Peacocks obtained by normalisation and strong peacocks

Antoine-Marie Bogso, Christophe Profeta, Bernard Roynette

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Antoine-Marie BOGSO[(1)], Christophe PROFETA[(1)], Bernard ROYNETTE[(1)]

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Abstract: This paper contains two parts:
Part I. Let $(V_t, t \geq 0)$ be an integrable right-continuous process such that $\mathbb{E}[|V_t|] < \infty$, for every $t \geq 0$. Let us consider the three types of processes:

1. $(C_t := V_t - \mathbb{E}[V_t], t \geq 0)$,
2. $\left(N_t := \dfrac{V_t}{\mathbb{E}[V_t]}, t \geq 0\right)$, with $\mathbb{E}[V_t] > 0$ for every $t \geq 0$,
3. $\left(Q_t := \dfrac{V_t}{\alpha(t)}, t \geq 0\right)$, where $\mathbb{E}[V_t] = 0$ for every $t \geq 0$ and, $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ is a Borel function which is strictly positive.

We shall give some classes of processes $(V_t, t \geq 0)$ such that C, N or Q are peacocks.
Part II. We introduce the notion of strong and very strong peacocks which leads to the study of new classes of processes.

Introduction

This article deals with processes which increase in the convex order. The investigation of this family of processes has gained renewed interest since the work of Carr, Ewald and Xiao. Indeed, they showed that in the Black-Scholes model, the price of an arithmetic average Asian call option increases with maturity, i.e: if $(B_s, s \geq 0)$ is a Brownian motion issued from 0, the process $\left(\dfrac{1}{t}\displaystyle\int_0^t e^{B_s - \frac{s}{2}}\, ds, t \geq 0\right)$ increases in the convex order. Since, many classes of processes which increase in the convex order have been described and studied (see e.g. [HPRY]). The aim of this paper is to complete the known results by exhibiting new families of processes which increase in the convex order. Let us start with some elementary definitions and results.

Definition 0.1. *Let U and V be two real-valued r.v.'s. U is said to be dominated by V for the convex order if, for every convex function $\psi : \mathbb{R} \to \mathbb{R}$ such that $\mathbb{E}[|\psi(U)|] < \infty$ and $\mathbb{E}[|\psi(V)|] < \infty$, one has:*

$$\mathbb{E}[\psi(U)] \leq \mathbb{E}[\psi(V)]. \tag{0.1}$$

We denote this order relation by:

$$U \overset{(c)}{\leq} V. \tag{0.2}$$

Definition 0.2. *We denote by $\mathbf{C}$ the class of convex $\mathcal{C}^2$-functions $\psi : \mathbb{R} \to \mathbb{R}$ such that ψ'' has a compact support, and by $\mathbf{C}_+$ the class of convex functions $\psi \in \mathbf{C}$ such that ψ is positive and increasing.*

We note that if $\psi \in \mathbf{C}$:

- $|\psi'|$ is a bounded function,
- there exist k_1 and $k_2 \geq 0$ such that:

$$|\psi(x)| \leq k_1 + k_2|x|. \tag{0.3}$$

The next result is proved in [HPRY].

Lemma 0.3. *Let U and V be two integrable real-valued r.v.'s. Then, the followings are equivalent:*

1) $U \overset{(c)}{\leq} V$

2) for every $\psi \in \mathbf{C}$: $\mathbb{E}[\psi(U)] \leq \mathbb{E}[\psi(V)]$

3) $\mathbb{E}[U] = \mathbb{E}[V]$ and for every $\psi \in \mathbf{C}_+$: $\mathbb{E}[\psi(U)] \leq \mathbb{E}[\psi(V)]$.

Definition 0.4.

1) A process $(Z_t, t \geq 0)$ is said to be integrable if, for every $t \geq 0$, $\mathbb{E}[|Z_t|] < \infty$.

2) A process $(Z_t, t \geq 0)$ is said to be increasing (resp. decreasing) in the convex order if, for every $s \leq t$, $Z_s \overset{(c)}{\leq} Z_t$ (resp. $Z_t \overset{(c)}{\leq} Z_s$).

3) An integrable process which is increasing (resp decreasing) in the convex order will be called a peacock (resp. a peadock).

If $(Z_t, t \geq 0)$ is a peacock, then it follows from Definitions 0.1 and 0.4, applied with $\psi(x) = x$ and $\psi(x) = -x$, that $\mathbb{E}[Z_t]$ does not depend on t.
In the sequel, when two processes $(U_t, t \geq 0)$ and $(V_t, t \geq 0)$ have the same 1-dimensional marginals, we shall write

$$U_t \overset{(1.d)}{=} V_t \tag{0.4}$$

and say that $(U_t, t \geq 0)$ and $(V_t, t \geq 0)$ are associated.
From Jensen's inequality, every martingale $(M_t, t \geq 0)$ is a peacock; conversely, a result due to Kellerer [Kel72] states that, for any peacock $(Z_t, t \geq 0)$, there exists (at least) a martingale $(M_t, t \geq 0)$ such that:

$$Z_t \overset{(1.d)}{=} M_t. \tag{0.5}$$

Many examples of peacocks with a description of associated martingales are given in [HPRY]. One may also refer to [BY09], [HRY09a] and [HRY09b] where the notions of Brownian and Lévy Sheet play an essential role in constructing associated martingales to certain peacocks.
On the contrary, we note that for most of the peacocks given in this article, the question of finding an associated martingale remains open.
We give below the plan of this paper; a sum-up of the main results is found at the end.

Contents

Part I
Peacocks obtained by normalisation, centering and quotient

1 Preliminaries

1.1 The aim of this part

Let $(V_t, t \geq 0)$ be an integrable right-continuous process such that $\mathbb{E}[|V_t|] < \infty$, for every $t \geq 0$ and, let us consider the three families of processes:

1. $(C_t := V_t - \mathbb{E}[V_t], t \geq 0)$,
2. $\left(N_t := \dfrac{V_t}{\mathbb{E}[V_t]}, t \geq 0\right)$, with $\mathbb{E}[V_t] > 0$ for every $t \geq 0$,
3. $\left(Q_t := \dfrac{V_t}{\alpha(t)}, t \geq 0\right)$, where $\mathbb{E}[V_t] = 0$ for every $t \geq 0$ and $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ is a Borel function which is strictly positive.

We adopt the notation C for centering, N for normalisation and Q for quotient.
We note that, for every $t \geq 0$, $\mathbb{E}[C_t] = \mathbb{E}[Q_t] = 0$ and $\mathbb{E}[N_t] = 1$. Since $\mathbb{E}[C_t]$, $\mathbb{E}[N_t]$ and $\mathbb{E}[Q_t]$ do not depend on t, it is a natural question to ask under which conditions on $(V_t, t \geq 0)$ the processes C, N and Q are peacocks.
Let us first recall the following elementary lemma (see [HPRY]).

Lemma 1.1. *Let U be a real-valued integrable random variable. Then, the following properties are equivalent:*

1) for every real c, $\mathbb{E}\left[1_{\{U \geq c\}} U\right] \geq 0$,

2) for every bounded and increasing function $h : \mathbb{R} \to \mathbb{R}_+$:

$$\mathbb{E}[h(U)U] \geq 0,$$

3) $\mathbb{E}[U] \geq 0$.

1.2 Some examples

We now deal with some examples.

Example 1.2.

i) If $V_t = \displaystyle\int_0^t e^{B_s - \frac{s}{2}} ds$, then Carr, Ewald and Xiao [CEX08] showed that:

$$\left(N_t := \frac{1}{t} \int_0^t e^{B_s - \frac{s}{2}} ds, t \geq 0\right) \text{ is a peacock,}$$

when $(B_s, s \geq 0)$ is a Brownian motion issued from 0. We note that $\mathbb{E}[V_t] = t$ for every $t \geq 0$.

ii) If $V_t = \int_0^t M_s d\alpha(s)$ (resp. $V_t = \int_0^t (M_s - M_0) d\alpha(s)$), where $(M_s, s \geq 0)$ is a martingale in H^1_{loc} and $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$, a continuous and increasing function such that $\alpha(0) = 0$, then it is shown in [HPRY], Chapter 1, that:

$$\left(N_t := \frac{1}{\alpha(t)} \int_0^t M_s \, d\alpha(s), t \geq 0 \right)$$

(resp.

$$\left(C_t := \int_0^t (M_s - M_0) \, d\alpha(s), t \geq 0 \right))$$

is a peacock. Let us note that, for every $t \geq 0$, one has $\mathbb{E}[V_t] = \alpha(t)\mathbb{E}[N_0]$ (resp. $\mathbb{E}[V_t] = 0$). We shall generalize this result in Theorem 5.5 thanks to the notion of very strong peacock.

Example 1.3.

i) If $V_t = tX$, where X is a centered real-valued r.v. such that $E[|X|] < \infty$, then $(C_t := tX, t \geq 0)$ is a peacock (see [HPRY], Chapter 1).

ii) If $V_t = e^{tX}$, where X is a real-valued r.v. such that, for every $t \geq 0$, $\mathbb{E}[e^{tX}] < \infty$, then, it is proved in [HPRY] (see also Example 1.5) that:

$$\left(N_t := \frac{e^{tX}}{\mathbb{E}[e^{tX}]}, t \geq 0 \right) \text{ is a peacock.}$$

In particular, if $(G_u, u \geq 0)$ is a centered Gaussian process with covariance function $K(s,t) := \mathbb{E}[G_s G_t]$ and with measurable paths, and if ν is a positive Radon measure on $\mathbb{R}_+$, then:

$$\left(N_t^{(\nu)} := \frac{\exp\left(\int_0^t G_u \nu(du) \right)}{\mathbb{E}\left[\exp\left(\int_0^t G_u \nu(du) \right) \right]}, t \geq 0 \right) \text{ is a peacock}$$

as soon as

$$t \longmapsto \gamma(t) := \int_0^t \int_0^t K(u,v)\nu(du)\nu(dv) \text{ is an increasing function.}$$

Indeed,

$$\int_0^t G_u \nu(du) \overset{\text{(law)}}{=} \sqrt{\gamma(t)} G,$$

with G a reduced normal r.v. If $(B_t, t \geq 0)$ is a Brownian motion issued from 0, then $\left(M_t := \exp\left(B_{\gamma(t)} - \frac{\gamma(t)}{2} \right), t \geq 0 \right)$ is a martingale associated to $\left(N_t^{(\nu)}, t \geq 0 \right)$.

Example 1.4. Let $(X_t, t \geq 0)$ be a centered and integrable process and $\alpha, \beta : \mathbb{R}_+ \to \mathbb{R}_+$ be two strictly positive Borel functions. We suppose that $\left(Q_t^{(\alpha)} := \frac{X_t}{\alpha(t)}, t \geq 0\right)$ is a peacock.

a) if $t \longmapsto \frac{\alpha(t)}{\beta(t)}$ is increasing, then:

$$\left(Q_t^{(\beta)} := \frac{X_t}{\beta(t)}, t \geq 0\right) \text{ is a peacock.}$$

b) if $\left(Q_t^{(\beta)} := \frac{X_t}{\beta(t)}, t \geq 0\right)$ is a peadock and if, for every $t \geq 0$, X_t is not identically 0, then

$$t \longmapsto \frac{\alpha(t)}{\beta(t)} \text{ is decreasing.}$$

<u>Proof.</u>

a) For every $\psi \in \mathbf{C}$ and every $0 \leq s < t$, we have:

$$\mathbb{E}\left[\psi\left(\frac{X_t}{\beta(t)}\right)\right] = \mathbb{E}\left[\psi\left(\frac{X_t}{\alpha(t)}\frac{\alpha(t)}{\beta(t)}\right)\right] \geq \mathbb{E}\left[\psi\left(\frac{X_s}{\alpha(s)}\frac{\alpha(t)}{\beta(t)}\right)\right]$$

(since $\left(Q_t^{(\alpha)} := \frac{X_t}{\alpha(t)}, t \geq 0\right)$ is a peacock)

$$\geq \mathbb{E}\left[\psi\left(\frac{X_s}{\alpha(s)}\frac{\alpha(s)}{\beta(s)}\right)\right]$$

(from point i) of Example 1.3 since X_s is centered and $t \longmapsto \frac{\alpha(t)}{\beta(t)}$ is increasing).

$$= \mathbb{E}\left[\psi\left(\frac{X_s}{\beta(s)}\right)\right]$$

b) Let us suppose that there exist $0 \leq s < t$ such that:

$$\frac{\alpha(s)}{\beta(s)} < \frac{\alpha(t)}{\beta(t)},$$

then, for every $\psi \in \mathbf{C}_+$, we should have:

$$\mathbb{E}\left[\psi\left(\frac{X_s}{\beta(s)}\right)\right] = \mathbb{E}\left[\psi\left(\frac{X_s}{\alpha(s)}\frac{\alpha(s)}{\beta(s)}\right)\right]$$

and we may choose ψ such that:

$$\mathbb{E}\left[\psi\left(\frac{X_s}{\alpha(s)}\frac{\alpha(s)}{\beta(s)}\right)\right] < \mathbb{E}\left[\psi\left(\frac{X_s}{\alpha(s)}\frac{\alpha(t)}{\beta(t)}\right)\right]$$

(from point i) of Example 1.3 since X_s is centered and not identically 0)

Hence,

$$\mathbb{E}\left[\psi\left(\frac{X_s}{\beta(s)}\right)\right] < \mathbb{E}\left[\psi\left(\frac{X_s}{\alpha(s)}\frac{\alpha(t)}{\beta(t)}\right)\right]$$
$$\leq \mathbb{E}\left[\psi\left(\frac{X_t}{\alpha(t)}\frac{\alpha(t)}{\beta(t)}\right)\right] \text{ (since } Q^{(\alpha)} \text{ is a peacock).}$$

This contradicts the fact that $Q^{(\beta)}$ is a peadock. □

Example 1.5. Let $(X_t, t \geq 0)$ be an $\mathbb{R}_+$-valued process with measurable paths such that $0 < \mathbb{E}[X_t] < \infty$:

a) if, for every $0 \leq s \leq t$, $x \longmapsto \frac{1}{x}\mathbb{E}[X_t | X_s = x]$ is an increasing function, then:

$$\left(\frac{X_t}{\mathbb{E}[X_t]}, t \geq 0 \right) \text{ is a peacock.}$$

b) if, for every $0 \leq s \leq t$, $x \longmapsto \frac{1}{x}\mathbb{E}[X_s | X_t = x]$ is an increasing function, then:

$$\left(\frac{X_t}{\mathbb{E}[X_t]}, t \geq 0 \right) \text{ is a peadock.}$$

Proof.

a) For every $\psi \in \mathbf{C}_+$ and every $0 \leq s < t$, we have:

$$\begin{aligned}
&\mathbb{E}\left[\psi\left(\frac{X_t}{\mathbb{E}[X_t]}\right)\right] - \mathbb{E}\left[\psi\left(\frac{X_s}{\mathbb{E}[X_s]}\right)\right] \\
&\geq \mathbb{E}\left[\psi'\left(\frac{X_s}{\mathbb{E}[X_s]}\right)\left(\frac{X_t}{\mathbb{E}[X_t]} - \frac{X_s}{\mathbb{E}[X_s]}\right)\right] \qquad \text{(since } \psi \text{ is convex)} \\
&= \mathbb{E}\left[\psi'\left(\frac{X_s}{\mathbb{E}[X_s]}\right)\frac{X_s}{\mathbb{E}[X_s]}\left(\frac{\mathbb{E}[X_s]}{\mathbb{E}[X_t]}\,\frac{\mathbb{E}[X_t|X_s]}{X_s} - 1\right)\right] \geq 0 \\
&\text{(since } x \longmapsto \frac{1}{x}\mathbb{E}[X_t|X_s = x] \text{ is increasing and using a slight} \\
&\quad \text{extension of Lemma 1.1.)}
\end{aligned}$$

b) For every $\psi \in \mathbf{C}_+$ and every $0 \leq s < t$, one has:

$$\begin{aligned}
&\mathbb{E}\left[\psi\left(\frac{X_t}{\mathbb{E}[X_t]}\right)\right] - \mathbb{E}\left[\psi\left(\frac{X_s}{\mathbb{E}[X_s]}\right)\right] \\
&\leq \mathbb{E}\left[\psi'\left(\frac{X_t}{\mathbb{E}[X_t]}\right)\left(\frac{X_t}{\mathbb{E}[X_t]} - \frac{X_s}{\mathbb{E}[X_s]}\right)\right] \qquad \text{(since } \psi \text{ is convex)} \\
&= \mathbb{E}\left[\psi'\left(\frac{X_t}{\mathbb{E}[X_t]}\right)\frac{X_t}{\mathbb{E}[X_t]}\left(1 - \frac{\mathbb{E}[X_t]}{\mathbb{E}[X_s]}\,\frac{\mathbb{E}[X_s|X_t]}{X_t}\right)\right] \leq 0 \\
&\text{(since } x \longmapsto \frac{1}{x}\mathbb{E}[X_s|X_t = x] \text{ is increasing and using a slight} \\
&\quad \text{extension of Lemma 1.1.)}
\end{aligned}$$

□

In particular,

i) Let $\phi : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}_+$ such that:

- for every $t \geq 0$, $x \mapsto \phi(t,x)$ is increasing,
- for every $s < t$, $x \mapsto \frac{\phi(t,x)}{\phi(s,x)}$ is increasing.

Then, if X is a r.v. such that $\mathbb{E}[\phi(t,X)] < \infty$ for every $t \geq 0$,

$$\left(N_t := \frac{\phi(t,X)}{\mathbb{E}[\phi(t,X)]}, t \geq 0 \right) \text{ is a peacock;}$$

ii) Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be an increasing $\mathcal{C}^1$-function such that $x \mapsto \dfrac{xf'(x)}{f(x)}$ is decreasing. If $(\gamma_t, t \geq 0)$ denotes the Gamma subordinator, then

$$\left(N_t := \frac{f(\gamma_t)}{\mathbb{E}[f(\gamma_t)]}, t \geq 0\right) \text{ is a peadock.}$$

This assertion follows from point b) above and from a well-known property of the Gamma subordinator: for every $t \geq 0$, the (Dirichlet) process $\left(\dfrac{\gamma_s}{\gamma_t}, 0 \leq s \leq t\right)$ is independent of the r.v. γ_t.

Example 1.6. Let $(X_t, t \geq 0)$ be a càdlàg $\mathbb{R}_+$-valued process such that

$$\mathbb{E}\left[\sup_{0 \leq u \leq t} |X_u|\right] < \infty$$

and let ν be a positive Radon measure on $\mathbb{R}_+$. Suppose that, for every $0 \leq s \leq t$, $\varphi_{s,t} : x \longmapsto \dfrac{1}{x}\mathbb{E}[X_s | X_t = x]$ is an increasing function. Then:

$$\left(N_t := \frac{\int_0^t X_u \nu(du)}{\mathbb{E}\left[\int_0^t X_u \nu(du)\right]}, t \geq 0\right) \text{ is a peadock.} \tag{1.1}$$

Proof of (1.1).
By approximation, we may assume that ν is absolutely continuous with respect to the Lebesgue measure and admits a continuous Radon-Nikodym density f:

$$\nu(dt) = f(t)dt. \tag{1.2}$$

Then, with $h(t) := \mathbb{E}\left[\int_0^t X_u \nu(du)\right]$ and $\psi \in \mathbf{C}$:

$$\begin{aligned}
&\frac{d}{dt}\mathbb{E}\left[\psi\left(\frac{\int_0^t X_u f(u)du}{h(t)}\right)\right] \\
&= \mathbb{E}\left[\psi'\left(\frac{\int_0^t X_u f(u)du}{h(t)}\right)\left(\frac{X_t f(t)}{h(t)} - \frac{h'(t)}{h^2(t)}\int_0^t X_u f(u)du\right)\right] \\
&\leq \frac{1}{h(t)}\mathbb{E}\left[\psi'\left(\frac{X_t f(t)}{h'(t)}\right)\left(X_t f(t) - \frac{h'(t)}{h(t)}\int_0^t X_u f(u)du\right)\right]
\end{aligned}$$

since ψ' is increasing and, if

$$\frac{X_t f(t)}{h(t)} \geq \frac{h'(t)}{h^2(t)}\int_0^t X_u f(u)du \quad \left(\text{resp. } \frac{X_t f(t)}{h(t)} \leq \frac{h'(t)}{h^2(t)}\int_0^t X_u f(u)du\right),$$

then

$$\frac{X_t f(t)}{h'(t)} \geq \frac{\int_0^t X_u f(u)du}{h(t)} \quad \left(\text{resp. } \frac{X_t f(t)}{h'(t)} \leq \frac{\int_0^t X_u f(u)du}{h(t)}\right).$$

Thus,

$$
\begin{aligned}
&\frac{d}{dt}\mathbb{E}\left[\psi\left(\frac{\int_0^t X_u f(u)du}{h(t)}\right)\right] \\
&\leq \frac{1}{h(t)}\mathbb{E}\left[\psi'\left(\frac{X_t f(t)}{h'(t)}\right)\left(X_t f(t) - \frac{h'(t)}{h(t)}\int_0^t X_u f(u)du\right)\right] \\
&= \frac{1}{h(t)}\mathbb{E}\left[\psi'\left(\frac{X_t f(t)}{h'(t)}\right)\left(X_t f(t) - \frac{h'(t)}{h(t)}\int_0^t \mathbb{E}[X_u|X_t] f(u)du\right)\right] \\
&= \frac{1}{h(t)}\mathbb{E}\left[\psi'\left(\frac{X_t f(t)}{h'(t)}\right) X_t \left(f(t) - \frac{h'(t)}{h(t)}\int_0^t \varphi_{u,t}(X_t) f(u)du\right)\right] \\
&\text{(where } \varphi_{u,t} \text{ is an increasing function)} \\
&\leq 0 \quad \text{(by Lemma 1.1).}
\end{aligned}
$$

□

In particular, if $f : \mathbb{R}_+ \to \mathbb{R}_+$ is an increasing $\mathcal{C}^1$-function such that $x \mapsto \dfrac{xf'(x)}{f(x)}$ is increasing and if $(\gamma_t, t \geq 0)$ stands for the Gamma subordinator, then:

$$
\left(N_t := \frac{\displaystyle\int_0^t f(\gamma_s)\, ds}{\mathbb{E}\left[\displaystyle\int_0^t f(\gamma_s)\, ds\right]}, t \geq 0 \right) \text{ is a peadock.}
$$

One may compare this result with the second point of Remark 5.6.

Example 1.7.

i) Let $L := (L_t, t \geq 0)$ be an integrable Lévy process and ν be a positive Radon measure on $\mathbb{R}_+$ such that $\nu(\{0\}) = 0$. Then:

a) If L is centered, then $\left(Q_t^{(\nu)} := \dfrac{1}{\nu([0,t])}\displaystyle\int_0^t L_u \nu(du), t \geq 0\right)$ is a peacock

b) $\left(N_t^{(\nu)} := \dfrac{\displaystyle\int_0^t L_u \nu(du)}{\displaystyle\int_0^t u\, \nu(du)}, t \geq 0 \right)$ is a peadock.

Proof.

The assertion a) is deduced from point ii) of Example 1.2 since $(L_t, t \geq 0)$ is a centered martingale. To prove b), we shall make some computations closed to those in the proof of Example 1.6 (although here, L does not take values in $\mathbb{R}_+$). We may suppose, without loss of generality, that L is centered and, as in the proof of Example 1.6, that $\nu(du) = f(u)du$, where f is continuous. Let us set $h(t) := \int_0^t uf(u)du$. Then, for every $\psi \in \mathbf{C}$:

$$\begin{aligned}
&\frac{d}{dt}\mathbb{E}\left[\psi\left(\frac{\int_0^t L_u f(u)du}{h(t)}\right)\right] \\
&= \mathbb{E}\left[\psi'\left(\frac{\int_0^t L_u f(u)du}{h(t)}\right)\left(\frac{L_t f(t)}{h(t)} - \frac{tf(t)}{h^2(t)}\int_0^t L_u f(u)du\right)\right] \\
&\leq \frac{tf(t)}{h(t)}\mathbb{E}\left[\psi'\left(\frac{L_t}{t}\right)\left(\frac{L_t}{t} - \frac{1}{h(t)}\int_0^t L_u f(u)du\right)\right] \\
&= \frac{tf(t)}{h(t)}\mathbb{E}\left[\psi'\left(\frac{L_t}{t}\right)\left(\frac{L_t}{t} - \frac{1}{h(t)}\int_0^t \mathbb{E}\left[L_u|\mathcal{F}_t^+\right] f(u)du\right)\right] \\
&\text{(where } \mathcal{F}_t^+ = \sigma(L_u, u \geq t)\text{).}
\end{aligned}$$

Observing that $\left(\frac{L_t}{t}, t \geq 0\right)$ is an inverse martingale with respect to the filtration $\left(\mathcal{F}_t^+, t \geq 0\right)$ (i.e., for every $0 < s \leq t$, $\mathbb{E}\left[\frac{L_s}{s}\middle|\mathcal{F}_t^+\right] = \frac{L_t}{t}$, see [JP88]), we obtain:

$$\begin{aligned}
&\frac{d}{dt}\mathbb{E}\left[\psi\left(\frac{\int_0^t L_u f(u)du}{h(t)}\right)\right] \\
&\leq \frac{tf(t)}{h(t)}\mathbb{E}\left[\psi'\left(\frac{L_t}{t}\right)\frac{L_t}{t}\left(1 - \frac{1}{h(t)}\int_0^t uf(u)du\right)\right] = 0.
\end{aligned}$$

ii) Therefore, the following question arises naturally: for which values of α $\left(\frac{1}{t^\alpha}\int_0^t L_u du, t \geq 0\right)$ is either a peacock (it is true for $\alpha \leq 1$) or a peadock (it is true for $\alpha \geq 2$) or neither of them?

a) If $(L_s, s \geq 0)$ is a Brownian motion (issued from 0), then, by scaling, $\left(\frac{1}{t^\alpha}\int_0^t L_u du, t \geq 0\right)$ is a peacock (resp. a peadock) if and only if $\alpha \leq \frac{3}{2}$ (resp. $\alpha \geq \frac{3}{2}$).

b) Let $(L_s, s \geq 0)$ be a Lévy process which is stable of index γ $(1 < \gamma \leq 2)$ and symmetric. Then $\left(\frac{1}{t^\alpha}\int_0^t L_u du, t \geq 0\right)$ is a peacock (resp. a peadock) if and only if $\alpha \leq 1 + \frac{1}{\gamma}$ (resp. $\alpha \geq 1 + \frac{1}{\gamma}$). Indeed, by scaling, $\int_0^t L_u du \overset{(1.d)}{=} t^{1+\frac{1}{\gamma}} S$, where the r.v. S is symmetric and stable of index γ (see point i) of Example 1.3).

c) Let $(L_s, s \geq 0)$ be a square integrable and centered Lévy process. We have

$$\mathbb{E}\left[\left(\frac{1}{t^\alpha}\int_0^t L_u du\right)^2\right] = c\, t^{3-2\alpha}, \text{ (where } c \text{ is constant).}$$

Hence:

• if $\alpha < \dfrac{3}{2}$, $\left(\dfrac{1}{t^\alpha}\displaystyle\int_0^t L_u du, t \geq 0\right)$ is not a peadock,

• if $\alpha > \dfrac{3}{2}$, $\left(\dfrac{1}{t^\alpha}\displaystyle\int_0^t L_u du, t \geq 0\right)$ is not a peacock.

In some specific situations, we may obtain simultaneously a peacock with one of its associated martingale. The results of the following Example are closed to those obtained in [HPRY]. Therefore, we state them without proof and refer the reader to ([HPRY], Chapter 2).

Example 1.8. Let $(L_t, t \geq 0)$ be a Lévy process such that:

$$\mathbb{E}\left[\exp\left(\int_0^t L_s\, ds\right)\right] < \infty, \text{ for every } t \geq 0.$$

Then:

1)

$$\left(N_t := \frac{\exp\left(\int_0^t L_s\, ds\right)}{\mathbb{E}\left[\exp\left(\int_0^t L_s\, ds\right)\right]}, t \geq 0\right) \text{ is a peacock}$$

and

$$\left(M_t := \frac{\exp\left(\int_0^t s\, dL_s\right)}{\mathbb{E}\left[\exp\left(\int_0^t s\, dL_s\right)\right]}, t \geq 0\right)$$

is a martingale associated to $(N_t, t \geq 0)$.

2)

$$\left(\widetilde{N}_t := \frac{\exp\left(\frac{1}{t}\int_0^t L_s\, ds\right)}{\mathbb{E}\left[\exp\left(\frac{1}{t}\int_0^t L_s\, ds\right)\right]}, t \geq 0\right) \text{ is a peacock}$$

and

$$\left(\widetilde{M}_t := \frac{\exp\left(\int_0^1 W_{u,t}^{(L)}\, du\right)}{\mathbb{E}\left[\exp\left(\int_0^1 W_{u,t}^{(L)}\, du\right)\right]}, t \geq 0\right)$$

is a $(\mathcal{G}_t^{(L)}, t \geq 0)$-martingale associated to $(\widetilde{N}_t, t \geq 0)$, where $\left(W_{u,t}^{(L)}, u \geq 0, t \geq 0\right)$ is the Lévy sheet associated to $(L_t, t \geq 0)$ and

$$\mathcal{G}_t^{(L)} = \sigma\left(W_{u,s}^{(L)}, u \geq 0, 0 \leq s \leq t\right) \text{ (see [HRY10]).}$$

1.3 Relation between the peacock properties of C and N

The peacock properties of C and N are linked as it is shown in the following:

Theorem 1.9. *Suppose that $(V_t, t \geq 0)$ is integrable, $\mathbb{E}[V_t] > 0$ for every $t \geq 0$ and $t \longmapsto \mathbb{E}[V_t]$ is monotone.*

1. *If $t \longmapsto \mathbb{E}[V_t]$ increases, we have the implication:*

$$(N_t, t \geq 0) \text{ is a peacock } \Rightarrow (C_t, t \geq 0) \text{ is a peacock.}$$

2. *If $t \longmapsto \mathbb{E}[V_t]$ decreases, we have the reverse implication:*

$$(C_t, t \geq 0) \text{ is a peacock } \Rightarrow (N_t, t \geq 0) \text{ is a peacock.}$$

We shall give two proofs of Theorem 1.9. In the first proof, we use Kellerer's theorem which is not necessary in the second one.

First proof of Theorem 1.9

1) We first assume that $t \longmapsto \mathbb{E}[V_t]$ is an increasing function and that $\left(N_t := \dfrac{V_t}{\mathbb{E}[V_t]}, t \geq 0\right)$ is a peacock. Then, from Kellerer's theorem, there exists a martingale $(M_t, t \geq 0)$ such that:

$$\frac{V_t}{\mathbb{E}[V_t]} \overset{(1.d)}{=} M_t, \quad \text{or, equivalently,} \quad V_t \overset{(1.d)}{=} M_t\, \mathbb{E}[V_t].$$

We note that $\mathbb{E}[M_t] = 1$ for every $t \geq 0$.
For every $\psi \in \mathbf{C}_+$ and every $0 < s \leq t$, one has:

$$\begin{aligned}
&\mathbb{E}[\psi(C_t)] - \mathbb{E}[\psi(C_s)] \\
&= \mathbb{E}[\psi\left(V_t - \mathbb{E}[V_t]\right)] - \mathbb{E}[\psi\left(V_s - \mathbb{E}[V_s]\right)] \\
&= \mathbb{E}[\psi\left((M_t - 1)\mathbb{E}[V_t]\right)] - \mathbb{E}[\psi\left((M_s - 1)\mathbb{E}[V_s]\right)] \\
&\geq \mathbb{E}\left[\psi'\left((M_s - 1)\mathbb{E}[V_s]\right)\left((M_t - 1)\mathbb{E}[V_t] - (M_s - 1)\mathbb{E}[V_s]\right)\right] \text{ (by convexity)} \\
&= \mathbb{E}\left[\psi'\left((M_s - 1)\mathbb{E}[V_s]\right)\left((M_s - 1)\mathbb{E}[V_t] - (M_s - 1)\mathbb{E}[V_s]\right)\right] \\
&\text{(taking the conditional expectation)} \\
&= \mathbb{E}\left[\psi'\left((M_s - 1)\mathbb{E}[V_s]\right)(M_s - 1)\right] \underbrace{(\mathbb{E}[V_t] - \mathbb{E}[V_s])}_{\geq 0} \\
&\geq \psi'(0)(\mathbb{E}[V_t] - \mathbb{E}[V_s])\mathbb{E}[M_s - 1] = 0 \text{ (since } \psi' \text{ is increasing).}
\end{aligned}$$

2) We now assume that $t \longmapsto \mathbb{E}[V_t]$ is a decreasing function and that $(C_t := V_t - \mathbb{E}[V_t], t \geq 0)$ is a peacock. From Kellerer's theorem, there exists a martingale $(M_t, t \geq 0)$ such that:

$$V_t - \mathbb{E}[V_t] \overset{(1.d)}{=} M_t \quad \text{or, equivalently,} \quad V_t \overset{(1.d)}{=} M_t + \mathbb{E}[V_t].$$

We note that $\mathbb{E}[M_t] = 0$ for every $t \geq 0$. Let $\psi \in \mathbf{C}_+$ and $0 < s \leq t$:

$$\mathbb{E}[\psi(N_t)] - \mathbb{E}[\psi(N_s)] = \mathbb{E}\left[\psi\left(\frac{V_t}{\mathbb{E}[V_t]}\right)\right] - \mathbb{E}\left[\psi\left(\frac{V_s}{\mathbb{E}[V_s]}\right)\right]$$
$$= \mathbb{E}\left[\psi\left(\frac{M_t}{\mathbb{E}[V_t]} + 1\right)\right] - \mathbb{E}\left[\psi\left(\frac{M_s}{\mathbb{E}[V_s]} + 1\right)\right]$$
$$\geq \mathbb{E}\left[\psi'\left(\frac{M_s}{\mathbb{E}[V_s]} + 1\right)\left(\frac{M_t}{\mathbb{E}[V_t]} - \frac{M_s}{\mathbb{E}[V_s]}\right)\right] \text{ (by convexity)}$$
$$= \mathbb{E}\left[\psi'\left(\frac{M_s}{\mathbb{E}[V_s]} + 1\right) M_s\right] \underbrace{\left(\frac{1}{\mathbb{E}[V_t]} - \frac{1}{\mathbb{E}[V_s]}\right)}_{\geq 0} \text{ (taking the conditional expectation)}$$
$$\geq \psi'(1)\left(\frac{1}{\mathbb{E}[V_t]} - \frac{1}{\mathbb{E}[V_s]}\right)\mathbb{E}[M_s] = 0 \text{ (since } \psi' \text{ is increasing).}$$

□

Second proof of Theorem 1.9
For every $t \geq 0$, we set $\alpha(t) = \mathbb{E}[V_t]$. Then, for every convex function $\psi \in \mathbf{C}$, we have:

$$\mathbb{E}[\psi\,(N_t)] = \mathbb{E}\left[\psi\left(\frac{1}{\alpha(t)}(V_t - \alpha(t)) + 1\right)\right] = \mathbb{E}\left[\widetilde{\psi}\left(\frac{C_t}{\alpha(t)}\right)\right], \tag{1.3}$$

where $\widetilde{\psi}(x) := \psi(x+1)$.

1) To prove the first point, we assume without loss of generality that

$$\widetilde{\psi}(0) = \widetilde{\psi}'(0) = 0 \tag{1.4}$$

Let us assume that $(N_t, t \geq 0)$ is a peacock. Then, for every $0 < s \leq t$, (1.3) implies that:

$$\mathbb{E}\left[\widetilde{\psi}\left(\frac{C_t}{\alpha(t)}\right)\right] = \mathbb{E}\left[\psi\,(N_t)\right] \geq \mathbb{E}\left[\psi\,(N_s)\right] = \mathbb{E}\left[\widetilde{\psi}\left(\frac{C_s}{\alpha(s)}\right)\right] \geq \mathbb{E}\left[\widetilde{\psi}\left(\frac{C_s}{\alpha(t)}\right)\right] \tag{1.5}$$

since $\dfrac{1}{\alpha(t)} \leq \dfrac{1}{\alpha(s)}$ and, from (1.4), $\widetilde{\psi}$ increases on $[0, +\infty[$ and decreases on $]-\infty, 0]$. Hence,

$$\mathbb{E}\left[\widetilde{\psi}\left(\frac{C_t}{\alpha(t)}\right)\right] \geq \mathbb{E}\left[\widetilde{\psi}\left(\frac{C_s}{\alpha(t)}\right)\right],$$

where $x \longmapsto \widetilde{\psi}\left(\dfrac{x}{\alpha(t)}\right)$ stands for any convex function.

2) The second point follows from (1.3). Indeed, if $(C_t, t \geq 0)$ is a peacock, then for every convex function $\widetilde{\psi} \in \mathbf{C}$ such that $\widetilde{\psi}(0) = \widetilde{\psi}'(0) = 0$ and every $0 < s \leq t$,

we have:

$$\begin{aligned}
&\mathbb{E}\left[\psi\left(N_t\right)\right] = \mathbb{E}\left[\widetilde{\psi}\left(\frac{C_t}{\alpha(t)}\right)\right] \\
&\geq \mathbb{E}\left[\widetilde{\psi}\left(\frac{C_s}{\alpha(t)}\right)\right] \text{ (since } (C_t, t \geq 0) \text{ is a peacock)} \\
&\geq \mathbb{E}\left[\widetilde{\psi}\left(\frac{C_s}{\alpha(s)}\right)\right] \text{ (since } \frac{1}{\alpha(t)} \geq \frac{1}{\alpha(s)}, C_s \text{ is centered and } \widetilde{\psi}(0) = \widetilde{\psi}'(0) = 0) \\
&= \mathbb{E}\left[\psi\left(N_s\right)\right].
\end{aligned}$$

□

Illustration of Theorem 1.9
The next example shows that N may be a peacock while C is not.

Example 1.10. Let X be a real-valued random variable and let μ be the law of X. Suppose that $\mathbb{E}\left[|X|e^{tX}\right] < \infty$ for every $t \geq 0$ and supp $\mu = \mathbb{R}$. Let $\alpha(t) := \mathbb{E}\left[e^{tX}\right]$. Then:

1. $\left(N_t := \dfrac{e^{tX}}{\alpha(t)}, t \geq 0\right)$ is a peacock.
2. $\left(C_t := e^{tX} - \alpha(t), t \geq 0\right)$ is a peacock if and only if α is increasing.

We note that, if $\mathbb{E}[X] > 0$, then, from Lemma 1.1,

$$\alpha'(t) = \mathbb{E}[Xe^{tX}] \geq 0. \tag{1.6}$$

Proof of Example 1.10
The first point is a particular case of Example 1.5. To prove the second point, we note that

$$0 = \frac{\partial}{\partial t}\mathbb{E}[C_t] = \mathbb{E}\left[Xe^{tX} - \alpha'(t)\right], \text{ for every } t \geq 0 \tag{1.7}$$

and, for every convex function $\psi \in \mathbf{C}_+$:

$$\frac{\partial}{\partial t}\mathbb{E}[\psi(C_t)] = \mathbb{E}[\psi'(e^{tX} - \alpha(t))(Xe^{tX} - \alpha'(t))]. \tag{1.8}$$

i) Let us suppose, on one hand, that α is increasing. The function $f_t : x \longmapsto xe^{tx} - \alpha'(t)$ has exactly one zero $a \geq 0$ and

$$f_t(x) > 0, \text{ for every } x > a.$$

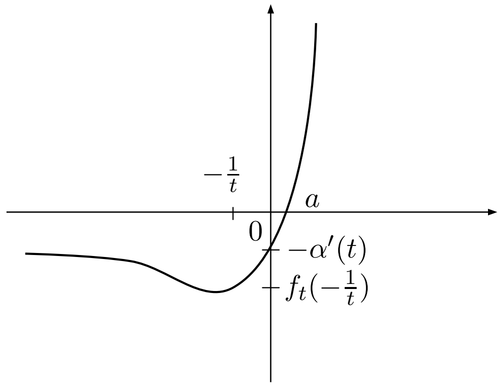

fig.1. Graph of f_t when α is strictly increasing and $t > 0$.

Indeed, the derivative function f'_t of f_t is strictly positive on $[0, \infty[$; hence, f_t is a continuous and strictly increasing function, i.e., a bijection map from $[0, \infty[$ to $[-\alpha'(t), \infty[$, $0 \in [-\alpha'(t), \infty[$ since $\alpha'(t) \geq 0$ for every $t \geq 0$ and, $f_t^{-1}(0) = a$; moreover, $f_t(x) < 0$ for every $x < 0$; therefore, distinguishing the cases $X \leq a$ and $X \geq a$, we have:

$$\frac{\partial}{\partial t}\mathbb{E}[\psi(C_t)] \geq \psi'\left(e^{ta} - \alpha(t)\right)\mathbb{E}\left[Xe^{tX} - \alpha'(t)\right] = 0.$$

Then, $\left(C_t := e^{tX} - \alpha(t), t \geq 0\right)$ is a peacock if α increases.
ii) On the other hand, if α is not increasing, then there exists $t_0 > 0$ such that $\alpha'(t_0) < 0$. The function $f_{t_0} : x \longmapsto xe^{t_0 x} - \alpha'(t_0)$ has exactly two zeros $a_1 < a_2 < 0$ and

$$f_{t_0}(x) \text{ is } \begin{cases} \text{strictly positive if } x < a_1, \\ \text{strictly negative if } a_1 < x < a_2, \\ \text{strictly positive if } x > a_2. \end{cases}$$

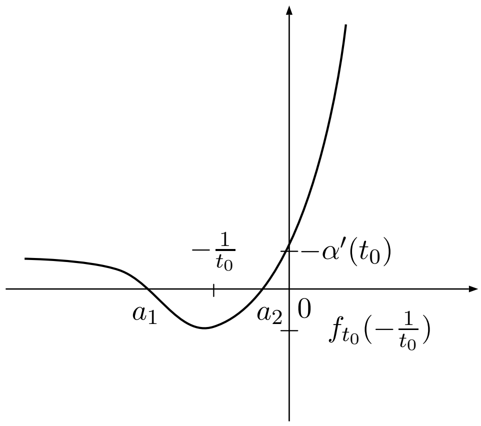

fig.2. Graph of f_{t_0}.

Denoting by μ the law of X, we then observe that:

$$\int_{-\infty}^{a_1} f_{t_0}(x)\mu(dx) > 0 \tag{1.9}$$

and, (1.7) implies:

$$\int_{-\infty}^{\infty} f_{t_0}(x)\mu(dx) = 0. \tag{1.10}$$

¿From (1.9) and (1.10), we deduce that:

$$\mathbb{E}\left[1_{\{X>a_1\}}\left(Xe^{t_0X} - \alpha'(t_0)\right)\right] = \int_{a_1}^{\infty} f_{t_0}(x)\mu(dx) < 0 \quad (\text{since supp } \mu = \mathbb{R}).$$

Then, the result follows by taking $\psi'\left(e^{tx} - \alpha(t)\right) = 1_{[a_1,\infty[}(x)$ in (1.8). □

Let us note that, if α is increasing and $\left(\dfrac{e^{tX}}{\alpha(t)}, t \geq 0\right)$ is a peacock, then, by Theorem 1.9, $\left(C_t := e^{tX} - \alpha(t), t \geq 0\right)$ is a peacock. This provides another proof of point **i)** of the preceding proof.

2 Peacocks obtained from conditionally monotone processes

2.1 Definition of conditionally monotone processes and examples

Let us first introduce the notion of conditional monotonicity, which already appears in [SS94, Chapter 4.B, p.114-126] and which is studied in [BPR12].

Definition 2.1 (Conditional monotonicity)**.** *A process $(X_\lambda, \lambda \geq 0)$ is said to be conditionally monotone if, for every $n \in \mathbb{N}^*$, every $i \in \{1, \dots, n\}$, every $0 < \lambda_1 < \cdots < \lambda_n$ and every bounded Borel function $\phi : \mathbb{R}^n \longrightarrow \mathbb{R}$ which increases (resp. decreases) with respect to each of its arguments, we have:*

$$\mathbb{E}[\phi(X_{\lambda_1}, X_{\lambda_2}, \dots, X_{\lambda_n})|X_{\lambda_i}] = \phi_i(X_{\lambda_i}), \tag{CM}$$

where $\phi_i : \mathbb{R} \longrightarrow \mathbb{R}$ is a bounded increasing (resp. decreasing) function.

To prove that a process is conditionally monotone, we can restrict ourselves to bounded Borel functions ϕ increasing with respect to each of their arguments. Indeed, replacing ϕ by $-\phi$, the result then also holds for bounded Borel functions decreasing with respect to each of their arguments.

Definition 2.2. *We denote by $\mathcal{E}_n$ the set of bounded Borel functions $\phi : \mathbb{R}^n \longrightarrow \mathbb{R}$ which are increasing with respect to each of their arguments.*

Remark 2.3.
1) Note that $(X_\lambda, \lambda \geq 0)$ is conditionally monotone if and only if $(-X_\lambda, \lambda \geq 0)$ is conditionally monotone.
2) Let $\theta : \mathbb{R} \longrightarrow \mathbb{R}$ be a strictly monotone and continuous function. It is not difficult to see that if the process $(X_\lambda, \lambda \geq 0)$ is conditionally monotone, then so is $(\theta(X_\lambda), \lambda \geq 0)$.

In [BPR12], the authors exhibited enough examples of processes enjoying the conditional monotonicity (CM) property. Among them are:

i) the processes with independent and log-concave increments,

ii) the Gamma subordinator,

iii) the well-reversible diffusions at a fixed time, such as, for example:

- the Brownian motion with drift ν,
- the Bessel processes of dimension $\delta \geq 2$,
- the Squared Bessel processes of dimension $\delta > 0$.

We refer the reader to [BPR12] and ([HPRY], Chapter 1, Section 4) for more details. The next lemma follows immediately from Definition 2.1.

Lemma 2.4. *Let $(X_t, t \geq 0)$ be a real-valued right-continuous process which is conditionally monotone and, let $q : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ be a continuous function such that, for every $s \geq 0$, $q_s : x \longmapsto q(s,x)$ is increasing. Then, for every positive function $\phi \in \mathcal{E}_1$, every positive Radon measure ν on $\mathbb{R}_+$ and every $t > 0$:*

$$\mathbb{E}\left[\phi\left(\int_0^t q(s,X_s)\,\nu(ds)\right)\middle| X_t\right] = \phi_t(X_t), \tag{2.1}$$

where ϕ_t is an increasing function.

2.2 Peacocks obtained by centering under a conditional monotonicity hypothesis

Theorem 2.5. *Let $(X_t, t \geq 0)$ be a real-valued right-continuous process which is conditionally monotone. Let $q : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}_+$ be a positive and continuous function such that, for every $s \geq 0$, $q_s : x \longmapsto q(s,x)$ is increasing and $\mathbb{E}[q(s,X_s)] > 0$. Let $\theta : \mathbb{R}_+ \to \mathbb{R}_+$ a positive, increasing and convex $\mathcal{C}^1$-function satisfying:*

$$\forall\, t \geq 0, \quad \mathbb{E}\left[\theta\left(\int_0^t q(s,X_s)ds\right)\right] < \infty \tag{2.2}$$

and

$$\forall\, a > 0, \quad \mathbb{E}\left[\sup_{0<t\leq a} q(t,X_t)\theta'\left(\int_0^t q(s,X_s)ds\right)\right] < \infty. \tag{2.3}$$

Then:

$$\left(C_t := \theta\left(\int_0^t q(s,X_s)ds\right) - h(t), t \geq 0\right) \text{ is a peacock,}$$

where $h(t) := \mathbb{E}\left[\theta\left(\int_0^t q(s,X_s)ds\right)\right]$.

Proof of Theorem 2.5
For every convex function $\psi \in \mathbf{C}_+$, we have:

$$\begin{aligned}
\frac{d}{dt}\mathbb{E}[\psi(C_t)] &= \mathbb{E}\left[\psi'(C_t)\left(q(t,X_t)\theta'\left(\int_0^t q(s,X_s)ds\right) - h'(t)\right)\right] \\
&= \mathbb{E}\left[\psi'(C_t)q(t,X_t)\left(\theta'\left(\int_0^t q(s,X_s)ds\right) - \frac{h'(t)}{\mathbb{E}[q(t,X_t)]}\right)\right] \\
&\quad + \mathbb{E}\left[\psi'(C_t)h'(t)\left(\frac{q(t,X_t)}{\mathbb{E}[q(t,X_t)]} - 1\right)\right] \\
&:= K_1(t) + K_2(t).
\end{aligned}$$

Let us prove that $K_1(t) \geq 0$.
We note that, for every $t \geq 0$:

$$\mathbb{E}\left[q(t, X_t)\left(\theta'\left(\int_0^t q(s, X_s)ds\right) - \frac{h'(t)}{\mathbb{E}[q(t, X_t)]}\right)\right] = 0 \tag{2.4}$$

since

$$\mathbb{E}\left[q(t, X_t)\theta'\left(\int_0^t q(s, X_s)ds\right)\right] = h'(t).$$

Then, since θ' is increasing, one has:

$$\begin{aligned} K_1(t) &= \mathbb{E}\left[\psi'(C_t)q(t, X_t)\left(\theta'\left(\int_0^t q(s, X_s)ds\right) - \frac{h'(t)}{\mathbb{E}[q(t, X_t)]}\right)\right] \\ &\geq \psi'\left(\theta \circ \theta'^{-1}\left(\frac{h'(t)}{\mathbb{E}[q(t, X_t)]}\right) - h(t)\right) \times \\ &\quad \mathbb{E}\left[q(t, X_t)\left(\theta'\left(\int_0^t q(s, X_s)ds\right) - \frac{h'(t)}{\mathbb{E}[q(t, X_t)]}\right)\right] = 0. \end{aligned}$$

Let us prove that $K_2(t) \geq 0$.
We have:

$$\begin{aligned} K_2(t) &= h'(t)\mathbb{E}\left[\psi'(C_t)\left(\frac{q(t, X_t)}{\mathbb{E}[q(t, X_t)]} - 1\right)\right] \\ &= h'(t)\mathbb{E}\left[\mathbb{E}[\psi'(C_t)|X_t]\left(\frac{q(t, X_t)}{\mathbb{E}[q(t, X_t)]} - 1\right)\right]. \end{aligned}$$

But, by Lemma 2.4,

$$\mathbb{E}[\psi'(C_t)|X_t] = \mathbb{E}\left[\psi'\left(\theta\left(\int_0^t q(s, X_s)ds\right) - h(t)\right)\Big| X_t\right] = \varphi_t(X_t),$$

where φ_t is an increasing function. Hence, from Lemma 1.1,

$$K_2(t) = h'(t)\mathbb{E}\left[\varphi_t(X_t)\left(\frac{q(t, X_t)}{\mathbb{E}[q(t, X_t)]} - 1\right)\right] \geq 0$$

since $q_t : x \longmapsto q(t, x)$ is increasing for every $t \geq 0$.

□

Example 2.6. Suppose $(X_t, t \geq 0)$ and $q : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}_+$ be chosen as in Theorem 2.5. If we take successively $\theta(x) = x$ and $\theta(x) = e^x$, we obtain:

$$\left(\int_0^t q(s, X_s)ds - \mathbb{E}\left[\int_0^t q(s, X_s)ds\right], t \geq 0\right) \text{ is a peacock} \tag{2.5}$$

and

$$\left(\exp\left(\int_0^t q(s, X_s)ds\right) - \mathbb{E}\left[\exp\left(\int_0^t q(s, X_s)ds\right)\right], t \geq 0\right) \text{ is a peacock.} \tag{2.6}$$

2.3 Peacocks obtained by normalisation from a particular class of conditionally monotone processes

We now consider the class of processes with independent and log-concave increments. (Note that these processes are conditionally monotone (see [BPR12])). Let us recall some definitions and properties.

Definition 2.7 ($\mathbb{R}$-valued log-concave r.v.'s)**.**
An $\mathbb{R}$-valued random variable X is said to be log*-concave if:*

1) *X admits a probability density g,*

2) *the function $\log g$ is concave; i.e., the second derivative of $\log g$ (in the distribution sense) is a negative measure.*

Definition 2.8 ($\mathbb{Z}$-valued log-concave r.v.'s)**.**
A $\mathbb{Z}$-valued random variable X is said to be log*-concave if, with $g(n) = \mathbb{P}(X = n)$ ($n \in \mathbb{Z}$), one has: for every $n \in \mathbb{Z}$,*

$$g^2(n) \geq g(n-1)g(n+1);$$

in other words, the discrete second derivative of $\log g$ is negative.

Example 2.9. Many common density functions on $\mathbb{R}$ (or $\mathbb{Z}$) are log-concave. Indeed, the normal density, the uniform density, the exponential density, the Poisson density and the geometric density are log-concave.

The following properties of log-concave random variables are well-known (see [Sch51]).

Lemma 2.10. *An $\mathbb{R}$-valued (resp. $\mathbb{Z}$-valued) random variable X is log-concave if and only if its probability density g satifies:*

1) *The support of g is an (finite or infinite) interval $I \subset \mathbb{R}$ (resp. $I \subset \mathbb{Z}$),*

2) *for every $x_2 \geq x_1$, $y_2 \geq y_1$,*

$$\det \begin{pmatrix} g(x_1 - y_1) & g(x_1 - y_2) \\ g(x_2 - y_1) & g(x_2 - y_2) \end{pmatrix} \geq 0. \tag{PF}$$

Lemma 2.11. *([Sch51])*

i) Every log-concave density is bounded.

ii) If g and h are two log-concave densities, then their convolution $g * h$ given by:

$$g * h(x) = \int_{-\infty}^{\infty} g(y)h(x-y)dy$$

is also log-concave, i.e: the sum of two independent log-concave random variables is log-concave.

The main result of this section is the following:

Theorem 2.12. *Let $(X_t, t \geq 0)$ be a right-continuous $\mathbb{R}$-valued process with independent and log-concave increments issued from 0, and $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ a right-continuous and increasing function satisfying $\alpha(0) = 0$. Let $q : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ be a continuous function such that, for every $t \geq 0$:*

i) the variable

$$\Theta_t := \exp\left(\alpha(t) \sup_{0 \leq s \leq t} q(s, X_s)\right) \text{ is integrable} \tag{INT1}$$

and,

$$\Delta_t := \mathbb{E}\left[\exp\left(\alpha(t) \inf_{0 \leq s \leq t} q(s, X_s)\right)\right] > 0, \tag{INT2}$$

ii) the function $x \longmapsto q(t, x)$ is increasing (resp. decreasing).

Then,

$$\left(N_t := \frac{\exp\left(\int_0^t q(s, X_s)\, d\alpha(s)\right)}{\mathbb{E}\left[\exp\left(\int_0^t q(s, X_s)\, d\alpha(s)\right)\right]}, t \geq 0\right) \text{ is a peacock.}$$

In particular, if $(B_t, t \geq 0)$ denotes a Brownian motion starting from 0, then

$$\left(N_t^{(B)} := \frac{\exp\left(\int_0^t q(s, B_s)\, d\alpha(s)\right)}{\mathbb{E}\left[\exp\left(\int_0^t q(s, B_s)\, d\alpha(s)\right)\right]}, t \geq 0\right) \text{ is a peacock.}$$

To prove Theorem 2.12, we need the following Lemma.

Lemma 2.13. *Let $(X_t, t \geq 0)$ be a $\mathbb{R}$-valued process with independent and log-concave increments and $(f_k : \mathbb{R} \to \mathbb{R}_+, k \in \mathbb{N}^*)$ a family of strictly positive Borel functions such that, for every $p \in \mathbb{N}^*$ and every $0 \leq t_1 < \cdots < t_p$:*

$$\mathbb{E}\left[\prod_{k=1}^{p} f_k(X_{t_k})\right] < \infty.$$

Then, for every $n \geq 2$, every $0 \leq \lambda_1 < \lambda_2 < \cdots < \lambda_n$ and every bounded Borel function $\phi : \mathbb{R}^{n-1} \to \mathbb{R}_+$ which increases (resp. decreases) with respect to each of its arguments:

$$K(n, z) = \frac{\mathbb{E}\left[\phi(X_{\lambda_1}, \ldots, X_{\lambda_{n-1}}) \prod_{k=1}^{n-1} f_k(X_{\lambda_k}) \,\middle|\, X_{\lambda_n} = z\right]}{\mathbb{E}\left[\prod_{k=1}^{n-1} f_k(X_{\lambda_k}) \,\middle|\, X_{\lambda_n} = z\right]}$$

is an increasing (resp. decreasing) function of z.

Proof of Lemma 2.13

i) We give the proof of Lemma 2.13 only in the continuous case (the proof in the discrete one is similar).

ii) By truncation and regularisation, it suffices to prove Lemma 2.13 when the functions f_k $(k \in \mathbb{N}^*)$ are bounded and when the support of all increments' densities is $\mathbb{R}$.

iii) In this proof we deal only, without loss of generality, with bounded Borel functions which increase with respect to each of its arguments.

We now prove this Lemma by induction on $n \geq 2$.

• **For n = 2:** we denote by g_1 (resp. $\widetilde{g}_2$) the density function of X_{λ_1} (resp. $X_{\lambda_2} - X_{\lambda_1}$). For every bounded and increasing Borel function $\phi : \mathbb{R} \to \mathbb{R}_+$:

$$\begin{aligned} K(2,z) &= \frac{\mathbb{E}\left[\phi(X_{\lambda_1}) f_1(X_{\lambda_1}) | X_{\lambda_2} = z\right]}{\mathbb{E}\left[f_1(X_{\lambda_1}) | X_{\lambda_2} = z\right]} \\ &= \frac{\displaystyle\int_{-\infty}^{\infty} \phi(u) f_1(u) g_1(u) \widetilde{g}_2(z-u)\, du}{\displaystyle\int_{-\infty}^{\infty} f_1(u) g_1(u) \widetilde{g}_2(z-u)\, du}. \end{aligned}$$

Since ϕ is bounded and increasing, then, by approximation, we can restrict ourselves to the case:

$$\phi = \sum_{i=1}^{l} c_i 1_{[x_i, \infty[}$$

where $l \in \mathbb{N}^*$ and, for every $i \in [\![1, l]\!]$, c_i is a positive constant and x_i a real number. Hence, the function $z \longmapsto K(2,z)$ increases if and only if, for every $x \in \mathbb{R}$,

$$z \longmapsto \frac{\displaystyle\int_{x}^{\infty} f_1(u) g_1(u) \widetilde{g}_2(z-u)\, du}{\displaystyle\int_{-\infty}^{\infty} f_1(u) g_1(u) \widetilde{g}_2(z-u)\, du} \quad \text{is increasing.}$$

This is also equivalent to say that: for every $x \in \mathbb{R}$,

$$L(x,z) = \frac{\displaystyle\int_{x}^{\infty} f_1(u) g_1(u) \widetilde{g}_2(z-u)\, du}{\displaystyle\int_{-\infty}^{x} f_1(u) g_1(u) \widetilde{g}_2(z-u)\, du}$$

is an increasing function of z.

Since $\widetilde{g}_2$ is log-concave, then, for every $x \in \mathbb{R}$, $z \in \mathbb{R}$ and $\eta > 0$, we have:

$$\frac{\widetilde{g}_2(z+\eta-u)}{\widetilde{g}_2(z+\eta-x)} \geq \frac{\widetilde{g}_2(z-u)}{\widetilde{g}_2(z-x)}, \text{ for every } u \geq x$$

and

$$\frac{\widetilde{g}_2(z+\eta-u)}{\widetilde{g}_2(z+\eta-x)} \leq \frac{\widetilde{g}_2(z-u)}{\widetilde{g}_2(z-x)}, \text{ for every } u \leq x.$$

Therefore, for every $x \in \mathbb{R}$, $z \in \mathbb{R}$ and $\eta > 0$, one has:

$$
\begin{aligned}
L(x,z+\eta) &= \frac{\displaystyle\int_x^\infty f_1(u)g_1(u)\widetilde{g}_2(z+\eta-u)\,du}{\displaystyle\int_{-\infty}^x f_1(u)g_1(u)\widetilde{g}_2(z+\eta-u)\,du}\\
&= \frac{\displaystyle\int_x^\infty f_1(u)g_1(u)\frac{\widetilde{g}_2(z+\eta-u)}{\widetilde{g}_2(z+\eta-x)}\,du}{\displaystyle\int_{-\infty}^x f_1(u)g_1(u)\frac{\widetilde{g}_2(z+\eta-u)}{\widetilde{g}_2(z+\eta-x)}\,du}\\
&\geq \frac{\displaystyle\int_x^\infty f_1(u)g_1(u)\frac{\widetilde{g}_2(z-u)}{\widetilde{g}_2(z-x)}\,du}{\displaystyle\int_{-\infty}^x f_1(u)g_1(u)\frac{\widetilde{g}_2(z-u)}{\widetilde{g}_2(z-x)}\,du}\\
&= L(x,z);
\end{aligned}
$$

which means that $L(x,z)$ increases with z. Then, $z \longmapsto K(2,z)$ increases for every bounded and increasing Borel function $\phi : \mathbb{R} \to \mathbb{R}_+$.

• **For $\mathbf{n \geq 3}$:** we assume that, for every bounded Borel function $\varphi : \mathbb{R}^{n-2} \to \mathbb{R}_+$ which increases with respect to each of its arguments, $z \longmapsto K(n-1,z)$ increases and, we denote by g_{n-1} (resp. $\widetilde{g}_n$) the density function of $X_{\lambda_{n-1}}$ (resp. $X_{\lambda_n} - X_{\lambda_{n-1}}$). Since the variables $X_{\lambda_n} - X_{\lambda_{n-1}}$ and $X_{\lambda_{n-1}}$ are independent, then, for every bounded Borel function $\phi : \mathbb{R}^{n-1} \to \mathbb{R}$ which increases with respect to each of its arguments, we have:

$$
\begin{aligned}
K(n,z) &= \frac{\mathbb{E}\left[\phi(X_{\lambda_1},\dots,X_{\lambda_{n-1}})\displaystyle\prod_{k=1}^{n-1} f_k(X_{\lambda_k})\,\middle|\, X_{\lambda_{n-1}} + X_{\lambda_n} - X_{\lambda_{n-1}} = z\right]}{\mathbb{E}\left[\displaystyle\prod_{k=1}^{n-1} f_k(X_{\lambda_k})\,\middle|\, X_{\lambda_{n-1}} + X_{\lambda_n} - X_{\lambda_{n-1}} = z\right]}\\
&= \frac{\displaystyle\int_{-\infty}^\infty \mathbb{E}\left[\phi(X_{\lambda_1},\dots,X_{\lambda_{n-2}},z-y)\prod_{k=1}^{n-2} f_k(X_{\lambda_k})\,\middle|\, X_{\lambda_{n-1}} = z-y\right] f_{n-1}(z-y)g_{n-1}(z-y)\widetilde{g}_n(y)\,dy}{\displaystyle\int_{-\infty}^\infty \mathbb{E}\left[\prod_{k=1}^{n-2} f_k(X_{\lambda_k})\,\middle|\, X_{\lambda_{n-1}} = z-y\right] f_{n-1}(z-y)g_{n-1}(z-y)\widetilde{g}_n(y)\,dy}.
\end{aligned}
$$

After the change of variable: $x = z - y$, we obtain:

$$
K(n,z) = \frac{\displaystyle\int_{-\infty}^\infty \mathbb{E}\left[\phi(X_{\lambda_1},\dots,X_{\lambda_{n-2}},x)\prod_{k=1}^{n-2} f_k(X_{\lambda_k})\,\middle|\, X_{\lambda_{n-1}} = x\right] f_{n-1}(x)g_{n-1}(x)\widetilde{g}_n(z-x)\,dx}{\displaystyle\int_{-\infty}^\infty \mathbb{E}\left[\prod_{k=1}^{n-2} f_k(X_{\lambda_k})\,\middle|\, X_{\lambda_{n-1}} = x\right] f_{n-1}(x)g_{n-1}(x)\widetilde{g}_n(z-x)\,dx}.
$$

For $x \in \mathbb{R}$, we define:

$$m(x) = \frac{\mathbb{E}\left[\phi(X_{\lambda_1}, \dots, X_{\lambda_{n-2}}, x) \prod_{k=1}^{n-2} f_k(X_{\lambda_k}) \middle| X_{\lambda_{n-1}} = x\right]}{\mathbb{E}\left[\prod_{k=1}^{n-2} f_k(X_{\lambda_k}) \middle| X_{\lambda_{n-1}} = x\right]}$$

and

$$f_*(x) = \mathbb{E}\left[\prod_{k=1}^{n-2} f_k(X_{\lambda_k}) \middle| X_{\lambda_{n-1}} = x\right].$$

Hence,

a) since $\phi : \mathbb{R}^{n-1} \to \mathbb{R}_+$ is bounded and increasing with respect to each of its arguments, the induction hypothesis implies that $m : \mathbb{R} \to \mathbb{R}_+$ is a bounded and increasing Borel function,

b) we have:

$$K(n, z) = \frac{\displaystyle\int_{-\infty}^{\infty} m(x) f_*(x) f_{n-1}(x) g_{n-1}(x) \widetilde{g}_n(z - x)\, dx}{\displaystyle\int_{-\infty}^{\infty} f_*(x) f_{n-1}(x) g_{n-1}(x) \widetilde{g}_n(z - x)\, dx}.$$

Using the log-concavity of g_n and the case $n = 2$ computed above, we have: for every $y \in \mathbb{R}$,

$$z \longmapsto \frac{\displaystyle\int_{y}^{\infty} f_*(x) f_{n-1}(x) g_{n-1}(x) \widetilde{g}_n(z - x)\, dx}{\displaystyle\int_{-\infty}^{y} f_*(x) f_{n-1}(x) g_{n-1}(x) \widetilde{g}_n(z - x)\, dx}$$

is an increasing function of z. Then, the function $z \longmapsto K(n, z)$ increases for every bounded Borel function $\phi : \mathbb{R}^{n-1} \to \mathbb{R}_+$ which increases with respect to each of its arguments.

□

Proof of Theorem 2.12

We prove this Theorem only in the case where $x \longmapsto q(\lambda, x)$ is increasing.

Let $T > 0$ be fixed.

1) We first consider the case

$$1_{[0,T]} d\alpha = \sum_{i=1}^{r} a_i \delta_{\lambda_i}$$

where $r \in [\![2, \infty[\![$, $a_1 \geq 0, a_2 \geq 0, \dots, a_r \geq 0$, $\sum_{i=1}^{r} a_i = \alpha(T)$ and $0 \leq \lambda_1 < \lambda_2 < \dots < \lambda_r \leq T$.

Let us prove that:

$$\left(N_n := \exp\left(\sum_{i=1}^{n} a_i q(\lambda_i, X_{\lambda_i}) - h(n)\right), n \in [\![1, r]\!]\right) \text{ is a peacock,}$$

where

$$h(n) := \log \mathbb{E}\left[\exp\left(\sum_{i=1}^{n} a_i q(\lambda_i, X_{\lambda_i})\right)\right], \text{ for every } n \in [\![1, r]\!].$$

We have:

$$\mathbb{E}[N_n - N_{n-1}] = 0, \text{ for every } n \in [\![2, r]\!]$$

with

$$N_n - N_{n-1} = N_{n-1}\left(e^{a_n q(\lambda_n, X_{\lambda_n}) - h(n) + h(n-1)} - 1\right),$$

and, for every convex function $\psi \in \mathbf{C}$,

$$\begin{aligned}
&\mathbb{E}[\psi(N_n)] - \mathbb{E}[\psi(N_{n-1})] \\
&\geq \mathbb{E}\left[\psi'(N_{n-1})N_{n-1}\left(e^{a_n q(\lambda_n, X_{\lambda_n}) - h(n) + h(n-1)} - 1\right)\right] \\
&= \mathbb{E}\left[\mathbb{E}[N_{n-1}|X_{\lambda_n}]K(n, X_{\lambda_n})\left(e^{a_n q(\lambda_n, X_{\lambda_n}) - h(n) + h(n-1)} - 1\right)\right],
\end{aligned}$$

where

$$K(n, z) = \frac{\mathbb{E}[\psi'(N_{n-1})N_{n-1}|X_{\lambda_n} = z]}{\mathbb{E}[N_{n-1}|X_{\lambda_n} = z]}.$$

The positive and bounded $\mathcal{C}^0$-function $\phi : \mathbb{R}^{n-1} \to \mathbb{R}_+$ given by:

$$\phi(x_1, \ldots, x_{n-1}) = \psi'\left[\exp\left(\sum_{i=1}^{n-1} a_i q(\lambda_i, x_i) - h(n-1)\right)\right]$$

increases with respect to each of its arguments. For $i \in \mathbb{N}^*$, let us define:

$$f_i(x) = e^{a_i q(\lambda_i, x)}, \text{ for every } x \in \mathbb{R};$$

then, for every $n \in [\![2, r]\!]$, we have:

$$N_{n-1} = e^{-h(n-1)} \prod_{k=1}^{n-1} f_k(X_{\lambda_k}).$$

Hence,

$$K(n, z) = \frac{\mathbb{E}\left[\phi(X_{\lambda_1}, \ldots, X_{\lambda_{n-1}}) \prod_{k=1}^{n-1} f_k(X_{\lambda_k}) \,\middle|\, X_{\lambda_n} = z\right]}{\mathbb{E}\left[\prod_{k=1}^{n-1} f_k(X_{\lambda_k}) \,\middle|\, X_{\lambda_n} = z\right]}.$$

Moreover, for every $n \in [\![1, r]\!]$,

$$\begin{aligned}
\mathbb{E}\left[\prod_{k=1}^{n} f_k(X_{\lambda_k})\right] &= \mathbb{E}\left[\exp\left(\sum_{k=1}^{n} a_i q(\lambda_i, X_{\lambda_i})\right)\right] \\
&\leq \mathbb{E}\left[\exp\left(\sup_{0\leq\lambda\leq T} q(\lambda, X_\lambda) \sum_{k=1}^{n} a_i\right)\right] \\
&\leq \mathbb{E}\left[\exp\left(\sup_{0\leq\lambda\leq T} q(\lambda, X_\lambda) \sum_{k=1}^{r} a_i\right) \vee 1\right] \\
&= \mathbb{E}\left[\exp\left(\alpha(T) \sup_{0\leq\lambda\leq T} q(\lambda, X_\lambda)\right) \vee 1\right] \\
&= \mathbb{E}[\Theta_T \vee 1] < \infty.
\end{aligned}$$

Therefore, thanks to Lemma 2.13, $K(n, z)$ is an increasing function of z. For $\lambda \geq 0$, we denote by q_λ^{-1}, the right-continuous inverse of $x \longmapsto q(\lambda, x)$. Let us also consider the variable:

$$V(n, X_{\lambda_n}) := K(n, X_{\lambda_n})\mathbb{E}[N_{n-1}|X_{\lambda_n}]\left(e^{a_n q(\lambda_n, X_{\lambda_n}) - h(n) + h(n-1)} - 1\right).$$

Then:

i) if
$$X_{\lambda_n} \leq q_{\lambda_n}^{-1}\left(\frac{h(n) - h(n-1)}{a_n}\right),$$
then
$$e^{a_n q(\lambda_n, X_{\lambda_n}) - h(n) + h(n-1)} - 1 \leq 0$$
and
$$V(n, X_{\lambda_n}) \geq \\ K\left(n, q_{\lambda_n}^{-1}\left(\frac{h(n) - h(n-1)}{a_n}\right)\right) \times \mathbb{E}[N_{n-1}|X_{\lambda_n}]\left(e^{a_n q(\lambda_n, X_{\lambda_n}) - h(n) + h(n-1)} - 1\right),$$

ii) if
$$X_{\lambda_n} \geq q_{\lambda_n}^{-1}\left(\frac{h(n) - h(n-1)}{a_n}\right),$$
then
$$e^{a_n q(\lambda_n, X_{\lambda_n}) - h(n) + h(n-1)} - 1 \geq 0$$
and
$$V(n, X_{\lambda_n}) \geq \\ K\left(n, q_{\lambda_n}^{-1}\left(\frac{h(n) - h(n-1)}{a_n}\right)\right) \mathbb{E}[N_{n-1}|X_{\lambda_n}]\left(e^{a_n q(\lambda_n, X_{\lambda_n}) - h(n) + h(n-1)} - 1\right).$$

Thus,

$$\begin{aligned}
&\mathbb{E}[\psi(N_n)] - \mathbb{E}[\psi(N_{n-1})] \geq \mathbb{E}V(n, X_{\lambda_n}) \\
&\geq K\left(n, q_{\lambda_n}^{-1}\left(\frac{h(n)-h(n-1)}{a_n}\right)\right)\mathbb{E}\left[\mathbb{E}[N_{n-1}|X_{\lambda_n}]\left(e^{a_n q(\lambda_n, X_{\lambda_n})-h(n)+h(n-1)}-1\right)\right] \\
&= K\left(n, q_{\lambda_n}^{-1}\left(\frac{h(n)-h(n-1)}{a_n}\right)\right)\mathbb{E}\left[N_{n-1}\left(e^{a_n q(\lambda_n, X_{\lambda_n})-h(n)+h(n-1)}-1\right)\right] \\
&= K\left(n, q_{\lambda_n}^{-1}\left(\frac{h(n)-h(n-1)}{a_n}\right)\right)\mathbb{E}\left[N_n - N_{n-1}\right] = 0.
\end{aligned}$$

Hence, for every $r \in [\![2, \infty[\![$:

$$\left(N_n := \exp\left(\sum_{i=1}^{n} a_i q(\lambda_i, X_{\lambda_i}) - h(n)\right), n \in [\![1, r]\!]\right) \text{ is a peacock.}$$

2) We now set $\mu = 1_{[0,T]}d\alpha$ and, for every $0 \leq t \leq T$,

$$N_t^{(\mu)} = \frac{\exp\left(\int_0^t q(u, X_u)\mu(du)\right)}{\mathbb{E}\left[\exp\left(\int_0^t q(u, X_u)\mu(du)\right)\right]}.$$

Since the function $\lambda \in [0, T] \longmapsto q(\lambda, X_\lambda)$ is right-continuous and bounded from above by $\sup_{0\leq\lambda\leq T} |q(\lambda, X_\lambda)|$ which is finite a.s., there exists a sequence $(\mu_n, n \geq 0)$ of measures of type considered in **1)**, with $\operatorname{supp} \mu_n \subset [0, T]$, $\int \mu_n(du) = \int \mu(du)$ and, for every $0 \leq t \leq T$,

$$\lim_{n\to\infty} \exp\left(\int_0^t q(u, X_u)\mu_n(du)\right) = \exp\left(\int_0^t q(u, X_u)\mu(du)\right) \text{ a.s.} \tag{2.7}$$

Moreover, for every $0 \leq t \leq T$ and every $n \geq 0$,

$$\begin{aligned}
&\sup_{n\geq 0} \exp\left(\int_0^t q(u, X_u)\mu_n(du)\right) \\
&\leq \exp\left(\sup_{0\leq\lambda\leq T} q(\lambda, X_\lambda)\int_0^t \mu_n(du)\right) \\
&= \exp\left(\sup_{0\leq\lambda\leq T} q(\lambda, X_\lambda)\int_0^T \mu_n(du)\right) \vee 1 \\
&= \exp\left(\sup_{0\leq\lambda\leq T} q(\lambda, X_\lambda)\int_0^T \mu(du)\right) \vee 1 = \Theta_T \vee 1
\end{aligned}$$

which is integrable from (INT1). Thus, the dominated convergence Theorem yields

$$\lim_{n\to\infty} \mathbb{E}\left[\exp\left(\int_0^t q(u, X_u)\mu_n(du)\right)\right] = \mathbb{E}\left[\exp\left(\int_0^t q(u, X_u)\mu(du)\right)\right]. \tag{2.8}$$

Using (2.7) and (2.8), we obtain:

$$\lim_{n\to\infty} N_t^{(\mu_n)} = N_t^{(\mu)} \text{ a.s., for every } 0 \leq t \leq T. \tag{2.9}$$

Now, from **1)**,

$$\left(N_t^{(\mu_n)}, 0 \leq t \leq T\right) \text{ is a peacock for every } n \geq 0, \tag{2.10}$$

i.e., for every $0 \leq s < t \leq T$ and for every $\psi \in \mathbf{C}$:

$$\mathbb{E}\left[\psi(N_s^{(\mu_n)})\right] \leq \mathbb{E}\left[\psi(N_t^{(\mu_n)})\right]. \tag{2.11}$$

Then, since

$$\sup_{0\leq t\leq T} \sup_{n\geq 0} \left|N_t^{(\mu_n)}\right| \leq \frac{\Theta_T \vee 1}{\Delta_T \wedge 1}, \tag{2.12}$$

which is integrable from (INT1) and (INT2), it remains to apply the dominated convergence Theorem in (2.11) to obtain that $(N_t^{(\mu)}, 0 \leq t \leq T)$ is a peacock for every $T > 0$.

□

3 Peacocks obtained from a diffusion by centering and normalisation

Let us consider two Borel functions $\sigma : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ and $b : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ such that $\sigma_s(x) := \sigma(s, x)$ and $b_s(x) := b(s, x)$ are Lipschitz continuous with respect to x, locally uniformly with respect to s and $(X_t, t \geq 0)$ a process with values in an interval $I \subset \mathbb{R}$ and which solves the SDE:

$$Y_t = x_0 + \int_0^t \sigma(s, Y_s) dB_s + \int_0^t b(s, Y_s) ds \tag{3.1}$$

where $x_0 \in I$ and $(B_s, s \geq 0)$ denotes a standard Brownian motion started at 0. For $s \geq 0$, let L_s denotes the second-order differential operator:

$$L_s := \frac{1}{2}\sigma^2(s, x)\frac{\partial^2}{\partial x^2} + b(s, x)\frac{\partial}{\partial x}. \tag{3.2}$$

The following results concern peacocks of C and N types.

3.1 Peacocks obtained by normalisation

Theorem 3.1. *Let $(X_t, t \geq 0)$ be a solution of (3.1) taking values in I. Let $\theta : I \to \mathbb{R}_+^*$ be an increasing $\mathcal{C}^2$-function such that:*

1. *for every $s \geq 0$:*

$$v_s : x \in I \longmapsto \frac{L_s\theta(x)}{\theta(x)} \textit{ is an increasing function} \tag{3.3}$$

2. the process

$$\left(M_t := \theta(X_t) - \theta(x_0) - \int_0^t L_s\theta(X_s)ds,\ t \geq 0\right)$$

is a martingale.

Then:

$$\left(N_t := \frac{\theta(X_t)}{\mathbb{E}[\theta(X_t)]}, t \geq 0\right) \text{ is a peacock.} \tag{3.4}$$

Proof of Theorem 3.1

For every $t \geq 0$, let $h(t) = \mathbb{E}[\theta(X_t)]$. We note that h is strictly positive. Let $\psi \in \mathbf{C}$ and $0 \leq s < t$. By Itô's formula

$$\begin{aligned}\psi\left(\frac{\theta(X_t)}{h(t)}\right) = \psi\left(\frac{\theta(X_s)}{h(s)}\right) &+ \int_s^t \psi'\left(\frac{\theta(X_u)}{h(u)}\right)\left[\frac{dM_u}{h(u)} + \frac{L_u\theta(X_u)du}{h(u)}\right] \\ &- \int_s^t \psi'\left(\frac{\theta(X_u)}{h(u)}\right)\frac{h'(u)\theta(X_u)}{h^2(u)}du \\ &+ \frac{1}{2}\int_s^t \psi''\left(\frac{\theta(X_u)}{h(u)}\right)\frac{1}{h^2(u)}d\langle M\rangle_u.\end{aligned}$$

Hence, it suffices to see that, for every $s \leq u < t$:

$$K(u) := \mathbb{E}\left[\psi'\left(\frac{\theta(X_u)}{h(u)}\right)\left[\frac{L_u\theta(X_u)}{h(u)} - \frac{h'(u)\theta(X_u)}{h^2(u)}\right]\right] \geq 0. \tag{3.5}$$

We note that:

$$\mathbb{E}\left[\frac{L_u\theta(X_u)}{h(u)} - \frac{h'(u)\theta(X_u)}{h^2(u)}\right] = 0 \tag{3.6}$$

since $u \longmapsto \dfrac{1}{h(u)}\mathbb{E}\left[\theta(X_u)\right]$ is constant and

$$\frac{d}{du}\mathbb{E}\left[\theta(X_u)\right] = \mathbb{E}[L_u\theta(X_u)]. \tag{3.7}$$

Hence, for every $s \leq u < t$, since, by hypothesis (3.3), $x \longmapsto v_u(x)$ is increasing, we have:

$$\begin{aligned}K(u) &= \mathbb{E}\left[\psi'\left(\frac{\theta(X_u)}{h(u)}\right)\frac{\theta(X_u)}{h(u)}\left(v_u(X_u) - \frac{h'(u)}{h(u)}\right)\right] \\ &\geq \psi'\left(\frac{\theta\left(v_u^{-1}\left(\frac{h'(u)}{h(u)}\right)\right)}{h(u)}\right)\mathbb{E}\left[\frac{\theta(X_u)}{h(u)}\left(v_u(X_u) - \frac{h'(u)}{h(u)}\right)\right] = 0.\end{aligned}$$

□

3.2 Peacocks obtained by centering

Theorem 3.2. *Let $(X_t, t \geq 0)$ be a solution of (3.1) taking values in I. Let $\theta : I \to \mathbb{R}_+$ be an increasing $\mathcal{C}^2$-function such that:*

1. *for every $s \geq 0$, $x \longmapsto L_s\theta(x)$ is increasing.*
2. *the process*
$$\left(M_t := \theta(X_t) - \theta(x_0) - \int_0^t L_s\theta(X_s)ds,\, t \geq 0\right)$$
is a martingale.

Then:
$$(C_t := \theta(X_t) - \mathbb{E}[\theta(X_t)], t \geq 0) \text{ is a peacock.}$$

Proof of Theorem 3.2
Let $\psi \in \mathbf{C}$, $h(t) = \mathbb{E}[\theta(X_t)]$ and $0 \leq s < t$. ¿From Itô's formula, we have:

$$\begin{aligned}&\psi(\theta(X_t) - h(t)) - \psi(\theta(X_s) - h(s)) = \\ &\int_s^t \psi'(\theta(X_u) - h(u))[dM_u + L_u\theta(X_u)du - h'(u)du] + \\ &\frac{1}{2}\int_s^t \psi''(\theta(X_u) - h(u))d\langle M\rangle_u.\end{aligned}$$

Hence, it is sufficient to show that, for every $s \leq u < t$:

$$\mathbb{E}\left[\psi'(\theta(X_u) - h(u))(L_u\theta(X_u) - h'(u))\right] \geq 0. \tag{3.8}$$

But, (3.8) follows from:

$$\mathbb{E}[L_u\theta(X_u) - h'(u)] = 0, \text{ for every } s \leq u < t \tag{3.9}$$

and

$$\begin{aligned}&\mathbb{E}\left[\psi'(\theta(X_u) - h(u))(L_u\theta(X_u) - h'(u))\right] \geq \\ &\psi'\left(\theta\left[(L_u\theta)^{-1}(h'(u))\right] - h(u)\right)\mathbb{E}\left[L_u\theta(X_u) - h'(u)\right] = 0.\end{aligned}$$

□

Remark 3.3. In Theorem 3.1, if we suppose futhermore that $L_s\theta(x) \geq 0$ for every $s \geq 0$ and $x \in I$, then

$$(C_t := \theta(X_t) - \mathbb{E}[\theta(X_t)], t \geq 0) \text{ is a peacock.} \tag{3.10}$$

Indeed, for every $t \geq 0$:

$$\theta(X_t) = \theta(x_0) + M_t + \int_0^t L_u\theta(X_u)du.$$

Thus, $h : t \longmapsto \mathbb{E}[\theta(X_t)]$ is increasing and the result follows from both Theorems 3.1 and 1.9.

3.3 Peacocks obtained from an additive functional by normalisation

Let A_s be the space-time differential operator given by:

$$A_s := \frac{\partial}{\partial s} + \frac{1}{2}\sigma^2(s,x)\frac{\partial^2}{\partial x^2} + b(s,x)\frac{\partial}{\partial x}. \tag{3.11}$$

We shall prove the following result:

Theorem 3.4. *Let $(X_t, t \geq 0)$ be a conditionally monotone process with values in I and which solves (3.1) and, let $q : \mathbb{R}_+ \times I \to \mathbb{R}_+$ be a strictly positive $\mathcal{C}^2$-function such that,*

1. *for every $s \geq 0$, $\mathbb{E}[q(s, X_s)] > 0$, $q_s : x \in I \longmapsto q(s,x)$ is increasing and*

$$f_s : x \in I \longmapsto \frac{A_s q(s,x)}{q(s,x)} \text{ is an increasing function} \tag{3.12}$$

2. *the process*

$$\left(Z_t := q(t, X_t) - q(0, x_0) - \int_0^t A_s q(s, X_s) ds,\ t \geq 0 \right)$$

   is a martingale.

Then, for every positive Radon measure ν on $\mathbb{R}_+$:

$$\left(N_t := \frac{\displaystyle\int_0^t q(s, X_s)\nu(ds)}{\mathbb{E}\left[\displaystyle\int_0^t q(s, X_s)\nu(ds)\right]}, t \geq 0 \right) \text{ is a peacock.}$$

One may find in [HPRY], Chapter 1, many examples of SDEs solutions which are conditionally monotone. This fact is related to the "well-reversible" property of these diffusions.

Proof of Theorem 3.4

We set:

$$\Gamma_u := \frac{1}{\mathbb{E}[q(u, X_u)]}, \text{ for every } u \geq 0. \tag{3.13}$$

For every $u \geq 0$, Itô's formula yields:

$$\begin{aligned} \Gamma_u q(u, X_u) &= 1 + \int_0^u \Gamma_v dZ_v + \int_0^u \left(\Gamma'_v q(v, X_v) + \Gamma_v A_v q(v, X_v) \right) dv \\ &:= M_u + H_u \end{aligned}$$

where

$$\left(M_u := 1 + \int_0^u \Gamma_v dZ_v, u \geq 0 \right) \text{ is a martingale} \tag{3.14}$$

and

$$\left(H_u := \int_0^u \left(\Gamma'_v q(v, X_v) + \Gamma_v A_v q(v, X_v) \right) dv, u \geq 0 \right) \text{ is a centered process} \tag{3.15}$$

since $\mathbb{E}[\Gamma_u q(u,X_u)] = 1$ for every $u \geq 0$ and

$$\frac{d}{du}\mathbb{E}[q(u,X_u)] = \mathbb{E}[A_u q(u,X_u)]. \tag{3.16}$$

Hence, by setting

$$h(t) := \int_0^t \frac{1}{\Gamma_u}\nu(du) = \mathbb{E}\left[\int_0^t q(u,X_u)\nu(du)\right], \tag{3.17}$$

one has:

$$\begin{aligned} N_t = \frac{1}{h(t)}\int_0^t q(u,X_u)\nu(du) &= \frac{1}{h(t)}\int_0^t \Gamma_u q(u,X_u)\frac{1}{\Gamma_u}\nu(du) \\ &= \frac{1}{h(t)}\int_0^t (M_u + H_u)dh(u). \end{aligned}$$

Thus, integrating by parts, we obtain:

$$dN_t = \frac{dh(t)}{h^2(t)}\left(M_t^{(h)} + H_t^{(h)}\right) \tag{3.18}$$

with

$$M_t^{(h)} = \int_0^t h(u)dM_u \text{ and } H_t^{(h)} = \int_0^t h(s)dH_s.$$

Then, for every $\psi \in \mathbf{C}_+$ and every $0 \leq s < t$, we have:

$$\begin{aligned} &\mathbb{E}[\psi(N_t)] - \mathbb{E}[\psi(N_s)] = \mathbb{E}\left[\int_s^t \psi'(N_u)dN_u\right] \\ &= \mathbb{E}\left[\int_s^t \psi'(N_u)\left(M_u^{(h)} + H_u^{(h)}\right)\frac{dh(u)}{h^2(u)}\right] \\ &= \int_s^t \frac{dh(u)}{h^2(u)}\mathbb{E}\left[\int_0^u \psi''(N_v)\left(M_v^{(h)} + H_v^{(h)}\right)^2\frac{dh(v)}{h^2(v)}\right] + \\ &\quad \int_s^t \frac{dh(u)}{h^2(u)}\mathbb{E}\left[\int_0^u \psi'(N_v)\left(dM_v^{(h)} + dH_v^{(h)}\right)\right]. \end{aligned}$$

Hence, it remains to see that, for every $u > 0$:

$$\begin{aligned} &\mathbb{E}\left[\int_0^u \psi'(N_v)dH_v^{(h)}\right] = \\ &\mathbb{E}\left[\int_0^u \psi'(N_v)h(v)\left(\Gamma_v A_v q(v,X_v) + \Gamma_v' q(v,X_v)\right)dv\right] \geq 0 \end{aligned} \tag{3.19}$$

But, for every $0 \leq v \leq u$,

$$\begin{aligned} K(v) &:= \mathbb{E}\left[\psi'(N_v)h(v)\left(\Gamma_v A_v q(v,X_v) + \Gamma_v' q(v,X_v)\right)\right] \\ &= h(v)\Gamma_v\,\mathbb{E}\left[\psi'(N_v)q(v,X_v)\left(f_v(X_v) + \frac{\Gamma_v'}{\Gamma_v}\right)\right] \quad \text{(with } f_v \text{ defined by (3.12))} \\ &= h(v)\Gamma_v\,\mathbb{E}\left[\mathbb{E}[\psi'(N_v)|X_v]q(v,X_v)\left(f_v(X_v) + \frac{\Gamma_v'}{\Gamma_v}\right)\right] \end{aligned}$$

But, by Lemma 2.4,

$$\mathbb{E}[\psi'(N_v)|X_v] = \mathbb{E}\left[\psi'\left(\frac{1}{h(v)}\int_0^v q(u, X_u)\nu(du)\right)\bigg| X_v\right] = \varphi_v(X_v), \tag{3.20}$$

where φ_v is an increasing function and,

$$\mathbb{E}\left[q(v, X_v)\left(f_v(X_v) + \frac{\Gamma_v'}{\Gamma_v}\right)\right] = 0. \tag{3.21}$$

Then,

$$K(v) \geq h(v)\Gamma_v\varphi_v\left(f_v^{-1}\left[-\frac{\Gamma_v'}{\Gamma_v}\right]\right)\mathbb{E}\left[q(v, X_v)\left(f_v(X_v) + \frac{\Gamma_v'}{\Gamma_v}\right)\right] = 0.$$

□

Part II
Strong and very strong peacocks

4 Strong peacocks

4.1 Definition and examples

Definition 4.1. *An integrable real-valued process* $(X_t, t \geq 0)$ *is said to be a strong peacock (resp. a strong peadock) if, for every* $0 \leq s < t$ *and every increasing and bounded Borel function* $\phi : \mathbb{R} \to \mathbb{R}$*:*

$$\mathbb{E}[(X_t - X_s)\phi(X_s)] \geq 0 \tag{SP}$$

(resp.

$$\mathbb{E}[(X_t - X_s)\phi(X_t)] \leq 0.)$$

Remark 4.2.
1) The definition of a peacock involves only its 1-dimensional marginals. On the other side, the definition of a strong peacock involves its 2-dimensional marginals.
2) If $(X_t, t \geq 0)$ is a strong peacock, then $\mathbb{E}[X_t]$ does not depend on t (it suffices to apply (SP) with $\phi = 1$ and $\phi = -1$). Every strong peacock is a peacock; indeed, if $\psi \in \mathbf{C}_+$, then:

$$\mathbb{E}[\psi(X_t)] - \mathbb{E}[\psi(X_s)] \geq \mathbb{E}[\psi'(X_s)(X_t - X_s)] \geq 0.$$

3) If $(X_t, t \geq 0)$ is a strong peacock such that $\mathbb{E}\left[X_t^2\right] < \infty$ for every $t \geq 0$, then:

$$\mathbb{E}[X_s(X_t - X_s)] \geq 0, \text{ for every } 0 \leq s < t. \tag{4.1}$$

4) For two processes X and Y having the same 1-dimensional marginals, it may be possible that X is a strong peacock while Y is not. For example, let us consider $(X_t := t^{\frac{1}{4}} B_1, t \geq 0)$ and $\left(Y_t := \dfrac{B_t}{t^{\frac{1}{4}}}, t \geq 0\right)$, where $(B_t, t \geq 0)$ is a Brownian motion started at 0. By Lemma 1.1, $(X_t, t \geq 0)$ is a strong peacock while $(Y_t, t \geq 0)$ is not. Indeed, for every $a \in \mathbb{R}$ and $0 < s \leq t$:

$$\mathbb{E}\left[1_{\left\{\frac{B_s}{s^{\frac{1}{4}}} > a\right\}}\left(\frac{B_t}{t^{\frac{1}{4}}} - \frac{B_s}{s^{\frac{1}{4}}}\right)\right] = \left(\frac{1}{t^{\frac{1}{4}}} - \frac{1}{s^{\frac{1}{4}}}\right)\mathbb{E}\left[1_{\left\{B_s > as^{\frac{1}{4}}\right\}} B_s\right] < 0.$$

More generally, for every martingale $(M_t, t \geq 0)$ and every increasing Borel function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$, $\left(\dfrac{M_t}{\alpha(t)}, t \geq 0\right)$ is not a strong peacock.
5) Theorem 1.9 remains true if one replaces peacock by strong peacock.

Example 4.3. Some examples of strong peacocks:

- Martingales: indeed, if $(M_t, t \geq 0)$ is a martingale with respect to some filtration $(\mathcal{F}_t, t \geq 0)$, then, for every increasing and bounded Borel function $\phi : \mathbb{R} \to \mathbb{R}$:

$$\mathbb{E}[\phi(M_s)(M_t - M_s)] = \mathbb{E}[\phi(M_s)(\mathbb{E}[M_t|\mathcal{F}_s] - M_s)] = 0.$$

- If $(M_u, u \geq 0)$ is a martingale belonging to H^1_{loc} and $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ is a strictly increasing Borel function such that $\alpha(0) = 0$, then

$$\left(\frac{1}{\alpha(t)} \int_0^t M_u d\alpha(u)\right) \text{ is a strong peacock}$$

  (see [HPRY])

- the process $(tX, t \geq 0)$ where X is a centered and integrable r.v.
- the process $\left(\frac{e^{tX}}{\mathbb{E}[e^{tX}]}, t \geq 0\right)$ (see [HPRY]).

In the case of Gaussian processes, we obtain a characterization of strong peacocks using the covariance function. Indeed, one has:

Proposition 4.4. *A centered Gaussian peacock* $(X_t, t \geq 0)$ *is strong if and only if, for every* $0 < s \leq t$*:*

$$\mathbb{E}[X_s X_t] \geq \mathbb{E}[X_s^2]. \tag{4.2}$$

We note that a centered gaussian process $(X_t, t \geq 0)$ is a peacock if and only if

$$t \longmapsto \mathbb{E}\left[X_t^2\right] \text{ is increasing} \tag{4.3}$$

and of course, (4.2) implies (4.3); indeed, for every $0 < s \leq t$:

$$\mathbb{E}\left[X_s^2\right] \leq \mathbb{E}\left[X_s X_t\right] \leq \mathbb{E}\left[X_s^2\right]^{\frac{1}{2}} \mathbb{E}\left[X_t^2\right]^{\frac{1}{2}}, \text{ (from Schwartz's Inequality)}$$

which implies (4.3).

Proof of Proposition 4.4

Let $(X_t, t \geq 0)$ be a centered Gaussian strong peacock.

1) By taking $\phi(x) = x$ in (SP), we have:

$$\mathbb{E}[X_s(X_t - X_s)] \geq 0, \text{ for every } 0 < s \leq t,$$

i.e.,

$$K(s,t) \geq K(s,s), \text{ for every } 0 < s \leq t.$$

2) Conversely, if (4.2) holds, then, for every $0 < s \leq t$ and every increasing Borel function $\phi : \mathbb{R} \to \mathbb{R}$:

$$\begin{aligned}\mathbb{E}[\phi(X_s)(X_t - X_s)] &= \mathbb{E}[\phi(X_s)(\mathbb{E}[X_t|X_s] - X_s)] \\ &= \left(\frac{K(s,t)}{K(s,s)} - 1\right) \mathbb{E}[\phi(X_s)X_s] \geq 0 \quad \text{(from Lemma 1.1).}\end{aligned}$$

□

Example 4.5. We give two examples:

- An Ornstein-Uhlenbeck process with parameter $c \in \mathbb{R}$:

$$X_t = B_t + c \int_0^t X_u du,$$

where $(B_t, t \geq 0)$ is a Brownian motion started at 0, is a peacock for every c and a strong peacock if and only if $c \geq 0$. Indeed, for every $t \geq 0$,

$$X_t = e^{ct} \int_0^t e^{-cs} dB_s$$

and, for every $0 < s \leq t$, since

$$\mathbb{E}[X_s X_t] = \frac{\sinh(cs)}{c} e^{ct},$$

we have:

$$\mathbb{E}[X_s X_t] - \mathbb{E}\left[X_s^2\right] = \frac{\sinh(cs)}{c}[e^{ct} - e^{cs}].$$

Thus,

$$\mathbb{E}[X_s X_t] - \mathbb{E}\left[X_s^2\right] \geq 0 \text{ if and only if } c \geq 0.$$

- A fractional Brownian motion $(X_t, t \geq 0)$ with index $H \in [0, 1]$ is a peacock for every H and a strong peacock if and only if $H \geq \frac{1}{2}$. This follows from the fact that,

$$K(s,t) - K(s,s) = \frac{1}{2}(t^{2H} - s^{2H} - (t-s)^{2H})$$

is positive for every $0 < s \leq t$ if and only if $H \geq \frac{1}{2}$, where K denotes the covariance function of $(X_t, t \geq 0)$.

4.2 Upper and lower orthant orders

Let $X = (X_1, X_2, \ldots, X_p)$ and $Y = (Y_1, Y_2, \ldots, Y_p)$ be two $\mathbb{R}^p$-valued random vectors. The following definitions are taken from M. Shaked and J. Shantikumar, [SS94], p.140.

Definition 4.6. ***(Upper orthant order).***
X is said to be smaller than Y in the upper orthant order (notation: $X \underset{u.o}{\leq} Y$) if one of the two following equivalent conditions is satisfied:

1) for every p-tuple $\lambda_1, \lambda_2, \ldots, \lambda_p$ of reals:

$$\mathbb{P}(X_1 > \lambda_1, X_2 > \lambda_2, \ldots, X_p > \lambda_p) \leq \mathbb{P}(Y_1 > \lambda_1, Y_2 > \lambda_2, \ldots, Y_p > \lambda_p) \quad (4.4)$$

2) for every p-tuple $l_1, l_2, \ldots, l_p$ of nonnegative increasing functions:

$$\mathbb{E}\left[\prod_{i=1}^p l_i(X_i)\right] \leq \mathbb{E}\left[\prod_{i=1}^p l_i(Y_i)\right] \quad (4.5)$$

Definition 4.7. ***(Upper orthant order for processes).***
A process $(X_t, t \geq 0)$ *is smaller than a process* $(Y_t, t \geq 0)$ *for the upper orthant order (notation:* $(X_t, t \geq 0) \underset{u.o}{\leq} (Y_t, t \geq 0)$*) if, for every integer* p *and every* $0 \leq t_1 < t_2 < \cdots < t_p$*:*

$$(X_{t_1}, X_{t_2}, \ldots, X_{t_p}) \underset{u.o}{\leq} (Y_{t_1}, Y_{t_2}, \ldots, Y_{t_p}). \tag{4.6}$$

If X *and* Y *are two càdlàg processes, (4.6) is equivalent to:*

$$\begin{aligned} &\text{for every } h : \mathbb{R} \longrightarrow \mathbb{R} \text{ càdlàg:} \\ &\mathbb{P}(\text{for every } t \geq 0, X_t \geq h(t)) \leq \mathbb{P}(\text{for every } t \geq 0, Y_t \geq h(t)). \end{aligned} \tag{4.7}$$

Definition 4.8. ***(Lower orthant order).***
X is said to be smaller than Y in the lower orthant order (notation: $X \underset{l.o}{\leq} Y$*) if one of the two following equivalent conditions is satisfied:*

1) for every p-tuple $\lambda_1, \lambda_2, \ldots, \lambda_p$ *of reals:*

$$\mathbb{P}(X_1 \leq \lambda_1, X_2 \leq \lambda_2, \ldots, X_p \leq \lambda_p) \geq \mathbb{P}(Y_1 \leq \lambda_1, Y_2 \leq \lambda_2, \ldots, Y_p \leq \lambda_p) \tag{4.8}$$

2) for every p-tuple $l_1, l_2, \ldots, l_p$ *of nonnegative decreasing functions:*

$$\mathbb{E}\left[\prod_{i=1}^{p} l_i(X_i)\right] \geq \mathbb{E}\left[\prod_{i=1}^{p} l_i(Y_i)\right] \tag{4.9}$$

Definition 4.9. ***(Lower orthant order for processes).***
A process $(X_t, t \geq 0)$ *is smaller than a process* $(Y_t, t \geq 0)$ *for the lower orthant order (notation:* $(X_t, t \geq 0) \underset{l.o}{\leq} (Y_t, t \geq 0)$*) if, for every integer* p *and every* $0 \leq t_1 < t_2 < \cdots < t_p$*:*

$$(X_{t_1}, X_{t_2}, \ldots, X_{t_p}) \underset{l.o}{\leq} (Y_{t_1}, Y_{t_2}, \ldots, Y_{t_p}). \tag{4.10}$$

If X *and* Y *are two càdlàg processes, (4.10) is equivalent to:*

$$\begin{aligned} &\text{for every } h : \mathbb{R} \longrightarrow \mathbb{R} \text{ càdlàg:} \\ &\mathbb{P}(\text{for every } t \geq 0, X_t \leq h(t)) \geq \mathbb{P}(\text{for every } t \geq 0, Y_t \leq h(t)). \end{aligned} \tag{4.11}$$

Remark 4.10. Observe that, if $X = (X_t, \geq 0)$ and $Y = (Y_t, \geq 0)$ are two processes such that: $X \underset{\text{l.o}}{\leq} Y \underset{\text{u.o}}{\leq} X$, then

$$(X_t, t \geq 0) \overset{(1.\text{d})}{=} (Y_t, t \geq 0)$$

Let $(X_t, t \geq 0)$ be a real-valued process with measurable paths and, for $t \geq 0$, let F_t denotes the distribution function of X_t. If U is uniformly distributed on $[0, 1]$, then

$$(X_t, t \geq 0) \overset{(1.\text{d})}{=} \left(F_t^{-1}(U), t \geq 0\right).$$

Moreover, we state the following:

Proposition 4.11. *Let $(X_t, t \geq 0)$ a real-valued process, and for $t \geq 0$, let F_t be the distribution function of X_t. Then, if U is uniformly distributed on $[0,1]$, one has:*

$$\left(F_t^{-1}(U), t \geq 0\right) \underset{l.o.}{\leq} (X_t, t \geq 0) \underset{u.o.}{\leq} (F_t^{-1}(U)).$$

Proof of Proposition 4.11

For every integer p, every p-tuple $\lambda_1, \lambda_2, \dots, \lambda_p$ of reals and every $0 \leq t_1 < t_2 < \dots < t_p$:

$$\begin{aligned}
\mathbb{P}(X_{t_1} > \lambda_1, X_{t_2} > \lambda_2, \dots, X_{t_p} > \lambda_p) &\leq \min_{i=1,2,\dots,p} \mathbb{P}(X_{t_i} > \lambda_i) \\
&= 1 - \max_{i=1,2,\dots,p} F_{t_i}(\lambda_i) \\
&= \mathbb{P}\left(U > \max_{i=1,2,\dots,p} F_{t_i}(\lambda_i)\right) \\
&= \mathbb{P}\left(U > F_{t_1}(\lambda_1), U > F_{t_2}(\lambda_2), \dots, U > F_{t_p}(\lambda_p)\right) \\
&= \mathbb{P}\left(F_{t_1}^{-1}(U) > \lambda_1, F_{t_2}^{-1}(U) > \lambda_2, \dots, F_{t_p}^{-1}(U) > \lambda_p\right).
\end{aligned}$$

On the other hand, one has:

$$\begin{aligned}
\mathbb{P}(X_{t_1} \leq \lambda_1, X_{t_2} \leq \lambda_2, \dots, X_{t_p} \leq \lambda_p) &\leq \min_{i=1,2,\dots,p} \mathbb{P}(X_{t_i} \leq \lambda_i) \\
&= \mathbb{P}\left(F_{t_1}^{-1}(U) \leq \lambda_1, F_{t_2}^{-1}(U) \leq \lambda_2, \dots, F_{t_p}^{-1}(U) \leq \lambda_p\right).
\end{aligned}$$

□

Let us introduce some definitions.

Definition 4.12. *For a given family of probability measures $\mu = (\mu_t, t \geq 0)$, we denote by $\mathcal{D}_\mu$ the set of real-valued processes which admit the family μ as one-dimensional marginals:*

$$\mathcal{D}_\mu := \{(X_t, t \geq 0);\ \text{such that for every } t \geq 0,\ X_t \sim \mu_t\}.$$

In particular, if the family μ increases in the convex order, then $\mathcal{D}_\mu$ is the set of peacocks associated to μ.

The next corollary follows immediately from Proposition 4.11.

Corollary 4.13. *Let μ be a family of probability measures. Then, the process $(F_t^{-1}(U), t \geq 0)$ is an absolute maximum of $\mathcal{D}_\mu$ for the upper orthant order and an absolute minimum of $\mathcal{D}_\mu$ for the lower orthant order.*

The following result is due to S. Cambanis, G. Simons and W. Stout [CSS76].

Theorem 4.14. *Let (X_1, X_2) and (Y_1, Y_2) be two $\mathbb{R}^2$-valued random vectors such that:*

$$X_1 \overset{(law)}{=} Y_1,\ X_2 \overset{(law)}{=} Y_2 \text{ and } (X_1, X_2) \underset{l.o}{\leq} (Y_1, Y_2) \tag{4.12}$$

Let $k : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be right-continuous and quasi-monotone, i.e:

$$k(x,y) + k(x',y') - k(x,y') - k(x',y) \geq 0, \text{ for every } x \leq x',\ y \leq y'. \tag{4.13}$$

Suppose that the expectations $\mathbb{E}[k(X_1, X_2)]$ and $\mathbb{E}[k(Y_1, Y_2)]$ exist (even if infinite valued) and either of the following conditions is satisfied:

(i) k is symmetric and the expectations $\mathbb{E}[k(X_1, X_1)]$ *and* $\mathbb{E}[k(X_2, X_2)]$ *are finite*

(ii) the expectations $\mathbb{E}[k(X_1, x_1)]$ *and* $\mathbb{E}[k(x_2, X_2)]$ *are finite for some* x_1 *and* x_2*.*

Then:

$$\mathbb{E}[k(X_1, X_2)] \geq \mathbb{E}[k(Y_1, Y_2)].$$

The next result is deduced from Proposition 4.11 and Theorem 4.14.

Corollary 4.15. *Let* $X := (X_t, t \geq 0)$ *be a peacock and, for every* $t \geq 0$*, let* F_t *be the distribution function of* X_t*. Let* U *be uniformly distributed on* $[0, 1]$*. Then:*

1) for every real-valued process $Y := (Y_t, t \geq 0)$ *such that* $Y_t \overset{(1.d)}{=} X_t$ *and every quasi-monotone function* $k : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *satisfying the same conditions as in Theorem 4.14, one has:*

$$\forall (s, t) \in \mathbb{R}_+ \times \mathbb{R}_+, \ \mathbb{E}\left[k\left(F_s^{-1}(U), F_t^{-1}(U)\right)\right] \geq \mathbb{E}[k(Y_s, Y_t)]. \tag{4.14}$$

In particular, for every $p \geq 1$ *such that* $\mathbb{E}[|X_u|^p] < \infty$*, for every* $u \geq 0$ *and every* $(s, t) \in \mathbb{R}_+ \times \mathbb{R}_+$*,*

$$\mathbb{E}\left[\left|F_t^{-1}(U) - F_s^{-1}(U)\right|^p\right] \leq \mathbb{E}\left[|Y_t - Y_s|^p\right], \tag{4.15}$$

2) $(F_t^{-1}(U), t \geq 0)$ *is a strong peacock.*

To prove Corollary 4.15 we may observe, for the first point, that for every $p \geq 1$, the function $k : (x, y) \longmapsto -|x - y|^p$ is quasi-monotone and, for the second point, that if $\phi : \mathbb{R} \to \mathbb{R}$ is increasing, then $k : (x, y) \mapsto \phi(x)(y - x)$ is a quasi-monotone function.

4.3 A peacocks' comparison Theorem

Let $(X_t, t \geq 0)$ be a real-valued process which is square integrable and which satisfies:

$$t \longmapsto X_t \text{ is a.s. measurable.} \tag{4.16}$$

For a probability measure ν on $\{(s, t); 0 \leq s \leq t\}$, let us define the 2-variability of $(X_t, t \geq 0)$ with respect to ν by the quantity:

$$\Pi_\nu(X) := \iint_{\{0 \leq s \leq t\}} \mathbb{E}[(X_t - X_s)^2]\nu(ds, dt).$$

Definition 4.16. *For a family of probability measures* $\mu := (\mu_t, t \geq 0)$*, let* $\mathcal{D}_\mu^+$ *denotes the set of strong peacocks which admit the family* μ *as one-dimensional marginals:*

$$\mathcal{D}_\mu^+ := \{(X_t, t \geq 0); X \text{ is a strong peacock such that, } \forall t \geq 0, \, X_t \sim \mu_t\}.$$

Given a family of probability measures $\mu := (\mu_t, t \geq 0)$ which increases in the convex order, we wish to determinate for which processes in $\mathcal{D}_\mu^+$, Π_ν attains his maximum (resp. his minimum).

Theorem 4.17. *Let ν be a probability measure on $\{(s,t);\, 0 \le s \le t\}$.*

1. *The maximum of $\Pi_\nu(X)$ in $\mathcal{D}_\mu^+$ is equal to:*

$$\max_{X \in \mathcal{D}_\mu^+} \Pi_\nu(X) = \iint_{\{0 \le s \le t\}} \left(\mathbb{E}\left[X_t^2\right] - \mathbb{E}\left[X_s^2\right]\right) \nu(ds, dt) \tag{4.17}$$

   and is attained when $(X_t, t \ge 0)$ is a martingale.

2. *The minimum of $\Pi_\nu(X)$ in $\mathcal{D}_\mu^+$ is equal to:*

$$\min_{X \in \mathcal{D}_\mu^+} \Pi_\nu(X) = \iint_{\{0 \le s \le t\}} \mathbb{E}\left[\left(F_t^{-1}(U) - F_s^{-1}(U)\right)^2\right] \nu(ds, dt) \tag{4.18}$$

   and is attained by $\left(X_t = F_t^{-1}(U), t \ge 0\right)$.

Proof of Theorem 4.17
1) Let $(X_t, t \ge 0)$ be a strong peacock. For every $0 \le s \le t$, one has:

$$\begin{aligned}
\mathbb{E}\left[(X_t - X_s)^2\right] &= \mathbb{E}\left[X_t^2\right] + \mathbb{E}\left[X_s^2\right] - 2\mathbb{E}[X_t X_s] \\
&= \mathbb{E}\left[X_t^2\right] - \mathbb{E}\left[X_s^2\right] - 2\mathbb{E}[(X_t - X_s)X_s] \\
&\le \mathbb{E}\left[X_t^2\right] - \mathbb{E}\left[X_s^2\right] \quad \text{(from (SP))}.
\end{aligned}$$

Hence, integrating against ν, we obtain:

$$\max_{X \in \mathcal{D}_\mu^+} \Pi_\nu(X) \le \iint_{\{0 \le s \le t\}} \left(\mathbb{E}\left[X_t^2\right] - \mathbb{E}\left[X_s^2\right]\right) \nu(ds, dt) := M(X)$$

and $M(X)$ is clearly attained when $(X_t, t \ge 0)$ is a martingale.
2) This point is a consequence of Theorem 4.14 and Corollary 4.15.

□

5 Very strong peacocks

5.1 Definition, examples and counterexamples

Definition 5.1. *An integrable real-valued process $(X_t, t \ge 0)$ is said to be a very strong peacock (VSP) if, for every $n \in \mathbb{N}^*$, every $0 < t_1 < \cdots < t_n < t_{n+1}$ and every $\phi \in \mathcal{E}_n$, we have:*

$$\mathbb{E}\left[\phi\left(X_{t_1}, \dots, X_{t_n}\right)\left(X_{t_{n+1}} - X_{t_n}\right)\right] \ge 0. \tag{VSP}$$

Remark 5.2.
1) The definition of a strong peacock involves its 2-dimensional marginals while the definition of a very strong peacock involves all its finite-dimensional marginals.
2) Every very strong peacock is a strong peacock. But, the converse is not true. Let us give two examples:

a) let G_1 and G_2 be two independent, centered Gaussian r.v.'s such that $\mathbb{E}[G_1^2] = \mathbb{E}[G_2^2] = 1$, α, β be two constants satisfying $1 + 2\alpha^2 \le \beta$ and (X_1, X_2, X_3) be the random Gaussian vector defined by:

$$X_1 = G_1 - \alpha G_2,\ X_2 = \beta G_1,\ X_3 = \beta G_1 + \alpha G_2. \tag{5.1}$$

Then, (X_1, X_2, X_3) is a strong peacock (from Proposition 4.4) which is not a very strong peacock since

$$\mathbb{E}[X_1(X_3 - X_2)] = -\alpha^2 \mathbb{E}\left[G_1^2\right] < 0.$$

b) Likewise, let G_1 and G_2 be two symmetric, independent and identically distributed r.v.'s such that $\mathbb{E}\left[G_i^2\right] = 1$ $(i = 1, 2)$. Then, for every $\beta \geq 3$, the random vector (X_1, X_2, X_3) given by:

$$X_1 = G_1 - G_2, \; X_2 = \beta G_1, \; X_3 = \beta G_1 + G_2. \tag{5.2}$$

is a strong peacock for which (VSP) does not hold.
Proof.
Since G_1 and G_2 are independent and centered, we first observe that:

$$\mathbb{E}\left[1_{\{X_2 \geq a\}}(X_3 - X_2)\right] = \mathbb{E}\left[1_{\{\beta G_1 \geq a\}} G_2\right] = 0.$$

Moreover,

$$\begin{aligned}
&\mathbb{E}\left[1_{\{X_1 \geq a\}}(X_2 - X_1)\right] = \mathbb{E}\left[1_{\{G_1 - G_2 \geq a\}}((\beta - 1)G_1 + G_2)\right] \\
&= (\beta - 1)\mathbb{E}\left[1_{\{G_1 - G_2 \geq a\}} G_1\right] + \mathbb{E}\left[1_{\{G_1 - G_2 \geq a\}} G_2\right] \\
&= \underbrace{(\beta - 1)\mathbb{E}\left[1_{\{G_2 - G_1 \geq a\}} G_2\right]}_{\text{(by interchanging } G_1 \text{ and } G_2)} + \mathbb{E}\left[1_{\{G_1 - G_2 \geq a\}} G_2\right] \\
&= \underbrace{(\beta - 2)\mathbb{E}\left[1_{\{G_2 - G_1 \geq a\}} G_2\right]}_{\geq 0 \text{ by Lemma 1.1, since } \beta > 2} + \mathbb{E}\left[1_{\{|G_1 - G_2| \geq a\}} G_2\right] \\
&\geq \mathbb{E}\left[1_{\{|G_1 - G_2| \geq a\}} G_2\right] = 0, \text{ (since } G_1 \text{ and } G_2 \text{ are symmetric)}
\end{aligned}$$

and similarly,

$$\begin{aligned}
&\mathbb{E}\left[1_{\{X_1 \geq a\}}(X_3 - X_1)\right] = \mathbb{E}\left[1_{\{G_1 - G_2 \geq a\}}((\beta - 1)G_1 + 2G_2)\right] \\
&= (\beta - 1)\mathbb{E}\left[1_{\{G_1 - G_2 \geq a\}} G_1\right] + 2\mathbb{E}\left[1_{\{G_1 - G_2 \geq a\}} G_2\right] \\
&= (\beta - 1)\mathbb{E}\left[1_{\{G_2 - G_1 \geq a\}} G_2\right] + 2\mathbb{E}\left[1_{\{G_1 - G_2 \geq a\}} G_2\right] \\
&= \underbrace{(\beta - 3)\mathbb{E}\left[1_{\{G_2 - G_1 \geq a\}} G_2\right]}_{\geq 0 \text{ by Lemma 1.1, since } \beta \geq 3} + 2\mathbb{E}\left[1_{\{|G_1 - G_2| \geq a\}} G_2\right] \\
&\geq 2\mathbb{E}\left[1_{\{|G_1 - G_2| \geq a\}} G_2\right] = 0.
\end{aligned}$$

Thus, (X_1, X_2, X_3) is a strong peacock. But, (X_1, X_2, X_3) is not a very strong peacock since

$$\mathbb{E}[X_1(X_3 - X_2)] = -\mathbb{E}\left[G_1^2\right] < 0.$$

Let us give some examples of very strong peacocks.

Example 5.3.
1) Each of the processes cited in Example 4.3 is a very strong peacock. We refer the reader to ([HPRY], Chapter 8) for further examples.
2) Let $(\tau_t, t \geq 0)$ be an increasing process with independent increments (for example a

subordinator) and $f : \mathbb{R} \to \mathbb{R}$ be a convex and increasing (or concave and decreasing) function such that $\mathbb{E}[|f(\tau_t)|] < \infty$, for every $t \geq 0$. Then, $(X_t := f(\tau_t) - \mathbb{E}[f(\tau_t)], t \geq 0)$ is a very strong peacock.

Proof.

Let f be a convex and increasing function and let $n \geq 1$, $0 < t_1 < t_2 < \cdots < t_n < t_{n+1}$ and $\phi \in \mathcal{E}_n$. We first note that:

$$\widetilde{\phi} : (x_1, \dots, x_n) \longmapsto \phi\left(f(x_1) - \mathbb{E}[f(\tau_{t_1})], \dots, f(x_n) - \mathbb{E}[f(\tau_{t_n})]\right) \text{ belongs to } \mathcal{E}_n \quad (5.3)$$

and, by setting $c_n := \mathbb{E}[f(\tau_{t_{n+1}})] - \mathbb{E}[f(\tau_{t_n})]$,

$$\mathbb{E}\left[\phi(X_{t_1}, \dots, X_{t_n})(X_{t_{n+1}} - X_{t_n})\right] = \mathbb{E}\left[\widetilde{\phi}(\tau_{t_1}, \dots, \tau_{t_n})(f(\tau_{t_{n+1}}) - f(\tau_{t_n}) - c_n)\right]. \quad (5.4)$$

Let us prove by induction that, for every $i \in [\![1, n]\!]$, there exists a function $\varphi_i \in \mathcal{E}_i$ such that:

$$\mathbb{E}\left[\phi(X_{t_1}, \dots, X_{t_n})(X_{t_{n+1}} - X_{t_n})\right] \geq \mathbb{E}\left[\varphi_i(\tau_{t_1}, \dots, \tau_{t_i})(f(\tau_{t_{n+1}}) - f(\tau_{t_n}) - c_n)\right]. \quad (5.5)$$

We note that, for $i = n$, we may choose $\varphi_n = \widetilde{\phi}$. On the other hand, let us suppose that (5.5) holds for some $i \in [\![1, n]\!]$. Then, since τ_{t_i} is independent of $\tau_{t_{n+1}} - \tau_{t_i}$ and $\tau_{t_n} - \tau_{t_i}$, one has:

$$\begin{aligned}
&\mathbb{E}\left[\phi(X_{t_1}, \dots, X_{t_n})(X_{t_{n+1}} - X_{t_n})\right] \\
&\geq \mathbb{E}\left[\varphi_i(\tau_{t_1}, \dots, \tau_{t_i})(f(\tau_{t_{n+1}}) - f(\tau_{t_n}) - c_n)\right] \quad \text{(by induction)} \\
&= \mathbb{E}\left[\varphi_i(\tau_{t_1}, \dots, \tau_{t_i})\left(f(\tau_{t_i} + \tau_{t_{n+1}} - \tau_{t_i}) - f(\tau_{t_i} + \tau_{t_n} - \tau_{t_i}) - c_n\right)\right] \\
&= \mathbb{E}\left[\varphi_i(\tau_{t_1}, \dots, \tau_{t_i})\left(\mathbb{E}[f(\tau_{t_i} + \tau_{t_{n+1}} - \tau_{t_i})|\mathcal{F}_{t_i}] - \mathbb{E}[f(\tau_{t_i} + \tau_{t_n} - \tau_{t_i})|\mathcal{F}_{t_i}] - c_n\right)\right] \\
&\quad \text{(where } \mathcal{F}_{t_i} := \sigma(\tau_s, 0 \leq s \leq t_i)) \\
&= \mathbb{E}\left[\varphi_i(\tau_{t_1}, \dots, \tau_{t_i})\widehat{f_i}(\tau_{t_i})\right], \\
&\quad \text{(where } \widehat{f_i}(x) = \mathbb{E}[f(x + \tau_{t_{n+1}} - \tau_{t_i})] - \mathbb{E}[f(x + \tau_{t_n} - \tau_{t_i})] - c_n).
\end{aligned}$$

But, the function $\widehat{f_i}$ is increasing since f is convex and $\tau_{t_{n+1}} \geq \tau_{t_n}$. Hence,

$$\begin{aligned}
\mathbb{E}\left[\phi(X_{t_1}, \dots, X_{t_n})(X_{t_{n+1}} - X_{t_n})\right] &\geq \mathbb{E}\left[\varphi_i\left(\tau_{t_1}, \dots, \tau_{t_{i-1}}, \tau_{t_i}\right)\widehat{f_i}(\tau_{t_i})\right] \\
&\geq \mathbb{E}\left[\varphi_i\left(\tau_{t_1}, \dots, \tau_{t_{i-1}}, \widehat{f_i}^{-1}(0)\right)\widehat{f_i}(\tau_{t_i})\right] \\
&= \mathbb{E}\left[\varphi_i\left(\tau_{t_1}, \dots, \tau_{t_{i-1}}, \widehat{f_i}^{-1}(0)\right)\left(f(\tau_{t_{n+1}}) - f(\tau_{t_n}) - c_n\right)\right],
\end{aligned}$$

i.e., (5.5) also holds for $i - 1$ with

$$\varphi_{i-1} : (x_1, \dots, x_{i-1}) \longmapsto \varphi_i\left(x_1, \dots, x_{i-1}, \widehat{f_i}^{-1}(0)\right).$$

Thus, (5.5) holds for every $i \in [\![1, n]\!]$. In particular, for $i = 1$, there exists $\varphi_1 \in \mathcal{E}_1$ such that:

$$\begin{aligned}
\mathbb{E}\left[\phi(X_{t_1}, \dots, X_{t_n})(X_{t_{n+1}} - X_{t_n})\right] &\geq \mathbb{E}\left[\varphi_1(\tau_{t_1})\widehat{f_1}(\tau_{t_1})\right] \\
&\geq \varphi_1\left(\widehat{f_1}^{-1}(0)\right)\mathbb{E}\left[\widehat{f_1}(\tau_{t_1})\right] \\
&= \varphi_1\left(\widehat{f_1}^{-1}(0)\right)\mathbb{E}\left[f(\tau_{t_{n+1}}) - f(\tau_{t_n}) - c_n\right] = 0.
\end{aligned}$$

□

5.2 Peacocks obtained by quotient under the very strong peacock hypothesis

Lemma 5.4. *An integrable real-valued process is a very strong peacock if and only if, for every $n \geq 1$, every $0 < t_1 < \cdots < t_n < t_{n+1}$, every $i \leq n$ and every $\phi \in \mathcal{E}_n$:*

$$\mathbb{E}\left[\phi\left(X_{t_1}, \ldots, X_{t_n}\right)\left(X_{t_{n+1}} - X_{t_i}\right)\right] \geq 0. \qquad (\widetilde{VSP})$$

Proof of Lemma 5.4

For every $n \geq 1$ and $i \leq n$, we shall prove by induction the following condition:

$$\mathbb{E}\left[\phi\left(X_{t_1}, \ldots, X_{t_n}\right)\left(X_{t_{n+1}} - X_{t_{n+1-i}}\right)\right] \geq 0 \qquad (5.6)$$

which, of course, is equivalent to $(\widetilde{VSP})$. If $i = 1$, we recover (VSP).
Now, let $1 \leq i \leq n-1$ be fixed and suppose that (VSP) is satisfied and that (5.6) holds for i. Let us prove that (5.6) is also true for $i+1$. One has:

$$\begin{aligned}
&\mathbb{E}\left[\phi\left(X_{t_1}, \ldots, X_{t_{n-1}}, X_{t_n}\right)\left(X_{t_{n+1}} - X_{t_{n+1-(i+1)}}\right)\right] \\
&= \underbrace{\mathbb{E}\left[\phi\left(X_{t_1}, \ldots, X_{t_{n-1}}, X_{t_n}\right)\left(X_{t_{n+1}} - X_{t_n}\right)\right]}_{\geq 0 \text{ (from (VSP))}} \\
&\quad + \mathbb{E}\left[\phi\left(X_{t_1}, \ldots, X_{t_{n-1}}, X_{t_n}\right)\left(X_{t_n} - X_{t_{n-i}}\right)\right] \\
&\geq \mathbb{E}\left[\phi\left(X_{t_1}, \ldots, X_{t_{n-1}}, X_{t_n}\right)\left(X_{t_n} - X_{t_{n-i}}\right)\right] \\
&\geq \mathbb{E}\left[\phi\left(X_{t_1}, \ldots, X_{t_{n-1}}, X_{t_{n-i}}\right)\left(X_{t_n} - X_{t_{n-i}}\right)\right] \geq 0
\end{aligned}$$

(since ϕ belongs to $\mathcal{E}_n$ and (5.6) holds for $1 \leq i \leq n-1$).

□

The importance of very strong peacocks lies in the following result.

Theorem 5.5. *Let $(X_t, t \geq 0)$ be a right-continuous and centered very strong peacock such that for every $t \geq 0$:*

$$\mathbb{E}\left[\sup_{s \in [0,t]} |X_s|\right] < \infty. \qquad (5.7)$$

Then, for every right-continuous and strictly increasing function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\alpha(0) = 0$:

$$\left(Q_t := \frac{1}{\alpha(t)} \int_0^t X_s \, d\alpha(s), t \geq 0\right) \text{ is a peacock.}$$

Remark 5.6.

1) Theorem 5.5 is a generalization of the case where $(X_s, s \geq 0)$ is a martingale (see Example 1.2).
2) Let $(\tau_s, s \geq 0)$ be a subordinator and $f : \mathbb{R}_+ \to \mathbb{R}$ be increasing, convex and such that $\mathbb{E}[|f(\tau_t)|] < \infty$, for every $t \geq 0$. Then, it follows from Theorem 5.5 and from the

second point of Example 5.3 that, for every right-continuous and strictly increasing function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $\alpha(0) = 0$:

$$\left(Q_t := \frac{1}{\alpha(t)} \int_0^t (f(\tau_s) - \mathbb{E}[f(\tau_s)])\, d\alpha(s), t \geq 0\right) \text{ is a peacock.}$$

Proof of Theorem 5.5

Let $T > 0$ be fixed.

1) Let us first suppose that $1_{[0,T]}d\alpha$ is a linear combination of Dirac measures and show that, for every $r \in [\![2, \infty]\!]$, every $a_1 > 0, a_2 > 0, \dots, a_r > 0$ such that

$$\alpha(r) := \sum_{i=1}^{r} a_i = \alpha(T)$$

and every $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n \leq T$:

$$\left(Q_n := \frac{1}{\alpha(n)} \sum_{i=1}^{n} a_i X_{\lambda_i}, n \in [\![1, r]\!]\right) \text{ is a peacock.} \tag{5.8}$$

Let $\psi \in \mathbf{C}_+$ and $n \geq 2$. For every $n \in [\![2, r]\!]$, one has:

$$\begin{aligned}
&\mathbb{E}[\psi(Q_n)] - \mathbb{E}[\psi(Q_{n-1})] \geq \mathbb{E}[\psi'(Q_{n-1})(Q_n - Q_{n-1})] \\
&= \mathbb{E}\left[\psi'\left(\frac{1}{\alpha(n-1)} \sum_{i=1}^{n-1} a_i X_{\lambda_i}\right)\left(\frac{1}{\alpha(n)} \sum_{i=1}^{n} a_i X_{\lambda_i} - \frac{1}{\alpha(n-1)} \sum_{i=1}^{n-1} a_i X_{\lambda_i}\right)\right] \\
&= \frac{a_n}{\alpha(n)\alpha(n-1)} \sum_{i=1}^{n-1} a_i \mathbb{E}\left[\phi\left(X_{\lambda_1}, \dots, X_{\lambda_{n-1}}\right)(X_{\lambda_n} - X_{\lambda_i})\right],
\end{aligned}$$

where

$$\phi : (x_1, \dots, x_{n-1}) \longmapsto \psi'\left(\frac{1}{\alpha(n-1)} \sum_{i=1}^{n-1} a_i x_i\right) \text{ belongs to } \mathcal{E}_{n-1}.$$

Then, the result follows from Lemma 5.4.

2) Let us set $\mu = 1_{[0,T]}d\alpha$ and, for every $0 \leq t \leq T$,

$$Q_t^{\mu} := \frac{1}{\mu([0,t])} \int_0^t X_u \mu(du).$$

Since the function $\lambda \in [0, T] \longmapsto X_\lambda$ is right-continuous and bounded from above by $\sup\limits_{0 \leq \lambda \leq T} |X_\lambda|$ which is finite a.s., then there exists a sequence $(\mu_n, n \geq 0)$ of measures of type used in **1)**, with supp $\mu_n \subset [0, T]$, $\int \mu_n(du) = \int \mu(du)$ and, for every $0 \leq t \leq T$:

$$\lim_{n\to\infty} \int_0^t X_u \mu_n(du) = \int_0^t X_u \mu(du) \text{ a.s.} \tag{5.9}$$

$$\lim_{n\to\infty} \mu_n([0,t]) = \mu([0,t]). \tag{5.10}$$

Then, from (5.9) and (5.10), it follows that:

$$\lim_{n\to\infty} Q_t^{(\mu_n)} = Q_t^{(\mu)} \text{ a.s., for every } 0 \leq t \leq T. \tag{5.11}$$

But, using **1)**,

$$\left(Q_t^{(\mu_n)}, 0 \leq t \leq T\right) \text{ is a peacock for every } n \geq 0, \tag{5.12}$$

i.e., for every $0 \leq s < t \leq T$, $\mathbb{E}\left[Q_s^{(\mu_n)}\right] = \mathbb{E}\left[Q_t^{(\mu_n)}\right]$ and, for every $\psi \in \mathbf{C}_+$:

$$\mathbb{E}\left[\psi(Q_s^{(\mu_n)})\right] = \mathbb{E}\left[\psi(Q_t^{(\mu_n)})\right]. \tag{5.13}$$

Moreover,

$$\sup_{0 \leq t \leq T} \sup_{n \geq 0} \left|Q_t^{(\mu_n)}\right| \leq \sup_{0 \leq \lambda \leq T} |X_\lambda| \tag{5.14}$$

which is integrable from (5.7).
Therefore, using (0.3), (5.11)-(5.13) and the dominated convergence Theorem, $\left(Q_t^{(\mu)}, 0 \leq t \leq T\right)$ is a peacock for every $T > 0$.

□

Table of the main peacocks studied in this paper

In this table:

- $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ is a right-continuous and increasing function such that $\alpha(0) = 0$,
- $q : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}_+$ is a continuous and positive function such that, for every $s \geq 0$,

$x \mapsto q(s, x)$ is increasing.

Main hypothesis	Peacocks	References
$(X_t, t \geq 0)$ is conditionally monotone and θ is positive, convex and increasing	$\left(C_t := \theta\left(\int_0^t q(s, X_s)ds\right) - \gamma(t), t \geq 0\right)$ with $\gamma(t) = \mathbb{E}\left[\theta\left(\int_0^t q(s, X_s)ds\right)\right]$	Theorem 2.5
$(X_t, t \geq 0)$ is a process with independent and log-concave increments	$\left(N_t := \dfrac{\exp\left(\int_0^t q(s, X_s)\, d\alpha(s)\right)}{\mathbb{E}\left[\exp\left(\int_0^t q(s, X_s)\, d\alpha(s)\right)\right]}, t \geq 0\right)$	Theorem 2.12
$(X_t, t \geq 0)$ solves an SDE with infinitesimal generator L_s, θ is positive, increasing and for every $s \geq 0$, $x \mapsto \dfrac{L_s\theta(x)}{\theta(x)}$ is increasing.	$\left(N_t := \dfrac{\theta(X_t)}{\mathbb{E}[\theta(X_t)]}, t \geq 0\right)$	Theorem 3.1
$(X_t, t \geq 0)$ solves an SDE with infinitesimal generator L_s, θ is positive and, for every $s \geq 0$, $x \mapsto L_s\theta(x)$ is increasing.	$(C_t := \theta(X_t) - \mathbb{E}[\theta(X_t)], t \geq 0)$	Theorem 3.2
$(X_t, t \geq 0)$ is conditionally monotone and solves an SDE and ν is a positive Radon measure on $\mathbb{R}_+$	$\left(N_t := \dfrac{\int_0^t q(s, X_s)\nu(ds)}{\mathbb{E}\left[\int_0^t q(s, X_s)\nu(ds)\right]}, t \geq 0\right)$	Theorem 3.4
$(X_t, t \geq 0)$ is a centered very strong peacock	$\left(Q_t := \dfrac{1}{\alpha(t)} \int_0^t X_s\, d\alpha(s), t \geq 0\right)$	Theorem 5.5
$(L_t, t \geq 0)$ is a Lévy process such that the variable $\exp\left(\int_0^t L_u\, du\right)$ is integrable	$\left(N_t := \dfrac{\exp\left(\int_0^t L_s\, ds\right)}{\mathbb{E}\left[\exp\left(\int_0^t L_s\, ds\right)\right]}, t \geq 0\right)$	Example 1.8
	$\left(\widetilde{N}_t := \dfrac{\exp\left(\frac{1}{t}\int_0^t L_s\, ds\right)}{\mathbb{E}\left[\exp\left(\frac{1}{t}\int_0^t L_s\, ds\right)\right]}, t \geq 0\right)$	

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