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Abstract. We show that computability of the Radon-Nikodym derivative of a measure μ absolutely continuous w.r.t. some other measure λ can be reduced to a single application of the non-computable operator EC, which transforms enumeration of sets (in $\mathbb{N}$) to their characteristic functions. We also give a condition on the two measures (in terms of the computability of the norm of a certain linear operator involving the two measures) which is sufficient to compute the derivative.

1 Introduction

Theorem 1 (Radon-Nikodym). *Let $(\Omega, \mathcal{A}, \lambda)$ be a measured space where λ is σ-finite. Let μ be a finite measure that is absolutely continuous w.r.t. λ. There exists a unique function $h \in L^1(\lambda)$ such that for all $f \in L^1(\mu)$,*

$$\int f \, \mathrm{d}\mu = \int fh \, \mathrm{d}\lambda.$$

h is called the **Radon-Nikodym derivative***, or* **density***, of μ w.r.t. ν.*

Is this theorem computable? Can h be computed from μ and λ? In [3] a negative answer was given.

In this paper we investigate to what extent this theorem is non-computable. We first give an upper bound for its non-computability, showing that it can be computed using a single application of the operator EC (which transforms enumerations of sets of natural numbers into their characteristic functions). In proving this result we use two classical theorems: Levy's zero-one law and Radon-Nikodym Theorem itself. We then give a sufficient condition on the measures to entail the computability of the RN derivative: this condition is the computability of the norm of a certain integral operator associated to the measures.

2 Preliminaries

2.1 Little bit of Computability via Representations

To carry out computations on infinite objects we encode those objects into infinite symbolic sequences, using representations (see [6] for a complete development). Let $\Sigma = \{0, 1\}$. A ***represented space*** is a pair (X, δ) where X is a set and $\delta \subset \Sigma^{\mathbb{N}} \to X$ is an onto partial map. Every $p \in \mathrm{dom}(\delta)$ such that $\delta(p) = x$ is called a ***δ-name*** of x (or ***name*** of x when δ is clear from the context).

Let (X, δ_X) be a represented space. An element $x \in X$ is ***computable*** if it has a computable name. Let (Y, δ_Y) be another represented space. A realizer for a function $f : X \to Y$ is a (partial) function $F : \Sigma^{\mathbb{N}} \to \Sigma^{\mathbb{N}}$ such that $f \circ \delta_X = \delta_Y \circ F$ (with the expected compatibilities between domains). f is ***computable*** if it has a computable realizer. Of course the image of a computable element by a computable function is computable.

Example 1. Let $2^{\mathbb{N}}$ be the powerset of $\mathbb{N}$. The classical notions of recursive and recursively enumerable sets can be grasped as the computable elements of the space $2^{\mathbb{N}}$ endowed with two representations, Cf and En respectively, defined by:

$$\mathrm{En}(p) = \{n \in \mathbb{N} : 100^n 1 \text{ is a subword of } p\}$$
$$\mathrm{Cf}(p) = \{n \in \mathbb{N} : p_n = 1\}$$

where $p \in \Sigma^{\mathbb{N}}$, p_n is the nth symbol in p. $\mathrm{En}, \mathrm{Cf} : \Sigma^{\mathbb{N}} \to 2^{\mathbb{N}}$ are total functions.

We define the operator EC as the identity from represented space $(2^{\mathbb{N}}, \mathrm{En})$ to represented space $(2^{\mathbb{N}}, \mathrm{Cf})$. It transforms an enumeration of a set into its characteristic function. EC is not computable.

The non-computability of functions f, g between represented spaces can be compared using a notion of reducibility introduced in [5]: $f \leq_W g$ if there are computable (partial) functions K, H such that for every realizer G of g, $p \mapsto K(p, G \circ H(p))$ is a realizer of f. In other words, $f \leq_W g$ if f can be computed using one single application of g (provided by an oracle) in the computation.[4]

2.2 Computable Measurable Spaces

We start by briefly recalling some basic definitions from measure theory. See for example [4,2,1] for a complete treatment. A ***ring*** $\mathcal{R}$ over a set Ω is a collection of subsets of Ω which contains the empty set and is closed under finite unions and relative complementation ($B \setminus A \in \mathcal{R}$, for $A, B \in \mathcal{R}$). A ***σ-algebra*** $\mathcal{A}$ (over the set Ω) is a collection of subsets of Ω which contains Ω and is closed under complementation and countable unions (and therefore also closed under countable intersections). A ring $\mathcal{R}$ generates a unique σ-algebra, denoted by $\sigma(\mathcal{R})$, and defined as the smallest σ-algebra containing $\mathcal{R}$.

In this paper we will work with the ***measurable space*** $(\Omega, \mathcal{A})$, where $\mathcal{A}$ is a σ-algebra generated by a countable ring $\mathcal{R}$. Members of $\mathcal{A} = \sigma(\mathcal{R})$ will be referred to as ***measurable sets***.

A ***measure*** over a collection $\mathcal{C}$ (which is at least closed by finite unions) of subsets of Ω is a function $\mu : \mathcal{C} \to \mathbb{R}^\infty$ ($= \mathbb{R} \cup \{\infty\}$) such that i) $\mu(\emptyset) = 0$, $\mu(E) \geq 0$ for all $E \in \mathcal{C}$, and ii) $\mu(\bigcup_i E_i) = \sum_i \mu(E_i)$ for pairwise disjoint sets $E_0, E_1, \ldots \in \mathcal{C}$ such that $\bigcup_i E_i \in \mathcal{C}$. A measure μ over a collection $\mathcal{C}$ is said to

[4] the relation is denoted $\leq_W$ as it is a generalization of Wadge reducibility.

be $\boldsymbol{\sigma}$***-finite***, if there are sets $E_0, E_1, \ldots \in \mathcal{C}$ such that $\mu(E_i) < \infty$ for all i and $\Omega = \bigcup_i E_i$.

It is well known that a σ-finite measure over ring $\mathcal{R}$ has a unique extension to a measure over the σ-algebra $\sigma(\mathcal{R})$. For measures μ and λ, we say that μ is ***absolutely continuous*** w.r.t. λ, and write $\boldsymbol{\mu \ll \lambda}$, if $(\lambda(A) = 0 \Longrightarrow \mu(E) = 0)$ for all measurable sets E.

We now introduce the effective counterparts.

Definition 1. *A* **computable measurable space** *is a tuple* $(\Omega, \mathcal{A}, \mathcal{R}, \alpha)$ *where*

1. $(\Omega, \mathcal{A})$ *is a measurable space,* $\mathcal{R}$ *is a countable ring such that* $\cup\mathcal{R} = \Omega$ *and* $\mathcal{A} = \sigma(\mathcal{R})$*,*
2. $\alpha : \mathbb{N} \to \mathcal{R}$ *is a computable enumeration such that the operations* $(A, B) \to A \cup B$ *and* $(A, B) \to A \setminus B$ *are computable w.r.t.* α*.*

In the following $(\Omega, \mathcal{A}, \mathcal{R}, \alpha)$ will be a computable measurable space. We will consider only measures $\mu : \mathcal{A} \to [0; \infty]$ such that $\mu(E) < \infty$ for every $E \in \mathcal{R}$. Observe that such a measure μ is σ-finite, and therefore well-defined by its values on the ring $\mathcal{R}$. Conversely if a measure μ over $\mathcal{A}$ is σ-finite, one can choose a countable generating $\mathcal{R}$ such that $\mu(E) < \infty$ for all $E \in \mathcal{R}$.

Computability on the space of measures over $(\Omega, \mathcal{A}, \mathcal{R})$ will be expressed via representations.

Definition 2. *Let* $\mathcal{M}$ *be the set of measures* μ *such that* $\mu(E) < \infty$ *for all* $E \in \mathcal{R}$ *and let* $\mathcal{M}_{<\infty}$ *be the set of all finite measures. Define representations* $\delta_{\mathcal{M}} : \Sigma^{\mathbb{N}} \to \mathcal{M}$ *and* $\delta_{\mathcal{M}_{<\infty}} : \Sigma^{\mathbb{N}} \to \mathcal{M}_{<\infty}$ *as follows:*

1. $\delta_{\mathcal{M}}(p) = \mu$*, iff* p *is (more precisely, encodes) a list of all* $(l, n, u) \in \mathbb{Q}^3$ *such that* $l < \mu(E_n) < u$*, for every* $E_n \in \mathcal{R}$*.*
2. $\delta_{\mathcal{M}_{<\infty}}(p) = \mu$*, iff* $p = \langle p_1, p_2 \rangle$ *such that* $\delta_{\mathcal{M}}(p_1) = \mu$ *and* p_2 *is (more precisely, encodes) a list of all* $(l, u) \in \mathbb{Q}^2$ *such that* $l < \mu(\Omega) < u$*.*

Thus, a $\delta_{\mathcal{M}}$-name p allows to compute $\mu(A)$ for every ring element A with arbitrary precision. A $\delta_{\mathcal{M}_{<\infty}}$-name allows additionally to compute $\mu(\Omega)$. Obviously, $\delta_{\mathcal{M}_{<\infty}} \leq \delta_{\mathcal{M}}$ and $\delta_{\mathcal{M}_{<\infty}} \equiv \delta_{\mathcal{M}}$ if $\Omega \in \mathcal{R}$. But in general, not even the restriction of $\delta_{\mathcal{M}}$ to the finite measures is reducible to $\delta_{\mathcal{M}_{<\infty}}$.

We will also work with the spaces L^1 of integrable functions and L^2 of square-integrable functions (w.r.t. some measure). A ***rational step function*** is a finite sum

$$s = \sum_{k=1}^{p} \mathbf{1}(E_{i_k}) q_{j_k},$$

where $E_{i_k} \in \mathcal{R}$ and $q_{j_k} \in \mathbb{Q}$.

The computable numberings of the ring $\mathcal{R} = (E_0, E_1, ...)$ and of the rational numbers $\mathbb{Q} = (q_0, q_1, ...)$ induce a canonical numbering of the the collection $\mathcal{RSF} = (s_0, s_1, ...)$ of rational step functions. Since the collection $\mathcal{RSF}$ is dense in the spaces L^1 and L^2, we can use it to handle computability over these spaces via the following representations:

Definition 3. *For every $\mu \in \mathcal{M}$ define Cauchy representations $\delta^k_\mu : \Sigma^{\mathbb{N}} \to L^k(\mu)$ of $L^k(\mu)$ $(k = 1, 2)$ by: $\delta^k_\mu(p) = f$ iff p is (encodes) a sequence $(s_{n_0}, s_{n_1}, \ldots)$ of rational step functions such that $\|s_{n_i} - f\|^k_\mu \leq 2^{-i}$ for all $i \in \mathbb{N}$.*

3 Effective Radon-Nikodym Theorem

3.1 An upper bound

In the following, we present our first main result which, in words, says that computability of the Radon-Nikodym derivative is reducible to a single application of the (non-computable) operator EC.

Theorem 2. *The function mapping every σ-finite measure $\lambda \in \mathcal{M}$ and every finite measure μ such that $\mu \ll \lambda$ to the function $h \in L^1(\lambda)$ such that $\mu(E) = \int_E h \, \mathrm{d}\lambda$ for all $E \in \sigma(\mathcal{R})$ is computable via the representations $\delta_{\mathcal{M}}$, $\delta_{\mathcal{M}_{<\infty}}$ and δ^1_λ with a single application of the operator* EC*.*

For the proof, we will use the following classical result on convergence of conditional expectations, which is a consequence of the more general Doob's martingale convergence theorems (see for example [2], Section 10.5). We recall that a filtration $(\mathcal{F}_n)_n$ is an increasing sequence of σ-algebras on a measurable space. In some sense, $\mathcal{F}_n$ represents the information available at time n. In words, the following result says that if we are learning gradually the information that determines the outcome of an event, then we will become gradually certain what the outcome will be.

Theorem 3 (Levy's zero-one law). *Consider a measured space $(\Omega, \sigma(\mathcal{R}), \lambda)$ with λ finite. Let $h \in L^1(\lambda)$. Let $(\mathcal{F}_n)_n$ be any filtration such that $\sigma(\mathcal{R}) = \sigma(\bigcup_n \mathcal{F}_n)$. Then*

$$\mathbb{E}(h|\mathcal{F}_n) \underset{n\to\infty}{\longrightarrow} h$$

both λ-almost everywhere and in $L^1(\lambda)$.

Proof (of Theorem 2). We start by proving the result for finite measures.

Lemma 1. *From descriptions of finite measures $\mu \ll \lambda$, one can compute a sequence $\{h_n\}_{n\in\mathbb{N}}$ of $L^1(\lambda)$-computable functions which converges in $L^1(\lambda)$ to the Radon-Nikodym derivative $h = \frac{d\mu}{d\lambda} \in L^1(\lambda)$.*

Proof. Consider the computable enumeration $E_0, E_1, \ldots$ of $\mathcal{R}$. For every n, consider the partition $\mathcal{P}_n$ of the space given by the cells:

- all the possible $E_0^* \cap E_1^* \cap \ldots \cap E_n^*$ where E^* is either E or $(E_0 \cup \ldots \cup E_n) \setminus E$,
- all the $E_{k+1} \setminus (E_0 \cup \ldots \cup E_k)$ for every $k \geq n$.

Observe that all the cells are elements of the ring $\mathcal{R}$. The partitions $\mathcal{P}_n$ induce a filtration which generates the σ-algebra $\sigma(\mathcal{R})$. Let h be the RN-derivative[5] of μ w.r.t. λ. $h \in L^1(\lambda)$ so by Levy's zero-one law,

$$\mathbb{E}(h|\mathcal{P}_n) \underset{n\to\infty}{\longrightarrow} h$$

both λ-almost everywhere and in $L^1(\lambda)$.

Now, $h_n := \mathbb{E}(h|\mathcal{P}_n)$ is constant on each cell C of $\mathcal{P}_n$, with value $\frac{\mu(C)}{\lambda(C)}$ if $\lambda(C) > 0$ and 0 otherwise. We now show that the functions h_n are (uniformly) computable elements of $L^1(\lambda)$. For a given ϵ, one can find cells $C_1, ..., C_k$ in $\mathcal{P}_n$ such that $\lambda(C_i) > 0$ for all $i = 1, ..., k$ and $\sum_{i=1}^k \mu(C_i) > \mu(\Omega)-\epsilon$. Define h_n^ϵ to be constant with value $\frac{\mu(C_i)}{\lambda(C_i)}$ for $i = 1, ..., k$ and 0 on the other cells. As these values are uniformly computable, the functions h_n^ϵ are uniformly $L^1(\lambda)$-computable. Moreover

$$\int |h_n - h_n^\epsilon| \,\mathrm{d}\lambda = \int_{\Omega\setminus\bigcup_{i=1}^k C_i} h_n \,\mathrm{d}\lambda < \epsilon$$

and thus, the functions h_n are (uniformly) computable elements of $L^1(\lambda)$, and converge to h in $L^1(\lambda)$. The lemma is proved.

Assume now that $\mu \in \mathcal{M}_{<\infty}$ and $\lambda \in \mathcal{M}$. Let once again $(E_0, E_1, \ldots)$ be the computable numbering of the ring $\mathcal{R}$. Let $F_0 := E_0$ and $F_{n+1} := E_{n+1} \setminus \{F_0, \ldots, F_n\}$. Then $(F_i)_i$ is a computable numbering of ring elements such that $F_i \cap F_j = \emptyset$ for $i \neq j$ and $\Omega = \bigcup_i F_i$.

Since $i \mapsto \mu(F_i)$ is computable, there is a computable function $d : \mathbb{N} \to \mathbb{N}$ such that $d(i) > \mu(F_i) \cdot 2^i$. Define a function $w : \Omega \to \mathbb{R}$ by $w(x) := 1/d(i)$ if $x \in F_i$. Then

$$\int w \,\mathrm{d}\lambda = \sum_i \mu(F_i)/d(i) < 2.$$

Define a new measure ν by

$$\nu(E) := \int_E w \,\mathrm{d}\lambda. \tag{1}$$

for all $E \in \mathcal{R}$. Then ν is a finite measure, which is equivalent to λ (i.e. $\nu \ll \lambda \ll \nu$) and such that a $\delta_{\mathcal{M}}$-name of ν can be computed from a $\delta_{\mathcal{M}}$-name of λ. Apply Lemma 1 to μ and ν: one can compute a sequence of $L^1(\nu)$-computable functions $\{h'_n\}$ whose limit (in $L^1(\nu)$) is the density $h' = \frac{\mathrm{d}\mu}{\mathrm{d}\nu}$. The sequence $\{h'_n w\}$ is computable and converges (in $L^1(\lambda)$) to $h = \frac{\mathrm{d}\mu}{\mathrm{d}\lambda} = wh'$. At this point we use the operator EC to extract a fast convergent subsequence, and hence to compute (a δ^1_λ-name of) h. The proof is complete.

The above theorem shows that the Radon-Nikodym theorem is reducible to the (non-computable) operator EC. On the other hand, it was shown in [3] that

[5] at this point we use the classical RN theorem

there exists a computable measure on the unit interval, absolutely continuous w.r.t. Lebesgue measure, and such that the operator EC can be reduced to the computation of the density (which is therefore not L^1-computable). This give us the following corollary.

Corollary 1. *For nontrivial computable measurable spaces, the Radon-Nikodym operator and EC are equivalent: $RN \equiv_W EC$.*

This result characterize the extent to which the Radon-Nikodym theorem is non-computable. However, it doesn't give us much information on "where" is the non-effectivity. In what follows we present a result which gives an explicit condition (in terms of the computability of the norm of a certain linear operator involving the two measures) allowing to compute the Radon-Nikodym derivative. The proof of this result is somewhat more involved, and some preparation will be required.

3.2 Locating the non-computability

Let $\mu \in \mathcal{M}_{<\infty}$ be a finite measure over $(\Omega, \mathcal{A}, \mathcal{R})$. Let $u : L^2(\mu) \to \mathbb{R}$ be a linear functional. Classically we have that the following are equivalent:

1. u is continuous,
2. u is uniformly continuous,
3. there exists c (a ***bound for*** u) such that for every $f \in L^2(\mu)$, $|u(f)| \leq c\|f\|_2$.

The smallest bound for u is called ***the norm*** of u and is denoted by $\|u\|$. That is,

$$\|u\| := \sup\{c \in \mathbb{R} : |u(f)| \leq c\|f\|_2\} = \sup_{\|f\|_2=1} |u(f)|.$$

Suppose that the operator u is computable in the sense that one can compute the real number $u(f)$ from (a $\delta_{\mathcal{M}}$-name of) μ and (a δ^2_μ-name of) f. Consider now the numbered collection $\mathcal{RSF} = \{r_n\}_{n\in\mathbb{N}}$. Since the sequence $s_i := \frac{r_i}{\|r_i\|_2}$ is uniformly computable (from μ) and dense in $\{f \in L^2(\mu) : \|f\|_2 = 1\}$, it follows that the norm $\|u\|$ of a computable operator u is always a lower-computable (from μ) number. It is not, in general, computable. In case it is computable, we will say that the operator u is ***computably normable***.

The following result is an effective version of Riesz-Fréchet Representation Theorem. For simplicity, we state the result in the particular Hilbert space $L^2(\mu)$, but the same proof works for any Hilbert Space provided that the inner product is computable.

Theorem 4. *Let u be a non-zero computably normable (from μ) linear functional over $L^2(\mu)$. Then from μ one can compute a name of (the unique) $g \in L^2(\mu)$ such that*

$$u(f) = \int fg \, \mathrm{d}\mu$$

for all $f \in L^2(\mu)$.

Proof. In the following, all what we will compute will be from a $\delta_{\mathcal{M}}$-name of μ. Chose any computable $x \in L^2(\mu)$ out of $ker(u)$. As $\|u\|$ is computable, from the classical formula $d(x, ker(u)) = \frac{|u(x)|}{\|u\|}$, it follows that $d(x, ker(u))$ is computable too. We can then enumerate a sequence of points $y_n \in E = ker(u) + x$ which is dense in E. Note that $d(0, E) = d(x, ker(u))$. Let $z_0 \in E$ be such that $\|z_0\| = d(0, E)$. Since

$$\|y_n - z_0\|^2 = \|y_n\|^2 - \|z_0\|^2$$

we can compute a subsequence y_{n_i} converging effectively to z_0 which is therefore computable. Put $z = \frac{z_0}{\|z_0\|}$ so that $\|z\| = 1$. All what remains is to show that $g = u(z)z$ (which is computable) satisfies the required property. This is done as in the classical proof, namely: z has the property of being orthogonal to $ker(u)$. That is, $\int zf \, \mathrm{d}\mu = 0$ for any $f \in ker(u)$. Put

$$r := u(f)z - u(z)f.$$

We have $u(r) = u(f)u(z) - u(z)u(f) = 0$ so that $r \in ker(u)$ and then $\int rz \, \mathrm{d}\mu = 0$. This gives,

$$u(f) = u(f) \int z^2 \, \mathrm{d}\mu = \int u(f)z^2 \, \mathrm{d}\mu = \int (r + u(z)f)z \, \mathrm{d}\mu = \int fu(z)z \, \mathrm{d}\mu$$

and hence $g = u(z)z$, as was to be shown.

Remark 1. Let μ be a finite measure. For f, g in $L^2(\mu)$, Hölder's inequality implies that $fg \in L^1(\mu)$ and:

$$\|fg\|_1 \leq \|f\|_2 \|g\|_2 \tag{2}$$

so that, in particular, since μ is finite, if $f \in L^2$ then $f \in L^1$ (taking $g \equiv 1$). Moreover, from a δ^2_μ-name of f one can compute a δ^1_μ-name.

Now, let μ and λ be finite measures and consider a new measure $\varphi := \mu + \nu$. Let $L_\mu : L^2(\varphi) \to \mathbb{R}$ be defined by $L_\mu(f) := \int f \, \mathrm{d}\mu$. This is a bounded operator and it is easy to see that from $\delta_{\mathcal{M}}$-names of μ and λ and from a δ^2_φ-name of f, one can compute the value $L_\mu(f)$.

Definition 4. *A finite measure μ is said to be* **computably normable relative to** *some other finite measure λ, if the norm of the operator L_μ (as defined above) is computable from μ and λ.*

At this point, we are ready to state our second main result.

Theorem 5. *Let $\mu, \lambda \in \mathcal{M}_{<\infty}$ be such that:*

(i) $\mu \ll \lambda$,
(ii) *μ is computably normable relative to λ.*

Then the Radon-Nikodym derivative $\frac{\mathrm{d}\mu}{\mathrm{d}\lambda}$ can be computed as an element of $L^1(\lambda)$, from μ and λ.

Proof. We follow Von Neumann's proof. The measure $\varphi := \mu + \lambda$ is computable and by hypothesis the operator $L_\mu : L^2(\varphi) \to \mathbb{R}$ defined by $L_\mu(f) := \int f \, \mathrm{d}\mu$ is computably normable. Hence, by Theor. 4 one can compute a name of $g \in L^2(\varphi)$ (and hence a name of g as a point in $L^1(\lambda)$) such that for all $f \in L^2(\varphi)$ the equality:

$$\int f \, \mathrm{d}\mu = \int f g \, \mathrm{d}\varphi = \int f g \, \mathrm{d}\lambda + \int f g \, \mathrm{d}\mu \tag{3}$$

holds. This relation can be rewritten as:

$$\int f g \, \mathrm{d}\lambda = \int f(1-g) \, \mathrm{d}\mu. \tag{4}$$

Note that (4) holds for any $f \geq 0$ (take $f_n = f\mathbf{1}_{\{f<n\}}$ and apply monotone convergence theorem). Taking $f = \mathbf{1}_{\{g=1\}}$ in (4) we see that $\lambda(\{g = 1\}) = 0$. Hence the following function is defined λ-almost everywhere:

$$h := \frac{g}{1-g}.$$

Taking $f = \mathbf{1}_{\{g<0\}}$ and $\mathbf{1}_{\{g>1\}}$ in (4) we see that $0 \leq g \leq 1$ λ-a.e., so $h \geq 0$ λ-a.e. Therefore,

$$\begin{aligned} \int f h \, \mathrm{d}\lambda &= \int \left(\frac{f}{1-g} \right) g \, \mathrm{d}\lambda \\ &= \int \left(\frac{f}{1-g} \right) (1-g) \, \mathrm{d}\mu && \text{by (4)} \\ &= \int f \, \mathrm{d}\mu. \end{aligned}$$

Now, taking f to be the constant function equal to 1, we conclude that $\int h \, \mathrm{d}\lambda = 1$ and then it is in $L^1(\lambda)$. This shows that h is the Radon-Nikodym derivative.

It remains to show that the function $h = \frac{g}{1-g}$ is $L^1(\lambda)$-computable.

As g is $L^1(\lambda)$-computable, we can effectively produce a sequence u_i of rational step functions such that $\|u_i - g\|_\lambda < 2^{-i}$. As $g \geq 0$ λ-a.e. we can assume w.l.o.g. that $u_i \geq 0$ (otherwise replace u_i with $\max(u_i, 0)$).

For $n \in \mathbb{N}$ let

$$\begin{aligned} g_n &:= \min(g, 1 - 2^{-n}) \qquad & h_n &:= g_n/(1-g_n) \\ u_{in} &:= \min(u_i, 1 - 2^{-n}) \qquad & v_{in} &:= u_{in}/(1-u_{in}) \end{aligned}$$

Since the function $x \mapsto x/(1-x)$ is nondecreasing over $(0, +\infty)$,

$$g_n \leq g_{n+1} \text{ and } \sup_n g_n = g,$$
$$h_n \leq h_{n+1} \text{ and } \sup_n h_n = h.$$

Given a rational number $\epsilon > 0$ we show how to compute n and i such that

$$\|h - v_{in}\|_\lambda < \epsilon.$$

v_{in} will then be a rational step function approximating h up to ϵ, in $L^1(\lambda)$. As a result, it will enable to compute a δ^1_μ-name of h.

To find n and i, we use the following inequality

$$\|h - v_{in}\|_\lambda \leq \|h - h_n\|_\lambda + \|h_n - v_{in}\|_\lambda \tag{5}$$

We first make the first term small. As for all n, $0 \leq h_n \leq h$, one has $\|h - h_n\|_\lambda = \|h\|_\lambda - \|h_n\|_\lambda$. As $\|h\|_\lambda = \mu(\Omega)$ is given as input and $\|h_n\|_\lambda$ can be computed from n, one can effectively find n such that $\|h\|_\lambda - \|h_n\|_\lambda < \epsilon/2$.

We then make the second term in (5) small:

$$\begin{aligned}
\|h_n - v_{in}\|_\lambda &= \left\| \frac{g_n}{1-g_n} - \frac{u_{in}}{1-u_{in}} \right\|_\lambda \\
&= \left\| \frac{g_n - u_{in}}{(1-g_n)(1-u_{in})} \right\|_\lambda \\
&\leq \|g_n - u_{in}\|_\lambda \cdot 2^{2n} \\
&\leq \|g_n - u_{in}\|^1_\varphi \cdot 2^{2n} \\
&\leq \|g - u_i\|^1_\varphi \cdot 2^{2n} \\
&\leq 2^{-i} \cdot 2^{2n}.
\end{aligned}$$

We then compute i such that $2^{-i} \cdot 2^{2n} < \epsilon/2$. One finally gets the expected inequality

$$\|h - v_{in}\|_\lambda \leq \|h - h_n\|_\lambda + \|h_n - v_{in}\|_\lambda < \epsilon$$

and the result follows.

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