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ANALYSIS OF A TWO-LEVEL SCHWARZ METHOD WITH COARSE SPACES BASED ON LOCAL DIRICHLET–TO–NEUMANN MAPS

VICTORITA DOLEAN, FRÉDÉRIC NATAF, ROBERT SCHEICHL, AND NICOLE SPILLANE

ABSTRACT. Coarse grid correction is a key ingredient in order to have scalable domain decomposition methods. For smooth problems, the theory and practice of such two-level methods is well established, but this is not the case for problems with complicated variation and high contrasts in the coefficients. Stable coarse spaces for high contrast problems are also important purely for approximation purposes, when it is not desirable to resolve all the fine scale variations in the problem. In a previous study, two of the authors introduced a coarse space adapted to highly heterogeneous coefficients using the low frequency modes of the subdomain DtN maps. In this work, we present a rigorous analysis of a two-level overlapping additive Schwarz method (ASM) with this coarse space, which provides an automatic criterion for the number of modes that need to be added per subdomain to obtain a convergence rate of the order of the constant coefficient case. Our method is suitable for parallel implementation and its efficiency is demonstrated by numerical examples on some challenging problems with high heterogeneities for automatic partitionings.

1. INTRODUCTION

When using an iterative method in a one-level domain decomposition framework, one may encounter a long stagnation or a slow convergence even in the case where physical coefficients are homogeneous: convergence deteriorates when the number of subdomains increases. This is a problem since, to perform well, one needs to have a scalable algorithm, i.e., an algorithm whose convergence rate is weakly dependent on the number of subdomains, see [34] and references therein. In order to achieve scalability, global information on the solution needs to be shared between subdomains. This process takes the form of a coarse space correction leading to a two-level algorithm or preconditioner. The BPS preconditioner introduced by Bramble, Paschiak and Schatz [1] is of this type. We can also mention the two-level overlapping Schwarz and the balancing Neumann-Neumann preconditioner, as well as the FETI algorithm, which have been extensively investigated, see [34] and references therein. The abstract analysis of two-level overlapping Schwarz methods, which we will use, is due to [9]. For symmetric systems the balancing preconditioner was proposed by Mandel [20]. The FETI algorithm was first introduced by [13].

The definition of a two-level preconditioner is closely related to its key ingredient: the choice of an appropriate coarse space. Our goal here is to build such a coarse space in the context of domain decomposition methods for elliptic problems with highly heterogeneous coefficients. When the jumps in the coefficients are across subdomain interfaces or inside the subdomains (and not near their boundaries), classical coarse spaces give good results, see

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e.g. [8, 21, 28, 26, 6, 7]. However, when the discontinuities are along subdomain interfaces, classical results break down. This is what we work to improve.

In previous work, [23], two of the authors proposed the construction of a coarse subspace, which leads to a two-level method that is observed to be robust with respect to heterogeneous coefficients for fairly arbitrary domain decompositions, e.g. provided by an automatic graph partitioner such as METIS or SCOTCH [19, 5]. This method was extensively studied from a numerical point of view in [24]. The construction is based on the low-frequency modes associated with the Dirichlet-to-Neumann (DtN) map on each subdomain. After obtaining the eigenvectors associated with the near-kernel components of the DtN operator, we use their harmonic extensions to the whole subdomain to build the coarse grid. With this method, even for discontinuities along (rather than across) the subdomain interfaces, the iteration counts are robust to arbitrarily large jumps of the coefficients leading to a very efficient, automatic method for these kinds of problems. It is also suitable for parallel implementation. Similar ideas to build stable coarse spaces, based on the solution of eigenvalue problems on the whole subdomain, have also been presented in other papers on Schwarz methods [15, 10], as well as in earlier work on algebraic multigrid methods [2, 4].

In this paper we will analyze the two-level preconditioner proposed in [23] theoretically. In order to do this, we will use the framework of weighted Poincaré inequalities, introduced and successfully applied to different domain decomposition methods in [29, 30, 27, 14, 32, 33]. Our analysis is inspired by the techniques in [15], as well as by the abstract framework developed in [33]. However, the DtN coarse space is better designed to deal with coefficient variations that are strictly interior to the subdomain, leading to a smaller dimension than the coarse space proposed in [15]. Only heuristic ideas are given in [15] to overcome this problem. For our coarse space we have a rigorous justification. The result that we obtain, generalizes the classical estimates of overlapping Schwarz methods to the case where the coarse space is richer than just the constant mode per domain [25], or other classical coarse spaces (cf. [34]).

The rest of the paper is organized as follows. In Section 2 we introduce the general framework. The model problem, its discretization and some weighted norms, depending on the coefficients, are introduced in Section 2.1. Then the decomposition into subdomains, the overlapping Schwarz method and some weighted Poincaré inequalities, including a new trace result, are presented in Sections 2.2–2.3. In Section 3 the new coarse space is presented, at local and global level, and theoretical stability results are shown (e.g. stability of the coarse space projectors), then a condition number estimate of the two-level method based on the existence of a stable decomposition is given in Theorem 3.5. Extensive numerical results for rather hard test cases are given in Section 4, followed by some conclusions in Section 5.

2. PRELIMINARIES AND NOTATION

2.1. **Model problem and discretization.** We consider the variational formulation of a second order, elliptic boundary value problem with Dirichlet boundary conditions: Find $u^* \in H_0^1(\Omega)$, for a given polygonal (polyhedral) domain $\Omega \subset \mathbb{R}^d$ ($d = 2$ or 3) and a source term $f \in L_2(\Omega)$, such that

$$(2.1) \qquad \underbrace{\int_\Omega \alpha(\boldsymbol{x})\,\nabla u^* \cdot \nabla v}_{\equiv\, a(u^*,\,v)} \;=\; \underbrace{\int_\Omega f(\boldsymbol{x})v(\boldsymbol{x})}_{\equiv\, (f,\,v)}, \quad \text{for all} \quad v \in H_0^1(\Omega).$$
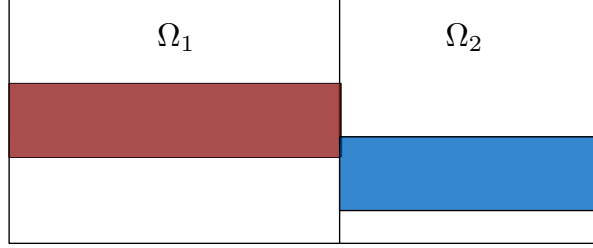
FIGURE 1. Coefficient α varying along and across the interface.

We are interested in the case where the diffusion coefficient $\alpha = \alpha(\boldsymbol{x})$ is a positive piecewise constant function that may have large variations within Ω, especially with discontinuities along and across the subdomain interfaces (see Figure 1). To be precise, we assume that the domain is a union of polygonal (polyhedral) subdomains $\mathcal{Y}_l$, such that:

$$\overline{\Omega} = \bigcup_{l=1}^{m} \overline{\mathcal{Y}}_l \quad \text{and} \quad \alpha(\boldsymbol{x}) = \alpha_l, \text{ for all } \boldsymbol{x} \in \mathcal{Y}_l \text{ and } l = 1, \ldots, m.$$

For any domain $D \subset \Omega$ we need the usual norms, with the standard notations $\|\cdot\|_{L_2(D)}$, $|\cdot|_{H^1(D)}$ and $\|\cdot\|_{H^1(D)}$, as well as the L_2 inner product $(v, w)_{L_2(D)}$. In addition to this, we need to define some related weighted quantities, which will be very useful in the following:

- the weighted H^1 (or energy) norm

(2.2)
$$|v|_{a,D}^2 = \int_D \alpha(\boldsymbol{x}) |\nabla v|^2.$$

  Note that if $v \in H_0^1(D)$ this is indeed a norm, and for $v \in H^1(D)$ this is only a seminorm. We denote the seminorm by $|\cdot|_{a,D}$.
- the weighted L_2 norm

(2.3)
$$\|v\|_{0,\alpha,D}^2 = \int_D \alpha(\boldsymbol{x}) v^2.$$

- the weighted L_2 inner product

$$(v, w)_{0,\alpha,D} = \int_D \alpha(\boldsymbol{x}) v w.$$

When $D = \Omega$ we omit the domain from the subscript and write $\|\cdot\|_a$ and $\|\cdot\|_{0,\alpha}$ instead of $\|\cdot\|_{a,\Omega}$ and $\|\cdot\|_{0,\alpha,\Omega}$, respectively.

Finally, we will also need averages and norms defined on $(d-1)$–dimensional manifolds $X \subset \mathbb{R}^d$, namely for any $v \in L_2(X)$ and for any $\beta \in L_\infty(X)$ we define

$$\overline{v}^X := \frac{1}{|X|} \int_X u \quad \text{and} \quad \|v\|_{0,\beta,X} := \int_X \beta v^2.$$

We consider a discretization of the variational problem (2.1) with continuous, piecewise linear finite elements (FE). To define the FE spaces and the approximate solution, we assume that we have a shape regular, simplicial triangulation $\mathcal{T}_h$ of Ω. We assume that this triangulation also resolves $\mathcal{Y}_l$, namely, for $l = 1, \ldots, m$ we have:

(2.4)
$$\overline{\Omega} = \bigcup_{\tau \in \mathcal{T}_h} \tau \quad \text{and} \quad \overline{\mathcal{Y}}_l = \bigcup_{\tau \in \mathcal{T}_{h,l}} \tau,$$

where $\mathcal{T}_{h,l} \subset \mathcal{T}_h$, for $l = 1, \ldots, m$. The standard space of continuous and piecewise linear (w.r.t $\mathcal{T}_h$) functions is then denoted by V_h, and the subspace of functions from V_h that vanish on the boundary of Ω by $V_{h,0}$. In our analysis, we will also need restrictions of FE functions into subdomains $D \subset \Omega$ that are resolved by $\mathcal{T}_h$. The space of restrictions of the functions in V_h to D is denoted by $V_h(D)$. Similarly, the space of restrictions of functions from V_h, which are supported in $\bar{D}$ is denoted by $V_{h,0}(D)$. Thus, $V_h(D) \subset H^1(D)$ and $V_{h,0}(D) \subset H_0^1(D)$. I_h denotes the standard nodal value interpolation operator from $C(\overline{D})$ to $V_h(D)$.

To finish this section let us write the discrete FE problem that we want to solve: Find $u_h \in V_{h,0}$ such that

$$(2.5) \qquad a(u_h, v_h) = (f, v_h), \quad \text{for all} \quad v_h \in V_{h,0}.$$

In the description and in the analysis of our preconditioners we will frequently switch between this variational point of view of the problem and a purely algebraic one. For that matter, let $\{\phi_i\}_{i=1}^n$ be the usual basis for $V_{h,0}$ consisting of nodal "hat" functions with $n := \dim(V_{h,0})$. Then (2.5) can be compactly written as

$$(2.6) \qquad A\boldsymbol{u} = \boldsymbol{f},$$

where $A_{i,j} := a(\phi_j, \phi_i)$, $f_i = (f, \phi_i)$, $i, j = 1, \ldots, n$ and $\boldsymbol{u}$ is the vector of coefficients corresponding to the unknown FE function u_h in (2.5). In the following we will often switch between matrices and their corresponding bilinear forms, e.g. A and $a(\cdot, \cdot)$, as well as between coefficient vectors in $\mathbb{R}^n$ and their corresponding FE functions in $V_{h,0}$, e.g. $\boldsymbol{u}$ and u_h. We use boldface for vectors and roman for FE functions.

Throughout the paper, the notation $E \lesssim F$ (for two quantities E, F) means that E/F is bounded above independently, not only of the mesh size h and the method specific parameters (such as $\text{diam}(\Omega_j)$ and δ_j, defined below for $j = 1, \ldots, J$), but also of the values of the coefficient $\alpha(\boldsymbol{x})$ at all $\boldsymbol{x} \in \Omega$. Moreover $E \eqsim F$ means that $E \lesssim F$ and $E \lesssim F$.

2.2. Automatic domain decomposition and partitions of unity.
In order to automatically construct robust two-level Schwarz type methods for (2.5) we first partition our domain Ω into a set of non-overlapping subdomains $\{\Omega'_j\}_{j=1}^J$ using for example a graph partitioner such as METIS or SCOTCH[19, 5]. Each subdomain Ω'_j is then extended to a domain Ω_j by adding one or several layers of fine grid elements, thus creating an overlapping decomposition $\{\Omega_j\}_{j=1}^J$ of Ω. For the moment we only assume that each point $\boldsymbol{x}$ in Ω is contained in at most N_0 subdomains.

We also need a partition of unity of functions $\{\chi_j\}_{j=1}^J$ defined on Ω, subordinate to the overlapping decomposition $\{\Omega_j\}_{j=1}^J$. To construct such a partition of unity automatically we define first, for each fine grid node $\boldsymbol{x}_i \in \Omega$, the index set $\mathcal{N}(\boldsymbol{x}_i)$ that contains the indices of all the domains Ω_j that contain $\boldsymbol{x}_i$. Then, for each subdomain Ω_j we define $\chi_j \in V_{h,0}(\Omega_j)$ by setting

$$(2.7) \qquad \chi_j(\boldsymbol{x}_i) := \frac{\text{dist}(\boldsymbol{x}_i, \partial\Omega_j)}{\sum_{k\in\mathcal{N}(\boldsymbol{x}_i)} \text{dist}(\boldsymbol{x}_i, \partial\Omega_k)}, \quad \text{at all fine grid nodes} \quad \boldsymbol{x}_i \in \Omega_j.$$

Clearly these functions form a partition of unity on Ω and satisfy $0 \le \chi_j \le 1$. Moreover, if

$$\Omega_j^\circ := \{\boldsymbol{x} \in \Omega_j : \chi_j(\boldsymbol{x}) < 1\}$$

denotes the boundary layer of Ω_j that is overlapped by neighbouring domains, then it is also easy to verify that

$$(2.8) \qquad\qquad |\nabla \chi_j| \lesssim \delta_j^{-1}$$

where δ_j denotes the width of Ω_j° at the narrowest place.

We now state the assumption that links the distribution of the values of the coefficient $\alpha(\boldsymbol{x})$ of the PDE and the domain partitioning.

Assumption 2.1. We assume that each Ω_j°, $j = 1, \ldots, J$, can be subdivided into regions D_{jk}, $k = 1, \ldots, K_j$, such that $\operatorname{diam}(D_{jk}) \eqsim \delta_j$ and $|D_{jk}| \eqsim \delta_j^d$, and such that, for each k, there exists a $(d-1)$–dimensional manifold $X_k \subset \partial\Omega_j \cap \overline{D}_{jk}$ with the following properties: (i) $|X_k| \eqsim \delta_j^{d-1}$, (ii) $\max_{\boldsymbol{x}, \boldsymbol{y} \in X_k} \alpha(\boldsymbol{x})/\alpha(\boldsymbol{y}) = \mathcal{O}(1)$ and (iii) there exists a path $P_{\boldsymbol{y}}$ from each $\boldsymbol{y} \in D_{jk}$ to X_k, such that $\alpha(\boldsymbol{x})$ is an increasing function along $P_{\boldsymbol{y}}$ (from $\boldsymbol{y}$ to X_k). Without loss of generality, we assume that the triangulation $\mathcal{T}_h$ resolves each of the regions D_{jk} and each of the manifolds X_k. See Figure 2 for some typical examples where the assumption is either verified or not verified.

Remark 2.2. Following [29, 30], if condition (iii) in Assumption 2.1 holds, $\alpha(x)$ is called quasi-monotone on D_{jk} with respect to X_k. It implies that the manifold X_k lies in the closure of the region $\mathcal{Y}_l$, where α takes its maximum on D_{jk}. This quasi-monotonicity is similar to the classical quasi-monotonicity defined in [8], if we can cluster the points in D_{jk} into shape-regular subregions Y of diameter $\eqsim \delta_j$, where the coefficient variation is $\mathcal{O}(1)$, and if we can then find a path P_Y with minimum diameter $\gtrsim \delta_j$ from Y to X_k such that $\alpha(x)$ is an increasing function along P_Y (from Y to X_k). If one of these paths P_Y is connected only via an m-dimensional manifold, $m < d - 1$, then we call $\alpha(x)$ type-m quasi-monotone on D_{jk} with respect to X_k. Assumption 2.1 provides only a sufficient condition for the theory below. It is not a necessary condition and various generalizations and extensions similar to those in [27] could be considered. We will not do this here.

The following two lemmas, based on Assumption 2.1, are crucial in the analysis below.

Lemma 2.3. *Let Assumption 2.1 be satisfied. Then there exists a uniform constant $C_P > 0$ independent of the values of $\alpha(x)$, such that the following weighted Poincaré–type inequality holds for all $j = 1, \ldots, J$ and $k = 1, \ldots, K_j$:*

$$\|v - \overline{v}^{X_k}\|_{0,\alpha,D_{jk}} \leq C_P \,\delta_j \,|v|_{a,D_{jk}}, \qquad \text{for all } \ v \in V_h(\mathcal{D}_{jk}).$$

The constant C_P is independent of h, δ_j and $\operatorname{diam}(\Omega_j)$, if $\alpha(x)$ satisfies the classical quasi-monotonicity condition (cf. Remark 2.2) on all D_{jk}. It depends on $\log(\delta_j/h)$ (resp. δ_j/h), if $\alpha(x)$ is only type-$(d-2)$ (resp. type-$(d-3)$) quasi-monotone on any D_{jk}, $k = 1, \ldots, K_j$.

Proof. Theorem 2.2 and Theorem 3.3 in [29]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

Lemma 2.4. *Let Assumption 2.1 be satisfied and $\operatorname{tr}_j \alpha(\boldsymbol{x}) := \lim\limits_{\boldsymbol{y} \in \Omega_j \to \boldsymbol{x}} \alpha(\boldsymbol{y})$, for a.e. $\boldsymbol{x} \in \partial\Omega_j$. Then*

$$\|u\|_{0,\alpha,\Omega_j^\circ}^2 \lesssim C_P^2 \delta_j^2 |u|_{a,\Omega_j^\circ}^2 + \delta_j \|u\|_{0,\operatorname{tr}_j\alpha,\partial\Omega_j}^2, \quad \text{for all } \ u \in V_h(\Omega_j^\circ).$$

If $\alpha(x)$ satisfies the classical quasi-monotonicity condition (cf. Remark 2.2) on all D_{jk}, then the statement holds for all $u \in H^1(\Omega_j^\circ)$.
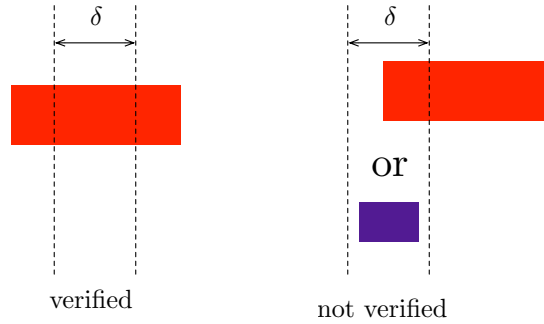
FIGURE 2. Overlap region between two subdomains with high-permeability inclusions, such that Assumption 2.1 is verified (left) and not verified (right).

Proof. Let $\{D_{jk}\}_{k=1}^{K_j}$ be the partitioning of Ω_j° in Assumption 2.1 and let X_k be the $(d-1)$–dimensional manifold associated with D_{jk}. Let $\|\alpha\|_{\infty,D_{jk}} := \mathrm{esssup}\{\alpha(\boldsymbol{x}) : \boldsymbol{x} \in D_{jk}\}$. Then it follows from Lemma 2.3, as well as the triangle and the Cauchy-Schwarz inequalities, that

$$
\begin{aligned}
\tfrac{1}{2}\|u\|_{0,\alpha,D_{jk}}^2 &\leq \|u - \overline{u}^{X_k}\|_{0,\alpha,D_{jk}}^2 + \|\overline{u}^{X_k}\|_{0,\alpha,D_{jk}}^2 \\
&\leq C_P^2 \delta_j^2 |u|_{a,D_{jk}}^2 + \frac{|D_{jk}|}{|X_k|^2}\|\alpha\|_{\infty,D_{jk}}\left(\int_{X_k} u\right)^2 \\
&\leq C_P^2 \delta_j^2 |u|_{a,D_{jk}}^2 + \frac{|D_{jk}|}{|X_k|}\|\alpha\|_{\infty,D_{jk}}\int_{X_k} u^2 \\
&\lesssim C_P^2 \delta_j^2 |u|_{a,D_{jk}}^2 + \delta_j \|u\|_{0,\alpha,X_k}^2 .
\end{aligned}
$$

In the last step, we have used Assumption 2.1, i.e. $|D_{jk}|/|X_k| \lesssim \delta_j$, $\alpha(x)$ attains its maximum on $\overline{D}_{jk}$ in a set containing X_k, and $\max_{\boldsymbol{x},\boldsymbol{y}\in X_k}\alpha(\boldsymbol{x})/\alpha(\boldsymbol{y}) = \mathcal{O}(1)$. The final result follows by summing over $k = 1,\ldots,K_j$. $\square$

2.3. **Two-level overlapping Schwarz method.** Having defined overlapping subdomains in Section 2.2, one-level Schwarz-type preconditioners for (2.5) or (2.6) can now simply be introduced by defining restriction operators R_j from functions in $V_{h,0}$ to functions in $V_{h,0}(\Omega_j)$ or from vectors in $\mathbb{R}^n$ to vectors in $\mathbb{R}^{n_j}$, where $n_j := \dim V_{h,0}(\Omega_j)$. As usual we use simple injection, i.e. for any $u \in V_{h,0}$ we set $(R_j u)(\boldsymbol{x}_i) = u(\boldsymbol{x}_i)$ for every grid node $\boldsymbol{x}_i \in \Omega_j$. In matrix notation the one-level overlapping additive Schwarz preconditioner is now simply

$$
M_{AS,1}^{-1} = \sum_{j=1}^J R_j^T A_j^{-1} R_j \quad \text{where} \quad A_j := R_j A R_j^T .
$$

This preconditioner is particularly well suited for preconditioning parallel iterative solvers, such as conjugate gradients (CG) for (2.6), since all the subdomain solves can be carried out independently of each other. However, the condition number and thus the number of CG iterations grow with the number of subdomains J. Thus, it is essential to include a second, coarse level in the preconditioner.

Let us assume that we have a subspace $V_H \subset V_{h,0}$ and a restriction operator R_H from $V_{h,0}$ to V_H. Then the two-level overlapping additive Schwarz preconditioner can be defined

similarly to $M_{AS,1}^{-1}$ as

$$M_{AS,2}^{-1} = R_H^T A_H^{-1} R_H + \sum_{j=1}^{J} R_j^T A_j^{-1} R_j \quad \text{where} \quad A_H := R_H A R_H^T.$$

In the classical algorithm V_H consists simply of FEs on a coarser triangulation $\mathcal{T}_H$ of Ω and R_H is the canonical restriction from $V_{h,0}$ to V_H, leading to a fully scalable iterative method (provided $\min_j \delta_j \approx H$). However, unfortunately this preconditioner is not fully robust to strong variations in the coefficient α and many recent articles, such as [16, 31, 15, 33, 35], have attempted to propose and analyse alternative choices for V_H and R_H that lead to a fully robust algorithm. In the next section, we will present a new, completely local approach to construct such a coarse space and such a restriction using eigenvectors of local Dirichlet-to-Neumann maps proposed in [23, 24].

Several alternative versions of these simple additive preconditioners exist, including the multiplicative Schwarz preconditioner, hybrid versions, deflation based coarse grid correction, or the restricted additive Schwarz (RAS) preconditioner [3]. We will only analyse the additive preconditioner, but we note that all the other symmetric versions can be analysed in the same way. The RAS variant is slightly different since it leads to a nonsymmetric iteration, but we will see in Section 4 that it behaves in a similar way and can often give slightly better results than the classical additive version above.

3. Coarse space construction based on local Dirichlet-to-Neumann maps

Our coarse space and the restriction operator will be constructed by defining a suitable set of basis functions $\{\Phi_k\}_{k=1}^{N}$ with N usually larger than J, such that $V_H = \text{span}\{\Phi_k\}_{k=1}^{N}$ and the restriction R_H is the canonical restriction associated with this basis. The construction we propose, is to some extent inspired by two observations, already made elsewhere, namely

- that robust coarse space basis functions can in many cases be obtained on standard simplicial meshes by harmonically extending suitable boundary data to the interior of coarse mesh elements [16],
- that local spectral information about the underlying differential operator can be used to obtain fully robust coarse spaces [15, 33].

While the coarse spaces in [16] are not able to deal with completely general piecewise constant coefficients, the spaces in [15] are often too large, especially when there are many small, isolated inclusions with large coefficient values. The coarse space presented here suffers from neither of these drawbacks since it is based on local spectral information (so it is parallelizable), and relies on minimal size interface information (only "absolutely necessary" modes are taken into account in the coarse space). This interface information is then further extended harmonically to the interior of the subdomains.

3.1. **Local coarse space construction.** To build a coarse space V_H for (2.5) we will first construct local basis functions on each subdomain Ω_j. To this end, let us fix $j \in \{1, \ldots, J\}$ and first consider at the continuous level the Dirichlet-to-Neumann map DtN_j on the boundary of Ω_j. Let $\Gamma := \partial \Omega_j$ and let $v_\Gamma : \Gamma \to \mathbb{R}$ (such that $v_\Gamma|_{\partial\Omega} = 0$ if $\Gamma \cap \partial\Omega \neq \emptyset$). We define

$$\text{DtN}_j(v_\Gamma) := \alpha \frac{\partial v}{\partial \boldsymbol{n}_j}\bigg|_\Gamma,$$
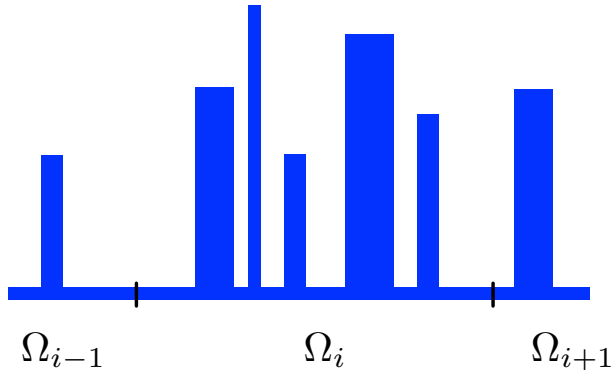
FIGURE 3. 1D example with many high coefficient inclusions per subdomain.

where $\boldsymbol{n}_j$ is the unit outward normal to Ω_j on Γ, and v satisfies

$$(3.1) \qquad \begin{aligned} -\mathrm{div}(\alpha\nabla v) &= 0 \quad \text{in } \Omega_j\,, \\ v &= v_\Gamma \quad \text{on } \Gamma\,. \end{aligned}$$

We see that DtN_j maps the Dirichlet data v_Γ on Γ to Neumann data (fluxes) on Γ. The function v is the α–harmonic extension of the boundary data v_Γ to the interior of Ω_j. To construct the (local) coarse basis functions, we use the low frequency modes of the Dirichlet to Neumann operator DtN_j with respect to the weighted L_2–norm on Γ, i.e. the smallest eigenvalues of

$$(3.2) \qquad \mathrm{DtN}_j(v_\Gamma) = \lambda\,\alpha\,v_\Gamma.$$

For simplicity let us consider an interior subdomain Ω_j that does not touch the (Dirichlet) boundary of the global domain Ω. The other case carries through in a similar way.

Instead of looking for an eigenpair of equation (3.2) and then compute v the harmonic extension of v_Γ, we directly search for the pair (v, λ). It is straightforward to check that it satisfies:

$$(3.3) \qquad \begin{aligned} -\mathrm{div}(\alpha\nabla v) &= 0 \quad \text{in } \Omega_j\,, \\ \alpha\frac{\partial v}{\partial \boldsymbol{n}_j} &= \lambda\,\alpha\,v \quad \text{on } \Gamma\,. \end{aligned}$$

The variational formulation of (3.3) is: Find $(v, \lambda) \in H^1(\Omega_j) \times \mathbb{R}$ such that

$$(3.4) \qquad \int_{\Omega_j} \alpha\nabla v \cdot \nabla w = \lambda \int_\Gamma \mathrm{tr}_j\alpha\,v\,w\,, \quad \forall w \in H^1(\Omega_j)$$

where $\mathrm{tr}_j\alpha$ is as defined in Lemma 2.4.

Remark 3.1. We note a clear relationship with the work of Galvis and Efendiev [15]. However, the significant difference lies in the fact that the latter uses local spectral basis functions associated with the eigenproblem $-\mathrm{div}(\alpha\nabla v) = \lambda\alpha v$, posed on the entire subdomain Ω_j, which will always lead to a larger number of potentially critical eigenmodes. We can easily illustrate this using the one-dimensional example in Figure 3: whatever the number of islands in the interior of Ω_j, our DtN coarse space consists of two vectors per subdomain at most, since the DtN map is a two by two matrix which has two eigenvectors.

To obtain the discrete form of the generalized eigenvalue problem (3.4), we consider the bilinear forms $a_j : H^1(\Omega_j) \times H^1(\Omega_j) \to \mathbb{R}$

$$a_j(v,w) := \int_{\Omega_j} \alpha \nabla v \cdot \nabla w \text{ and } m(v,w) := \int_{\Gamma} \text{tr}_j \alpha \, v \, w \,, \quad \forall w \in H^1(\Omega_j) \,.$$

With a finite element basis $\{\phi_k\}$, the coefficient matrix of the variational form a_j is

$$A_{kl}^{(j)} = \int_{\Omega_j} \alpha \nabla \phi_k \cdot \nabla \phi_l.$$

Note that, for the whole domain Ω, the coefficient matrix is given by

$$A_{kl} = \int_{\Omega} \alpha \nabla \phi_k \cdot \nabla \phi_l.$$

Let I (resp. Γ) be the set of indices corresponding to the interior (resp. boundary) degrees of freedom and $n_\Gamma := \#\Gamma$ the number of interface degrees of freedom. With block notations, we have

$$A_{\text{II}}^{(j)} = A_{\text{II}}, \quad A_{\Gamma\text{I}}^{(j)} = A_{\Gamma\text{I}} \quad \text{and} \quad A_{\text{I}\Gamma}^{(j)} = A_{\text{I}\Gamma} \,.$$

The matrix $A_{\Gamma\Gamma}^{(j)} \neq A_{\Gamma\Gamma}$, since it refers to the matrix prior to assembly with the neighboring subdomains. Therefore, it cannot simply be extracted from the coefficient matrix A.

Let M be the weighted mass matrix on Γ corresponding to the variational form m i.e.,

$$(M)_{kl} := \int_{\Gamma} \text{tr}_j \alpha \phi_k \, \phi_l \,.$$

Then we can write down the finite element approximation to generalized eigenproblem (3.4) in matrix notation

(3.5) $$A^{(j)} \mathbf{V} = \lambda M \mathbf{V}$$

where $\mathbf{V}$ denotes the degrees of freedom of the finite element approximation to v.

To define the discrete action of the DtN map, let now $v_{\Gamma,h} := \sum_{k \in \Gamma} V_{\Gamma,k} \phi_k$ be some predescribed piecewise linear boundary data on Γ with coefficient vector $\mathbf{V}_\Gamma \in \mathbb{R}^{n_\Gamma}$, and let $V_h := \sum_{k \in \text{I}} V_{\text{I},k} \phi_k + \sum_{l \in \Gamma} V_{\Gamma,l} \phi_l \in V_h(\Omega_j)$ be its discrete harmonic extension. In matrix notation, the finite element approximation top (3.1) reads $A_{\text{II}} \mathbf{U}_\text{I} + A_{\text{I}\Gamma} \mathbf{U}_\Gamma = \mathbf{0}$, leading to the usual discrete α-harmonic extension

(3.6) $$\mathbf{U}_\text{I} = -A_{\text{II}}^{-1} A_{\text{I}\Gamma} \mathbf{U}_\Gamma.$$

We introduce

$$(M_\Gamma)_{kl} := \int_{\Gamma} \text{tr}_j \alpha \phi_k \, \phi_l \,, \quad \text{for all } k, l \in \Gamma \,.$$

Thus we have,

$$A_{\Gamma\Gamma}^{(j)} \mathbf{V}_\Gamma + A_{\Gamma\text{I}} \mathbf{V}_\text{I} = \lambda M_\Gamma \mathbf{V}_\Gamma \,,$$

and using the discrete α-harmonic extension (3.6), the discrete form of (3.2) is a generalised eigenvalue problem

(3.7) $$S_\Gamma \mathbf{V}_\Gamma \; = \; \lambda \, M_\Gamma \mathbf{V}_\Gamma \,.$$

for the Schur complement $S_\Gamma := A_{\Gamma\Gamma}^{(j)} - A_{\Gamma\text{I}} A_{\text{II}}^{-1} A_{\text{I}\Gamma}$. Let the n_Γ eigenpairs $(\lambda_\ell^{(j)}, \mathbf{V}_{\Gamma,\ell})_{\ell=1}^{n_\Gamma}$ corresponding to (3.7) be numbered in increasing order of $\lambda_\ell^{(j)}$. Since matrices S_Γ and M_Γ are symmetric the eigenvectors $\mathbf{V}_{\Gamma,\ell}$, $\ell = 1, \ldots, n_\Gamma$, are orthogonal in the S_Γ and in the M_Γ

inner products. Note that S_Γ is symmetric positive semidefinite and that in the case of an interior subdomain, there is exactly one eigenvalue that is 0 with constant eigenvector.

Note that

$$(3.8) \qquad A^{(j)}\mathbf{V}_\ell^{(j)} = \lambda_\ell^{(j)} \begin{bmatrix} 0 & 0 \\ 0 & M_\Gamma \end{bmatrix} \mathbf{V}_\ell^{(j)} = M \, \mathbf{V}_\ell^{(j)}$$

and so $\mathbf{V}_\ell^{(j)}$ is an eigenvector corresponding to the eigenvalue $\lambda_\ell^{(j)}$ of generalized eigenproblem (3.5). Moreover, since

$$\left(\mathbf{V}_k^{(j)}\right)^T A^{(j)}\mathbf{V}_\ell^{(j)} \;=\; \begin{bmatrix} -A_{II}^{-1}A_{I\Gamma}\mathbf{V}_{\Gamma,k} \\ \mathbf{V}_{\Gamma,k} \end{bmatrix}^T \begin{bmatrix} \mathbf{0} \\ S_\Gamma \mathbf{V}_{\Gamma,\ell} \end{bmatrix} \;=\; \mathbf{V}_{\Gamma,k}^T S_\Gamma \mathbf{V}_{\Gamma,\ell} \;=\; 0,$$

the vectors $\{\mathbf{V}_\ell^{(j)}\}_{\ell=1}^{n_\Gamma}$ are $A^{(j)}$-orthogonal. Let us assume further, that they are normalised in the $A^{(j)}$-norm, such that $\|\mathbf{V}_\ell^{(j)}\|_{A^{(j)}} = 1$. Since $(A^{(j)})^{-1}\begin{bmatrix} 0 & 0 \\ 0 & M_\Gamma \end{bmatrix}$ has only got n_Γ nonzero eigenvalues, all the remaining eigenvalues of (3.8) are ∞, and so the smallest eigenvalues in (3.7) are also the smallest eigenvalues of (3.8).

The local coarse space is now defined as the span of the finite element functions $v_\ell^{(j)} \in V_h(\Omega_j)$, $\ell \le m_j \le n_\Gamma$, corresponding to the first m_j eigenpairs of (3.8). For any $u \in V_h(\Omega_j)$, we can define the projection on $\text{span}\{v_\ell^{(j)}\}_{\ell=1}^{m_j}$ by

$$(3.9) \qquad \Pi_j u \;:=\; \sum_{\ell=1}^{m_j} a_j\left(v_\ell^{(j)}, u\right) v_\ell^{(j)},$$

which is stable and satisfies a weak approximation property, as the following theorem shows.

Theorem 3.2. *Let Assumption 2.1 hold. Then, for any $u \in V_h(\Omega_j)$,*

$$(3.10) \qquad |\Pi_j u|_{a,\Omega_j} \;\le\; |u|_{a,\Omega_j} \quad and$$

$$(3.11) \qquad \|u - \Pi_j u\|_{0,\alpha,\Omega_j^\circ} \;\lesssim\; \sqrt{c_j(m_j)}\, \delta_j\, |u|_{a,\Omega_j},$$

where $c_j(m_j) := C_P^2 + \left(\delta_j \lambda_{m_j+1}^{(j)}\right)^{-1}$.

Proof. The stability estimate (3.10) follows immediately from the fact that Π_j is an $a^{(j)}$-orthogonal projection. To prove (3.11) let us first apply Lemma 2.4, i.e.

$$(3.12) \qquad \|u - \Pi_j u\|_{0,\alpha,\Omega_j^\circ}^2 \;\lesssim\; C_P \delta_j^2 |u - \Pi_j u|_{a,\Omega_j^\circ}^2 \;+\; \delta_j \|u - \Pi_j u\|_{0,\mathrm{tr}_j\alpha,\Gamma}^2,$$

It follows from the triangle inequality and (3.10) that

$$(3.13) \qquad |u - \Pi_j u|_{a,\Omega_j^\circ}^2 \;\le\; |u - \Pi_j u|_{a,\Omega_j}^2 \;\lesssim\; |u|_{a,\Omega_j}^2$$

and so it only remains to bound $\|u - \Pi_j u\|_{0,\mathrm{tr}_j\alpha,\Gamma}^2$.

To do this, let us extend the set $\{v_\ell^{(j)}\}_{\ell=1}^{n_\Gamma}$ to an a_j-orthonormal basis of $V_h(\Omega_j)$ by adding $n_j = \dim V_{h,0}(\Omega_j) = \#I$ suitable functions $v_{n_\Gamma+\ell}^{(j)}$, $\ell = 1, \ldots, n_j$, and write

$$(3.14) \qquad u \;=\; \sum_{\ell=1}^{n_j+n_\Gamma} a_j\left(v_\ell^{(j)}, u\right) v_\ell^{(j)}.$$
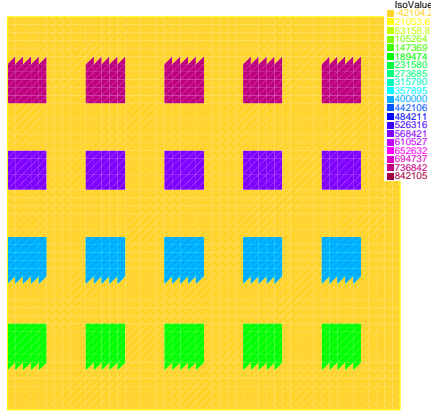
FIGURE 4. Coefficient distribution on a subdomain Ω_j for a two-dimensional model problem with high-permeability inclusions.

The restriction of the functions $\{v_\ell^{(j)}\}_{\ell=1}^{n_\Gamma}$ to the boundary Γ forms a complete basis of $V_h(\Gamma)$. This implies that $v_{n_\Gamma+\ell}^{(j)} \equiv 0$ on Γ, for all $\ell = 1, \ldots, n_j$. Moreover, it follows from the definition of the eigenproblem (3.7) that the functions $\{v_\ell^{(j)}\}_{\ell=1}^{n_\Gamma}$ are orthogonal also in the $(\cdot,\cdot)_{0,\mathrm{tr}_j\alpha,\Gamma}$ inner product. Therefore

$$(3.15) \quad \|u - \Pi_j u\|_{0,\mathrm{tr}_j\alpha,\Gamma}^2 = \sum_{\ell=m_j+1}^{n_\Gamma} a\left(v_\ell^{(j)}, u\right)^2 \|v_\ell^{(j)}\|_{0,\mathrm{tr}_j\alpha,\Gamma}^2 = \sum_{\ell=m_j+1}^{n_\Gamma} \frac{1}{\lambda_\ell^{(j)}} a\left(v_\ell^{(j)}, u\right)^2$$

$$\leq \frac{1}{\lambda_{m_j+1}^{(j)}} \sum_{\ell=1}^{n_\Gamma} a\left(v_\ell^{(j)}, u\right)^2 = \frac{1}{\lambda_{m_j+1}^{(j)}} |u|_{a,\Omega_j}^2$$

and the result follows from (3.12), (3.13) and (3.15). $\qquad\square$

As explained in previous work (cf. [24]), we only include eigenvectors with eigenvalues smaller than $\mathrm{diam}(\Omega_j)^{-1}$ in the construction of the coarse space. In the constant coefficient case, the smallest eigenvalue of the DtN map is zero and it corresponds to the constant function 1. For a shape regular subdomain, the first positive eigenvalue is of order $\mathrm{diam}(\Omega_j)^{-1}$, see [12]. In the heterogeneous case, due to the variation of the coefficients we may possibly have positive eigenvalues smaller than $\mathcal{O}\big(\mathrm{diam}(\Omega_j)^{-1}\big)$. Thus, in order to have a convergence behavior similar to that in the constant coefficient case, it is natural to keep all eigenvectors with eigenvalues smaller than $\mathrm{diam}(\Omega_j)^{-1}$.

Remark 3.3. We consider now the two-dimensional permeability field α of a subdomain shown in Figure 4. We show in Figure 5 a typical eigenvector used in [15] (left plot) and a typical generalized eigenvector introduced in our paper (right plot). We note that the eigenvectors used in the coarse space construction in the DtN case are related only to the high-permeability islands that cross the boundaries of the local domain. Whereas for the coarse space considered in [15] it is related to the islands in the interior of the subdomain and thus their number is substantially larger.

3.2. **Global coarse space construction, stability and interpolation estimates.** Using the partition of unity defined in Section 2.2, we now combine the local basis functions constructed in the previous section to obtain a coarse space $V_H \subset V_{h,0}$ on all of Ω.
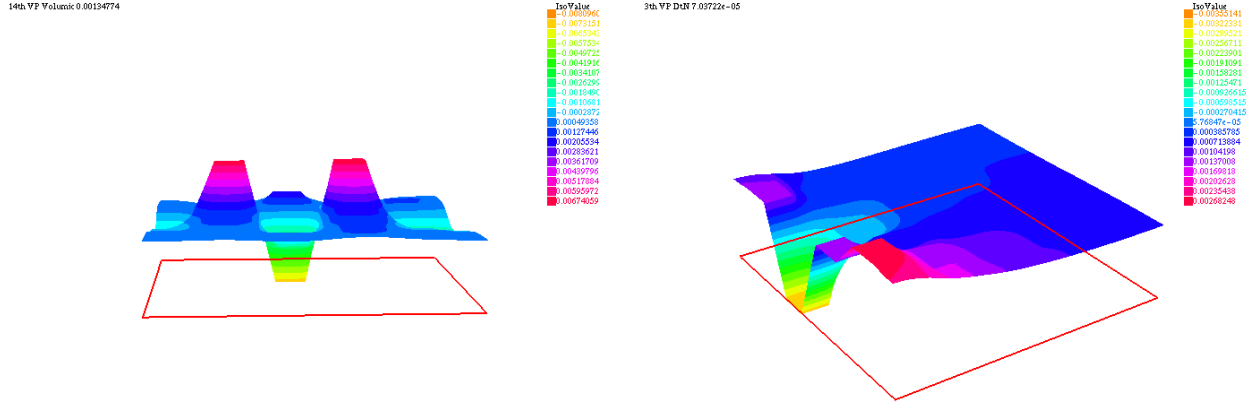
FIGURE 5. Eigenvector of the full subdomain eigenproblem for the model problem in Figure 4 used in the coarse space construction in [15] (left) and eigenvector of the interface-DtN map used in our construction here (right).

For any Ω_j, $1 \leq j \leq J$, let Π_j be the projection onto the first m_j local DtN eigenvectors defined in (3.9) and let χ_j be the partition of unity function associated with Ω_j defined in (2.7). The new coarse space is now defined as

$$V_H := \operatorname{span}\{\Phi^H_{j,\ell} : 1 \leq j \leq J \text{ and } 1 \leq \ell \leq m_j\}, \quad \text{where} \quad \Phi^H_{j,\ell} := I_h(\chi_j v_\ell^{(j)})$$

and I_h is the standard nodal interpolant onto $V_{h,0}(\Omega)$. The dimension of V_H is $\sum_{j=1}^J m_j$. By construction each of the functions $\Phi_{j,\ell} \in V_{h_0}$, so that as required $V_H \subset V_{h,0}$.

For a given function $u \in V_{h,0}$, we introduce the coarse interpolant of u as

$$(3.16) \qquad u_0 := I_h\left(\sum_{j=1}^J \chi_j \Pi_j u|_{\Omega_j}\right) \in V_H.$$

In the following, to ease the presentation when there is no confusion and it is clear from the context, we will simply denote the restriction $u|_{\Omega_j}$ of u onto Ω_j by u, and write, e.g., $\Pi_j u$ instead of $\Pi_j u|_{\Omega_j}$.

The following theorem is key to proving the robustness of this coarse space construction.

Theorem 3.4. *Let $u \in V_{h,0}$ be given and let $u_0 \in V_H$ be the coarse grid interpolant of u, defined in (3.16). Then there are functions $u_j \in V_{h,0}(\Omega_j)$, $j = 1, \ldots, J$, such that*

$$u = \sum_{j=0}^J u_j \quad and \quad \sum_{j=0}^J \|u_j\|_a^2 \lesssim \max_{j=1}^J \{c_j(m_j)\} \|u\|_a^2,$$

i.e. there exists a stable splitting of u.

Proof. We choose $u_j := I_h(\chi_j(u - \Pi_j u))$. Then clearly, since by definition $u = \sum_{j=1}^J \chi_j u$,

$$\sum_{j=1}^J u_j = \sum_{j=1}^J I_h(\chi_j(u - \Pi_j u)) = I_h\left(\sum_{j=1}^J \chi_j u\right) - \sum_{j=1}^J I_h(\chi_j \Pi_j u) = u - u_0.$$

It follows from a simple application of the triangle inequality that

$$
(3.17) \qquad \sum_{j=0}^{J} \|u_j\|_a^2 \;\le\; \|u - u_0\|_a^2 + \|u\|_a^2 + \sum_{j=1}^{J} \|u_j\|_a^2
$$

Since the interpolant I_h is stable with respect to the a-norm (cf. [33, Lemma 2.3]) and since each point is overlapped by at most N_0 domains, we have

$$
\|u - u_0\|_a^2 = \left\| I_h\Big(u - \sum_{j=1}^{J} \chi_j \Pi_j u \Big) \right\|_a^2 \;\lesssim\; \left\| u - \sum_{j=1}^{J} \chi_j \Pi_j u \right\|_a^2 \;\lesssim\; \sum_{j=1}^{J} \|\chi_j (u - \Pi_j u)\|_a^2
$$

Substituting this into (3.17) and using the definition of u_j as well as the a-stability of the interpolant I_h we get

$$
(3.18) \qquad \sum_{j=0}^{J} \|u_j\|_a^2 \;\lesssim\; \sum_{j=1}^{J} \|\chi_j (u - \Pi_j u)\|_a^2 + \|u\|_a^2
$$

Note that $\operatorname{supp}\{\chi_j\} = \overline{\Omega}_j$ and $\operatorname{supp}\{\nabla \chi_j\} = \overline{\Omega}_j^{\circ}$. Thus, using triangle inequality and product rule

$$
(3.19) \qquad
\begin{aligned}
\sum_{j=1}^{J} \|\chi_j (u - \Pi_j u)\|_a^2 &\;\lesssim\; \sum_{j=1}^{J} \|\chi_j\|_\infty^2 |u - \Pi_j u|_{a,\Omega_j}^2 + \|\nabla \chi_j\|_\infty^2 \|u - \Pi_j u\|_{0,\alpha,\Omega_j^{\circ}}^2 \\
&\;\lesssim\; \sum_{j=1}^{J} |u|_{a,\Omega_j}^2 + |\Pi_j u|_{a,\Omega_j}^2 + \delta_j^{-2} \|u - \Pi_j u\|_{0,\alpha,\Omega_j^{\circ}}^2 \,.
\end{aligned}
$$

Note that in the last part we used the property of the partition of unity (2.8).

Since it was assumed that each point $\boldsymbol{x} \in \Omega$ is contained in at most N_0 subdomains, we can now use Theorem 3.2 to deduce from (3.18) and (3.19) that

$$
\sum_{j=0}^{J} \|u_j\|_a^2 \;\lesssim\; \|u\|_a^2 + \sum_{j=1}^{J} |u|_{a,\Omega_j}^2 c_j(m_j) \;\lesssim\; \max_{j=1}^{J}\{c_j(m_j)\} \|u\|_a^2,
$$

which ends the proof. $\qquad\square$

As usual (see e.g. [34]), the existence of a stable splitting established in Theorem 3.4 together with the hypothesis that each point $\boldsymbol{x} \in \Omega$ is contained in at most N_0 subdomains is sufficient to deduce the following bound on the condition number of $M_{AS,2}^{-1}A$ from the abstract Schwarz theory.

Theorem 3.5. *Let Assumption 2.1 be satisfied. Then the condition number of the two-level Schwarz algorithm with a coarse space based on local DtN maps can be bounded by*

$$
\kappa(M_{AS,2}^{-1}A) \;\lesssim\; \max_{j=1}^{J}\{c_j(m_j)\} \lesssim \left(C_P^2 + \max_{j=1}^{J} \frac{1}{\delta_j \lambda_{m_j+1}^{(j)}} \right).
$$

The hidden constant is independent of h, δ_j, and $\operatorname{diam}(\Omega_j)$, as well as any jumps in α.
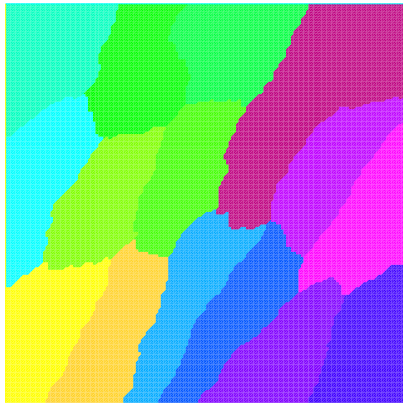
FIGURE 6. Subdomain partitioning into 16 subdomains using METIS.

Remark 3.6. Note that, by choosing the number m_j of modes per subdomain in the coarse space as discussed at the end of Section 3.1, i.e. such that $\lambda_{m_j+1}^{(j)} \geq \text{diam}(\Omega_j)^{-1}$, the method we implement in the next section satisfies

$$\kappa(M_{AS,1}^{-1}A) \lesssim \left(C_P^2 + \max_{j=1}^{J} \frac{\text{diam}(\Omega_j)}{\delta_j} \right).$$

Hence, provided the weighted Poincaré constant C_P in Assumption 2.1 is uniformly bounded, independently of any jumps in the coefficients, we retrieve the classical estimate for the Additive Schwarz Method. An interesting observation is that the bound depends only in an additive way on the constant C_P and on the ratio of subdomain diameter to overlap.

4. Numerical Results

We solve the model problem (2.1) on the domain $\Omega = [0,1]^2$ using standard continuous, piecewise linear (P_1) finite elements. The diffusion α is a function of $\boldsymbol{x}$. The boundary condition is $u = 0$ on the left side boundary and $\frac{\partial u}{\partial n} = 0$ on the remainder. The corresponding discretizations and data structures were obtained by using the software FreeFem++ [18] in connection with the METIS partitioner [19]. We will test the standard additive Schwarz (AS) and the restricted additive Schwarz (RAS) preconditioners with and without coarse space, in particular comparing the new coarse space based on harmonic extensions of eigenvectors of the local DtN operators with the standard coarse space that is the piecewise constant space of Nicolaides [25]. We test the method on (fairly irregular) overlapping partitions into J subdomains. These overlapping partitions are built by adding layers to non-overlapping ones obtained, e.g., via graph partitioner METIS (see Figure 6). Extensive numerical results have been presented in a previous paper [24]. Here we present only a selection of difficult test cases, with so called inclusions and channels.

In order to compare our method to existing codes we solve two test cases with known difficulties. The diffusion coefficient α is highly heterogeneous, it takes values between 1 and approximately 2×10^6 and contains both high-permeability inclusions and channels. First of all we will analyze the performance of the method by increasing the number of channels and then by increasing the number of inclusions.
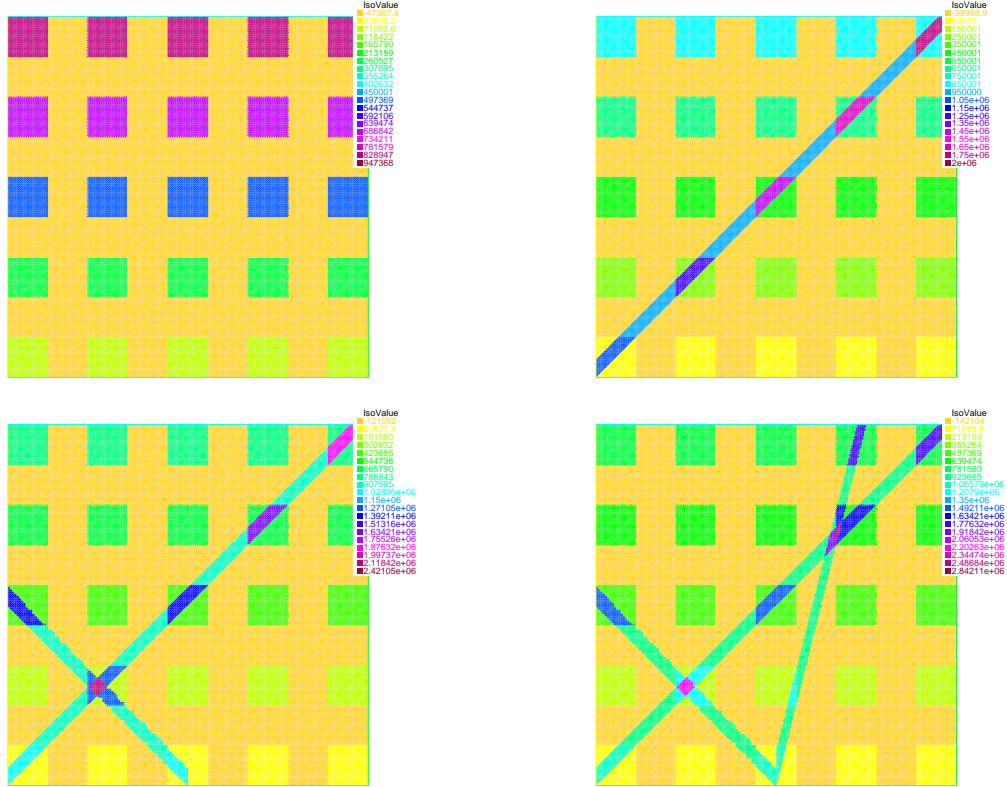
FIGURE 7. Test Problem 1: Successively adding channels.

	AS	AS+Nicolaides	AS+DtN	RAS	RAS+Nicolaides	RAS+DtN
no channel	529	1000	57	243	245	41
1 channel	619	520	64	227	228	46
2 channels	> 1000	516	68	226	226	47
3 channels	585	697	76	212	213	44

TABLE 1. Number of iterations for Test Problem 1 (additive coarse grid correction).

We use a uniform triangulation with 160×160 nodes and a partition into 16 (irregular) subdomains (see Figure 6). Each subdomain is extended by one layer, leading to an overlap of 2 layers, such that $\delta_j = 2h$ for all $j = 1, \ldots, J$. . We use the AS preconditioner within conjugate gradients (CG) and the RAS preconditioner within GMRES, and in each case we stop the iteration process, when the relative residual is smaller than 10^{-6}.

We start with only inclusions and add the channels one by one as shown in Figure 7 (Test Problem 1). When there are no channels, α varies between 1 and 10^6, as indicated by the colours in Figure 7. With all three channels present, α varies between 1 and 2.8×10^6. The corresponding convergence results are given in Table 1. Our algorithm performs significantly better. The piecewise constant coarse space has virtually no effect on the performance of either AS or RAS, leading to iteration numbers that differ little from the results without any coarse grid in all four cases. Our new coarse space, on the other hand, is fully robust

	AS	AS+Nicolaides	AS+DtN	RAS	RAS+Nicolaides	RAS+DtN
no channel	529	656	39	243	231	25
1 channel	619	538	41	227	215	28
2 channels	>1000	808	47	226	211	27
3 channels	585	641	47	212	199	28

TABLE 2. Number of iterations for Test Problem 1 (deflation-based coarse grid correction).

Over $J = 16$ subdomains	Total number $n_{\Gamma j}$ of eigenvalues on Γ_j	Number m_j of functions included in V_H from Ω_j			
		no channel	1 channel	2 channels	3 channels
Minimum	70	1	1	1	1
Maximum	191	4	4	4	4
Average	138.8	2.75	2.88	2.94	2.94
Sum	2220	44	46	47	47

TABLE 3. Size of the coarse space for Test Problem 1.

to the coefficient variation and to the addition of channels, and it leads to a gain of at least a factor 8 compared to the one-level method in all cases. The situation is similar, if we use deflation-based coarse grid correction [22] with the same coarse spaces (see Table 2). However, the absolute numbers of iterations are reduced almost by a factor 2 in this case. Our theory applies equally to this case (see e.g. [17] for details), but we will not include any further numerical results with deflation-based coarse grid correction.

Table 3 gives some information on the size of the coarse space that we build with our automatic selection strategy: for each number of channels we give $\min_j m_j$ and $\max_j m_j$, as well as the global coarse space size $n_H = \sum_j m_j$ and the average number of modes included per subdomain n_H/J. For comparison, we also include information on the total number $n_{\Gamma j}$ of eigenmodes of the discrete DtN operator on each subdomain. We note that adding channels does not have a big influence on the size of the coarse space; we only need three additional eigenvectors in the case of three channels compared to the case of no channels.

Then, using the same domain and the same partition we successively add inclusions without any channels present as shown in Figure 8 (Test Problem 2). The results are in Table 4. Again, the piecewise constant coarse space is not working at all for this test problem. The DtN-based coarse space is almost completely robust to an increase in the number of inclusions and requires again significantly less iterations than the one-level method in all cases. Note that the subdomain partition (cf. Figure 6) is not aligned with the inclusions at all (cf. Figure 8). In Table 5 we see that also in this test problem, the coarse space size grows only very slowly with the number of inclusions and even in the hardest case n_H is only 53 (cf. the dimension n of $V_{h,0}$, and thus of A is 25600).

The last series of tests, in Table 6, aims to prove that the number m_j of eigenvectors per subdomain chosen by our automatic algorithm is indeed optimal in some sense. For Test Problem 1 with one channel, we first reduce the number of coarse basis functions per subdomain by one, this has a huge influence on the iteration count. Then we add one basis
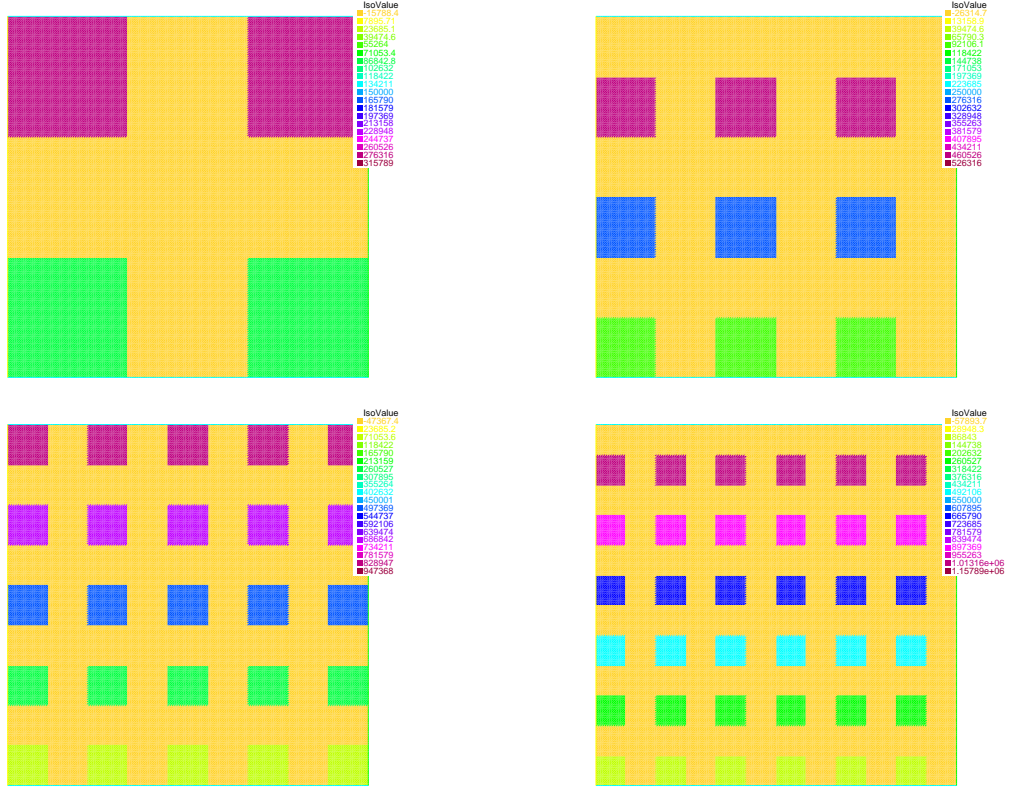
FIGURE 8. Test Problem 2: Successively adding inclusions.

	AS	AS+Nicolaides	AS+DtN	RAS	RAS+Nicolaides	RAS+DtN
2×2 inclusions	108	80	51	100	81	41
3×3 inclusions	194	342	58	154	153	46
5×5 inclusions	529	> 1000	57	243	245	41
6×6 inclusions	835	823	71	266	267	51

TABLE 4. Number of iterations for Test Problem 2 (additive coarse grid correction).

Over $J = 16$ subdomains	Total number $n_{\Gamma j}$ of eigenvalues on Γ_j	Number m_j of functions included in V_H from Ω_j			
		no channel	1 channel	2 channels	3 channels
Minimum	70	1	1	1	1
Maximum	191	3	3	4	5
Average	138.8	1.63	2.06	2.75	3.31
Sum	2220	26	33	44	53

TABLE 5. Size of the coarse space for Test Problem 2.

function per subdomain and notice that this has much less effect. This suggests that the selection process we have designed is indeed the best compromise between enriching the coarse grid and solving a reasonably sized coarse problem.

	AS	RAS
No coarse space	619	227
Nicolaides' piecewise constant coarse space	520	228
DtN space with $\max\{m_j - 1, 1\}$ functions from Ω_j	446	177
DtN space with m_j functions from Ω_j	64	46
DtN space with $m_j + 1$ functions from Ω_j	37	32

TABLE 6. Iteration numbers when reducing or increasing the number m_j of coarse basis functions per subdomain given by the automatic selection strategy.

5. Conclusions

In this paper we have given a rigorous analysis of the coarse space proposed in [23, 24]. We have seen that a robust coarse space can be chosen automatically with the method proposed there, if all the eigenmodes of local DtN maps that are smaller than the inverse of the diameter of the respective subdomain are included in the coarse space. Moreover, our results show that this selection criterion is to some extent optimal with respect to cost and robustness. It remains to be seen whether the proposed coarse space has also got good approximation properties in the context of multiscale approximation techniques from problems, where it is not desirable to resolve all the fine scale variation. For the coarse space based on subdomain eigenproblems developed in the context of two-level Schwarz methods in [15], this has already been demonstrated to some extent in [11]. We are also currently working on the extension of the ideas in this paper to linear elasticity problems.

We included a small number of numerical tests to confirm the theoretical results on some hard test problems with a complicated coefficient distribution (channels and inclusions) and a high contrast in the values of the coefficient. We did not study the dependence on geometrical parameters in detail, and because we only considered small overlap it was also not possible to verify how important Assumption 2.1 is in practice. Figure 2 gives typical situations where the assumption is not satisfied, but we believe that in practice a much more complicated coefficient distribution and a larger overlap would be necessary to really observe any loss of robustness of the proposed method. The extensive tests in [23, 24] do also suggest this.

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