Inherent Two-Way Ambiguity in 2D Projective Reconstruction from Three Uncalibrated 1D Images

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Abstract

It is shown that there always exists a two-way ambiguity for 2D projective reconstruction from three uncalibrated 1D views independent of the number of point correspondences. It is also shown that the two distinct projective reconstructions are exactly related by a quadratic transformation with the three camera centers as the fundamental points. The unique reconstruction exists only for the case where the three camera centers are aligned. The theoretical results are demonstrated on numerical examples.

1. Introduction

A usual CCD camera is commonly modeled as a 2D projective device that projects a point in $\mathcal{P}^3$ (the projective space of dimension 3) to a point in $\mathcal{P}^2$. By analogy, we can consider what we call a 1D projective camera which projects a point in $\mathcal{P}^2$ to a point in $\mathcal{P}^1$. This 1D projective camera may seem very abstract, but many imaging systems using laser beams, infra-red or ultra-sounds acting only on a source plane can be modeled this way. What is less obvious, but more interesting, is that in some situations, the usual 2D camera model is also closely related to this 1D camera model. The first example is the case of the 2D affine camera model operating on line segments: The direction vectors of lines in 3D space and in the image correspond to each other via this 1D projective camera model [12, 11]. It has been shown [6] that a 2D camera undergoing a planar motion is reduced to a 1D camera on the trifocal line of the 2D cameras.

The geometry of multiple 1D views is completely and nicely characterised by its associated trifocal tensor which has the interesting properties of uniqueness and minimality that the 2D trifocal tensor [16, 5] does not have. Although the tensor could be estimated linearly, explicit 2D reconstruction from the tensor has a two-way ambiguity [11]. In this paper, we will prove new results that the two distinct projective reconstructions are exactly related by a quadratic transformation with the three camera centers as the fundamental points. The unique reconstruction is possible when the three camera centers are aligned.

The paper is organised as follows. In Section 2 and 3, we review the 1D projective camera, its trifocal tensor and 2D projective reconstruction from the trifocal tensor. We then prove the major result that the two distinct projective reconstructions are related by a quadratic transformation in Section 4. The numerical simulation examples are given in Section 5 to support the theretical developement. Finally, some concluding remarks and future directions are given in Section 6.

Throughout the paper, vectors are denoted in lower case boldface $\mathbf{x}$, $\mathbf{u}$ $\ldots$, matrices and tensors in upper case boldface $\mathbf{A}$, $\mathbf{T}$ $\ldots$; Scalars are any plain letters or lower case Greek a, u, A, λ $\ldots$. Some basic tensor notation is used: Covariant indices are written as subscripts and contravariant indices as superscripts. e.g. the coordinates of a point $\mathbf{x}$ in $\mathcal{P}^3$ are written with an upper index $\mathbf{x} = (x^1, x^2, x^3, x^4)^T$. A matrix $\mathbf{A}$ may also be written with two indices like A^i_j, where i indexes rows and j columns. The implicit summation convention is also adopted.

2. 1D projective camera and its trifocal tensor

We will first review the one-dimensional camera which was abstracted from the study of the geometry of lines under affine cameras [12, 11]. We can also introduce it directly by analogy to a 2D projective camera.

A 1D projective camera projects a point $\mathbf{x} = (x^1, x^2, x^3)^T$ in $\mathcal{P}^2$ (projective plane) to a point $\mathbf{u} = (u^1, u^2)^T$ in $\mathcal{P}^1$ (projective line). This projection may be described by a 2×3 homogeneous matrix $\mathbf{M}$ as follows:

$$\lambda \mathbf{u} = \mathbf{M}_{2\times 3}\mathbf{x}. \tag{1}$$

We now examine the geometric constraints available for points seen in multiple views similar to the 2D camera case [14, 15, 7, 2, 3, 1, 8, 16, 5]. There is a constraint only in the case of 3 views, as there is no any constraint for 2 views (two projective lines always intersect in a point in a projective plane).

Let the three views of the same point $\mathbf{x}$ be given as follows:

$$\begin{cases} \lambda \mathbf{u} &= \mathbf{M}\mathbf{x}, \\ \lambda' \mathbf{u}' &= \mathbf{M}'\mathbf{x}, \\ \lambda'' \mathbf{u}'' &= \mathbf{M}''\mathbf{x}. \end{cases} \tag{2}$$

These can be rewritten in matrix form as

$$\begin{pmatrix} \mathbf{M} & \mathbf{u} & 0 & 0 \\ \mathbf{M}' & 0 & \mathbf{u}' & 0 \\ \mathbf{M}'' & 0 & 0 & \mathbf{u}'' \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ -\lambda \\ -\lambda' \\ -\lambda'' \end{pmatrix} = 0. \tag{3}$$

The vector $(\mathbf{x}, -\lambda, -\lambda', -\lambda'')^T$ cannot be zero, so

$$\begin{vmatrix} \mathbf{M} & \mathbf{u} & 0 & 0 \\ \mathbf{M}' & 0 & \mathbf{u}' & 0 \\ \mathbf{M}'' & 0 & 0 & \mathbf{u}'' \end{vmatrix} = 0. \tag{4}$$

The expansion of this determinant produces a trifocal constraint for the three views

$$T_{ijk} u^i u'^j u''^k = 0, \tag{5}$$

where T_{ijk} is a $2 \times 2 \times 2$ homogeneous tensor.

It can be easily seen that any constraint obtained by adding further views reduces to a trilinearity. This proves the uniqueness of the trifocal constraint. Moreover, the $2\times2\times2$ homogeneous tensor has $7 = 2 \times 2 \times 2 - 1$ d.o.f., so it is a minimal parametrization of three views in the uncalibrated setting since three views have exactly $3 \times (2 \times 3 - 1) - (3 \times 3 - 1) = 7$ d.o.f., up to a projective transformation in $\mathcal{P}^2$.

This result for the one-dimensional projective camera is very interesting. The trifocal tensor encapsulates exactly the information needed for projective reconstruction in $\mathcal{P}^2$. Namely, it is the unique matching constraint, it minimally parametrizes the three views and it can be estimated linearly. Contrast this to the 2D image case in which the multilinear constraints are algebraically redundant and the linear estimation is only an approximation based on over-parametrization.

3. Two way ambiguity of 2D projective reconstruction

According to Triggs [16], the projective reconstruction in $\mathcal{P}^3$ can be viewed as being equivalent to the rescaling of the image points in $\mathcal{P}^2$.

For each 1D image point across three views (*cf.* Equation (2)), the scale factors λ, λ' and λ''–taken individually–are arbitrary: However, taken as a whole $(\lambda\mathbf{u}, \lambda'\mathbf{u}', \lambda''\mathbf{u}'')^T$, they encode the projective structure of the points $\mathbf{x}$ in $\mathcal{P}^2$. One way to explicitly recover the scale factors $(\lambda, \lambda', \lambda'')^T$ is to notice that the rescaled image coordinates $(\lambda\mathbf{u}, \lambda'\mathbf{u}', \lambda''\mathbf{u}'')^T$ should lie in the joint image, or alternatively to observe the following matrix identity:

$$\begin{pmatrix} \mathbf{M} & \lambda\mathbf{u} \\ \mathbf{M}' & \lambda'\mathbf{u}' \\ \mathbf{M}'' & \lambda''\mathbf{u}'' \end{pmatrix} = \begin{pmatrix} \mathbf{M} \\ \mathbf{M}' \\ \mathbf{M}'' \end{pmatrix} \begin{pmatrix} \mathbf{I}_{3\times3} & \mathbf{x} \end{pmatrix}.$$

The rank of the left matrix is therefore at most 3. All 4×4 minors vanish. Expanding by cofactors in the last column gives homogeneous linear equations in the components of $\lambda\mathbf{u}$, $\lambda'\mathbf{u}'$ and $\lambda''\mathbf{u}''$ with coefficients that are 3×3 minors of the joint projection matrix:

$$\mathbf{T}_{ijk}(\lambda\mathbf{u}^i) - \mathbf{e}_1''(\lambda'\mathbf{u}')^T + (\lambda''\mathbf{u}'')\mathbf{e}_1'^{\,T} = \mathbf{0}_{2\times2}. \tag{6}$$

There are two types of minors: Those involving three views with one row from each view and those involving two views with two rows from one view and one from the other. The first type gives the 8 components of the tensor $\mathbf{T}_{2\times2\times2}$ and the second type gives 12 components of the "*epipoles*" $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_1', \mathbf{e}_2', \mathbf{e}_1'', \mathbf{e}_2''$. The epipoles are defined by analogy with the 2D camera case, as the projection of one projection center onto another view.

At present we only know T_{ijk}–the epipoles are still unknown. To find the rescaling factors for projective reconstruction, we need to solve for the epipoles. One way to proceed is as follows. Taking $\mathbf{x}$ to be the projection center of the second view $\mathbf{o}'$, and projecting into the three views, Equation (6) reduces to

$$\lambda\mathbf{T}_{ijk}\mathbf{e}_2^i = -\lambda''\mathbf{e}_2''\mathbf{e}_1'^T.$$

As $\mathbf{e}_1'\mathbf{e}''^T$ has rank 1, so does $\mathbf{T}_{ijk}\mathbf{e}_2^i$. Its 2×2 determinant must vanish, *i.e.*

$$\det(\mathbf{T}_{ijk}\mathbf{e}_2^i) = 0.$$

As each entry of the 2×2 matrix is homogeneous linear in $\mathbf{e}_2 = (u, v)^T$, the expansion of $\det(\mathbf{T}_{\cdot jk}\mathbf{e}_2)$ gives a homogeneous quadratic

$$\alpha u^2 + \beta uv + \gamma v^2 = 0, \tag{7}$$

where α, β, γ are known in terms of T_{ijk}.

Doing the same thing with the projection center of the third view $\mathbf{o}''$ gives

$$\lambda \mathbf{T}_{ijk}\mathbf{e}_3^i = \lambda' \mathbf{e}_1'' \mathbf{e}_3'^T.$$

and hence

$$\det(\mathbf{T}_{\cdot jk}\mathbf{e}_3) = 0.$$

In other words, it leads to exactly the same quadratic equation (7) with $\mathbf{e}_3$ replacing $\mathbf{e}_2$. The two solutions of the quadratic (7) are $\mathbf{e}_2$ and $\mathbf{e}_3$—only the ordering remains ambiguous.

The other epipoles are easily obtained, $\mathbf{e}_1'$ and $\mathbf{e}_3'$ by factorizing the matrix $\mathbf{T}_{ijk}\mathbf{e}_1'^j$ and $\mathbf{e}_1''$ and $\mathbf{e}_2''$ by factorizing $\mathbf{T}_{ijk}\mathbf{e}_1''^i$.

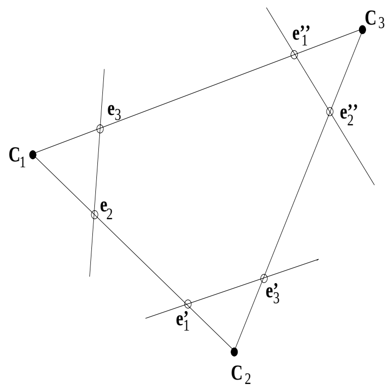

Figure 1. The geometry of three 1D images with the associated epipoles.

We therefore obtain the first possible solution set of epipoles for 3 views which are three ordered pairs:

$$\{\mathbf{e}_2, \mathbf{e}_3\}, \{\mathbf{e}_1', \mathbf{e}_3'\}, \{\mathbf{e}_1'', \mathbf{e}_2''\}.$$

The second solution is obtained by permuting each pair as

$$\{\mathbf{e}_3, \mathbf{e}_2\}, \{\mathbf{e}_3', \mathbf{e}_1'\}, \{\mathbf{e}_2'', \mathbf{e}_1''\}.$$

4. Quadratic transformation between two solutions

We are now ready to prove that the two solutions of the epipoles indeed turn out two distinct projective reconstructions which are exactly related by a quadratic transformation. To construct the quadratic transformation in an explicit manner, we need to choose a canonical projective basis on each 1D image. The two epipoles on the image line provide two natural reference points. One more reference point is still necessary to fix the projective basis. The choice of this third point is described in the following paragraph.

4.1. Invariant points of three 1D images

Let look for the set of invariant points for three 1D images. By its definition, if $\mathbf{f} \leftrightarrow \mathbf{f}' \leftrightarrow \mathbf{f}''$ are invariant point in three images, then

$$\lambda \mathbf{f} = \lambda' \mathbf{f}' = \lambda'' \mathbf{f}''.$$

The triplet of corresponding points $\mathbf{f} \leftrightarrow \mathbf{f}' \leftrightarrow \mathbf{f}''$ satisfies the trilinear constraint (5) as all corresponding points do, therefore,

$$T_{ijk} f^i f'^j f''^k = 0, \quad \text{i.e.} \quad T_{ijk} f^i f^j f^k = 0.$$

This yields the following cubic equation in the unknown $x = f^1/f^2$:

$$T_{111}x^3 + (T_{211}+T_{112}+T_{121})x^2 + (T_{212}+T_{221}+T_{122})x + T_{222} = 0.$$

As all the coefficients of the cubic equation are real, it has in general either three real roots or one real root and a pair of complex conjugate roots. This gurantees the existence of a real point in P^2 which has invariant image $\mathbf{f}$ in all three views.

4.2. Quadratic transformation between two distinct projective reconstructions

In this section, we prove that the two solutions for projective reconstruction are exactly related by a quadratic transformation in P^2. The idea of relating two distinct projective reconstruction by a quadratic transformation comes from discussions with Maybank.

A quadratic transformation from $\mathcal{P}^2$ to $\mathcal{P}^2$ is the simplest type of non-linear polynomial transformations between projective spaces, they are also called Cremona transformation [13]. Inversion with respect to a circle in the euclidean plane is an example of a special type of quadratic transformations.

These tools have been used by Maybank in [9] for studing the ambiguity of structure from motion [10, 4]. We will first briefly review some key properties of quadratic transformation necessary for our developement. One can refer to [13, 9] for more details on quadratic transformations.

Quadratic transformation The correspondence set up by a quadratic transformation $\Phi : \mathbf{x} \rightarrow \mathbf{y}$ between two projective planes ($\mathbf{x}$-plane and $\mathbf{y}$-plane) is one-one, except at three special points for which the defining quadratic polynomials $\phi_i, i = 1, 2, 3$ are vanishing, therefore the transformed point is not uniquely defined. These three points are called the fundamental points of Φ. Let the three fundamental points be $\mathbf{x}_i$ and $\mathbf{y}_i$ and assign the canonical projective coordinates as: $\mathbf{x}_1 = \mathbf{y}_1 = (1, 0, 0)^T, \mathbf{x}_2 = \mathbf{y}_2 = (0, 1, 0)^T, \mathbf{x}_3 = \mathbf{y}_3 = (0, 0, 1)^T$.

The canonical quadratic transformation Φ_0 is then given by

$$(y^1, y^2, y^3)^T = (x^2x^3, x^1x^3, x^1x^2)^T.$$

The inverse $\Phi^{-1} = \Phi$ is therefore also a canonical quadratic transformation. The three lines through the fundamental points are called the fundamental lines and transform into the three fundamental points of Φ^{-1}. A line through a fundamental point yields another line through the corresponding fundamental point. A line which does not pass through any fundamental point transforms into a conic through the fundamental points. A conic through these three points transforms into a line.

Any general quadratic transformation Φ of the projective plane can be reduced to the standard quadratic transformation by suitable collineations A in **x**-plane such as $A \circ \Phi_0$.

Choose a canonical basis for the first solution As we are working with uncalibrated images in a projective framework, without loss of generality, we may choose the two epipoles and the fixed point in each image. For the first set of epipoles $\{\mathbf{e}_2, \mathbf{e}_3\}, \{\mathbf{e}'_1, \mathbf{e}'_3\}, \{\mathbf{e}''_1, \mathbf{e}''_2\}$, we assign

$$\mathbf{e}_2 = (1,0)^T, \mathbf{e}_3 = (0,1)^T \text{ and } \mathbf{f} = (1,1)^T$$

for the first image line. For the second and the third image lines, we take $\mathbf{e}'_1, \mathbf{e}'_3, \mathbf{f}$ and $\mathbf{e}''_1, \mathbf{e}''_2, \mathbf{f}$ in order.

The canonical coordinates of 1D image points are becoming $\theta_i = \lambda_i/\mu_i = \{\mathbf{u}_i, \mathbf{f}; \mathbf{e}_2, \mathbf{e}_3\}$ as

$$\mathbf{u}_i = \lambda_i \mathbf{e}_2 + \mu_i \mathbf{e}_3.$$

This transformation to the canonical basis can be described by **A** such that $(\theta, 1)^T = A_i^j u^i$.

Similarly $\mathbf{u}'$ and $\mathbf{u}''$ have new canonical coordinates θ' and θ'' by **B** and **C**.

The trifocal tensor expressed in these canonical bases is the canonical trifocal tensor T^0 which is

$$T^0_{ijk} = T_{abc} A_i^a B_j^b C_k^c.$$

In 2D space, we can equally choose a canonical basis defined by the three camera centers $\mathbf{o}$, $\mathbf{o}'$, $\mathbf{o}''$ and the point $\mathbf{y}$ which has the invariant images $\mathbf{f}$: $\mathbf{o} = (1,0,0)^T, \mathbf{o}' = (0,1,0)^T, \mathbf{o}'' = (0,0,1)^T$, and $\mathbf{y} = (1,1,1)^T$. Obviously the plane collineation that brings to this canonical basis does not change the trifocal tensor T^0_{ijk}.

Consider the second set of epipoles The second set of epipoles $\{\mathbf{e}_3, \mathbf{e}_2\}, \{\mathbf{e}'_3, \mathbf{e}'_1\}, \{\mathbf{e}''_2, \mathbf{e}''_1\}$ is obtained by interchanging the pair of the epipoles of the first solution. Such a permutation of the epipoles gives a permutation of reference points on each image line, the canonical projective parameter—canonical coordinate of the image point θ—inverses as

$$\theta' = \{\mathbf{x}, \mathbf{f}; \mathbf{e}_1, \mathbf{e}_2\} = 1/\{\mathbf{x}, \mathbf{f}; \mathbf{e}_2, \mathbf{e}_1\} = 1/\theta.$$

This is equivalent to applying a homography $\bar{I} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ in each image line. Such a homography transforms the canonical trifocal tensor T^0_{ijk} into $T^0_{\bar{i}\bar{j}\bar{k}}$ where $^-$ is the operator permuting the index 1, 2 into 2 and 1, as $T^0_{\bar{i}\bar{j}\bar{k}} = T^0_{abc} \bar{I}_i^a \bar{I}_j^b \bar{I}_k^c$.

We see indeed that the second set of epipoles would have given a different tensor if no other things happened on the plane.

Now in 2D space if applying a quadratic transformation with the camera centers $\mathbf{o}, \mathbf{o}'$ and $\mathbf{o}''$ as the fundamental points. This plane quadratic transformation has consequence that any pencil of lines parameterized by $(\lambda, \mu)^T$ through a fundamental point tranforms homographically into a pencil of $(\mu, \lambda)^T$. As 1D image points could be equally viewed as a pencil of rays through the camera center. So the plane quadratic transformation induces homographic transformation on three image lines (which are also the three fundamental lines of the quadratic transformation) of form $\bar{I} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Again, this homographic transformation permutes the indices of the tensor which undoes the first permutation of indices by interchanging the epipoles. Two steps combined, permutation of the epipoles and the quadratic transformation, we come up with the initial canonical trifocal tensor T^0 since $T^0_{\bar{\bar{i}}\bar{\bar{j}}\bar{\bar{k}}} = T^0_{abc}$.

In summary, a permutation of epipoles provided by selecting the second solution for epipoles inverses the 1D image coordinates from θ to $1/\theta$. A quadratic transformation again inverses the 1D image coordinates from $1/\theta$ back to $1/1/\theta = \theta$, therefore undoes the inversion. But to acheive this, a non-linear transformation is necessary on the plane. This proves:

Theorem 1 *There always exist two distinct 2D projective reconstructions from three 1D images. This is independent of the number of point correspondences. And the two solutions are exactly related by a quadratic transformation in* P^2 *with the three camera centers as the fundamental points.*

When the three camera centers are aligned, the reconstruction is unique since there is only a unique solution for the epipoles, two solutions are identical. Geometrically, for instance, in the first image, the epipoles $\mathbf{e}_2$ and $\mathbf{e}_3$ are confused. This could also be understood that the quadratic transformation is not defined for collinear fundamental points.

5. Numerical simulation

The theoretical results developed in this paper are experimented on numerical simulations. We take a regular grid as illustrated in Figure 2, then simulate a 1D camera and take three images of this regualr grid in three different positions. The two 2D projective reconstructions are illustrated in Figure 3 and 4. We can see that the first solution in Figure 3 is indeed a projective transformation of the original regular grid and the second in Figure 4 is a quadratic transformation of the grid—all collinear points are transformed into points lying on conic sections.

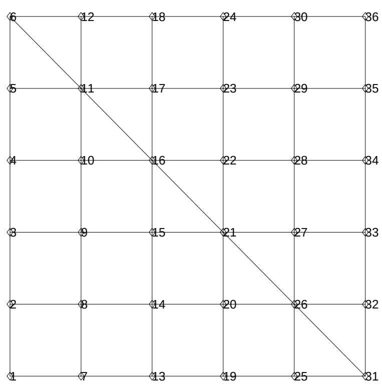

Figure 2. The original planar regular grid.

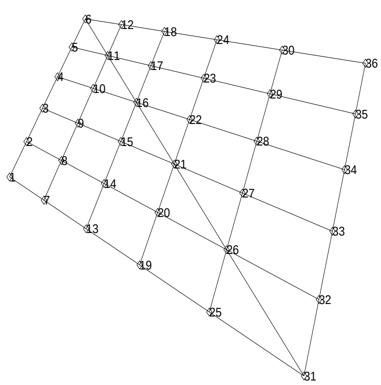

Figure 3. b. The first 2D projective reconstruction of the grid.

Another experiment is conducted by placing the three camera centers along a line. The resulting reconstruction confirms the unicity of the solution illustrated in Figure 5.

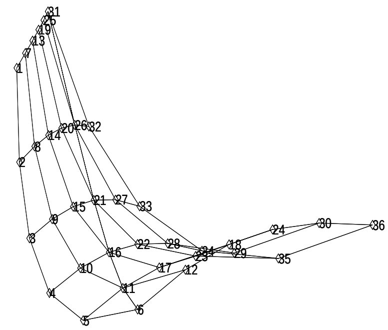

Figure 4. The second 2D projective reconstruction which is deformed by a planar quadratic transformation.

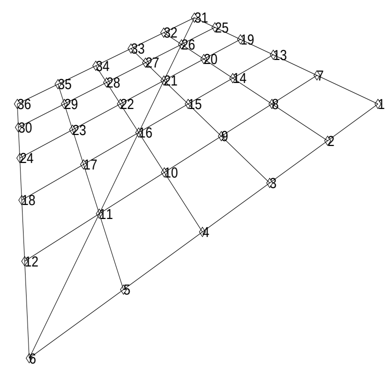

Figure 5. The unique 2D projective reconstruction of the grid when the three camera centers are aligned.

All these numerical experiments confirmed the theorems proved in this paper.

6. Discussion

We have shown that there exists always a two way ambiguity for 2D projective reconstruction from three uncalibrated 1D images no matter how many point correspondences (at least seven) are available. More exactly, we establish that two distinct projective reconstructions are related by a quadratic transformation with the three camera centers as the fundamental points of the quadratic transformation. All these theoretical results are also validated on numerical simulations. This gives a new insight into the intrinsic structure of the projective reconstruction and may provide interesting hints for the study of the multiple solutions of 3D projective reconstruction from 2D cameras.

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