# 5-choosability of graphs with 2 crossings

Victor Campos, Frédéric Havet

INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET EN AUTOMATIQUE

# *5-choosability of graphs with 2 crossings*

Victor Campos — Frédéric Havet

**N° 7618**

Mai 2011

Thème COM

Rapport de recherche

ISSN 0249-6399 ISRN INRIA/RR--7618--FR+ENG

INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET EN AUTOMATIQUE
INRIA
centre de recherche SOPHIA ANTIPOLIS - MÉDITERRANÉE

# 5-choosability of graphs with 2 crossings *

Victor Campos† , Frédéric Havet ‡

Thème COM — Systèmes communicants
Équipe-Projet Mascotte

Rapport de recherche n° 7618 — Mai 2011 — 19 pages

**Abstract:** We show that every graph with two crossings is 5-choosable. We also prove that every graph which can be made planar by removing one edge is 5-choosable.

**Key-words:** list colouring, choosability, crossing number

† Universidade Federal do Ceará, Departamento de Computaçao, Bloco 910, Campus do Pici, Fortaleza, Ceará, CEP 60455-760, Brasil. campos@lia.ufc.br; Partially supported by CNPq/Brazil.

‡ Projet Mascotte, I3S(CNRS, UNSA) and INRIA, 2004 route des lucioles, BP 93, 06902 Sophia-Antipolis Cedex, France. Frederic.Havet@inria.fr; Partially supported by the ANR Blanc International ANR-09-blan-0373-01.

* This work was partially supported by Equipe Associée EWIN.

# 5-choisissabilité des graphes ayant deux croisements

**Résumé :** Nous montrons que tout graphe ayant deux croisements est 5-choisissable. Nous prouvons également que tout graphe qui peut être rendu planaire par la suppression d'une arête est 5-choisissable.

**Mots-clés :** coloration sur listes, choisissabilité, nombre de croisements

# 1 Introduction

The crossing number of a graph $G$, denoted by $\mathrm{cr}(G)$, is the minimum number of crossings in any drawing of $G$ in the plane.

The Four Colour Theorem states that, if a graph has crossing number zero (i.e. is planar), then it is 4-colourable. Deleting one vertex per crossing, it follows that $\chi(G) \leq 4 + cr(G)$. So it is natural to ask for the smallest integer $f(k)$ such that every graph $G$ with crossing number at most $k$ is $f(k)$-colourable? Settling a conjecture of Albertson [1], Schaefer [8] showed that $f(k) = O\left(k^{1/4}\right)$. This upper bound is tight up to a constant factor since $\chi(K_n) = n$ and $\mathrm{cr}(K_n) \leq \binom{|E(K_n)|}{2} = \binom{\binom{n}{2}}{2} \leq \frac{1}{8}n^4$.

The values of $f(k)$ are known for a number of small values of $k$. The Four Colour Theorem states $f(0) = 4$ and implies easily that $f(1) \leq 5$. Since $\mathrm{cr}(K_5) = 1$, we have $f(1) = 5$. Oporowski and Zhao [7] showed that $f(2) = 5$. Since $\mathrm{cr}(K_6) = 3$, we have $f(3) = 6$. Further, Albertson et al. [2] showed that $f(6) = 6$. Albertson then conjectured that if $\chi(G) = r$, then $\mathrm{cr}(G) \leq \mathrm{cr}(K_r)$. This conjecture was proved by Barát and Tóth [3] for $r \leq 16$.

A *list assignment* of a graph $G$ is a function $L$ that assigns to each vertex $v \in V(G)$ a list $L(v)$ of available colours. An *L-colouring* is a function $\varphi : V(G) \to \bigcup_v L(v)$ such that $\varphi(v) \in L(v)$ for every $v \in V(G)$ and $\varphi(u) \neq \varphi(v)$ whenever $u$ and $v$ are adjacent vertices of $G$. If $G$ admits an $L$-colouring, then it is *L-colourable*. A graph $G$ is *k-choosable* if it is $L$-colourable for every list assignment $L$ such that $|L(v)| \geq k$ for all $v \in V(G)$. The *choose number* of $G$, denoted by $\mathrm{ch}(G)$, is the minimum $k$ such that $G$ is $k$-choosable.

Similarly to the chromatic number, one may seek for bounds on the choose number of a graph with few crossings or with independent crossings. Thomassen's Five Colour Theorem [10] states that if a graph has crossing number zero (i.e. is planar) then it is 5-choosable. A natural question is to ask whether the chromatic number is bounded in terms of its crossing number. Erman et al. [5] observed that Thomassen's result can be extended to graphs with crossing number at most 1. Deleting one vertex per crossing yields $\mathrm{ch}(G) \leq 4 + cr(G)$. Hence, what is the smallest integer $g(k)$ such that every graph $G$ with crossing number at most $k$ is $g(k)$-choosable? Obviously, since $\chi(G) \leq \mathrm{ch}(G)$, we have $f(k) \leq g(k)$.

In this paper, we extend Erman et al. result in two ways. We first show that every graph which can be made planar by the removal of an edge is 5-choosable (Theorem 3). We then prove that is $g(2) = 5$. In other words, every graph with crossing number 2 is 5-choosable[1]. This generalizes the result of Oporowski and Zhao [7] to list colouring.

# 2 Planar graphs plus an edge

In order to prove its Five Colour Theorem, Thomassen [10] showed a stronger result.

**Definition 1.** An *inner triangulation* is a plane graph such that every face of $G$ is bounded by a triangle except its outer face which is bounded by a cycle.

Let $G$ be a plane graph and $x$ and $y$ two consecutive vertices on its outer face $F$. A list assignment $L$ of $G$ is $\{x,y\}$*-suitable* if

- $|L(x)| \geq 1$, $|L(y)| \geq 2$,

[1] While writing this paper, we discovered that Dvořák et al. [4] independently proved this result. Their proof has some similarity to ours but is different. They prove by induction a stronger result, while we use the existence of a shortest path between the two crossings which satisfies some given properties.

- for every $v \in V(F) \setminus \{x,y\}$, $|L(v)| \geq 3$, and
- for every $v \in V(G) \setminus V(F)$, $|L(v)| \geq 5$.

A list assignment of $G$ is *suitable* if it is $\{x,y\}$-suitable for some vertices $x$ and $y$ on the outer face of $G$.

The following theorem is a straightforward generalization of Thomassen's five colour Theorem which holds for non-separable plane graphs.

**Theorem 2** (Thomassen [10]). *If $L$ is a suitable list assignment of a plane graph $G$ then $G$ is $L$-colourable.*

This result is the cornerstone of the following proof.

**Theorem 3.** *Let $G$ be a graph. If $G$ has an edge such that $G \setminus e$ is planar then* $\mathrm{ch}(G) \leq 5$.

*Proof.* Let $e = uv$ be an edge of $G$ such that $G \setminus e$ is planar. Let $G'$ be a planar triangulation containing $G \setminus e$ as a subgraph. Without loss of generality, we may assume that $u$ is on the outer triangle of $G'$. The graph $G' - u$ has an outer cycle $C'$ whose vertices are the neighbours of $u$ in $G'$.

Let $L$ be a 5-list assignment of $G$. Let $\alpha, \beta \in L(u)$. Let $L'$ be the list-assignment of $G' - u$ defined by $L'(w) = L(w) \setminus \{\alpha, \beta\}$ if $w \in V(C')$ and $L'(w) = L(w)$ otherwise. Then $L'$ is suitable. So $G' - u$ admits an $L'$-colouring by Theorem 2. This colouring may be extended into an $L$-colouring of $G$ by assigning to $u$ a colour in $\{\alpha, \beta\}$ different from the colour of $v$.

Hence $G$ is 5-choosable. □

# 3 Graphs with two crossings

## 3.1 Preliminaries

We first recall the celebrated characterization of planar graphs due to Kuratowski [6]. See also [9] for a nice proof.

**Theorem 4** (Kuratowski [6]). *A graph is planar if and only if it contains no minor isomorphic to either $K_5$ or $K_{3,3}$.*

Let $G$ be a plane graph and $x$, $y$ and $z$ three distinct vertices on the outer face $F$ of $G$. A list assignment $L$ of $G$ is $(x,y,z)$*-correct* if

- $|L(x)| = 1 = |L(y)|$ and $L(x) \neq L(y)$,
- $|L(z)| \geq 3$,
- for every $v \in V(F) \setminus \{x,y,z\}$, $|L(v)| \geq 4$, and
- for every $v \in V(G) \setminus V(F)$, $|L(v)| \geq 5$.

If $L$ is $(x,y,z)$-correct and $L(z) \geq 4$, we say that $L$ is $\{x,y\}$*-correct*.

**Lemma 5.** *Let $G$ be an inner triangulation and $x$ and $y$ two distinct vertices on the outer face of $G$. If $L$ is an $(x,y,z)$-correct list assignment of $G$ then $G$ is $L$-colourable.*

*Proof.* We prove the result by induction on the number of vertices, the result holding trivially when $|V(G)| = 3$.

Suppose first that $F$ has a chord $xt$. Then $xt$ lies in two unique cycles in $F \cup xt$, one $C_1$ containing $y$ and the other $C_2$. For $i = 1,2$, let $G_i$ denote the subgraph induced by the vertices lying on $C_i$ or inside it. By the induction hypothesis, there exists an $L$-colouring $\phi_1$ of $G_1$. Let $L_2$ be the list assignment on $G_2$ defined by $L_2(t) = \{\phi_1(t)\}$ and $L_2(u) = L(u)$ if $u \in V(G_2) \setminus \{t\}$. Let $z' = z$ if $z \in V(C_2)$ and $z'$ be any vertex of $V(C_2) \setminus \{x,t\}$ otherwise. Then $L_2$ is $(x,t,z')$-correct for $G_2$ so $G_2$ admits an $L_2$-colouring $\phi_2$ by induction hypothesis. The union of $\phi_1$ and $\phi_2$ is an $L$-colouring of $G$.

Suppose now that $x$ has exactly two neighbours $u$ and $v$ on $F$. Let $u,u_1,u_2\dots,u_m,v$ be the neighbours of $x$ in their natural cyclic order around $x$. As $G$ is an inner triangulation, $uu_1u_2\cdots u_m,v = P$ is a path. Hence the graph $G-x$ has $F' = P \cup (F-x)$ as outer face.

Assume first that $z \notin \{u,v\}$. Then let $L'$ be the list assignment on $G-x$ defined by $L'(w) = L(w) \setminus L(x)$ if $w \in N_G(x)$ and $L'(w) = L(w)$ otherwise. Clearly, $|L'(w)| \geq 3$ if $w \in F'$ and $|L'(w)| \geq 5$ otherwise. Hence, by Theorem 2, $G-x$ admits an $L'$-colouring. Colouring $x$ with the colour of its list, we obtain an $L$-colouring of $G$.

Assume now that $z \in \{u,v\}$, say $z = u$. Let $\alpha$ be a colour of $L(z) \setminus (L(x) \cup L(y))$. Let $L'$ be the list assignment on $G-x$ defined by $L'(z) = \{\alpha\}$, $L'(w) = L(w) \setminus L(x)$ if $w \in N_G(x) \setminus \{z\}$ and $L'(w) = L(w)$ otherwise. Clearly, $L'$ is $(y,z,v)$-correct. Hence, by the induction hypothesis, $G-x$ admits an $L'$-colouring. Colouring $x$ with the colour of its list, we obtain an $L$-colouring of $G$. □

### 3.2 Nice, great and good paths

Let $G$ be a graph and $H$ an induced subgraph of $G$.

We denote by $Z_H$ the set of vertices of $G$ which are adjacent to at least 3 vertices of $H$. For every vertex $v$ in $V(G)$, we denote by $N_H(v)$ the set of vertices of $H$ adjacent to $v$, and we set $d_H(v) = |N_H(v)|$.

Let $L$ be a list assignment of $G$. For any $L$-colouring $\phi$ of $H$, we denote by $L_\phi$ the list assignment of $G-H$ defined by $L_\phi(z) = L(z) \setminus \phi(N_H(z))$. A vertex $z \in V(G-H)$ is *safe* (with respect to $\phi$), if $|L_\phi(z)| \geq 3$. An $L$-colouring of $H$ is *safe* if all vertices of $z \in V(G-H)$ are safe. Observe that if $L$ is a 5-list assignment, then for any $L$-colouring $\phi$ of $H$, every vertex $z$ not in $Z_H$ has at most two neighbours in $H$ and therefore $|L_\phi(z)| \geq 3$. Hence $\phi$ is safe if and only if every vertex in $Z_H$ is safe.

Let $P = v_1 \cdots v_p$ be an induced path in $G$. For $2 \leq i \leq p-1$, we denote by $[v_i]_P$, or simply $[v_i]$ if $P$ is clear from the context, the set $\{v_{i-1}, v_i, v_{i+1}\}$. We say that a vertex $z$ is adjacent to $[v_i]$ if it is adjacent to all vertices in the set $[v_i]$. Note that if $z$ is adjacent to $[v_i]$ then $z$ is not in $P$ as $P$ is induced.

**Lemma 6.** *Let $P = v_1 \cdots v_p$ be an induced path in G, x a vertex such that $N_P(x) = [v_{i+1}]$, $1 \leq i \leq p-1$, and $\phi$ a colouring of $P - v_i$. If $i = 1$ or $\phi(v_{i-1}) = \phi(v_{i+1})$, then one can extend $\phi$ to $v_i$ such that x is safe.*

*Proof.* If $\{\phi(v_{i+1}), \phi(v_{i+2})\} \not\subset L(x)$, then assigning to $v_i$ any colour distinct from $\phi(v_{i+1})$, we get a colouring of $P$ such that $x$ is safe. So we may assume that $\{\phi(v_{i+1}), \phi(v_{i+2})\} \subset L(x)$.

If $\phi(v_{i+2}) \in L(v_i)$, then setting $\phi(v_i) = \phi(v_{i+2})$, we have a colouring $\phi$ such that $x$ is safe. If not, there is a colour $\alpha$ in $L(v_i) \setminus L(x)$. Necessarily, $\alpha \neq \phi(v_{i+1})$ and so one can colour $v_i$ with $\alpha$. Doing so, we obtain a colouring such that $x$ is safe. □

Let $P = v_1 \cdots v_p$ be an induced path. It is a *nice path* in $G$ if the following are true.

(a) for every $z \in Z_P$, $N_P(z) = [v_i]$ for some $2 \leq i \leq p-1$;

(b) for every $2 \leq i \leq p-1$, there are at most two vertices adjacent to $[v_i]$ and, if there are two such vertices, then the number of vertices adjacent to $[v_{i-1}]$ or $[v_{i+1}]$ is at most 1.

It is a *great path* in $G$ if is is nice and satisfies the following extra property.

(c) for any $i < j$, if there are two vertices adjacent to $[v_i]$ and two vertices adjacent to $[v_j]$, then the number of vertices adjacent to $[v_{i+1}]$ or $[v_{j-1}]$ is at most 1.

A safe colouring of a path $P = v_1 \cdots v_p$ is $\alpha$-*safe* if $\phi(v_1) = \alpha$.

**Lemma 7.** *If $P$ is a great path and $L$ is a 5-list assignment of $G$, then for any $\alpha \in L(v_1)$, there exists an $\alpha$-safe $L$-colouring $\phi$ of $P$.*

*Proof.* We prove this result by induction on $p$, the number of vertices of $P$, the result holding trivially when $p \leq 2$.

Assume now that $p \geq 3$. Since $P$ is great then every vertex of $Z_P$ adjacent to $v_1$ is also adjacent to $v_2$ and there are at most two vertices of $Z_P$ adjacent to $[v_2]$.

Set $\phi(v_1) = \alpha$.

1. If there is no vertex adjacent to $[v_2]$, then by induction, for any $\beta \in L(v_2) \setminus \{\alpha\}$, there is a $\beta$-safe $L$-colouring $\phi$ of $v_2 \cdots v_p$. Since $\phi(v_1) = \alpha$, $\phi$ is an $\alpha$-safe $L$-colouring of $P$.

2. Assume now that there is a unique vertex $z$ adjacent to $[v_2]$.

   If $\alpha \notin L(z)$, then by Case 1, there is an $\alpha$-safe $L$-colouring $\phi$ of $P$ in $G - z$. It is also an $\alpha$-safe $L$-colouring of $P$ in $G$ since $z$ is safe as $\alpha \notin L(z)$. Hence we may assume that $\alpha \in L(z)$.

   Assume there is a colour $\beta$ in $L(v_2) \setminus \{\alpha\}$. By induction there is a $\beta$-safe $L$-colouring $\phi$ of $v_2 \cdots v_p$. Since $\phi(v_1) = \alpha$, we obtain an $\alpha$-safe $L$-colouring of $P$ because $z$ is safe as $\beta \notin L(z)$. Hence we may assume that $L(v_2) = L(z)$. In particular, $\alpha \in L(v_2)$. Let $\gamma$ be $\alpha$ if $\alpha \in L(v_3)$, and a colour in $L(v_3) \setminus L(v_2)$ otherwise. We set $\phi(v_3) = \gamma$. Observe that whatever colour is assigned to $v_2$, the vertex $z$ will be safe.

   2.1. Assume that no vertex is adjacent to $[v_3]$. By induction hypothesis, there is a $\gamma$-safe $L$-colouring $\phi$ of $v_3 \cdots v_p$. Choosing $\phi(v_2)$ in $L(v_2) \setminus \{\alpha, \gamma\}$, we obtain an $\alpha$-safe $L$-colouring of $P$.

   2.2. Assume that exactly one vertex $t$ is adjacent to $[v_3]$. By induction hypothesis, there is a $\gamma$-safe $L$-colouring $\phi$ of $v_3 \cdots v_p$. So far all the vertices except $t$ will be safe. So we just need to choose $\phi(v_2)$ so that $t$ is safe.

   Observe that if $\{\gamma, \phi(v_4)\} \not\subset L(t)$, choosing any colour of $L(v_2) \setminus \{\alpha, \gamma\}$ will do the job. So we may assume that $\{\gamma, \phi(v_4)\} \subset L(t)$. If there is a colour $\beta \in L(v_2) \setminus (L(t) \cup \{\alpha\})$, then setting $L(v_2) = \beta$ will make $t$ safe. So we may assume that $L(v_2) \setminus \{\alpha\} \subset L(t)$ and so $L(t) = L(v_2) \cup \{\gamma\} \setminus \{\alpha\}$. Thus $\phi(v_4) \in L(v_2) \setminus \{\alpha, \gamma\}$. Then setting $\phi(v_2) = \phi(v_4)$ makes $t$ safe.

   2.3. Assume that two vertices $t_1$ and $t_2$ are adjacent to $[v_3]$. Then no vertex is adjacent to $[v_4]$. Therefore, it suffices to prove that there is an $\alpha$-safe $L$-colouring of $v_1v_2v_3v_4$. Indeed, if we have such a colouring $\phi$, then by induction, $v_4 \cdots v_p$ admits a $\phi(v_4)$-safe $L$-colouring $\phi'$. The union of these two colourings is an $\alpha$-safe $L$-colouring of $P$.

   If there exists $\beta \in L(v_4) \cap L(v_2) \setminus \{\alpha, \gamma\}$, then setting $\phi(v_2) = \phi(v_4) = \beta$, we obtain an $\alpha$-safe $L$-colouring of $v_1v_2v_3v_4$. Otherwise, $L(v_4) \setminus \{\gamma\}$ and $L(v_2) \setminus \{\alpha\}$ are disjoint. Hence one can choose $\beta$ in $L(v_2) \setminus \{\alpha\}$ and $\delta$ in $L(v_4) \setminus \{\gamma\}$ so that $|\{\beta, \gamma, \delta\} \cap L(t_i)| \leq 2$ for $i = 1, 2$. Setting $\phi(v_2) = \beta$ and $\phi(v_4) = \delta$, we obtain an $\alpha$-safe $L$-colouring of $v_1v_2v_3v_4$.

3. Assume that two vertices $z_1$ and $z_2$ are adjacent to $[v_2]$.

   We claim that it suffices to prove that there is an $\alpha$-safe $L$-colouring of $v_1v_2v_3$.

   Let $j$ be the smallest index such that no vertex is adjacent to $[v_j]$. For the definition of $j$, consider there is no vertex adjacent to $[v_p]$ so that $j \leq p$. By the property (c) of great path, for all $3 \leq i < j$, there is exactly one vertex $z_i$ adjacent to $[v_i]$. For $i = 3, \ldots, j-1$, one after another, one can use Lemma 6 in the path $v_{i+1} \cdots v_1$ to extend $\phi$ to $v_{i+1}$, so that $z_i$ is safe. Then applying induction on the path $v_j \cdots v_p$, we obtain an $\alpha$-safe $L$-colouring. This proves the claim.

   Let us now prove that an $\alpha$-safe $L$-colouring of $v_1v_2v_3$ exists.

   If $\alpha \notin L(z_i)$, then any $\alpha$-safe $L$-colouring of $v_1v_2v_3$ in $G - z_i$ will be an $\alpha$-safe $L$-colouring in $G$. By Case 2, one can find such a colouring in $G - z_i$, so we may assume that $\alpha \in L(z_i)$.

   If there is a colour $\beta \in L(v_2) \setminus L(z_1)$, then set $\phi(v_2) = \beta$. By Lemma 6 in the path $v_3v_2v_1$, one can choose $\phi(v_3)$ in $L(v_3)$ to obtain an $\alpha$-safe $L$-colouring of $v_1v_2v_3$. Hence we may assume that $L(z_1) = L(v_2)$. Similarly, we may assume that $L(z_2) = L(v_2)$. Therefore, any $\alpha$-safe $L$-colouring of $v_1v_2v_3$ in $G - z_2$ will be an $\alpha$-safe $L$-colouring in $G$. We can find such a colouring using Case 2.

□

We say that an induced path $P = v_1 \cdots v_p$ is *good* path if either $P$ is great or $p \geq 4$ and there is a vertex $z \in Z_P$ adjacent to $v_1$ such that $\{v_1, v_4\} \subset N_P(z) \subseteq \{v_1, v_2, v_3, v_4\}$ satisfying the following conditions:

- $P$ is a great path in $G \setminus v_1z$.
- if two vertices distinct from $z$ are adjacent to $[v_2]$, then $N_P(z) = \{v_1, v_3, v_4\}$ and there is no vertex adjacent to $[v_3]$; and
- if two vertices distinct from $z$ are adjacent to $[v_3]$, then $N_P(z) = \{v_1, v_2, v_4\}$ and there is no vertex adjacent to $[v_2]$.

Note that since $P$ is induced, then $z$ is not in $P$.

**Lemma 8.** *If $P = v_1 \cdots v_p$ is a good path and $L$ is a 5-list assignment of $G$, then there exists a safe $L$-colouring of $P$.*

*Proof.* If $P$ is great, then the result follows from Lemma 7. So we may assume that $P$ is not great. Let $z$ be the vertex of $Z_P$ such that $\{v_1, v_4\} \subset N_P(z) \subseteq \{v_1, v_2, v_3, v_4\}$.

If there is a colour $\alpha \in L(v_1) \setminus L(z)$, then let $\phi(v_1) = \alpha$ and use Lemma 7 to colour $v_1 \cdots v_p$ in $G \setminus v_1z$. The obtained colouring $\phi$ is a safe $L$-colouring of $P$. For any $z' \in Z_P \setminus \{z\}$, we have $|L_\phi(z')| \geq 3$ because $z'$ has the same neighbourhood in $G$ and $G \setminus v_1z$. Now $|L_\phi(z)| \geq 3$ since $\alpha \notin L(z)$, so $\phi$ is safe. Henceforth, we assume that $L(v_1) = L(z)$.

1. Assume first that $N_P(z) = \{v_1, v_2, v_3, v_4\}$.

   By the properties of a good path, at most one vertex $z'$ different from $z$ is adjacent to $[v_2]$.

   1.1. Assume first that $z$ is the unique vertex adjacent to $[v_3]$.
   If there is a colour $\alpha \in L(z) \cap L(v_3)$, then set $\phi(v_1) = \phi(v_3) = \alpha$. By Lemma 7, one can extend $\phi$ to $v_3 \cdots v_p$ so that all vertices of $Z_P$ but $z$ are safe. Then by Lemma 6 applied to

$v_2 \cdots v_p$, one can choose $\phi(v_2) \in L(v_2)$ so that $z$ is safe for $P - v_1$. Since $\phi(v_1) = \phi(v_3)$, then $\phi$ is a proper colouring and $z$ is safe for $P$. Hence $\phi$ is a safe $L$-colouring of $P$. So we may assume that $L(z) \cap L(v_3) = \emptyset$.

If there exists $\beta \in L(v_2) \setminus L(z)$, then set $\phi(v_2) = \beta$. By Lemma 7, one can extend $\phi$ to $v_2 \cdots v_p$ so that all vertices of $Z_P$ but $z$ and $z'$ are safe. Observe that necessarily $z$ will be safe because $\phi(v_2) \notin L(z)$ and $\phi(v_3) \notin L(z)$. By Lemma 6, one can extend $\phi$ to $v_1$ so that $z'$ is safe, thus getting a safe $L$-colouring of $P$. So we may assume that $L(v_2) = L(z)$.

We have $|L(v_2) \cup L(v_3)| = 10 \geq |L(z')|$. So we can find $\alpha \in L(v_2)$ and $\beta \in L(v_3)$ so that $|\{\alpha, \beta\} \cap L(z')| \leq 1$. Using Lemma 7 take a $\beta$-safe $L$-colouring $\phi$ of the path $v_3 v_4 \ldots v_p$ and set $\phi(v_2) = \alpha$. If $\phi(v_4) \in L(z) \setminus \{\alpha\}$, then colour $v_1$ with $\phi(v_4)$, otherwise colour it with any colour distinct from $\alpha$. This gives a safe $L$-colouring of $P$.

1.2 Assume now that a vertex $y \neq z$ is adjacent to $[v_3]$.

* Suppose that a vertex $t$ is adjacent to $[v_4]$. Then $z'$ does not exist.

  If there is a colour $\alpha \in L(v_2) \setminus L(z)$, then using Lemma 7 take an $\alpha$-safe $L$-colouring $\phi$ of $v_2 \cdots v_p$. If $\phi(v_3) \notin L(z)$, then $z$ would be safe whatever colour we assign to $v_1$, so there is a safe $L$-colouring of $P$. If If $\phi(v_3) \in L(z)$, then setting $\phi(v_1) = \phi(v_3)$, we obtain a safe $L$-colouring of $P$. So we may assume that $L(v_2) = L(z)$.

  If there is a colour $\alpha$ in $L(z) \cap L(v_4)$, then set $\phi(v_2) = \phi(v_4) = \alpha$. Then $y$ will be safe. Extend $\phi$ to $v_4 \cdots v_p$ by Lemma 7. Then all the vertices are safe except $t$ and $z$. By Lemma 6, one can choose $\phi(v_3)$ so that $t$ is safe. If $\phi(v_3) \in L(z)$, then setting $\phi(v_1) = \phi(v_3)$, we get a safe $L$-colouring of $P$. If $\phi(v_3) \notin L(z)$, then whatever colour we assign to $v_1$, we obtain a safe colouring of $P$. Hence we may assume that $L(z) \cap L(v_4) = \emptyset$. By Lemma 7, there is a safe $L$-colouring of $P$ in $G \setminus zv_4$. This colouring is also a safe colouring of $P$ in $G$, since $\phi(v_4)$ is not in $L(z)$.

* If no vertex is adjacent to $[v_4]$, then $z'$ may exist. In this case, it is sufficient to prove that there exists a safe $L$-colouring of $v_1 v_2 v_3 v_4$. Indeed, if there is such a colouring $\phi$, then by Lemma 7, it can be extended to a safe $L$-colouring of $P$.

  Symmetrically to the way we proved the result when $L(v_1) \neq L(z)$, one can prove it when $L(v_4) \neq L(z)$. Hence we may assume that $L(v_4) = L(z)$.

  Assume that there is a colour $\alpha \in L(v_2) \cap L(z)$. Set $\phi(v_2) = \phi(v_4) = \alpha$. If there is a colour $\beta \in L(v_3) \setminus L(z)$, then set $\phi(v_3) = \beta$ so that $z$ will be safe and extend $\phi$ with Lemma 6 so that $z'$ is safe to obtain a safe colouring of $v_1 v_2 v_3 v_4$ in $G$. If $L(v_3) = L(z)$, then assign to $v_1$ and $v_3$ a same colour in $L(z) \setminus \{\alpha\}$ to get a safe colouring of $v_1 v_2 v_3 v_4$. Hence we may assume that $L(v_2) \cap L(z) = \emptyset$. Symmetrically, we may assume that $L(v_3) \cap L(z) = \emptyset$. By Lemma 7, there exists a safe colouring $\phi$ of $v_1 v_2 v_3 v_4$ in $G - z$. It is also a safe colouring of $v_1 v_2 v_3 v_4$ in $G$ because $\phi(v_2)$ and $\phi(v_3)$ cannot be in $L(z)$.

2. Assume now that $N_P(z) = \{v_1, v_3, v_4\}$.

   If no vertex is adjacent to $[v_2]$, then using Lemma 7 take a safe $L$-colouring of $v_2 \ldots v_p$. If $\phi(v_3) \in L(z)$, then set $\phi(v_1) = \phi(v_3)$. If not colour $v_3$ with any colour in $L(z) \setminus \{\phi(v_2)\}$. This gives a safe $L$-colouring of $P$. Hence we may assume that a vertex $t$ is adjacent to $[v_2]$.

   By the properties of a good path, we know that at most one vertex, say $u$, is adjacent to $v_3$. If $L(v_3) \cap L(z)$ is empty, then any safe $L$-colouring of $P$ given by Lemma 7 in $G \setminus zv_1$ would be a safe $L$-colouring of $P$. Hence we may assume that there is a colour $\alpha$ in $L(v_3) \cap L(z)$. Set $\phi(v_1) = \phi(v_3) = \alpha$ and apply Lemma 7 to $v_3 \ldots v_p$. Then by Lemma 6, we can choose $\phi(v_2)$ so that the possible vertex $u$ is safe. This gives a safe colouring of $P$.

3. Assume that $N_P(z) = \{v_1, v_2, v_4\}$.

   Suppose no vertex is adjacent to $[v_2]$. By Lemma 7, there is a safe$L$- colouring of $v_2 \dots v_p$. Set $\phi(v_1) = \phi(v_4)$ if $\phi(v_4) \in L(z) \setminus \{\phi(v_2)\}$, and let $\phi(v_1)$ be any colour of $L(v_1) \setminus \{\phi(v_2)\}$ otherwise. Doing so $z$ is safe and so $\phi$ is a safe $L$-colouring of $P$. Hence we may assume that a vertex $u$ is adjacent to $[v_2]$. By definition of good path, it is the unique vertex adjacent to $[v_2]$.

   Suppose that there exists a colour $\beta$ in $L(v_2) \setminus L(z)$. By Lemma 7, there is a safe colouring $\phi$ of $v_2 \dots v_p$ such that $\phi(v_2) = \beta$. By Lemma 6, it can be extended to $v_1$ so that $u$ is safe. This yields a safe $L$-colouring of $P$. Hence we may assume that $L(v_2) = L(z)$.

   If $L(v_4) \cap L(z) = \emptyset$, then in every colouring of $P$, the vertex $z$ will be safe. Hence any safe colouring of $P$ in $G - z$, (there is one by Lemma 7) is a safe $L$-colouring of $P$ in $G$. So we may assume that there exists a colour $\alpha \in L(v_4) \cap L(z)$.

   Assume that at most one vertex $s$ is adjacent to $[v_4]$. Set $\phi(v_2) = \phi(v_4) = \alpha$ so that $z$ and all the vertices adjacent to $[v_3]$ will be safe. By Lemma 7, there is an $\alpha$-safe colouring of $v_4 \dots v_p$. Now by Lemma 6, one can extend $\phi$ to $v_3$ so that $s$ (if it exists) is safe, and then again by Lemma 6 extend it to $v_1$ so that $u$ is safe. This gives a safe $L$-colouring of $P$. So we may assume that two vertices $s$ and $s'$ are adjacent to $[v_4]$.

   Assume that there is a vertex $t$ adjacent to $[v_3]$, then there is no vertex adjacent to $[v_5]$. Hence it suffices to find a safe $L$-colouring of $v_1v_2v_3v_4v_5$. Indeed, if we have such a colouring $\phi$, then using Lemma 7, one can extend it to a safe $L$-colouring of $P$. Set $\phi(v_2) = \phi(v_4) = \alpha$. Doing so $t$ and $z$ will be safe. If $\alpha$ or some colour $\beta \in L(v_5) \setminus \{\alpha\}$ is not contained in one of lists $L(s)$ and $L(s')$, say $L(s')$. Then colouring $v_5$ with $\beta$, if it exists, or any other colour otherwise, the vertex $s'$ will also be safe. By Lemma 6, one can colour $v_3$ so that $s$ is safe. By Lemma 6, one can then colour $v_1$ to obtain a colouring for which $u$ is safe. This $L$-colouring of $v_1v_2v_3v_4v_5$ is safe. Hence, we may assume that $L(s) = L(s') = L(v_5)$. Colour $v_5$ with any colour in $L(v_5) \setminus \{\alpha\}$. Using Lemma 6, colour $v_3$ so that $s$ is safe. Then $s'$ will be also safe because $L(s) = L(s')$. Again by Lemma 6, colour $v_1$ so that $u$ is safe to obtain a safe colouring of $v_1v_2v_3v_4v_5$.

   Assume finally that no vertex is adjacent to $[v_3]$. By Lemma 7, there is a safe $L$-colouring $\phi$ of $v_3 \dots v_p$. If $\phi(v_4) \notin L(z)$, then assign to $v_2$ any colour in $L(v_2) \setminus \{\phi(v_3)\}$. If not, then set $\phi(v_2) = \phi(v_4)$. (This is possible since $L(v_2) = L(z)$.) Then $z$ will be safe. By Lemma 6, colour $v_1$ so that $u$ is safe to obtain a safe $L$-colouring of $P$.

□

### 3.3 Main theorem

A drawing of $G$ is *nice* if two edges intersect at most once. It is well known that every graph with crossing number $k$ has a nice drawing with at most $k$ crossings. (See [5] for example.) In this paper, we will only consider nice drawings. Thus a crossing is uniquely defined by the pair of edges it belongs to. Henceforth, we will confound a crossing with this set of two edges. The *cluster* of a crossing $C$ is the set of endvertices of its two edges and is denoted $V(C)$.

**Theorem 9.** *Let $G$ be a graph having a drawing in the plane with two crossings. Then* $\mathrm{ch}(G) \leq 5$.

*Proof.* By considering a counter-example $G$ with the minimum number of vertices. Let $L$ be a 5-list assignment of $G$ such that $G$ is not $L$-colourable.

Let $C_1$ and $C_2$ be the two crossings. By Theorem 3, $C_1$ and $C_2$ have no edge in common. Set $C_i = \{v_i w_i, t_i u_i\}$. Free to add edges and to redraw them along the crossing, we may assume that $v_i u_i$, $u_i w_i$, $w_i t_i$ and $t_i v_i$ are edges and that the 4-cycle $v_i u_i w_i t_i$ has no vertex inside but the two edges of $C_i$. In addition, we assume that $u_1 v_1 t_1 w_1$ appear in clockwise order around the crossing point of $C_1$ and that $u_2 v_2 t_2 w_2$ appear in counter-clockwise order around the crossing point of $C_2$. Free to add edges, we may also assume that $G \setminus \{v_1 w_1, v_2 w_2\}$ is a triangulation of the plane. In the rest of the proof, for convenience, we will refer to this fact by writing that $G$ is *triangulated*.

**Claim 9.1.** *Every vertex of G has degree at least* 5.

*Proof.* Suppose not. Then $G$ has a vertex $x$ of degree at most 4. By minimality of $G$, $G - x$ has an $L$-colouring $\phi$. Now assigning to $x$ a colour in $L(x) \setminus \phi(N(x))$ we obtain an $L$-colouring of $G$, a contradiction. □

A cycle is *separating* if none of its edges is crossed and both its interior and exterior contain at least one vertex. A cycle is *nicely separating* if it is separating and its interior or its exterior has no crossing.

**Claim 9.2.** *G has no nicely separating triangle.*

*Proof.* Assume, by way of contradiction, that a triangle $T = x_1 x_2 x_3$ is nicely separating. Let $G_1$ (resp. $G_2$) be the subgraph of $G$ induced by the vertices on $T$ or outside $T$ (resp. inside $T$). Without loss of generality, we may assume that $G_2$ is a plane graph.

By minimality of $G$, $G_1$ has an $L$-colouring $\phi_1$. Let $L_2$ be the list assignment of $G_2$ defined by $L_2(x_1) = \{\phi_1(x_1)\}$, $L_2(x_2) = \{\phi_1(x_1), \phi_1(x_2)\}$, $L_2(x_3) = \{\phi_1(x_1), \phi_1(x_2), \phi_1(x_3)\}$, and $L_2(x) = L(x)$ for every vertex inside $T$. Then $L_2$ is a suitable list assignment of $G_2$, so by Theorem 2, $G_2$ admits an $L_2$-colouring $\phi_2$. Observe that necessarily $\phi_2(x_i) = \phi_1(x_i)$. Hence the union of $\phi_1$ and $\phi_2$ is an $L$-colouring of $G$, a contradiction. □

**Claim 9.3.** *Let $C = abcd$ be a* 4*-cycle with no crossing inside it. If a and c have no common neighbour inside C then C has no vertex in its interior.*

*Proof.* Assume by way of contradiction that the set $S$ of vertices inside $C$ is not empty.

Then $ac$ is not an edge otherwise one of the triangles $abc$ and $acd$ would be nicely separating. Since $G$ is triangulated, the neighbours of $a$ (resp. $c$) inside $C$ plus $b$ and $d$ (in cyclic order around $a$ (resp. $c$)) form a $(b,d)$-path $P_a$ (resp. $P_c$). The paths $P_a$ and $P_c$ are internally disjoint because $a$ and $c$ have no common neighbour inside $C$. Hence $P_a \cup P_c$ is a cycle $C'$. Furthermore $C'$ is the outerface of $G' = G\langle S \cup \{b,d\}\rangle$.

By minimality of $G$, $G_1 = (G - S) \cup bd$ admits an $L$-colouring $\phi$. Let $L'$ be the list-colouring of $G'$ defined by $L'(b) = \{\phi(b)\}$, $L'(d) = \{\phi(d)\}$, $L'(x) = L(x) \setminus \{\phi(a)\}$ if $x$ is an internal vertex of $P_a$, $L'(x) = L(x) \setminus \{\phi(c)\}$ if $x$ is an internal vertex of $P_c$, and $L'(x) = L(x)$ if $x \in V(G' - C')$. Then $L'$ is a $\{b,d\}$-correct list assignment of $G'$. Hence, by Lemma 5, $G'$ admits an $L'$-colouring $\phi'$. The union of $\phi$ and $\phi'$ is an $L$-colouring of $G$, a contradiction. □

**Claim 9.4.** *G has no nicely separating* 4*-cycle.*

*Proof.* Suppose not. Then there exists a nicely separating 4-cycle $abcd$. Let $b = z_1, z_2, \dots, z_{p+1} = d$ be the common neighbours of $a$ and $c$ in clockwise order around $a$. By Claim 9.3, we have $p \geq 2$. Each of the 4-cycles $a z_i c z_{i+1}$, $1 \leq i \leq p$ has empty interior by Claim 9.3. So $z_2$ has degree at most 4. This contradicts Claim 9.1. □

A path $P$ is *friendly* if there are two adjacent vertices $x$ and $y$ such that $|N_P(x)| \leq 4$, $|N_P(y)| \leq 3$ and $P$ is good in $G - \{x,y\}$. A path $P$ *meets* a crossing if it contains at least one endvertex of each of the two crossed edges. A *magic path* is a friendly path meeting both crossings.

**Claim 9.5.** *G has no magic path Q.*

*Proof.* Suppose for a contradiction that $G$ has a magic path $Q$. Then there exists two adjacent vertices $x$ and $y$ such that $|N_Q(x)| \leq 4$, $|N_Q(y)| \leq 3$ and $P$ is good in $G - \{x,y\}$. Lemma 8, there in a $L$-colouring $\phi$ of $Q$ such that every vertex $z$ of $(G-Q) - \{x,y\}$ satisfies $|L_\phi(z)| \geq 3$. Now $|L_\phi(x)| \geq 1$ and $|L_\phi(y)| \geq 2$, because $|N_Q(x)| \leq 4$ and $N_Q(y) \leq 3$ Since $Q$ meets the two crossings, $G-Q$ is planar. Furthermore, $G-Q$ may be drawn in the plane such that all the vertices on the outer face are those of $N(Q)$. Hence $L_\phi$ is a suitable assignment of $G-Q$. Hence by Theorem 2, $G-Q$ is $L_\phi$-colourable and so $G$ is $L$-colourable, a contradiction. □

In the remaining of the proof, we shall prove that $G$ contains a magic path, thus getting a contradiction. Therefore, we consider *shortest* $(C_1,C_2)$*-paths*, that are paths joining $C_1$ and $C_2$ with the smallest number of edges. We first consider the cases when the distance between $C_1$ and $C_2$ is 0 or 1. We then deal with the general case when $dist(C_1,C_2) \geq 2$.

**Claim 9.6.** $dist(C_1,C_2) > 0$.

*Proof.* Assume for a contradiction that $dist(C_1,C_2) = 0$. Then, without loss of generality, $v_1 = v_2$. Note that $u_1 \neq u_2$ as otherwise the path $u_1v_1$ would be magic, contradicting Claim 9.5. Similarly, we have $t_1 \neq t_2$.

Note that $w_1$ is not adjacent to $u_2$ for otherwise both the interior and exterior of $w_1u_1v_1u_2$ would contain at least one neighbour of $u_1$ by Claim 9.1. Thus this 4-cycle would be nicely separating, a contradiction to Claim 9.4. Henceforth, by symmetry, $w_1$ is not adjacent to $u_2$ nor $t_2$ and $w_2$ is not adjacent to $u_1$ nor $t_1$.

If $u_1$ is not adjacent to $u_2$, then consider the induced path $Q = u_1v_1u_2$. Since $w_1$ and $w_2$ are not adjacent to $u_2$ and $u_1$, respectively, then $\{w_1,w_2\} \cap Z_Q = \emptyset$. The vertices $t_1$ and $t_2$cannot be both in $Z_Q$ for otherwise $u_1t_2$ and $u_2t_1$ would cross. Furthermore, if $z_1$ and $z_2$ are distinct vertices in $Z_Q \setminus \{t_1,t_2\}$, then either $u_1v_1u_2z_1$ nicely separates $z_2$ or $u_1v_1u_2z_2$ nicely separates $z_1$ contradicting Claim 9.4. Thus, $|Z_Q| \leq 2$ and $Q$ is magic contradicting Claim 9.5. Henceforth, $u_1$ is adjacent to $u_2$, and, by a symmetrical argument, $t_1$ is adjacent to $t_2$.

If $u_1$ is adjacent to $t_2$, then both the interior and exterior of $u_1u_2w_2t_2$ contain at least one neighbour of $w_2$ by Claim 9.1. Thus this 4-cycle would be nicely separating, a contradiction to Claim 9.4. Henceforth, $u_1$ is not adjacent to $t_2$, and symmetrically $t_1$ is not adjacent to $u_2$.

Therefore $Q = u_1v_1t_2$ is an induced path. Note that $Z_Q \subseteq N(v_1)$. The triangles $v_1u_1u_2$ and $v_1t_1t_2$ together with Claim 9.2 imply that $N(v_1) = \{u_1,u_2,t_1,t_2,w_1,w_2\}$. Since $w_1$ is not adjacent to $t_2$ and $w_2$ is not adjacent to $u_1$, then $Z_Q = \{u_2,t_1\}$. Thus $Q$ is magic contradicting Claim 9.5. □

**Claim 9.7.** *Let* $i \in \{1,2\}$ *and* $x$ *a vertex not in* $C_i$*. Then at most one vertex in* $\{u_i,t_i\}$ *is adjacent to* $x$ *and at most one vertex in* $\{v_i,w_i\}$ *is adjacent to* $x$*.*

*Proof.* Assume for a contradiction that $x$ is adjacent to both $u_i$ and $t_i$. Observe that the edges $u_ix$ and $t_ix$ are not crossed since $dist(C_1,C_2) \geq 1$. Then one of the two 4-cycles $u_iv_it_ix$ and $u_iw_it_ix$ is nicely separating. Thus the region bounded by this cycle has no vertex by Claim 9.4. Hence either $d(v_i) \leq 4$ or $d(w_i) \leq 4$. This contradicts Claim 9.1.

Similarly, one shows that at most one vertex in $\{v_i,w_i\}$ is adjacent to $x$. □

**Claim 9.8.** $dist(C_1,C_2) > 1$.

*Proof.* Assume for a contradiction that $dist(C_1,C_2) = 1$. Without loss of generality, we may assume that $v_1v_2 \in E(G)$.

Let us first show that without loss of generality, we may assume that $u_1$ is not adjacent to $v_2$ and $u_2$ is not adjacent to $v_1$. By symmetry, if $t_1$ is not adjacent to $v_2$ and $t_2$ is not adjacent to $v_1$, then we get the result by renaming swapping the names of $u_i$ and $t_i$, $i = 1,2$. Thus by symmetry and by Claim 9.7, if it not the case, then $u_1v_2 \in E(G)$ and $v_1t_2 \in E(G)$. Moreover $w_1v_2$ is not an edge by Claim 9.7. Hence renaming $u_1$, $v_1$, $t_1$, $w_1$ into $v_1$, $t_1$, $w_1$, $u_1$ respectively, we are in the desired configuration.

The vertices $u_1$ and $u_2$ are not adjacent, for otherwise the cycle $u_1v_1v_2u_2$ would be nicely separating since $G$ is triangulated and $u_1v_2$ and $u_2v_1$ are not edges. So $Q$ is an induced path.

A vertex of $Z_Q$ is *goofy* if it is adjacent to $u_1$ and $u_2$.

- Suppose first that there is a goofy vertex $z'$ not in $C_1 \cup C_2$.

  Without loss of generality, we may assume that $z'$ is adjacent to $u_1$, $v_1$ and $u_2$. If the crossing $C_1$ is inside $z'u_1v_1$, then consider the path $R = t_1v_1v_2u_2$. It is induced since $z'u_1v_1$ separates $t_1$ from $v_2$ and $u_2$. Moreover all the neighbours of $t_1$ are inside $z'u_1v_1$, so they have at most two neighbours in $R$ except for $u_1$ which is not adjacent to $v_2$ nor to $u_2$. Hence the vertices of $Z_R$ are all adjacent to $\{v_1,v_2,u_2\}$. Moreover $w_2 \notin Z_R$ because $w_2v_1$ is not an edge by Claim 9.7. Hence by planarity of $G-\{w_1,w_2\}$, there are at most two vertices adjacent to $\{v_1,v_2,u_2\}$. Thus $R$ is magic, a contradiction.

  Hence we may assume that $C_1$ is outside $z'u_1v_1$. The 4-cycle $z'v_1v_2u_2$ is not nicely separating by Claim 9.4, and $G$ is triangulated. So $z'v_2 \in E(G)$ because $v_1$ is not adjacent to $u_2$. So $z'$ is adjacent to all vertices of $Q$.

  Then there is no other vertex $z''$ in $Z_Q \setminus \{C_1 \cup C_2\}$, for otherwise one of the crossing $C_i$ is inside $u_iv_iz''$ and as above, we obtain the contradiction that $R$ is magic.

  Now $w_1u_2$ is not an edge, for otherwise $w_1u_1z'u_2$ would be separating since $d(u_1) \geq 5$, a contradiction to Claim 9.4. Similarly, $w_2u_1$ is not an edge. Hence $Z_Q \subset \{z',t_1,t_2\}$. Now one of the edges $t_1u_2$ and $t_2u_1$ is not in $E(G)$, since otherwise they would cross. Without loss of generality, $t_1$ is not adjacent to $u_2$. Then $Q$ is good in $G-t_2$, and so $Q$ is magic. This contradicts Claim 9.5.

- Suppose now that all the goofy vertices of $Z_Q$ are in $C_1 \cup C_2$.

  Suppose first that $w_1$ is in $Z_Q$, then $w_1u_2$ is an edge because $w_1$ is not adjacent to $v_2$ according to Claim 9.7. Thus $t_2$ and $w_2$ are not adjacent to $u_1$. So $w_2 \notin Z_Q$ and $N_Q(t_2) \subset \{v_1,v_2,u_2\}$, so $t_2$ is not goofy. Moreover by planarity of $G-\{w_1,w_2\}$, there is at most two vertices adjacent $\{v_1,v_2,u_2\}$. Furthermore, all the vertices distinct from $t_1$ and adjacent to $\{u_1,v_1,v_2\}$ are in the region bounded by $w_1v_1v_2u_2$ containing $u_1$. Therefore there is at most one such vertex. Hence $Q$ is good in $G-\{w_1,t_1\}$. Thus $Q$ is magic and contradicts Claim 9.5.

  Similarly, we get a contradiction if $w_2 \in Z_Q$. So $Z_Q \cap (C_1 \cup C_2) \subseteq \{t_1,t_2\}$. Then easily $Q$ is good in $G-t_2$ and so $Q$ is magic. This contradicts Claim 9.5.

□

**Claim 9.9.** *Some of the shortest $(C_1,C_2)$-paths is nice.*

*Proof.* Let $P = x_1x_2\cdots x_p$ be any shortest $(C_1,C_2)$-path. Then no vertex in $C_1$ is adjacent to a vertex in $P-\{x_1,x_2\}$. Therefore, $V(C_1)\cap Z_P = \emptyset$. Similarly, we have $V(C_2)\cap Z_P = \emptyset$. Hence the graph $G'$ induced by $V(P)\cup Z_P$ is planar as it contains exactly one vertex from each crossing.

Any vertex not in $P$ can be adjacent only to vertices of $P$ at distance at most two from each other, otherwise there would be a $(C_1,C_2)$-path shorter than $P$. Thus, if $z\in Z_P$, then $z$ has precisely three neighbours in $P$. Moreover, there exists an $i\in\{2,\dots,p-1\}$ such that $N_P(z)=[x_i]$.

If there are distinct vertices $z_1,z_2,z_3\in Z_P$ such that $N_P(z_1)=N_P(z_2)=N_P(z_2)=[x_i]$ for some value of $i$, then the subgraph of $G'$ induced by $\{z_1,z_2,z_3\}\cup\{x_{i-1},x_i,x_{i+1}\}$ contains a $K_{3,3}$. By Kuratowski's Theorem, this contradicts the fact that $G'$ is planar. Therefore, for every $2\le i\le p-1$, there are at most two vertices in $Z_P$ adjacent to $[x_i]$.

Let $z_1,z_2\in Z_P$ be such that $N_P(z_1)=N_P(z_2)=[x_i]$. The edges of $H=G[\{z_1,z_2\}\cup[x_i]]$ separate the plane into five regions $R_1,\dots,R_5$ as follows. Let $R_1$ be the region bounded by $x_{i-1}x_iz_1$ not containing the vertex $z_2$, $R_2$ be the region bounded by $x_ix_{i+1}z_1$ not containing the vertex $z_2$, $R_3$ be the region bounded by $x_{i-1}x_iz_2$ not containing the vertex $z_1$, $R_4$ be the region bounded by $x_ix_{i+1}z_2$ not containing the vertex $z_1$ and $R_5$ be the region bounded by $x_{i-1}z_1x_{i+1}z_2$ not containing $x_i$ (see Figure 1). Since $(V(C_1)\cup V(C_2))\cap Z_P=\emptyset$ and $P$ is a shortest $(C_1,C_2)$-path, then no edge in $H$ is crossed.

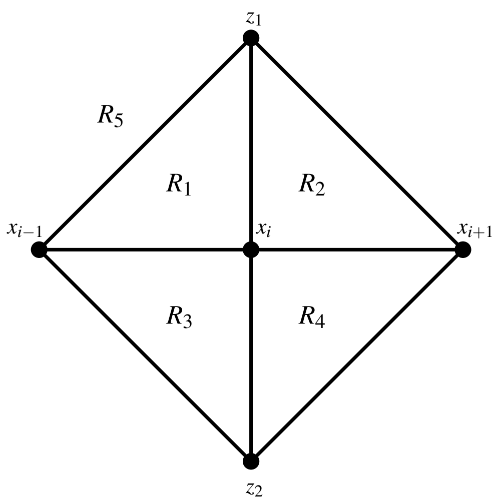


Figure 1: Regions $R_1$, $R_2$, $R_3$, $R_4$ and $R_5$.

Let $J_P$ be the subset of $\{3,\dots,p-2\}$ such that for $j\in J_P$, there are two vertices in $Z_P$ adjacent to $[x_j]$ and at least one vertex adjacent to $[x_{j-1}]$ and another adjacent to $[x_{j+1}]$. The path $P$ is said to be *semi-nice* if $J_P=\emptyset$.

Let us first prove that some of the shortest $(C_1,C_2)$-paths is semi-nice.

Suppose for a contradiction that no shortest $(C_1,C_2)$-path is semi-nice. Let $P$ be a shortest $(C_1,C_2)$-path that maximizes the smallest index $i$ in $J_P$. Let $z_1,z_2\in Z_P$ be such that $N_P(z_1)=N_P(z_2)=[x_i]$.

Let $z\in Z_P$ be a vertex adjacent to $[x_{i+1}]$. If $C_2$ is in $R_5$, then so is $x_{i+2}$ and we get a contradiction from the fact that either $zx_i$ or $zx_{i+2}$ must cross an edge of $H$. Since $P$ defines a path between $x_{i+1}$ and $V(C_2)$, then $C_2$ must be either in $R_2$ or in $R_4$ (say $R_4$). Similarly, $C_1$ is either in $R_1$ or in $R_3$. The cycle $x_{i-1}x_ix_{i+1}z_2$ is not be a nicely separating cycle by Claim 9.4, so $C_1$ must be in $R_1$.

Now, by Claim 9.2, $R_2$ and $R_3$ are empty, and, by Claim 9.4, there is no vertex in $R_5$. Since $P$ is a shortest path, $x_{i-1}x_{i+1}$ is not an edge and therefore $z_1$ is adjacent to $z_2$ as $G$ is triangulated.

Now, consider the path $P'$ obtained from $P$ by replacing $x_i$ with $x'_i = z_2$. Note that $P'$ is also a shortest path and that both $z_1$ and $x_i$ are adjacent to $[x'_i]$. Since no edge in $H$ is crossed, for any $v \in V(G) \setminus (\{z_1, z_2\} \cup [x_i])$, if $v$ is adjacent to $x_{i-1}$ then it must be in $R_1$ and if $v$ is adjacent to $z_2$ then it must be in $R_4$. Therefore, there is no vertex in $Z_{P'}$ adjacent to $\{x_{i-2}, x_{i-1}, z_2\}$. This implies that if $j \in J_{P'}$, then either $j \leq i-3$ or $j \geq i+1$. Note that if $j \in J_{P'}$ and $j \leq i-3$, then $j \in J_P$. As $i$ is the minimum of $J_P$, the minimum of $J_{P'}$ is at least $i+1$. This contradicts our choice of $P$.

Let $K_P$ be the subset of $\{2, \ldots, p-1\}$ such that for $k \in K_P$, there are two vertices in $Z_P$ adjacent to $[x_k]$ and two vertices adjacent to $[x_{k+1}]$. Observe that a nice path $P$ is a semi-nice path such that $K_P$ is empty, that is a path such that $J_P$ and $K_P$ are empty.

Suppose, by way of contradiction, that every $(C_1, C_2)$-shortest path is not nice. Then consider the semi-nice $(C_1, C_2)$-shortest path that maximizes the minimum of $K_P$.

Let $z_1, z_2, z_3, z_4 \in Z_P$ be such that $N_P(z_1) = N_P(z_2) = [x_i]$ and $N_P(z_3) = N_P(z_4) = [x_{i+1}]$, where $i$ is the smallest index in $K_P$. Recall that the edges of $H = G[\{z_1, z_2\} \cup [x_i]]$ separate the plane into the five above-described regions $R_1, \ldots, R_5$. Again, we can use $z_3$ or $z_4$ to prove that $C_2$ is either in $R_2$ or in $R_4$ (say $R_4$). Therefore, $x_{i+2}$ is in $R_4$ which implies $z_3$ and $z_4$ are also in $R_4$. Thus, $z_1$ is not adjacent to $z_3$ nor $z_4$. Furthermore, $z_2$ cannot be adjacent to both $z_3$ and $z_4$ for otherwise we can obtain a $K_5$ in the subgraph of $G'$ induced by $[x_{i+1}] \cup \{z_2, z_3, z_4\}$ by contracting the edge $z_4x_{i+2}$ (see Figure 2). Thus, without loss of generality, suppose $z_2$ and $z_3$ are not adjacent.

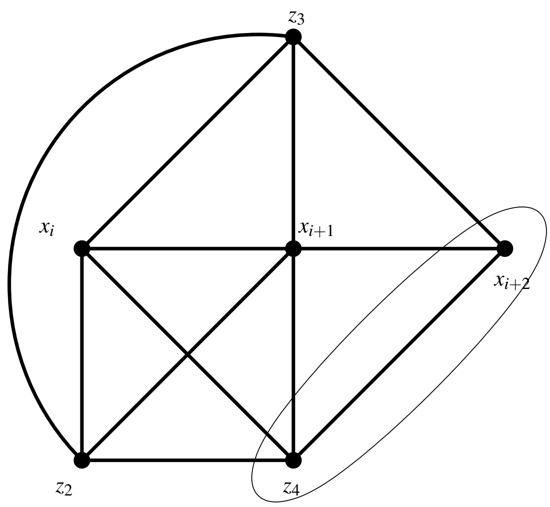


Figure 2: $K_5$ minor of $G'$ is obtained by contracting $z_4x_{i+2}$.

Consider the path $P'$ obtained from $P$ by replacing $x_{i+1}$ with $x'_{i+1} = z_3$. Since no edge in $H$ is crossed, for any $v \in V(G) \setminus (\{z_1, z_2\} \cup [x_i])$, if $v$ is adjacent to $x_{i-1}$ then it is not in $R_4$, and if $v$ is adjacent to $z_3$ then it must be in $R_4$. Since neither $z_1$ nor $z_2$ are adjacent to $z_3$ and $x_{i+1}$ is not adjacent to $x_{i-1}$, there is no vertex in $Z_{P'}$ adjacent to $\{x_{i-1}, x_i, z_3\}$. This implies that if $k \in K_{P'}$, then either $k \leq i-2$ or $k \geq i+1$. Note that if $k \in K_{P'}$ and $k \leq i-2$, then $k \in K_P$. This implies that the minimum index in $K_{P'}$ is strictly greater than $i$. Hence by our choice of $P$, the path $P'$ is not semi-nice, that is $J_{P'} \neq \emptyset$.

Observe that if $j \in J_{P'}$, then either $j \leq i-2$ or $j \geq i+2$. Note that if $j \in J_{P'}$ and either $j \leq i-2$ or $j \geq i+4$, then $j \in J_P$. Since $J_P$ is empty, then $J_{P'} \subseteq \{i+2, i+3\}$. Let $z'_1, z'_2 \in Z_{P'}$ be such that $N_{P'}(z'_1) = N_{P'}(z'_2) = [x'_j]$, for some $j \in J_{P'}$ with $J_{P'} \subseteq \{i+2, i+3\}$. Note that for the two possible values of $j$, both $z'_1$ and $z'_2$ are adjacent to $x_{i+3}$. Since $P$ is a shortest $(C_1, C_2)$-path, neither $z_2$ nor $x_{i+1}$ are adjacent to $x_{i+3}$ and therefore $z'_1$ and $z'_2$ are in $R_4$. Let $R'_1$ be the region bounded by $x'_{j-1}x'_jz'_1$ not containing the vertex $z'_2$ and $R'_3$ be the region bounded by $x'_{j-1}x'_jz'_2$ not containing the vertex $z'_1$. Both of these regions are contained in $R_4$. With the same argument used above in the proof of existence of a semi-nice path, one shows that if $j \in J_{P'}$, then $C_1$ is either contained in $R'_1$ or in $R'_3$. We get a contradiction as the path $P$ from $V(C_1)$ to $x_{i-1}$ crosses an edge of $H$. □

**Claim 9.10.** *There exists an induced path $Q = x_0x_1 \cdots x_px_{p+1}$ with the following properties:*

- $P_1$. *$P = x_1 \cdots x_p$ is a shortest $(C_1, C_2)$-path and is a nice path;*
- $P_2$. *$x_0 \in V(C_1)$ and $x_{p+1} \in V(C_2)$ but $x_0x_1$ and $x_px_{p+1}$ are not crossed edges; and*
- $P_3$. *there is at most one vertex in $Z_Q$ adjacent to both vertices in $\{x_0, x_3\}$ and at most one vertex in $Z_Q$ adjacent to both vertices in $\{x_{p-2}, x_{p+1}\}$.*
- $P_4$. *for any $i < j$, if there are two vertices adjacent to $[v_i]$ and two vertices adjacent to $[v_j]$, then the number of vertices adjacent to $[v_{i+1}]$ or to $[v_{j-1}]$ is at most* 1.

*Proof.* By Claim 9.9 there exists a shortest $(C_1, C_2)$-path $P = x_1 \cdots x_p$ which is nice. Without loss of generality, we may assume that $x_1 = v_1$ and $x_p = v_2$. According to Claim 9.7, we can choose vertices $x_0 \in \{u_1, t_1\}$ and $x_{p+1} \in \{u_2, t_2\}$ such that $Q$ is induced. Therefore, we have at least one path satisfying properties $P_1$ and $P_2$. We say that $x_0$ is a *valid endpoint* if there is at most one vertex in $Z_Q$ adjacent to both vertices in $\{x_0, x_3\}$ and $x_{p+1}$ is a *valid endpoint* if there is at most one vertex in $Z_Q$ adjacent to both vertices in $\{x_{p-2}, x_{p+1}\}$.

Let $Q$ be a path satisfying properties $P_1$ and $P_2$ which maximizes the number of valid endpoints of $Q$.

Let us first show that $Q$ has only valid endpoints, and satisfies property $P_4$. By contradiction, suppose that $Q$ has an invalid endpoint. Without loss of generality, $x_0$ is invalid.

Let $z_1, z_2 \in Z_Q$ be two vertices adjacent to both vertices in $\{x_0, x_3\}$. Since $P$ is a shortest $(C_1, C_2)$-path, no vertex of $C_1$ is adjacent to $x_3$. Therefore, no edge of $x_0x_1x_2x_3z_1$ and $x_0x_1x_2x_3z_2$ is crossed. Let $R_1$ be the region bounded by $x_0x_1x_2x_3z_1$ that does not contain $z_2$ and $R_2$ be the region bounded by $x_0x_1x_2x_3z_2$ that does not contain $z_1$. Since the edges bounding the regions $R_1$ and $R_2$ are not crossed, then the crossing $C_1$ is contained in one of the regions $R_1$ or $R_2$ (say $R_1$). Let $\hat{x}_0$ be the vertex of $\{u_1, t_1\} \setminus \{x_0\}$ (see Figure 3).

Assume first that $\hat{x}_0$ is not adjacent to $x_2$. Let $\hat{Q}$ be the path obtained from $Q$ by replacing $x_0$ with $\hat{x}_0$. Clearly the path $\hat{Q}$ is induced and satisfies properties $P_1$ and $P_2$. By definition of $Q$, $\hat{x}_0$ must be an invalid endpoint. Hence, there is a vertex $\hat{z}$ in $Z_{\hat{Q}} \setminus \{z_1\}$ which is adjacent to $\hat{x}_0$ and $x_3$. This vertex in necessarily inside $R_1$ because it is adjacent to $x_0$. But then, by planarity, $z_1$ cannot be adjacent to $x_1$ and $x_2$, a contradiction to $z_1 \in Z_Q$.

Assume now that $\hat{x}_0$ is adjacent to $x_2$. Let $Q'$ be the path obtained from $Q$ by replacing $x_0$ with $w_1$ and $x_1$ with $\hat{x}_0$. Note that $Q'$ is induced as $w_1$ is not adjacent to $x_2$ by Claim 9.7.

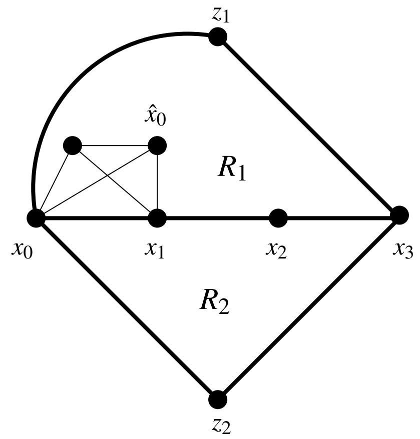


Figure 3: Regions $R_1$ and $R_2$ and the vertex $\hat{x}_0$.

Note that property $P_2$ is valid for $Q'$. The path $P' = \hat{x}_0 x_2 \cdots x_p$ is a $(C_1, C_2)$ shortest path. Let us prove that $P'$ is nice and so that $P'$ satisfies property $P_1$. If $p = 3$, then, since no vertex in the cluster of $C_1$ is adjacent to $x_3$, at most two vertices are in $Z_{P'}$ for otherwise we would get a $K_{3,3}$ in $G - \{w_1, w_2\}$, which is impossible as this graph is planar. Thus $P'$ is nice. Suppose now that $p \geq 4$. By planarity, $z_1$ is not adjacent to $x_1$, so $z_1$ is adjacent to $x_2$ as $z_1 \in Z_Q$. In addition, $z_1 x_2$ is contained in $R_1$. Thus, any vertex in $Z_{P'}$ adjacent to $\hat{x}_0$ must be in region $R_1$ and cannot be adjacent to $x_3$. Hence no vertex is adjacent to $[x_2]_{P'}$ so, since $P$ is a nice path, $P'$ is also a nice path.

By definition of $Q$, $w_1$ must be an invalid endpoint of $Q'$. Hence, there is a vertex $z'$ in $Z_{Q'} \setminus \{z_1\}$ which is adjacent to $w_1$ and $x_3$. This vertex in necessarily inside $R_1$ because neither $x_0$ nor $x_1$ are adjacent to $x_3$. But then, by planarity, $z_1$ cannot be adjacent to $x_1$ and $x_2$, a contradiction to $z_1 \in Z_Q$.

Let us now prove that $Q$ satisfies property $P_4$. By contradiction, suppose $Q$ does not. Let $z_1, z_2, z'_1, z'_2 \in Z_Q$ be such that both $z_1$ and $z_2$ are adjacent to $[x_i]$ and $z'_1$ and $z'_2$ are adjacent to $[x_j]$. Consider the regions $R_1, \ldots, R_5$ related to $z_1$ and $z_2$ used in Figure 1. Consider the regions $R'_1, \ldots, R'_5$ related to $z'_1$ and $z'_2$ used in Figure 1 for $i = j$.

Let $z \in Z_Q$ be adjacent to $[x_{i+1}]$. Note that we can have $\{z_1, z_2\} \cap \{u_1, t_1\} \neq \emptyset$ if $i = 1$. But since $dist(C_1, C_2) \geq 2$, the edges $z_1 x_{i+1}$ and $z_2 x_{i+1}$ are not crossed. Furthermore, since no vertex in the cluster of $C_1$ is adjacent to $x_3$ and not vertex in the cluster of $C_2$ is adjacent to $x_1$ ($P$ is a shortest $(C_1, C_2)$-path), then $z$ is not in the cluster of either crossing.

Therefore, since $z$ is adjacent to both $x_i$ and $x_{i+2}$, we must have that both $z$ and $x_3$ are in $R_2$ or in $R_4$ (say $R_2$). This also implies that $C_2$ is in $R_2$. Note also that, by our choice of $x_0$, the edges $z_1 x_i$ and $z_2 x_i$ are not crossed. Therefore, $C_1$ is contained in $R_1 \cup R_3 \cup R_5$. With a symmetric argument, we have that $C_1$ is either in $R'_1$ or in $R'_3$ (say $R_1$). Since both $z'_1$ and $z'_2$ are also in $R_2$, then $R'_1 \cup R'_3$ are contained in $R_2$ and we get a contradiction. □

Let $Q$ be a path given by Claim 9.10. Without loss of generality, suppose $x_1 = v_1$ and $x_p = v_2$. Note also that Claim 9.7 implies $w_1$ and $w_2$ are not in $Z_Q$ and therefore $G[V(Q) \cup Z_Q]$ is planar.

**Claim 9.11.** *$dist(C_1, C_2) = 2$ and there is a vertex adjacent to $x_0$ and $x_4$.*

*Proof.* Suppose not. Then no vertex in $Z_Q$ is adjacent to vertices at distance at least four in $Q$. Observe that this is the case when $dist(C_1, C_2) \geq 3$, since $x_1 \ldots x_p$ is a shortest $(C_1, C_2)$-path.

Since $P$ is a nice and shortest $(C_1,C_2)$-path, then the only vertices in $Z_Q$ adjacent to vertices at distance at least three in $Q$ must be adjacent to both $x_0$ and $x_3$ or to both $x_{p-2}$ and $x_{p+1}$. By the property $P_3$ of Claim 9.10, there is at most one vertex, say $z$, adjacent to $x_0$ and $x_3$ and at most one vertex, say $z'$, adjacent to $x_{p-2}$ and $x_{p+1}$.

Let us make few observations.

Obs. 1 If two vertices $z_1$ and $z_2$ distinct from $z$ are adjacent to $[x_2]$, then no vertex is adjacent to $[x_1]$ and $N_Q(z) = \{x_0,x_1,x_3\}$. Indeed $z$ must be in the region $R_5$ in Figure 1 because it is adjacent to $x_0$ and $x_3$. By the planarity of $G[V(Q)\cup Z_Q]$ and since $z$ is adjacent to $x_0$, $x_0$ must also be in $R_5$. Again by planarity, $z$ is not adjacent to $x_2$ and, therefore, must be adjacent to $x_1$ as $z \in Z_Q$.

Obs. 2 If two vertices $z_1$ and $z_2$ distinct from $z$ are adjacent to $[x_1]$, then no vertex is adjacent to $[x_2]$ and $N_Q(z) = \{x_0,x_2,x_3\}$. This argument is symmetric to Observation 1.

Suppose that $z$ exists.

If $z'$ exists, by Observations 1 and 2 (and their analog for $z'$) and the properties of $Q$ from Claim 9.10, the path $Q$ is good in $G-z'$ because it is great in $G-\{z,z'\}$. Hence $Q$ is magic, a contradiction to Claim 9.5. Hence $z'$ does not exists.

By Claim 9.7, $w_2$ is not adjacent to $x_{p-1}$ and $w_1$ is not adjacent to $x_p$ since $dist(C_1,C_2) \geq 2$. So, by planarity of $G-\{w_1,w_2\}$, at most two vertices are adjacent to $[x_p]$. Let $y$ be a vertex adjacent to $[x_p]$. The path $Q$ is not great in $G-\{y,z\}$, for otherwise it would be magic. Hence, according to the properties of $Q$ and the above observations, there must be two vertices adjacent to $[x_p]$, two vertices adjacent to $[x_{p-1}]$ and one vertex adjacent to $[x_{p-2}]$. Let $z_1$ and $z_2$ be the two vertices adjacent to $[x_{p-1}]$ and $R_1 \ldots R_5$ be the regions as in Figure 1 with $i = p-1$. Since there is a vertex adjacent to $[x_{p-2}]$, then $C_1$ is in $R_1$ or $R_3$, and $C_2$ is in $R_2$ or $R_4$ because a vertex is adjacent to $[x_p]$. But by Claim 9.4 the 4-cycle $z_1x_pz_2x_{p-2}$ is not nicely separating, so there is no vertex inside $R_5$. Since $G$ is triangulated, and $x_{p-2}x_p$ is not an edge because $P$ is a shortest $(C_1,C_2)$-path, $z_1z_2 \in E(G)$. Now the path $Q$ is good in $G-\{z_1,z_2\}$ and so is magic. This contradicts Claim 9.5.

Hence we may assume that $z$ does not exists and by symmetry that $z'$ does not exist. We get a contradiction similarly by considering a vertex $w$ adjacent to $[x_1]$ in place of $z$. □

**Claim 9.12.** *There is precisely one vertex $z \in Z_Q$ adjacent to both $x_0$ and $x_4$.*

*Proof.* Observe that there are at most two vertices adjacent to $x_0$ and $x_4$. Indeed such vertices cannot be in the crossings because $dist(C_1,C_2) = 2$. Thus if there were three such vertices, together with contracting the path $x_1x_2x_3$ we would get $K_{3,3}$ minor in $G-\{w_1,w_2\}$, a contradiction.

Suppose by contradiction that two distinct vertices $z_1,z_2 \in Z_Q$ adjacent to vertices $x_0$ and $x_4$. The edges of $Q$ are contained in the same region of the plane bounded by the cycle $x_0z_1x_4z_2$. Therefore, both crossings are also in the region containing the edges of $Q$. By Claim 9.3, the region bounded by the cycle $x_0z_1x_4z_2$ that does not contain the crossings has no vertex in its interior. Since $G$ is triangulated, $z_1z_2 \in E(G)$ as $x_0$ because $x_4$ are not adjacent as $dist(C_1,C_2) = 2$.

By the property $P_3$ of Claim 9.10, $z_1$ and $z_2$ cannot be both adjacent to the five vertices in $Q$. Therefore, without loss of generality, suppose $|N_Q(z_2)| \leq 4$. Let us prove that $Q$ is great in $H = (G-z_2)\setminus\{z_1x_0,z_1x_4\}$.

(i) If a vertex $t$ in $G-\{z_1,z_2\}$ is adjacent to at least four vertices of $Q$, then without loss of generality it is adjacent to $\{x_0,x_1,x_2,x_3\}$ as it cannot be adjacent to $x_0$ and $x_4$. Now by property $P_3$, $z_1$ and $z_2$ are not adjacent to $x_3$. Hence one of them (the one such that $x_0x_1x_2x_3x_4z_i$ separates $t$ from $z_{3-i}$) cannot be adjacent to any vertex of $\{x_1,x_2,x_3\}$, a contradiction to the fact that it is in $Z_Q$. Hence $Q$ satisfies (a) in $H$.

(ii) If two vertices $t_1$ and $t_2$ of $H$ are adjacent to $[x_2]$, then necessarily $x_1t_1x_2t_2$ is a nicely separating, a contradiction to Claim 9.4. Hence there is at most one vertex of $H$ adjacent to $[x_2]$. Thus $Q$ satisfies (b) in $H$.

(iii) If two vertices $r_1$ and $r_2$ of $H$ are adjacent to $[x_1]$, then no vertex is adjacent to $[x_2]$. Indeed suppose for a contradiction that a vertex $t$ is adjacent to $[v_2]$ none of $\{r_1,r_2,t\}$ is in $\{w_1,w_2\}$ by Claim 9.7 and because $dist(C_1,C_2) \geq 2$. Now contracting the path $tx_3x_4z_2$ into a vertex $w$, we obtain a $K_{3,3}$ with parts $\{r_1,r_2,w\}$ and $\{x_0,x_1,x_2\}$. This contradicts the planarity of $G$.

Symmetrically, if two vertices of $H$ are adjacent to $[x_3]$, then no vertex is adjacent to $[x_2]$. Therefore $Q$ satisfies (c) in $H$.

It follows that $Q$ is a good path in $H' = (G-z_2)\setminus z_1x_4$. Let $\phi$ be a safe $L$-colouring of $Q$ in $H'$ obtained by Lemma 8. Since $Q$ meets the two crossings, $G-Q$ is planar. Furthermore, $G-Q$ can be drawn in the plane such that all vertices on the outer face are those in $N(Q)$. Every vertex of $Z_Q\setminus\{z_1,z_2\}$ is safe in $H'$ and so in $G$, so $|L_\phi(v)| \geq 3$. In $H'$, $z_1$ is safe and in $G$, $z_1$ has one more neighbour in $Q$ in $G$ than $H'$, namely $x_4$. Thus in $G$, $|L_\phi(z_1)| \geq 2$ because $z_1$ was safe in $H'$. Since $z_2$ has at most four neighbours in $Q$, we have $|L_\phi(z_2)| \geq 1$. Now $z_1$ is adjacent to $z_2$, so $L_\phi$ is a $\{z_1,z_2\}$-suitable assignment for $G-Q$. Hence by Theorem 2, $G-Q$ is $L_\phi$-colourable and so $G$ is $L$-colourable, a contradiction. □

- Assume first that $|N_Q(z)| = 5$. Let $H = G\setminus\{zx_0,zx_4\}$. $z$ is the unique vertex adjacent to $x_0$ and $x_4$. Moreover by property $P_3$ $z$ is the unique vertex adjacent to $x_0$ and $x_3$ and the unique one adjacent to $x_1$ and $x_4$. Hence $Q$ satisfies (a) in $H$. Moreover, for $1 \leq i \leq 3$, there is at most one vertex distinct form $z$ adjacent to $[x_i]$ otherwise $G[V(Q)\cup Z_Q]$ would contain a $K_{3,3}$. Hence $Q$ also satisfies (b) and (c) in $H$. Therefore $Q$ is great in $H$. By Lemma 7, there exists a safe $L$-colouring $\phi$ of $Q$ in $H$. Thus in $G$, every vertex in $Z_Q\setminus\{z\}$ satisfies $|L_\phi(v)| \geq 3$ while $|L_\phi(z)| \geq 1$. Hence $L_\phi$ is suitable for $G-Q$. Therefore, by Theorem 2, $G-Q$ is $L_\phi$-colourable and so $G$ is $L$-colourable, a contradiction.

- Assume now that $|N_Q(z)| \leq 4$.

  Suppose that there are two distinct vertices $z_1,z_2 \in Z_Q$ with $z_1$ adjacent to $x_0$ and $x_3$ and $z_2$ adjacent to $x_1$ and $x_4$. Let $R_1$ be the region bounded by the cycle $x_0x_1x_2x_3z_1$ not containing $z_2$ and $R_2$ be the region bounded by the cycle $x_1x_2x_3x_4z_2$ not containing $z_1$ (see Figure 4). Now, note that any vertex adjacent to both $x_0$ and $x_4$ is not in $R_1\cup R_2$ and any vertex adjacent to $x_2$ must be in $R_1\cup R_2$. Therefore, $z \in \{z_1,z_2\}$. Indeed if this was not true, then by property $P_3$ $z$ is not adjacent to $x_1$ nor $x_3$. Thus $z$ must be adjacent to $x_2$ as it is in $Z_Q$. So $z$ is inside $R_1\cup R_2$, which contradicts the fact that it is adjacent to $x_0$ and $x_4$.

  Thus, at most one other vertex $z'$ in $Z_Q\setminus\{z\}$ is adjacent to vertices at distance three in $Q$. By symmetry, we may assume that $z'$ is adjacent to $x_0$ and $x_3$. Hence all vertices in $Z_Q\setminus\{z,z'\}$ are adjacent to some $[x_i]$ for $1 \leq i \leq 3$. Similarly to (ii) and (iii) in Claim 9.12, one shows that $Q$ also satisfies (a) and (b) in $(G-z)\setminus z'x_0$. Hence $Q$ is a good path in $G-z$. Then $Q$ is magic, a contradiction to Claim 9.5.

□

# Acknowledgement

The authors would like to thank Claudia Linhares Sales for stimulating discussions.

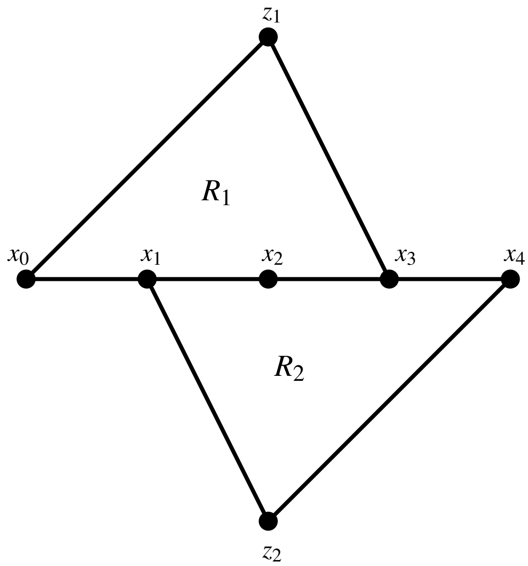


Figure 4: Regions $R_1$ and $R_2$.

## References

[1] M. O. Albertson. Chromatic Number, Independence Ratio, and Crossing Number. *Ars Mathematica Contemporanea* 1:1–6, 2008.

[2] M. O. Albertson, M. Heenehan, A. McDonough, and J. Wise. Coloring graphs with given crossing patterns. *manuscript*.

[3] J. Barát and G. Tóth. Towards the Albertson Conjecture. *Electronic Journal of Combinatorics* 17: R-73, 2010.

[4] Z. Dvořák, B. Lidický, and R. Škrekovski. Graphs with two crossings are 5-choosable. (arXiv:1103.1801v1 [math.CO]).

[5] R. Erman, F. Havet, B. Lidicky, and O. Pangrac. 5-colouring graphs with 4 crossings. *SIAM J. Discrete Math.* 25(1):401–422, 2011.

[6] C. Kuratowski. Sur le problème des courbes gauches en topologie. *Fund. Math.* 15: 271–283, 1930.

[7] B. Oporowski and D. Zhao. Coloring graphs with crossing. *Discrete Mathematics* 309: 2948–2951, 2009.

[8] M. Schaefer. *personal communication to M. O. Albertson.*

[9] C. Thomassen. Kuratowski's theorem. *J. Graph Theory* 5:225–241, 1981.

[10] C. Thomassen. Every planar graph is 5-choosable. *J. Comb. Theory B* 62:180–181, 1994.

Centre de recherche INRIA Sophia Antipolis – Méditerranée
2004, route des Lucioles - BP 93 - 06902 Sophia Antipolis Cedex (France)

Centre de recherche INRIA Bordeaux – Sud Ouest : Domaine Universitaire - 351, cours de la Libération - 33405 Talence Cedex
Centre de recherche INRIA Grenoble – Rhône-Alpes : 655, avenue de l'Europe - 38334 Montbonnot Saint-Ismier
Centre de recherche INRIA Lille – Nord Europe : Parc Scientifique de la Haute Borne - 40, avenue Halley - 59650 Villeneuve d'Ascq
Centre de recherche INRIA Nancy – Grand Est : LORIA, Technopôle de Nancy-Brabois - Campus scientifique
615, rue du Jardin Botanique - BP 101 - 54602 Villers-lès-Nancy Cedex
Centre de recherche INRIA Paris – Rocquencourt : Domaine de Voluceau - Rocquencourt - BP 105 - 78153 Le Chesnay Cedex
Centre de recherche INRIA Rennes – Bretagne Atlantique : IRISA, Campus universitaire de Beaulieu - 35042 Rennes Cedex
Centre de recherche INRIA Saclay – Île-de-France : Parc Orsay Université - ZAC des Vignes : 4, rue Jacques Monod - 91893 Orsay Cedex

Éditeur
INRIA - Domaine de Voluceau - Rocquencourt, BP 105 - 78153 Le Chesnay Cedex (France)
http://www.inria.fr
ISSN 0249-6399