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Sangnam Nam*, Michael E. Davies†, Michael Elad‡ and Rémi Gribonval*
*INRIA Rennes – Bretagne Atlantique, France
†IDCOM & Joint Research Institue for Signal and Image Processing, Edinburgh University, UK
‡Department of Computer Science, The Technion – Israel Institute of Technology, Israel

***Abstract*—In the past decade there has been a great interest in a synthesis-based model for signals, based on sparse and redundant representations. This work considers an alternative *analysis-based* model, where an analysis operator multiplies the signal, leading to a *cosparse* outcome. We consider this analysis model, in the context of a generic missing data problem. Our work proposes a uniqueness result for the solution of this problem, based on properties of the analysis operator and the measurement matrix. A new greedy algorithm for solving the missing data problem is proposed along with theoretical study of the success of the algorithm and experimental results.**

I. Introduction

Given a set of incomplete linear observation $\mathbf{y} = \mathbf{M}\mathbf{x}_0 \in \mathbf{R}^m$ of a signal $\mathbf{x}_0 \in \mathbf{R}^d$, $m < d$, the assumption that $\mathbf{x}_0$ admits a *sparse* representation $\mathbf{z}_0$ in some *synthesis dictionary* $\mathbf{D}$ is known to be of significant help in recovering the original signal $\mathbf{x}_0$. Indeed, it is now well understood that under incoherence assumptions on the matrix $\mathbf{MD}$, one can recover vectors $\mathbf{x}_0$ with sufficiently sparse representations by solving the optimization problem:

$$\hat{\mathbf{x}}_S := \mathbf{D}\hat{\mathbf{z}}; \quad \hat{\mathbf{z}} := \arg\min_{\mathbf{z}} \|\mathbf{z}\|_\tau \text{ subject to } \mathbf{y} = \mathbf{MDz} \tag{1}$$

for $0 \le \tau \le 1$.

An alternative to (1) which has also been used successfully in practice is to consider the *analysis* ℓ_τ-optimization [2], [6], [7]:

$$\hat{\mathbf{x}}_A := \arg\min_{\mathbf{x}} \|\mathbf{\Omega x}\|_\tau \text{ subject to } \mathbf{y} = \mathbf{Mx}, \tag{2}$$

where $\mathbf{\Omega} : \mathbf{R}^d \to \mathbf{R}^p$ is an *analysis operator*. Typically the dimensions are $m < d \le p, n$.

The fact that $\mathbf{z}_0$ contains few zeros, i.e., is *sparse* may be thought of as the principal reason why one can recover the so-called *sparse* signals via (1). We show that while the optimization (2) has similar look to (1), a different model, which we name the *cosparse analysis model*, is more closely linked to (2) than the sparse synthesis model. In particular, contrary to the sparse model, we are more interested in the signals $\mathbf{x}_0$ whose analysis representation $\mathbf{\Omega x}_0$ contains *many zeros*. We call such signals *cosparse* and the quantity $\ell := p - \|\mathbf{\Omega x}_0\|_0$ the *cosparsity*.

II. Uniqueness

Based on the existing work [1], [3], we establish [4] the uniqueness of cosparse signals in the context of linear inverse problems above. The result we have derived has simple forms for two particular classes of analysis operators: $\mathbf{\Omega}$ that is in general position, which means that the rows of $\mathbf{\Omega}$ has no non-trivial linear dependencies, and the popular 2D TV analysis operators $\mathbf{\Omega}$ that consists of all the vertical and horizontal one-step differences in a 2D image. For these two types of $\mathbf{\Omega}$, we have:

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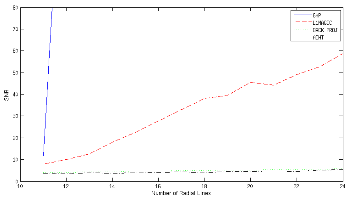

Fig. 1. SNR vs Number of radial observation lines in the Shepp Logan phantom recovery. The line for GAP is clipped because the SNR was over 150 from 12 radial lines.

Proposition 1. *1) Let $\mathbf{\Omega}$ be in general position. Then, the problem $\mathbf{y} = \mathbf{Mx}$ has at most one ℓ-cosparse solution if and only if $m \ge 2(d - \ell)$.*

2) Let $\mathbf{\Omega}$ be the 2D TV analysis operator. Then, the problem $\mathbf{y} = \mathbf{Mx}$ has at most one ℓ-cosparse solution if $m + \ell \ge 2d$.

III. Algorithm, Theory, and Experimental Result

With the uniqueness property established, we propose a new greedy algorithm which aims to recover cosparse signals based on incomplete linear observations. This algorithm, named the Greedy Analysis Pursuit (GAP), may be considered as the counterpart of the Orthogonal Matching Pursuit (OMP) in the sparse model. However, the GAP tries to detect the elements *outside* the locations of the zeros of analysis representations, this way carving its way towards the index set of zeros in the end.

We then provide a theoretical condition that guarantees the success of both the GAP and the analysis ℓ_1-minimization in cosparse signal recovery. Finally, we run some synthetic experiments to demonstrate the effectiveness of the proposed algorithm. Interestingly, we observe that GAP performs better than the analysis ℓ_1-minimization in the given tasks. In particular, Fig. 1 shows SNR vs the number of radial observation lines in the Shepp Logan phantom recovery problem.

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