

$$[z_1,z_2,z_3,z_4] = \frac{\Delta(z_1-z_3)}{\Delta(z_2-z_3)} : \frac{\Delta(z_1-z_4)}{\Delta(z_2-z_4)}$$

where Δ is the determinant function of V (extented to $V_{\mathbb{C}}$).

The cross-ratio is invariant under the group G^c (when it is well defined) : The invariance under translations is clear. The invariance under the group G_0 follows from the relation $\Delta(gz)=\chi(g)\Delta(z)$ where χ is a character of G_0. The invariance under the inversion follows from the Hua identity $\Delta(w^{-1}-z^{-1})=\Delta(z)^{-1}\Delta(z-w)\Delta(w)^{-1}$.

On the Cayley type symmetric space $M \simeq S \times S \setminus \Delta_S$ there exists a $G-$invariant measure

$$\underline{\mathbf{g}} = |\Delta(z-w)|^{-2\frac{n}{r}} d\sigma(z) d\sigma(w),$$

where n is the dimension of V and r its rank.

Let $\langle \cdot, \cdot \rangle$ be the inner product of the Euclidean Jordan algebra V extented to a Hermitian inner product of $V_{\mathbb{C}}$.

Let $f : V_{\mathbb{C}} \to V_{\mathbb{C}}$ be a map of class C^3. Let $z_j = z + ta_j u$ be four points tending to $z \in \bar{D}$, where $t \in \mathbb{R}$ and $a_i \in \mathbb{R}$ for $j = 1, 2, 3, 4$.

Theorem 5.1. *For any $\alpha \in \mathbb{R}$ we have*

$$\frac{[f(z_1), f(z_2), f(z_3), f(z_4)]^\alpha}{[z_1, z_2, z_3, z_4]^\alpha} - 1 = \alpha t^2 (a_1 - a_2)(a_3 - a_4) S(f)(z) + o(t^3)$$

where

$$S(f) = \frac{1}{6}\langle f''', f'^{-1}\rangle - \frac{1}{4}\langle P(f'') f'^{-1}, f'^{-1}\rangle$$

with $f' = Df(z)u$, $f'' = D^2 f(z)(u, u)$ and $f''' = D^3 f(z)(u, u, u)$

One can also prove

Theorem 5.2. *Let f be an orientation-preserving diffeomorphism of $(M, \underline{\boldsymbol{g}})$. Then*

(1) *$\boldsymbol{c}_f(z, w) \to 1$ as $z \to w$ and $\boldsymbol{c}_f$ extends smoothly to $S \times S$ and has, moreover, Δ_S as its critical set.*
(2) *The Schwarzian $S(f)$ completely determines $\boldsymbol{c}_f$.*

The complete proofs will appear in a forthcoming paper.

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