Mild solutions for the one dimensional Nordstrom-Vlasov system

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Abstract

The Nordström-Vlasov system describes the evolution of a population of self-gravitating collisionless particles. We study the existence and uniqueness of mild solution for the Cauchy problem in one dimension. This approach does not require any derivative for the initial particle density. For any initial particle density uniformly bounded with respect to the space variable by some function with finite kinetic energy and any initial smooth data for the field equation we construct a global solution, preserving the total energy. Moreover the solution propagates with finite speed. The propagation speed coincides with the light speed.

Keywords: Nordström equation, Vlasov equation, weak/mild solutions, global existence.

AMS classification: 35B35, 35A05, 35B45, 82D10.

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1 Introduction

Consider a population of particles interacting by fields created collectively. We assume that the collisions are so rare such that we can neglect them meaning that there is no direct interaction between particles. The fields acting on the particles depend on the physical model. Typical examples of such collisionless gases occur in plasma physics and in astrophysics. In plasma physics the particles interact by electro-magnetic forces and the dynamics of the system is described by the Vlasov-Maxwell equations. If the particle velocities are small compared to the light speed then we can neglect the magnetic field and use the quasi-electrostatic approximation given by Vlasov-Poisson equations, cf. [11], [22], [3]. If the particles interact by gravitational forces the evolution of the system is given by the Einstein-Vlasov equations. The Cauchy problem of the former system is well understood, cf. [5], [12], [13], [14], [15], [17]. The Einstein-Vlasov system is much more difficult. The reader can refer to [1] for a recent review, see also [19], [20], [21]. The main application of these equations concern the stellar dynamics: the systems to be considered are galaxies and the particles are stars.

We analyze here a different relativistic model obtained by coupling the Vlasov equation to the Nordström scalar gravitation theory [18]. Let $F = F(t, x, p) \geq 0$ denote the density of particles in the phase-space, where $t \in \mathbb{R}$ represents time, $x \in \mathbb{R}^N$ position and $p \in \mathbb{R}^N$ momentum, with $N \in \{1, 2, 3\}$. This density satisfies the Vlasov equation

$$\partial_t F + v(p) \cdot \nabla_x F - \Big((\partial_t \phi + v(p) \cdot \nabla_x \phi) p + (1 + |p|^2)^{-\frac{1}{2}} \nabla_x \phi \Big) \cdot \nabla_p F = 0,$$

coupled to the wave equation

$$\partial_t^2 \phi - \Delta_x \phi = -e^{(N+1)\phi(t,x)} \int_{\mathbb{R}^N} \frac{F(t, x, p)}{(1 + |p|^2)^{\frac{1}{2}}} \, dp.$$

Here $v(p) = \frac{p}{(1+|p|^2)^{1/2}}$ denotes the relativistic velocity of a particle with momentum p. We normalize the physical units such that the rest mass of each particle, the gravitational constant and the speed of the light are all equal to unity. For more details on this model see [6]. It is convenient to rewrite the system in terms of the unknowns (f, ϕ) where $f(t,x,p) = e^{(N+1)\phi(t,x)}F(t,x,p)$. The system becomes

$$\partial_t f + v(p)\cdot\nabla_x f - \Big((S\phi)p + (1+|p|^2)^{-\frac{1}{2}}\nabla_x\phi\Big)\cdot\nabla_p f = (N+1)f(S\phi), \qquad (1)$$

$$\partial_t^2\phi - \Delta_x\phi = -\mu(t,x), \qquad (2)$$

$$\mu(t,x) = \int_{\mathbb{R}^N} \frac{f(t,x,p)}{(1+|p|^2)^{\frac{1}{2}}}\,dp, \qquad (3)$$

where $S = \partial_t + v(p)\cdot\nabla_x$ is the free-transport operator. We consider the initial conditions

$$f(0,\cdot,\cdot) = f_0, \;\; \phi(0,\cdot) = \varphi_0, \;\; \partial_t\phi(0,\cdot) = \varphi_1. \qquad (4)$$

The system (1), (2), (3), (4) will be referred to as the Nordström-Vlasov system. Recently Calogero and Rein proved in [8] that classical solutions for the initial value problem with regular data exist at least locally in time in the three dimensional case. They give also a condition for global existence. The main tool consists in deriving representation formula for the time and spatial derivatives of ϕ, following the ideas in [15]. In one dimension they obtained the global existence and uniqueness for smooth compactly supported initial particle density $f_0 \in C^1_c(\mathbb{R}^2)$ and smooth initial conditions $\varphi_0 \in C^2_b(\mathbb{R})$, $\varphi_1 \in C^1_b(\mathbb{R})$ for the potential (the subscript b indicates that the functions are bounded together with their derivatives up to the indicated order). Assuming that the characteristic particle velocity is small compared to the light speed leads to the gravitational Vlasov-Poisson system, *i.e.,* a Vlasov equation coupled to a Poisson equation with opposite sign in front of the mass density. This comes by the fact that the forces are repulsive in the electromagnetic case and attractive in the gravitational case. This asymptotic regime was justified recently in [7]. The existence of global weak solution for the Nordström-Vlasov system in three dimensions has been studied as well, see [9]. One of the key points is the smoothing effect due to momentum averaging, see [12], [16].

The aim of this paper is to prove a more general existence and uniqueness result for the Nordström-Vlasov system in one dimension under less restrictive hypotheses.

Our method allows us to remove the support compactness of the initial density f_0 and the C^1 regularity of it. We assume that f_0 is bounded uniformly with respect to x by some function $g_0 = g_0(p)$ having finite kinetic energy and we construct a unique global solution (f, ϕ) where f is solution by characteristics (or mild solution) of (1) and ϕ is a classical solution of (2). We call such a solution (f, ϕ) a mild solution for the Nordström-Vlasov equations. The same method was used recently for studying a reduced Vlasov-Maxwell system for laser-plasma interaction, which shares some common features with the Nordström-Vlasov system, see [4], [10]. Results for initial boundary value problems can be obtained as well by this method, see [2] for an existence and uniqueness result of the mild solution for the one dimensional Vlasov-Poisson system. Our existence and uniqueness result is the following

Theorem 1.1 *Assume that* $\varphi_0 \in W^{2,\infty}(\mathbb{R})$, $\varphi_1 \in W^{1,\infty}(\mathbb{R})$, $f_0 \in L^1(\mathbb{R}^2)$ *and that there is some function* $g_0 \in L^\infty(\mathbb{R})$ *such that* $0 \le f_0(x,p) \le g_0(p), \quad \forall\ (x,p) \in \mathbb{R}^2$, $\mathrm{sign}(\cdot)g_0(\cdot)$ is nonincreasing on $\mathbb{R}$, *and* $\int_{\mathbb{R}} (1+|p|)g_0(p)\ dp < +\infty$. *Then there is a unique mild solution* $(f \ge 0, \phi) \in L^\infty(]0,T[; L^1(\mathbb{R}^2)) \times W^{2,\infty}(]0,T[\times\mathbb{R})$, $\forall\ T > 0$ *of the Nordström-Vlasov system* (1), (2), (3), (4) *with* $N = 1$. *Moreover* $\int_{\mathbb{R}} (1 + |p|)f(\cdot,\cdot,p)\ dp$ *belongs to* $L^\infty(]0,T[\times\mathbb{R})$, $\forall\ T > 0$.

Another motivation for this approach is that the proofs of our existence and uniqueness result provide, after minor changes, the finite speed propagation of the solution. This property inherit from the relativistic feature of the problem in hand. To our knowledge this is the first theoretical result in this direction.

Theorem 1.2 *Assume that the initial conditions* $(f_0^k, \varphi_0^k, \varphi_1^k)_{k\in\{1,2\}}$ *satisfy the hypotheses of Theorem* 1.1 *and denote by* $(f_k, \phi_k)_{k\in\{1,2\}}$ *the corresponding global solutions of the one dimensional Nordström-Vlasov system. Moreover suppose that* $\int_{\mathbb{R}}\ |p|^2 g_0^k(p)\ dp < +\infty,\ k \in \{1,2\}$. *Then for any* $R > 0$ *there is a constant* C_R *depending on* R, $\|\varphi_0^k\|_{W^{2,\infty}(\mathbb{R})}$, $\|\varphi_1^k\|_{W^{1,\infty}(\mathbb{R})}$, $\|f_0^k\|_{L^1(\mathbb{R}^2)}$, $\sum_{m=0}^{2} \|\ |p|^m g_0^k\|_{L^1(\mathbb{R})}$, $k \in \{1,2\}$ *such that*

$$\max_{t\in[0,R],|x|\le R-t} \left(|\phi_1 - \phi_2| + |\partial_x\phi_1 - \partial_x\phi_2| + |\partial_t\phi_1 - \partial_t\phi_2|\right)(t,x) \le C_R D_0^R,$$

where $D_0^R = \|\varphi_0^1 - \varphi_0^2\|_{W^{1,\infty}([-R,R])} + \|\varphi_1^1 - \varphi_1^2\|_{L^\infty([-R,R])} + \int_{-R}^R \int_{\mathbb{R}} (1+|p|)|f_0^1 - f_0^2| dpdx.$

Corollary 1.1 *Assume that the initial conditions* $(f_0^k, \varphi_0^k, \varphi_1^k)_{k\in\{1,2\}}$ *satisfy the hypotheses of Theorem* 1.1 *and denote by* $(f_k, \phi_k)_{k\in\{1,2\}}$ *the corresponding global solutions of the one dimensional Nordström-Vlasov system. Suppose also that* $f_0^1(x,p) = f_0^2(x,p)$, $\varphi_0^1(x) = \varphi_0^2(x), \varphi_1^1(x) = \varphi_1^2(x), x \in [-R,R], p \in \mathbb{R}$. *Then we have* $f_1(t,x,p) = f_2(t,x,p), \phi_1(t,x) = \phi_2(t,x), t \in [0,R], |x| \leq R - t, p \in \mathbb{R}$.

Hence, the motivation of the question we address is two-fold. First we justify the well posedness of the one dimensional Nordström-Vlasov system in a larger class of solutions. Roughly speaking we deal with particle densities with finite mass and kinetic energy (uniformly in space) which are closer to the typical physical candidates, than the compactly supported smooth functions. Second, we highlight the finite propagation speed of the solutions. For the numerical point of view this property has important consequences: it allows us to localize in space when computing them.

Our paper is organized as follows. In Section 2 we analyze the Vlasov equation. We introduce the equations of characteristics and we recall the notion of mild solution. We state also some important properties of the characteristics. The details of proof can be found in the Appendix. In Section 3 we construct the fixed point application and we obtain estimates for the first and second order derivatives. In Section 4 we prove the existence and uniqueness of mild solution for the Nordström-Vlasov equations in one dimension. We check also that this solution preserves the total energy. In the last section we establish the finite speed propagation property.

2 The Vlasov equation

In this section we assume that $\phi = \phi(t,x)$ is a given smooth function and we recall the notion of solution by characteristics for the Vlasov equation (1) in one dimension

$$\partial_t f + v(p)\cdot\partial_x f - \left((S\phi)p + (1+p^2)^{-\frac{1}{2}}\partial_x\phi\right)\cdot\partial_p f = 2f(S\phi),\ (t,x,p)\in]0,T[\times\mathbb{R}^2,\ (5)$$

with the initial condition

$$f(0,x,p) = f_0(x,p),\quad (x,p)\in\mathbb{R}^2. \tag{6}$$

For any $(t,x,p)\in[0,T]\times\mathbb{R}^2$ we introduce the system of characteristics for (5)

$$\frac{dX}{ds} = v(P(s)),\quad \frac{dP}{ds} = -P(s)\left(\partial_t\phi + v(P(s))\partial_x\phi\right)(s,X(s)) - \frac{\partial_x\phi(s,X(s))}{(1+|P(s)|^2)^{\frac{1}{2}}}, \tag{7}$$

with the conditions

$$X(s=t) = x,\quad P(s=t) = p. \tag{8}$$

Notice that if ϕ is a C^1 function and $\partial_t\phi, \partial_x\phi$ are Lipschitz with respect to $x\in\mathbb{R}$ uniformly on t in $[0,T]$, then there is a unique C^1 solution for (7), (8), denoted $(X(s;t,x,p), P(s;t,x,p))$ or simply $(X(s),P(s))$. The divergence of the field appearing in the right hand side terms of (7) is given by

$$\mathrm{Div}_{(x,p)}\left(v(p), -(S\phi)p - (1+p^2)^{-\frac{1}{2}}\partial_x\phi\right) = -\partial_t\phi - v(p)\partial_x\phi, \tag{9}$$

and by observing that $\frac{d}{ds}\phi(s,X(s)) = \partial_t\phi(s,X(s)) + v(P(s))\partial_x\phi(s,X(s))$ we deduce that the jacobian matrix $J(s) := \frac{\partial(X(s;t,x,p),P(s;t,x,p))}{\partial(x,p)}$ has the determinant given by

$$\det J(s) = \det\left(\frac{\partial(X(s;t,x,p),P(s;t,x,p))}{\partial(x,p)}\right) = e^{-\phi(s,X(s;t,x,p))+\phi(t,x)} \neq 0. \tag{10}$$

Multiplying (5) by $e^{-2\phi(t,x)}$ and by taking into account that

$$S(fe^{-2\phi}) = e^{-2\phi}Sf + fS(e^{-2\phi}) = e^{-2\phi}Sf - 2fe^{-2\phi}S\phi,$$

we obtain

$$\partial_t(fe^{-2\phi}) + v(p)\partial_x(fe^{-2\phi}) - \left((S\phi)p + (1+p^2)^{-\frac{1}{2}}\partial_x\phi\right)\partial_p(fe^{-2\phi}) = 0.$$

We deduce formally that $fe^{-2\phi}$ is constant along any characteristic of (7) and we have the usual definition

Definition 2.1 *Assume that* $\phi \in C^1([0,T] \times \mathbb{R})$, $\partial_t\phi, \partial_x\phi \in L^\infty(]0,T[;W^{1,\infty}(\mathbb{R}))$ *and* f_0 *is a measurable function on* $\mathbb{R}^2$. *The solution by characteristics (or mild solution) of* (5), (6) *is given by*

$$f(t,x,p) = f_0(X(0;t,x,p), P(0;t,x,p))e^{2\phi(t,x)-2\phi(0,X(0;t,x,p))}, \quad (t,x,p) \in [0,T] \times \mathbb{R}^2.$$

In the following proposition we recall some immediate properties of the mild solution

Proposition 2.1 *Assume that* $\phi \in C^1([0,T] \times \mathbb{R}) \cap L^\infty(]0,T[\times\mathbb{R})$, $\partial_t\phi, \partial_x\phi \in L^\infty(]0,T[;W^{1,\infty}(\mathbb{R}))$, f_0 *is measurable. Denote by* f *the mild solution of* (5), (6).

1) If f_0 *is nonnegative then* f *is nonnegative ;*

2) If f_0 *is bounded then* f *is bounded and we have*

$$\|f\|_{L^\infty(]0,T[\times\mathbb{R}^2)} \leq e^{4\|\phi\|_{L^\infty(]0,T[\times\mathbb{R})}}\|f_0\|_{L^\infty(\mathbb{R}^2)} \ ;$$

3) If $f_0 \in L^1(\mathbb{R}^2)$ *then* $f \in L^\infty(]0,T[;L^1(\mathbb{R}^2))$ *and*

$$\int_{\mathbb{R}}\int_{\mathbb{R}} |f(t,x,p)| \, dp \, dx \leq e^{2\|\phi\|_{L^\infty(]0,T[\times\mathbb{R})}}\|f_0\|_{L^1(\mathbb{R}^2)}.$$

In particular the conservation of the total mass holds

$$\int_{\mathbb{R}}\int_{\mathbb{R}} |f(t,x,p)|e^{-\phi(t,x)} \, dp \, dx = \int_{\mathbb{R}}\int_{\mathbb{R}} |f_0(x,p)|e^{-\phi(0,x)} \, dp \, dx, \quad t \in [0,T] \ ;$$

4) If $f_0 \in L^1(\mathbb{R}^2)$ *then for any continuous bounded function* ψ *we have*

$$\begin{aligned}\int_0^T\int_{\mathbb{R}}\int_{\mathbb{R}} f(t,x,p)\psi(t,x,p) \, dp \, dx \, dt &= \int_{\mathbb{R}}\int_{\mathbb{R}} f_0(x,p) \int_0^T \psi(t, X(t;0,x,p), P(t;0,x,p)) \\ &\times \ e^{\phi(t,X(t;0,x,p))-\phi(0,x)} \, dt \, dp \, dx \ ;\end{aligned}$$

5) If $(1+|p|)f_0 \in L^1(\mathbb{R}^2)$ *then* $(1+|p|)f \in L^\infty(]0,T[;L^1(\mathbb{R}^2))$ *and*

$$\begin{aligned}\int_{\mathbb{R}}\int_{\mathbb{R}}(1+|p|)|f(t,x,p)| \, dp \, dx &\leq e^{2\|\phi\|_{L^\infty}}\left(1 + e^{2\|\phi\|_{L^\infty}}\int_0^t \|\partial_x\phi(s)\|_{L^\infty} ds\right)\|f_0\|_{L^1(\mathbb{R}^2)} \\ &+ \ e^{4\|\phi\|_{L^\infty}}\int_{\mathbb{R}}\int_{\mathbb{R}} |p||f_0(x,p)| \, dp \, dx \ ;\end{aligned}$$

6) If $f_0 \in L^1(\mathbb{R}^2)$ *then the mild solution* f *satisfies the following weak formulation*

$$\begin{aligned}\int_0^T\int_{\mathbb{R}}\int_{\mathbb{R}} f(t,x,p)e^{-\phi(t,x)} &\left(\partial_t\theta + v(p)\partial_x\theta - \left(pS\phi + (1+p^2)^{-\frac{1}{2}}\partial_x\phi\right)\partial_p\theta\right) dp \, dx \, dt \\ &+ \int_{\mathbb{R}}\int_{\mathbb{R}} f_0(x,p)e^{-\phi(0,x)}\theta(0,x,p) \, dp \, dx = 0, \qquad (11)\end{aligned}$$

for any $\theta \in C_c^1([0,T[\times\mathbb{R}^2)$. *In particular, if* $(1+|p|)f_0 \in L^1(\mathbb{R}^2)$, *the above formulation holds for any* $\theta \in C_b^1([0,T]\times\mathbb{R}^2)$ *such that* $\theta(T,\cdot,\cdot)=0$.

Proof. The first and second statements are trivial.

3) Assume now that $f_0 \in L^1(\mathbb{R}^2)$ and for any $t \in [0,T]$ consider the change of variables $x = X(t;0,y,q)$, $p = P(t;0,y,q)$. By using (10) we have

$$\begin{aligned}\int_{\mathbb{R}}\int_{\mathbb{R}} |f(t,x,p)|\,dp\,dx &= \int_{\mathbb{R}}\int_{\mathbb{R}} |f(t,X(t),P(t))|e^{-\phi(t,X(t))+\phi(0,y)}\,dq\,dy\\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} |f_0(y,q)|e^{\phi(t,X(t))-\phi(0,y)}\,dq\,dy\\ &\le e^{2\|\phi\|_{L^\infty}}\|f_0\|_{L^1(\mathbb{R}^2)}.\end{aligned}$$

Similarly we obtain

$$\begin{aligned}\int_{\mathbb{R}}\int_{\mathbb{R}} |f(t,x,p)|e^{-\phi(t,x)}\,dp\,dx &= \int_{\mathbb{R}}\int_{\mathbb{R}} |f(t,X(t),P(t))|e^{-2\phi(t,X(t))+\phi(0,y)}\,dq\,dy\\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} |f_0(y,q)|e^{-\phi(0,y)}\,dq\,dy.\end{aligned}$$

4) Take now $\psi \in C_b^0([0,T]\times\mathbb{R}^2)$. By the previous point we already know that $f(t) \in L^1(\mathbb{R}^2)$ for any $t \in [0,T]$. By change of variables we obtain as before

$$\begin{aligned}\int_0^T\int_{\mathbb{R}}\int_{\mathbb{R}} f\psi\,dp\,dx\,dt &= \int_0^T\int_{\mathbb{R}}\int_{\mathbb{R}} f(t,X(t),P(t))\psi(t,X(t),P(t))e^{-\phi(t,X(t))+\phi(0,y)}\,dq\,dy\,dt\\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} f_0(y,q)\int_0^T \psi(t,X(t),P(t))e^{\phi(t,X(t))-\phi(0,y)}\,dt\,dq\,dy.\end{aligned}$$

5) By using one more time the change of variables $x = X(t;0,y,q), p = P(t;0,y,q)$ and (10) one gets

$$\begin{aligned}\int_{\mathbb{R}}\int_{\mathbb{R}}(1+|p|)|f|\,dp\,dx &= \int_{\mathbb{R}}\int_{\mathbb{R}}(1+|P(t)|)|f(t,X(t),P(t))|e^{-\phi(t,X(t))+\phi(0,y)}\,dq\,dy\\ &= \int_{\mathbb{R}}\int_{\mathbb{R}}(1+|P(t)|)|f_0(y,q)|e^{\phi(t,X(t))-\phi(0,y)}\,dq\,dy. \qquad (12)\end{aligned}$$

By (7) we have

$$\frac{d}{ds}\{P(s)e^{\phi(s,X(s))}\} = -\frac{e^{\phi(s,X(s))}}{(1+|P(s)|^2)^{\frac{1}{2}}}\partial_x\phi(s,X(s)), \qquad (13)$$

and therefore

$$P(t)e^{\phi(t,X(t))} = qe^{\phi(0,y)} - \int_0^t \frac{e^{\phi(s,X(s))}}{(1+|P(s)|^2)^{\frac{1}{2}}} \partial_x \phi(s,X(s)) ds.$$

We deduce that for any $t \in [0,T]$ we have

$$\begin{aligned} |P(t)| &\leq |q| e^{\phi(0,y)-\phi(t,X(t))} + e^{-\phi(t,X(t))} \int_0^t e^{\phi(s,X(s))} |\partial_x \phi(s,X(s))| \, ds \\ &\leq |q| e^{2\|\phi\|_{L^\infty}} + e^{2\|\phi\|_{L^\infty}} \int_0^t \|\partial_x \phi(s)\|_{L^\infty}. \end{aligned} \tag{14}$$

Combining (12), (14) yields

$$\begin{aligned} \int_{\mathbb{R}}\int_{\mathbb{R}} (1+|p|)|f(t,x,p)| \, dp \, dx \leq e^{2\|\phi\|_{L^\infty}} \left(1 + e^{2\|\phi\|_{L^\infty}} \int_0^t \|\partial_x \phi(s)\|_{L^\infty} ds\right) \|f_0\|_{L^1(\mathbb{R}^2)} \\ + e^{4\|\phi\|_{L^\infty}} \int_{\mathbb{R}}\int_{\mathbb{R}} |p||f_0(x,p)| \, dp \, dx. \end{aligned}$$

6) For any $\theta \in C_c^1([0,T[\times \mathbb{R}^2)$ consider

$$\psi(t,x,p) = e^{-\phi(t,x)} \left(\partial_t \theta + v(p) \partial_x \theta - \left(pS\phi + (1+p^2)^{-\frac{1}{2}} \partial_x \phi \right) \partial_p \theta \right).$$

Notice that $\psi(s,X(s),P(s)) = e^{-\phi(s,X(s))} \frac{d}{ds}\{\theta(s,X(s),P(s))\}$. By applying 4) one gets

$$\begin{aligned} \int_0^T\int_{\mathbb{R}}\int_{\mathbb{R}} & f e^{-\phi(t,x)} \left(\partial_t \theta + v(p) \partial_x \theta - \left(pS\phi + (1+p^2)^{-\frac{1}{2}} \partial_x \phi \right) \partial_p \theta \right) \, dp \, dx \, dt \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} f_0(x,p) \int_0^T e^{-\phi(0,x)} \frac{d}{ds}\{\theta(s,X(s),P(s))\} \, ds \, dp \, dx \\ &= -\int_{\mathbb{R}}\int_{\mathbb{R}} f_0(x,p) e^{-\phi(0,x)} \theta(0,x,p) \, dp \, dx. \end{aligned}$$

If $(1+|p|)f_0 \in L^1(\mathbb{R}^2)$ we know that $(1+|p|)f \in L^\infty(]0,T[; L^1(\mathbb{R}^2))$. Take $\chi \in C_c^1(\mathbb{R})$, $\chi(u) = 0$ if $|u| \geq 2$, $\chi(u) = 1$ if $|u| \leq 1$, $\chi \geq 0$ and $\chi_R(\cdot) = \chi(\frac{\cdot}{R})$, $R > 0$. In order to prove (11) for $\theta \in C_b^1([0,T] \times \mathbb{R}^2)$, $\theta(T,\cdot,\cdot) = 0$, apply them with $\theta_R = \theta \chi_R(x) \chi_R(p)$ and let $R \to +\infty$. □

Remark 2.1 *Notice that* (5) *can be written*

$$\partial_t(fe^{-\phi}) + \partial_x(v(p)fe^{-\phi}) - \partial_p \left((pS\phi + (1+p^2)^{-\frac{1}{2}} \partial_x \phi) f e^{-\phi} \right) = 0,$$

which justifies formally the weak formulation (11). *In particular the total mass* $\int_{\mathbb{R}}\int_{\mathbb{R}} f(t,x,p) e^{-\phi(t,x)} \, dp \, dx$ *is preserved for any* t.

We end this section with a continuous dependence result of the characteristics with respect to ϕ. It will be used in the next section. The proof is standard and it is postponed to the Appendix. We use the notation $\|u\|_{1,\infty} = \|u\|_\infty + \|u'\|_\infty$ for any function $u \in W^{1,\infty}(\mathbb{R})$.

Proposition 2.2 *Assume that* $\phi_k \in C^1([0,T] \times \mathbb{R}) \cap L^\infty(]0,T[\times\mathbb{R})$, $\partial_t\phi_k, \partial_x\phi_k \in L^\infty(]0,T[;W^{1,\infty}(\mathbb{R}))$, $k \in \{1,2\}$. *For any* $(x,p) \in \mathbb{R}^2$ *consider* $(X_k(t), P_k(t)) = (X_k(t;0,x,p), P_k(t;0,x,p))$ *the characteristics corresponding to* ϕ_k, $k \in \{1,2\}$. *Then for any* $t \in [0,T]$ *there is a constant* C *depending on* $\sup_{s\in[0,t],k\in\{1,2\}}\{\|\phi_k(s)\|_{L^\infty} + \|\partial_x\phi_k(s)\|_{L^\infty} + \|\partial_x^2\phi_k(s)\|_{L^\infty}\}$ *such that for any* $s \in [0,t]$ *we have*

$$\begin{aligned}|X_1(s) - X_2(s)| + |P_1(s)e^{\phi_1(s,X_1(s))} - P_2(s)e^{\phi_2(s,X_2(s))}| &\leq C\,|p(\phi_1 - \phi_2)(0,x)| \\ &+ C\int_0^t \|\phi_1(\tau) - \phi_2(\tau)\|_{1,\infty}\, d\tau.\end{aligned}$$

The following result is a direct consequence of the above proposition (see the Appendix for proof details)

Corollary 2.1 *Under the hypotheses of Proposition* 2.2*, there is a constant* C *depending on*

$$\sup_{s\in[0,T],k\in\{1,2\}}\{\|\phi_k(s)\|_\infty + \|\partial_x\phi_k(s)\|_\infty + \|\partial_t\phi_k(s)\|_\infty + \|\partial_x^2\phi_k(s)\|_\infty + \|\partial_{xt}^2\phi_k(s)\|_\infty\}$$

such that we have for any $t \in [0,T]$

$$\begin{aligned}\left|\sum_{k=1}^2(-1)^k\frac{e^{\phi_k(t,X_k(t))}}{(1+|P_k(t)|^2)^{\frac12}}\right| &\leq C\left(\|\phi_1(t) - \phi_2(t)\|_\infty + \int_0^t \|\phi_1(s) - \phi_2(s)\|_{1,\infty}\, ds\right) \\ &+ C\,|p(\phi_1 - \phi_2)(0,x)|,\end{aligned}$$

$$\begin{aligned}\left|\sum_{k=1}^2\frac{(-1)^k e^{\phi_k(t,X_k(t))}}{(1+|P_k(t)|^2)^{\frac12}}\partial_x\phi_k(t,X_k(t))\right| &\leq C(\|\phi_1(t) - \phi_2(t)\|_{1,\infty} + \int_0^t\|\phi_1(s) - \phi_2(s)\|_{1,\infty}ds) \\ &+ C\,|p(\phi_1 - \phi_2)(0,x)|,\end{aligned}$$

$$\begin{aligned}\left|\sum_{k=1}^2\frac{(-1)^k e^{\phi_k(t,X_k(t))}}{(1+|P_k(t)|^2)^{\frac12}}\partial_t\phi_k(t,X_k(t))\right| &\leq C(\|\phi_1(t) - \phi_2(t)\|_\infty + \|\partial_t\phi_1(t) - \partial_t\phi_2(t)\|_\infty \\ &+ \int_0^t\|\phi_1(s) - \phi_2(s)\|_{1,\infty}ds + |p(\phi_1 - \phi_2)(0,x)|).\end{aligned}$$

3 Fixed point application

Assume that $\varphi_0 \in W^{2,\infty}(\mathbb{R}), \varphi_1 \in W^{1,\infty}(\mathbb{R}),\ f_0 \in L^1(\mathbb{R}^2),\ f_0 \geq 0$ and take an arbitrary $T > 0$. Since ϕ satisfies a one dimensional wave equation, a natural definition for the fixed point application is $\mathcal{F}\phi = \tilde{\phi}$ for any ϕ in the set

$$\mathcal{D}_1 = \{\phi \in C^1([0,T]\times\mathbb{R}) : (\phi, \partial_t\phi)(0,\cdot) = (\varphi_0, \varphi_1), \partial_t\phi, \partial_x\phi \in L^\infty(]0,T[; W^{1,\infty}(\mathbb{R}))\},$$

where $\tilde{\phi}$ is given by d'Alembert formula

$$\begin{aligned} \tilde{\phi}(t,x) &= \frac{1}{2}\{\varphi_0(x+t) + \varphi_0(x-t)\} + \frac{1}{2}\int_{x-t}^{x+t} \varphi_1(y)\, dy \\ &- \frac{1}{2}\int_0^t \int_{x-(t-s)}^{x+(t-s)} \mu(s,y)\, dy\, ds, \quad (t,x) \in [0,T] \times \mathbb{R}, \end{aligned} \tag{15}$$

the function μ is given by

$$\mu(t,x) = \int_{\mathbb{R}} \frac{f(t,x,p)}{(1+p^2)^{\frac{1}{2}}}\, dp, \quad (t,x) \in [0,T] \times \mathbb{R}, \tag{16}$$

and f is the mild solution of (5), (6) corresponding to ϕ.

3.1 Estimate of $\mathcal{F}$

Since the initial density f_0 is nonnegative, we know that any mild solution f is nonnegative and by (15) we deduce the a priori estimate

$$\tilde{\phi}(t,x) \leq \|\varphi_0\|_\infty + t\|\varphi_1\|_\infty, \quad (t,x) \in [0,T] \times \mathbb{R}. \tag{17}$$

Actually we can restrict the application $\mathcal{F}$ to the set

$$\begin{aligned} \mathcal{D}_2 &= \mathcal{D}_1 \cap \{\phi \ :\ -\|\varphi_0\|_\infty - T\|\varphi_1\|_\infty - \frac{T}{2}e^{2\|\varphi_0\|_\infty + T\|\varphi_1\|_\infty}\|f_0\|_{L^1} \leq \phi(t,x) \\ &\leq \|\varphi_0\|_\infty + T\|\varphi_1\|_\infty,\ \forall (t,x) \in [0,T] \times \mathbb{R}\}, \end{aligned}$$

and we check easily that for any $\phi \in \mathcal{D}_2$ we have for $(t,x) \in [0,T] \times \mathbb{R}$

$$-\|\varphi_0\|_\infty - T\|\varphi_1\|_\infty - \frac{T}{2}e^{2\|\varphi_0\|_\infty + T\|\varphi_1\|_\infty}\|f_0\|_{L^1} \leq \mathcal{F}\phi(t,x) \leq \|\varphi_0\|_\infty + T\|\varphi_1\|_\infty. \tag{18}$$

Indeed, for any $\phi \in \mathcal{D}_2$, by using the definition of the mild solution f and the conservation of the total mass, cf. Proposition 2.1, we obtain for any $0 \le s \le t \le T$

$$\begin{aligned}
\int_{x-(t-s)}^{x+(t-s)} \mu(s,y)\, dy &\le \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{f(s,y,p)}{\sqrt{1+p^2}}\, dp\, dy \\
&\le e^{\sup \phi(s,\cdot)} \int_{\mathbb{R}}\int_{\mathbb{R}} f(s,y,p) e^{-\phi(s,y)}\, dp\, dy \\
&= e^{\sup \phi(s,\cdot)} \int_{\mathbb{R}}\int_{\mathbb{R}} f_0(y,p) e^{-\varphi_0(y)}\, dp\, dy \\
&\le e^{2\|\varphi_0\|_\infty + T\|\varphi_1\|_\infty} \|f_0\|_{L^1}.
\end{aligned}$$

Therefore (15), (17) imply (18).

3.2 Estimate of $D\mathcal{F}\phi$

The d'Alembert formula (15) implies

$$\begin{aligned}
\partial_t \tilde{\phi}(t,x) &= \frac{1}{2}\{\varphi_0'(x+t) - \varphi_0'(x-t)\} + \frac{1}{2}\{\varphi_1(x+t) + \varphi_1(x-t)\} \\
&- \frac{1}{2}\int_0^t \{\mu(s, x+t-s) + \mu(s, x-(t-s))\}\, ds, \qquad (19)
\end{aligned}$$

and

$$\begin{aligned}
\partial_x \tilde{\phi}(t,x) &= \frac{1}{2}\{\varphi_0'(x+t) + \varphi_0'(x-t)\} + \frac{1}{2}\{\varphi_1(x+t) - \varphi_1(x-t)\} \\
&- \frac{1}{2}\int_0^t \{\mu(s, x+t-s) - \mu(s, x-(t-s))\}\, ds. \qquad (20)
\end{aligned}$$

In order to estimate the L^∞ norms of $\partial_t \tilde{\phi}, \partial_x \tilde{\phi}$ we need to estimate the L^∞ norm of μ. In particular we have to assume that $\mu_0(\cdot) = \int_{\mathbb{R}} \frac{f_0(\cdot,p)}{(1+p^2)^{\frac{1}{2}}}\, dp \in L^\infty(\mathbb{R})$. It is convenient to suppose that there is some function $g_0 \in L^\infty(\mathbb{R})$ such that

$$0 \le f_0(x,p) \le g_0(p), \;\; (x,p) \in \mathbb{R}^2, \;\; \mathrm{sign}(\cdot) g_0(\cdot) \text{ is nonincreasing on } \mathbb{R}, \qquad (21)$$

and

$$\int_{\mathbb{R}} \frac{g_0(p)}{(1+p^2)^{\frac{1}{2}}}\, dp < +\infty. \qquad (22)$$

Proposition 3.1 *Assume that* $\varphi_0 \in W^{2,\infty}(\mathbb{R}), \varphi_1 \in W^{1,\infty}(\mathbb{R}), f_0 \in L^1(\mathbb{R}^2)$ *and* (21), (22) *hold. Then for any* $\phi \in \mathcal{D}_2$ *we have*

$$\|\mu(t)\|_{L^\infty} \leq C_1 + C_2 \int_0^t \|\partial_x \phi(\tau)\|_{L^\infty} \, d\tau, \ t \in [0,T],$$

where

$$C_1 = \frac{e^{6M}}{\sqrt{\min(1, e^{4M})}} \int_{\mathbb{R}} \frac{g_0(p)}{(1+p^2)^{\frac{1}{2}}} \, dp, \quad C_2 = 2e^{6M} \|g_0\|_{L^\infty},$$

and

$$M = \|\varphi_0\|_\infty + T \ \|\varphi_1\|_\infty + \frac{T}{2} e^{2\|\varphi_0\|_\infty + T \ \|\varphi_1\|_\infty} \|f_0\|_{L^1}.$$

Proof. For $(t,x,p) \in [0,T] \times \mathbb{R}^2$ consider $(X(s), P(s)) = (X(s;t,x,p), P(s;t,x,p))$, $s \in [0,T]$. By (13) we deduce that for any $s \in [0,T]$ we have

$$e^{\phi(s,X(s))} P(s) = e^{\phi(t,x)} p - \int_t^s \frac{e^{\phi(\tau, X(\tau))}}{(1+|P(\tau)|^2)^{\frac{1}{2}}} \partial_x \phi(\tau, X(\tau)) \, d\tau. \tag{23}$$

In particular we obtain

$$e^{\phi(0,X(0))} P(0) = e^{\phi(t,x)} p + \int_0^t \frac{e^{\phi(\tau, X(\tau))}}{(1+|P(\tau)|^2)^{\frac{1}{2}}} \partial_x \phi(\tau, X(\tau)) \, d\tau. \tag{24}$$

We denote by M a bound for $\|\phi\|_\infty$, for example

$$M = \|\varphi_0\|_\infty + T \ \|\varphi_1\|_\infty + \frac{T}{2} \ e^{2\|\varphi_0\|_\infty + T\|\varphi_1\|_\infty} \|f_0\|_{L^1},$$

and for any $t \in [0,T]$ we introduce the notation $R(t) = \int_0^t \|\partial_x \phi(\tau)\|_\infty \, d\tau$. Observe that for any $p \geq e^{2M} R(t)$ formula (24) implies $e^{\phi(0,X(0))} P(0) \geq e^{-M} p - e^M R(t) \geq 0$, and thus

$$P(0;t,x,p) \geq e^{-2M} p - R(t) \geq 0. \tag{25}$$

Similarly, if $p \leq -e^{2M} R(t)$, formula (24) implies $-e^{\phi(0,X(0))} P(0) \geq e^{-M}(-p) - e^M R(t) \geq 0$. In this case we obtain

$$-P(0;t,x,p) \geq e^{-2M}(-p) - R(t) \geq 0. \tag{26}$$

We can estimate μ as follows

$$\begin{aligned} \mu(t,x) &= \int_{\mathbb{R}} \frac{f(t,x,p)}{(1+p^2)^{\frac{1}{2}}} \mathbf{1}_{\{|p| < e^{2M} R(t)\}} \, dp + \int_{\mathbb{R}} \frac{f(t,x,p)}{(1+p^2)^{\frac{1}{2}}} \mathbf{1}_{\{|p| \geq e^{2M} R(t)\}} \, dp \\ &= \mu_1(t,x) + \mu_2(t,x), \quad (t,x) \in [0,T] \times \mathbb{R}. \end{aligned} \tag{27}$$

By the definition of the mild solution we have

$$\begin{aligned}
\mu_1(t,x) &= \int_{\mathbb{R}} \frac{f_0(X(0;t,x,p),P(0;t,x,p))}{(1+p^2)^{\frac{1}{2}}} e^{2\phi(t,x)-2\phi(0,X(0;t,x,p))} \mathbf{1}_{\{|p|<e^{2M}R(t)\}}\, dp \\
&\le e^{4M} \int_{\mathbb{R}} \frac{g_0(P(0;t,x,p))}{(1+p^2)^{\frac{1}{2}}} \mathbf{1}_{\{|p|<e^{2M}R(t)\}}\, dp \\
&\le 2e^{6M}\|g_0\|_\infty R(t), \quad (t,x) \in [0,T]\times\mathbb{R}. \qquad (28)
\end{aligned}$$

In order to estimate μ_2 observe that (25), (26) and the monotonicity of g_0 yields

$$f_0(X(0;t,x,p),P(0;t,x,p)) \le g_0(P(0;t,x,p)) \le g_0(e^{-2M}p - R(t)),\ \forall\, p \ge e^{2M}R(t),$$

respectively

$$f_0(X(0;t,x,p),P(0;t,x,p)) \le g_0(P(0;t,x,p)) \le g_0(e^{-2M}p + R(t)),\ \forall\, p \le -e^{2M}R(t).$$

Therefore we have as before

$$\begin{aligned}
\mu_2(t,x) &= \int_{\mathbb{R}} \frac{f(t,x,p)}{(1+p^2)^{\frac{1}{2}}} \mathbf{1}_{\{p\le -e^{2M}R(t)\}}\, dp + \int_{\mathbb{R}} \frac{f(t,x,p)}{(1+p^2)^{\frac{1}{2}}} \mathbf{1}_{\{p\ge e^{2M}R(t)\}}\, dp \\
&\le e^{4M} \sum_{k=1}^{2} \int_{\mathbb{R}} \frac{g_0(P(0;t,x,p))}{(1+p^2)^{\frac{1}{2}}} \mathbf{1}_{\{(-1)^k p \ge e^{2M}R(t)\}}\, dp \\
&\le e^{4M} \sum_{k=1}^{2} \int_{\mathbb{R}} \frac{g_0(e^{-2M}p - (-1)^k R(t))}{(1+p^2)^{\frac{1}{2}}} \mathbf{1}_{\{(-1)^k p \ge e^{2M}R(t)\}}\, dp \\
&= I^-(t) + I^+(t). \qquad (29)
\end{aligned}$$

By direct computations (use the changes of variable $e^{-2M}p + R(t) = q \le 0$, respectively $e^{-2M}p - R(t) = q \ge 0$) one gets

$$I^\pm(t) \le \frac{\pm e^{6M}}{\sqrt{\min(1,e^{4M})}} \int_0^{\pm\infty} \frac{g_0(p)}{(1+p^2)^{\frac{1}{2}}}\, dp. \qquad (30)$$

Finally from (27), (28), (29), (30) one gets

$$\|\mu(t)\|_\infty \le \frac{e^{6M}}{\sqrt{\min(1,e^{4M})}} \int_{\mathbb{R}} \frac{g_0(p)}{(1+p^2)^{\frac{1}{2}}}\, dp + 2e^{6M}\|g_0\|_\infty R(t), \quad t\in[0,T]. \qquad (31)$$

□

Now we obtain easily estimates for the L^∞ norms of $\partial_t \tilde{\phi}, \partial_x \tilde{\phi}$. Indeed, (20) and Proposition 3.1 imply for any $t \in [0,T]$

$$\begin{aligned}
\|\partial_x \tilde{\phi}(t)\|_\infty &\leq \|\varphi_0'\|_\infty + \|\varphi_1\|_\infty + \int_0^t \|\mu(s)\|_\infty ds \\
&\leq \|\varphi_0'\|_\infty + \|\varphi_1\|_\infty + \int_0^t \{C_1 + C_2 R(s)\} ds \\
&\leq \|\varphi_0'\|_\infty + \|\varphi_1\|_\infty + C_1 T + C_2 T \int_0^t \|\partial_x \phi(s)\|_\infty \, ds. \qquad (32)
\end{aligned}$$

We restrict the application $\mathcal{F}$ to

$$\mathcal{D}_3 = \mathcal{D}_2 \cap \{\phi \;:\; \max(\|\partial_x \phi(t)\|_\infty, \|\partial_t \phi(t)\|_\infty) \leq (\|\varphi_0'\|_\infty + \|\varphi_1\|_\infty + C_1 T) e^{C_2 T t}, t \in [0,T]\}.$$

By using (32) we check easily that for any $\phi \in \mathcal{D}_3$ we have

$$\|\partial_x \mathcal{F}\phi(t)\|_\infty = \|\partial_x \tilde{\phi}(t)\|_\infty \leq (\|\varphi_0'\|_\infty + \|\varphi_1\|_\infty + C_1 T) e^{C_2 T t}, \; \forall \, t \in [0,T]. \qquad (33)$$

Notice that for any $\phi \in \mathcal{D}_3$ we have by (19)

$$\begin{aligned}
\|\partial_t \mathcal{F}\phi(t)\|_\infty &\leq \|\varphi_0'\|_\infty + \|\varphi_1\|_\infty + \int_0^t \|\mu(s)\|_\infty ds \\
&\leq \|\varphi_0'\|_\infty + \|\varphi_1\|_\infty + C_1 T + C_2 T \int_0^t \|\partial_x \phi(s)\|_\infty \, ds \\
&\leq (\|\varphi_0'\|_\infty + \|\varphi_1\|_\infty + C_1 T) e^{C_2 T t}, \; \forall \, t \in [0,T]. \qquad (34)
\end{aligned}$$

By (31) we deduce also that for any $\phi \in \mathcal{D}_3$ we have

$$\begin{aligned}
\|\mu(t)\|_\infty &\leq C_1 + C_2 \int_0^t (\|\varphi_0'\|_\infty + \|\varphi_1\|_\infty + C_1 T) e^{C_2 T s} \, ds \\
&= C_1 + (\|\varphi_0'\|_\infty + \|\varphi_1\|_\infty + C_1 T) \frac{e^{C_2 T t} - 1}{T}. \qquad (35)
\end{aligned}$$

3.3 Estimate of $D^2 \mathcal{F}\phi$

We intend to estimate the second derivatives of $\tilde{\phi} = \mathcal{F}\phi$, for any $\phi \in \mathcal{D}_3$. We have

$$\begin{aligned}
\partial_x^2 \tilde{\phi}(t,x) &= \frac{1}{2}\{\varphi_0''(x+t) + \varphi_0''(x-t)\} + \frac{1}{2}\{\varphi_1'(x+t) - \varphi_1'(x-t)\} \\
&- \frac{1}{2}\partial_x D^+(t,x) + \frac{1}{2}\partial_x D^-(t,x), \qquad (36)
\end{aligned}$$

where $D^{\pm}(t,x) = \int_0^t \mu(s, x \pm (t-s))\, ds$. Obviously we have for any $t \in [0,T]$

$$\|\partial_x^2 \tilde{\phi}(t)\|_\infty \le \|\varphi_0''\|_\infty + \|\varphi_1'\|_\infty + \frac{1}{2}\|\partial_x D^+(t)\|_\infty + \frac{1}{2}\|\partial_x D^-(t)\|_\infty, \qquad (37)$$

and thus it remains to estimate the L^∞ norms of $\partial_x D^{\pm}$. For any test function $\psi \in C_c^1(\mathbb{R})$ we can write

$$\begin{aligned}\int_{\mathbb{R}} D^{\pm}(t,x)\psi'(x)\, dx &= \int_0^t\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{f(s, x\pm(t-s), p)}{(1+p^2)^{\frac{1}{2}}}\psi'(x)\, dp\, dx\, ds \\ &= \int_0^t\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{f(s,x,p)}{(1+p^2)^{\frac{1}{2}}}\psi'(x\mp(t-s))\, dp\, dx\, ds. \qquad (38)\end{aligned}$$

By Proposition 2.1 we can write

$$\int_{\mathbb{R}} D^{\pm}(t,x)\psi'(x)\, dx = \int_{\mathbb{R}}\int_{\mathbb{R}} f_0(x,p)e^{-\varphi_0(x)}\int_0^t e^{\phi(s,X(s))}\frac{\psi'(X(s)\mp(t-s))}{(1+|P(s)|^2)^{\frac{1}{2}}}\, ds\, dp\, dx, \qquad (39)$$

where $(X(s),P(s)) = (X(s;0,x,p), P(s;0,x,p))$, $(s,x,p) \in [0,t]\times\mathbb{R}^2$. We need to estimate the integrals

$$\begin{aligned}I^{\pm}(t,x,p) &= \int_0^t \frac{e^{\phi(s,X(s))}}{(1+|P(s)|^2)^{\frac{1}{2}}}\psi'(X(s)\mp(t-s))\, ds \\ &= \int_0^t \frac{e^{\phi(s,X(s))}}{(1+|P(s)|^2)^{\frac{1}{2}}}\frac{1}{v(P(s))\pm 1}\frac{d}{ds}\{\psi(X(s)\mp(t-s))\}\, ds \\ &= \int_0^t e^{\phi(s,X(s))}(-P(s)\pm(1+|P(s)|^2)^{\frac{1}{2}})\frac{d}{ds}\{\psi(X(s)\mp(t-s))\}\, ds \\ &= e^{\phi(s,X(s))}(-P(s)\pm(1+|P(s)|^2)^{\frac{1}{2}})\psi(X(s)\mp(t-s))|_0^t \\ &- \int_0^t \frac{d}{ds}\{e^{\phi(s,X(s))}(-P(s)\pm(1+|P(s)|^2)^{\frac{1}{2}})\}\psi(X(s)\mp(t-s))\, ds \\ &= I_1^{\pm}\psi(X(t)) - I_2^{\pm}\psi(x\mp t) - \int_0^t I_3^{\pm}(s)\psi(X(s)\mp(t-s))\, ds, \qquad (40)\end{aligned}$$

where

$$I_1^{\pm} := e^{\phi(t,X(t))}(-P(t)\pm(1+|P(t)|^2)^{\frac{1}{2}}), \quad I_2^{\pm} = e^{\varphi_0(x)}(-p\pm(1+p^2)^{\frac{1}{2}}),$$

and

$$I_3^{\pm}(s) = \frac{d}{ds}\{e^{\phi(s,X(s))}(-P(s)\pm(1+|P(s)|^2)^{\frac{1}{2}})\}.$$

The terms $I_1^\pm, I_2^\pm$ can be estimated by

$$|I_1^\pm| \leq 2e^{\|\phi\|_\infty}(1+|P(t)|), \ \ |I_2^\pm| \leq 2e^{\|\varphi_0\|_\infty}(1+|p|). \tag{41}$$

By using (7) and the bounds for the L^∞ norms of $\phi, \partial_x\phi, \partial_t\phi$ one gets by direct computations that

$$|I_3^\pm(s)| \leq C(1+|P(s)|), \ \forall \ 0 \leq s \leq t \leq T, \tag{42}$$

for some constant C depending only on the initial conditions $\varphi_0, \varphi_1, f_0$ and T. Therefore combining (39), (40), (41), (42) yields

$$\begin{aligned} \left|\int_{\mathbb{R}} D^\pm(t,x)\psi'(x)\ dx\right| \ &\leq \ C\int_{\mathbb{R}}\int_{\mathbb{R}} f_0(x,p)(1+|p|)|\psi(x\mp t)|\ dp\, dx \\ &+ \ C\int_{\mathbb{R}}\int_{\mathbb{R}} f_0(x,p)(1+|P(t)|)|\psi(X(t))|\ dp\, dx \\ &+ \ C\int_{\mathbb{R}}\int_{\mathbb{R}} f_0(x,p)\int_0^t (1+|P(s)|)|\psi(X(s)\mp(t-s))|\ dp\, dx, \end{aligned} \tag{43}$$

for any $t \in [0,T]$, $\psi \in C_c^1(\mathbb{R})$ and for some constant depending only on the initial conditions and T. We will use the following lemma

Lemma 3.1 *Assume that* $\varphi_0 \in W^{2,\infty}(\mathbb{R})$, $\varphi_1 \in W^{1,\infty}(\mathbb{R})$, $f_0 \in L^1(\mathbb{R}^2)$ *and that* (21) *holds for some function* $g_0 \in L^\infty(\mathbb{R})$ *satisfying*

$$\int_{\mathbb{R}}(1+|p|)g_0(p)\ dp < +\infty. \tag{44}$$

Then for any $\phi \in \mathcal{D}_3$ *we have* $\left\|\int_{\mathbb{R}}(1+|p|)f(\cdot,\cdot,p)\ dp\right\|_{L^\infty(]0,T[\times\mathbb{R})} \leq C_3$, *where* f *is the mild solution of* (5), (6) *corresponding to* ϕ *and* C_3 *depends only on the initial conditions and* T.

Proof. It is very similar to those of Proposition 3.1. For any $(t,x) \in [0,T]\times\mathbb{R}$ let $k(t,x) = \int_{\mathbb{R}}(1+|p|)f(t,x,p)\ dp$. We have

$$\begin{aligned} k(t,x) \ &= \ \int_{\mathbb{R}}(1+|p|)f(t,x,p)\mathbf{1}_{\{|p|<e^{2M}R(t)\}}\ dp + \int_{\mathbb{R}}(1+|p|)f(t,x,p)\mathbf{1}_{\{|p|\geq e^{2M}R(t)\}}\ dp \\ &= \ k_1(t,x)+k_2(t,x), \end{aligned} \tag{45}$$

where $M = \|\varphi_0\|_\infty + T\|\varphi_1\|_\infty + \frac{T}{2}e^{2\|\varphi_0\|_\infty + T\|\varphi_1\|_\infty}\|f_0\|_{L^1}$ and $R(t) = \int_0^t \|\partial_x \phi(s)\|_\infty \, ds$. Note that since ϕ belongs to $\mathcal{D}_3$, $\sup_{t\in[0,T]} R(t)$ can be bounded by some constant depending only on the initial conditions (observe that hypothesis (44) is stronger than (22)) and T. For the first term k_1 we can write

$$\begin{aligned} k_1(t,x) &= \int_{\mathbb{R}} f_0(X(0;t,x,p), P(0;t,x,p)) e^{2\phi(t,x)-2\phi(0,X(0))}(1+|p|)\mathbf{1}_{\{|p|<e^{2M}R(t)\}} \, dp \\ &\leq 2e^{6M}(1+e^{2M}R(t))R(t)\|g_0\|_\infty. \end{aligned} \tag{46}$$

For the second term k_2 we have

$$\begin{aligned} k_2(t,x) &\leq e^{4M}\sum_{k=1}^{2}\int_{\mathbb{R}} g_0(P(0;t,x,p))(1+|p|)\mathbf{1}_{\{(-1)^k p \geq e^{2M}R(t)\}} \, dp \\ &\leq e^{4M}\sum_{k=1}^{2}\int_{\mathbb{R}} g_0(e^{-2M}p - (-1)^k R(t))(1+|p|)\mathbf{1}_{\{(-1)^k p \geq e^{2M}R(t)\}} \, dp \\ &\leq e^{6M}\max\{e^{2M}, 1+e^{2M}R(t)\}\int_{\mathbb{R}}(1+|p|)g_0(p) \, dp. \end{aligned} \tag{47}$$

Our conclusion follows from (45), (46), (47). □

We prove also a more general result for later use

Lemma 3.2 *Assume that $\varphi_0 \in W^{2,\infty}(\mathbb{R})$, $\varphi_1 \in W^{1,\infty}(\mathbb{R})$, $f_0 \in L^1(\mathbb{R}^2)$ and that (21) holds for some function $g_0 \in L^\infty(\mathbb{R})$ satisfying $\int_{\mathbb{R}} |p|^{m+1} g_0(p) \, dp < +\infty$ with $m \in \mathbb{N}$. Consider $r : \mathbb{R}^2 \to [0,+\infty[$ verifying $\int_{\mathbb{R}} r(x,p) \, dx \leq \alpha|p| + \beta$, $\forall\, p \in \mathbb{R}$. Then for any $\phi \in \mathcal{D}_3$ we have $\left\| \int_{\mathbb{R}}\int_{\mathbb{R}} |p|^m f(\cdot,x,p) r(x,p) \, dp \, dx \right\|_{L^\infty(]0,T[)} \leq (\alpha+\beta)C$, where f is the mild solution of (5), (6) corresponding to ϕ and C depends only on the initial conditions and T.*

Proof. With the same notations as before we have for any $t \in [0,T]$

$$\begin{aligned} \int_{\mathbb{R}}\int_{\mathbb{R}} |p|^m f(t,x,p) r(x,p) \, dp \, dx &= \int_{\mathbb{R}}\int_{\mathbb{R}} |p|^m f(t,x,p) r(x,p) \mathbf{1}_{\{|p|<e^{2M}R(t)\}} \, dp \, dx \\ &+ \int_{\mathbb{R}}\int_{\mathbb{R}} |p|^m f(t,x,p) r(x,p) \mathbf{1}_{\{|p|\geq e^{2M}R(t)\}} \, dp \, dx \\ &= I_1^m(t) + I_2^m(t). \end{aligned} \tag{48}$$

For the first term we have

$$\begin{aligned} I_1^m(t) &\leq e^{4M}\|g_0\|_\infty \int_{\mathbb{R}} |p|^m \mathbf{1}_{\{|p|<e^{2M}R(t)\}} \left(\int_{\mathbb{R}} r(x,p)\,dx\right)\,dp \\ &\leq (\alpha+\beta)C_1^m, \end{aligned} \tag{49}$$

with $C_1^m = 2e^{2M(3+m)}(R(T))^{m+1}\max\{1, e^{2M}R(T)\}\|g_0\|_\infty$. For the second term we can write

$$\begin{aligned} I_2^m(t) &\leq e^{4M}\sum_{k=1}^{2}\int_{\mathbb{R}}\int_{\mathbb{R}} |p|^m g_0(P(0;t,x,p))r(x,p)\mathbf{1}_{\{(-1)^k p\geq e^{2M}R(t)\}}\,dp\,dx \\ &\leq e^{4M}\sum_{k=1}^{2}\int_{\mathbb{R}}\int_{\mathbb{R}} |p|^m g_0(e^{-2M}p-(-1)^k R(t))r(x,p)\mathbf{1}_{\{(-1)^k p\geq e^{2M}R(t)\}}\,dp\,dx \\ &\leq (\alpha+\beta)C_2^m, \end{aligned} \tag{50}$$

with $C_2^m = e^{2M(3+m)}\max\{\int_{\mathbb{R}}(|p|+R(T))^m g_0(p)\,dp, e^{2M}\int_{\mathbb{R}}(|p|+R(T))^{m+1}g_0(p)\,dp\}$. The conclusion follows by (48), (49), (50). □

Remark 3.1 *If $\alpha = 0$ the conclusion of Lemma* 3.2 *holds true by assuming only $\int_{\mathbb{R}} |p|^m g_0(p)\,dp < +\infty$.*

Now, under the hypotheses of Lemma 3.1, by using (43) we can find estimates for the L^∞ norms of $\partial_x D^\pm$. Indeed, for the first term in the right hand side of (43) we have for any $t \in [0,T]$

$$\begin{aligned} \int_{\mathbb{R}}\int_{\mathbb{R}}(1+|p|)f_0(x,p)|\psi(x\mp t)|\,dp\,dx &= \int_{\mathbb{R}}|\psi(x\mp t)|\int_{\mathbb{R}}(1+|p|)f_0(x,p)\,dp\,dx \\ &\leq \int_{\mathbb{R}}|\psi(x\mp t)|\int_{\mathbb{R}}(1+|p|)g_0(p)\,dp\,dx \\ &= \int_{\mathbb{R}} g_0(p)(1+|p|)\,dp\,\|\psi\|_{L^1}. \end{aligned} \tag{51}$$

For the other terms in (43) let us estimate $\int_{\mathbb{R}}\int_{\mathbb{R}} f_0(x,p)(1+|P(s)|)|\psi(X(s)\mp(t-s))|\,dp\,dx$ for any $0 \leq s \leq t \leq T$. By changing variables along characteristics (see

Proposition 2.1) we obtain

$$\begin{aligned}
&\int_{\mathbb{R}}\int_{\mathbb{R}} f_0(x,p)(1+|P(s)|)|\psi(X(s)\mp(t-s))|\,dp\,dx \\
&\quad= \int_{\mathbb{R}}\int_{\mathbb{R}} f(s,X(s),P(s))e^{2\varphi_0(x)-2\phi(s,X(s))}(1+|P(s)|)|\psi(X(s)\mp(t-s))|\,dp\,dx \\
&\quad= \int_{\mathbb{R}}\int_{\mathbb{R}} f(s,X(s),P(s))e^{\varphi_0(x)-\phi(s,X(s))}(1+|P(s)|)|\psi(X(s)\mp(t-s))|\det J(s)\,dp\,dx \\
&\quad\le e^{2\|\phi\|_\infty}\int_{\mathbb{R}}|\psi(X\mp(t-s))|\int_{\mathbb{R}} f(s,X,P)(1+|P|)\,dP\,dX \\
&\quad\le e^{2\|\phi\|_\infty} C_3\|\psi\|_{L^1}. \qquad (52)
\end{aligned}$$

Combining (43), (51), (52) we obtain that $\left|\int_{\mathbb{R}} D^{\pm}(t,x)\psi'(x)\,dx\right| \le C_4\|\psi\|_{L^1}$, $\forall\ \psi \in C_c^1(\mathbb{R})$, for some constant C_4 depending only on the initial conditions and T. We deduce that $\|\partial_x D^{\pm}\|_\infty \le C_4$ and therefore (37) implies

$$\|\partial_x^2\tilde{\phi}\|_\infty \le \|\varphi_0''\|_\infty + \|\varphi_1'\|_\infty + C_4 =: C_5. \qquad (53)$$

Observe that from (19) we have

$$\begin{aligned}
\partial_{xt}^2\tilde{\phi}(t,x) &= \frac{1}{2}\{\varphi_0''(x+t)-\varphi_0''(x-t)\} + \frac{1}{2}\{\varphi_1'(x+t)+\varphi_1'(x-t)\} \\
&- \frac{1}{2}\partial_x D^+(t,x) - \frac{1}{2}\partial_x D^-(t,x),
\end{aligned}$$

and therefore we obtain the same bound $\|\partial_{xt}^2\tilde{\phi}\|_\infty \le \|\varphi_0''\|_\infty + \|\varphi_1'\|_\infty + C_4 = C_5$. For estimating the second derivative $\partial_t^2\tilde{\phi}$ we can write

$$|\partial_t^2\tilde{\phi}(t,x)| = |\partial_x^2\tilde{\phi}(t,x) - \mu(t,x)| \le C_5 + \|\mu\|_\infty.$$

By using (35) one gets

$$\|\partial_t^2\tilde{\phi}\|_\infty \le C_6, \qquad (54)$$

for some constant depending only on the initial conditions and T.

Remark 3.2 *The computations of the paragraphs* 3.1, 3.2, 3.3 *show that under the hypotheses of Lemma* 3.1 $\mathcal{D}_3$ *is left invariant by the application* $\mathcal{F}$ *and we have*

$$\sup_{\phi\in\mathcal{D}_3}\left\{\|\partial_x^2\tilde{\phi}\|_\infty, \|\partial_{xt}^2\tilde{\phi}\|_\infty, \|\partial_t^2\tilde{\phi}\|_\infty, \left\|\int_{\mathbb{R}} f(\cdot,\cdot,p)(1+|p|)\,dp\right\|_\infty\right\} < +\infty,$$

where $\tilde{\phi} = \mathcal{F}\phi$ *and* f *is the mild solution of* (5), (6). *It is convenient to restrict one more time the application* $\mathcal{F}$ *to the set*

$$\mathcal{D}_4 = \mathcal{D}_3 \cap \{\phi \;:\; \max\{\|\partial_x^2 \phi\|_\infty, \|\partial_{xt}^2 \phi\|_\infty\} \leq C_5, \|\partial_t^2 \phi\|_\infty \leq C_6\},$$

which is also left invariant by $\mathcal{F}$.

3.4 Estimate of $\mathcal{F}\phi_1 - \mathcal{F}\phi_2$

The following step is to estimate $\mathcal{F}\phi_1 - \mathcal{F}\phi_2$ in terms of $\phi_1 - \phi_2$ for any $\phi_1, \phi_2 \in \mathcal{D}_4$. Consider $\phi_k \in \mathcal{D}_4$, $k \in \{1, 2\}$ and denote by $f_k, k \in \{1, 2\}$ the mild solutions of (5), (6) corresponding to ϕ_k, $k \in \{1, 2\}$. For any $\psi \in L^1(\mathbb{R})$ and $t \in [0, T]$ we have

$$\begin{aligned} \int_{\mathbb{R}} (\tilde{\phi}_1(t,y) - \tilde{\phi}_2(t,y))\psi(y)\, dy &= -\frac{1}{2} \int_{\mathbb{R}} \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{(f_1 - f_2)(s,x,p)}{(1+p^2)^{\frac{1}{2}}} \psi(y) \\ &\times \mathbf{1}_{\{|x-y|<t-s\}}\, dp\, dx\, ds\, dy \qquad (55) \\ &= -\frac{1}{2} \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} (f_1 - f_2)(s,x,p) \frac{\theta(s,x)}{(1+p^2)^{\frac{1}{2}}}\, dp\, dx\, ds, \end{aligned}$$

where $\theta(s,x) = \int_{x-(t-s)}^{x+(t-s)} \psi(y)\, dy$. Denote by $(X_k(s), P_k(s))$ the characteristics corresponding to ϕ_k, $k \in \{1, 2\}$. We obtain by Proposition 2.1

$$\begin{aligned} 2 \int_{\mathbb{R}} (\tilde{\phi}_1(t,y) - \tilde{\phi}_2(t,y))\psi(y)\, dy &= \sum_{k=1}^{2} (-1)^k \int_{\mathbb{R}} \int_{\mathbb{R}} f_0(x,p) e^{-\varphi_0(x)} \int_0^t \frac{\theta(s, X_k(s))}{(1+|P_k(s)|^2)^{\frac{1}{2}}} \\ &\times e^{\phi_k(s, X_k(s))}\, ds\, dp\, dx. \qquad (56) \end{aligned}$$

By using Remark 3.2 and Corollary 2.1 (with ϕ_1, ϕ_2 satisfying $\phi_1(0) = \phi_2(0) = \varphi_0$) we deduce that there is a constant C depending only on the initial conditions and T such that

$$\left| \sum_{k=1}^{2} (-1)^k \frac{e^{\phi_k(s, X_k(s))}}{(1+|P_k(s)|^2)^{\frac{1}{2}}} \right| \leq C \left(\|\phi_1(s) - \phi_2(s)\|_\infty + \int_0^s \|\phi_1(\tau) - \phi_2(\tau)\|_{1,\infty}\, d\tau \right). \qquad (57)$$

We deduce that

$$
\begin{aligned}
|\sum_{k=1}^{2}(-1)^k \frac{e^{\phi_k(s,X_k(s))}}{(1+|P_k(s)|^2)^{\frac{1}{2}}}\theta(s,X_k(s))| &\le |\theta(s,X_1(s))| \left|\sum_{k=1}^{2}(-1)^k \frac{e^{\phi_k(s,X_k(s))}}{(1+|P_k(s)|^2)^{\frac{1}{2}}}\right| \\
&+ \; e^{\|\phi_2(s)\|_\infty} \left|\sum_{k=1}^{2}(-1)^k\theta(s,X_k(s))\right| \\
&\le \; C\|\psi\|_{L^1}\left(\|\phi_1(s)-\phi_2(s)\|_\infty + \int_0^s \|\phi_1(\tau)-\phi_2(\tau)\|_{1,\infty}\, d\tau\right) \\
&+ \; C\left|\int_{X_1(s)-(t-s)}^{X_2(s)-(t-s)} \psi(y)\; dy\right| + C\left|\int_{X_1(s)+(t-s)}^{X_2(s)+(t-s)} \psi(y)\; dy\right|. \qquad (58)
\end{aligned}
$$

Combining (56), (58) we obtain

$$
\begin{aligned}
2\left|\int_{\mathbb{R}}(\tilde{\phi}_1-\tilde{\phi}_2)(t,y)\psi(y)\; dy\right| \;&\le\; C\int_0^t\int_{\mathbb{R}}\int_{\mathbb{R}} f_0\|\psi\|_{L^1}(\|\phi_1(s)-\phi_2(s)\|_\infty \\
&+\; \int_0^s \|\phi_1(\tau)-\phi_2(\tau)\|_{1,\infty}\, d\tau)\; dp\, dx\, ds \\
&+\; C\int_0^t\int_{\mathbb{R}}\int_{\mathbb{R}} f_0 \left|\int_{X_1(s)-(t-s)}^{X_2(s)-(t-s)}\psi(y)\; dy\right|\; dp\, dx\, ds \\
&+\; C\int_0^t\int_{\mathbb{R}}\int_{\mathbb{R}} f_0 \left|\int_{X_1(s)+(t-s)}^{X_2(s)+(t-s)}\psi(y)\; dy\right|\; dp\, dx\, ds \\
&=\; I_1(t)+I_2(t)+I_3(t). \qquad (59)
\end{aligned}
$$

For the first term in the right hand side of (59) we have for some constant C (depending on the initial conditions and T)

$$
I_1(t) \le C\|\psi\|_{L^1}\|f_0\|_{L^1}\int_0^t \|\phi_1(s)-\phi_2(s)\|_{1,\infty}\, ds. \qquad (60)
$$

In order to estimate the second term in the right hand side of (59) observe that for any $0 \le s \le t \le T$ we have

$$
\begin{aligned}
\int_{\mathbb{R}}\int_{\mathbb{R}} f_0 \left|\int_{X_1(s)-(t-s)}^{X_2(s)-(t-s)}\psi(y)\; dy\right|\; dp\, dx &\le \int_{\mathbb{R}}\int_{\mathbb{R}} f_0|\psi(y)|\mathbf{1}_{\{|y-(X_1(s)-(t-s))|<|X_1(s)-X_2(s)|\}} dy\; dp\, dx \\
&= \int_{\mathbb{R}}|\psi(y)|\int_{\mathbb{R}}\int_{\mathbb{R}} f_0\mathbf{1}_{\{|y-(X_1(s)-(t-s))|<|X_1(s)-X_2(s)|\}}\; dp\, dx\; dy. \qquad (61)
\end{aligned}
$$

Since $\phi_1(0) = \phi_2(0) = \varphi_0$ we deduce by Proposition 2.2 that $|X_1(s) - X_2(s)| \le CR(s)$ where $R(s) = \int_0^s \|\phi_1(\tau) - \phi_2(\tau)\|_{1,\infty}\, d\tau$. By using Lemma 3.1 we obtain after the change of variables along characteristics (cf. Proposition 2.1)

$$\begin{aligned}
\int_{\mathbb{R}}\int_{\mathbb{R}} f_0\ \mathbf{1}_{\{|y-(X_1(s)-(t-s))|<|X_1(s)-X_2(s)|\}}\, dp\, dx &\le \int_{\mathbb{R}}\int_{\mathbb{R}} f_0 \mathbf{1}_{\{|y-(X_1(s)-(t-s))|<CR(s)\}}\, dp\, dx \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} f_1(s, X_1(s), P_1(s)) e^{\varphi_0(x)-\phi_1(s,X_1(s))} \mathbf{1}_{\{|y-(X_1(s)-(t-s))|<CR(s)\}} \det J_1(s)\, dp\, dx \\
&\le e^{\|\varphi_0\|_\infty + \|\phi_1\|_\infty} \int_{\mathbb{R}}\int_{\mathbb{R}} f_1(s, X, P) \mathbf{1}_{\{|y-(X-(t-s))|<CR(s)\}}\, dP\, dX \\
&= e^{\|\varphi_0\|_\infty + \|\phi_1\|_\infty} \int_{\mathbb{R}} \mathbf{1}_{\{|y-(X-(t-s))|<CR(s)\}} \int_{\mathbb{R}} f_1(s, X, P)\, dP\, dX \\
&\le 2Ce^{\|\varphi_0\|_\infty + \|\phi_1\|_\infty} \left\| \int_{\mathbb{R}} f_1(\cdot, \cdot, p)\, dp \right\|_\infty \int_0^s \|\phi_1(\tau) - \phi_2(\tau)\|_{1,\infty}\, d\tau. \qquad (62)
\end{aligned}$$

From (61), (62) we obtain for any $t \in [0, T]$

$$I_2(t) \le C\|\psi\|_{L^1} \int_0^t \|\phi_1(s) - \phi_2(s)\|_{1,\infty}\, ds, \qquad (63)$$

for some constant depending on the initial conditions and T. A similar estimate holds for the term $I_3(t)$ and finally (59) implies

$$\left| \int_{\mathbb{R}} (\tilde{\phi}_1 - \tilde{\phi}_2)(t, y)\psi(y)\, dy \right| \le C\|\psi\|_{L^1} \int_0^t \|\phi_1(s) - \phi_2(s)\|_{1,\infty}\, ds, \qquad (64)$$

for any $t \in [0, T]$ and $\psi \in L^1(\mathbb{R})$. We deduce that for any $\phi_1, \phi_2 \in \mathcal{D}_4$

$$\|\mathcal{F}\phi_1(t) - \mathcal{F}\phi_2(t)\|_\infty \le C \int_0^t \|\phi_1(s) - \phi_2(s)\|_{1,\infty}\, ds. \qquad (65)$$

3.5 Estimate of $\partial_x \mathcal{F}\phi_1 - \partial_x \mathcal{F}\phi_2$

We need to estimate also the difference of spatial derivatives $\partial_x \mathcal{F}\phi_1$, $\partial_x \mathcal{F}\phi_2$, in terms of $\phi_1 - \phi_2$ for any $\phi_1, \phi_2 \in \mathcal{D}_4$. With the notations of the previous paragraphs, by using (20), we have for any function $\psi \in L^1(\mathbb{R})$

$$\begin{aligned}
\int_{\mathbb{R}} (\partial_x \tilde{\phi}_1 - \partial_x \tilde{\phi}_2)\psi \, dx &= \frac{1}{2}\int_0^t \int_{\mathbb{R}} \{\mu_1(s, x-(t-s)) - \mu_2(s, x-(t-s))\}\psi(x)\, dxds \\
&- \frac{1}{2}\int_0^t \int_{\mathbb{R}} \{\mu_1(s, x+(t-s)) - \mu_2(s, x+(t-s))\}\psi(x)\, dxds \\
&= \frac{1}{2}\int_0^t \int_{\mathbb{R}}\int_{\mathbb{R}} (f_1 - f_2)(s, x-(t-s), p)\frac{\psi(x)}{(1+p^2)^{\frac{1}{2}}}\, dp\, dx\, ds \\
&- \frac{1}{2}\int_0^t \int_{\mathbb{R}}\int_{\mathbb{R}} (f_1 - f_2)(s, x+(t-s), p)\frac{\psi(x)}{(1+p^2)^{\frac{1}{2}}}\, dp\, dx\, ds \\
&= \frac{1}{2} I_4(t) - \frac{1}{2} I_5(t). \qquad (66)
\end{aligned}$$

We analyze only the term $I_4(t)$. The estimate for $I_5(t)$ follows in similar way. By Proposition 2.1 we have

$$\begin{aligned}
I_4(t) &= \int_0^t \int_{\mathbb{R}}\int_{\mathbb{R}} (f_1 - f_2)(s, x, p)\frac{\psi(x+(t-s))}{(1+p^2)^{\frac{1}{2}}}\, dp\, dx\, ds \\
&= -\int_{\mathbb{R}}\int_{\mathbb{R}} f_0(x,p)e^{-\varphi_0(x)} \int_0^t \sum_{k=1}^{2} (-1)^k \frac{e^{\phi_k(s, X_k(s))}}{(1+|P_k(s)|^2)^{\frac{1}{2}}}\psi(X_k(s)+(t-s)) ds\, dp\, dx \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} f_0(x,p)e^{-\varphi_0(x)} \int_0^t \sum_{k=1}^{2} (-1)^k ((1+|P_k(s)|^2)^{\frac{1}{2}} + P_k(s)) e^{\phi_k(s, X_k(s))} \\
&\times \frac{d}{ds}\int_{X_k(t)}^{X_k(s)+(t-s)} \psi(u)\, du\, ds\, dp\, dx \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} f_0(x,p)e^{-\varphi_0(x)} I_6(t,x,p)\, dp\, dx. \qquad (67)
\end{aligned}$$

After integration by parts with respect to the variable s one gets

$$\begin{aligned}
I_6(t,x,p) &= \sum_{k=1}^{2} (-1)^k ((1+p^2)^{\frac{1}{2}} + p) e^{\varphi_0(x)} \int_{x+t}^{X_k(t)} \psi(u) du \qquad (68) \\
&- \int_0^t \sum_{k=1}^{2} (-1)^k \frac{d}{ds}\left\{((1+|P_k(s)|^2)^{\frac{1}{2}} + P_k(s)) e^{\phi_k(s, X_k(s))}\right\} \int_{X_k(t)}^{X_k(s)+(t-s)} \psi(u) du\, ds.
\end{aligned}$$

By direct computations we obtain

$$\frac{d}{ds}\left\{\left((1+|P_k(s)|^2)^{\frac{1}{2}} + P_k(s)\right) e^{\phi_k(s, X_k(s))}\right\} = \frac{e^{\phi_k(s, X_k(s))}}{(1+|P_k(s)|^2)^{\frac{1}{2}}}(\partial_t \phi_k - \partial_x \phi_k). \quad (69)$$

We can write

$$
\begin{aligned}
& \left|\sum_{k=1}^{2}(-1)^k \frac{d}{ds}\left\{\left((1+|P_k(s)|^2)^{\frac{1}{2}}+P_k(s)\right)e^{\phi_k(s,X_k(s))}\right\}\int_{X_k(t)}^{X_k(s)+t-s}\psi(u)du\right| \\
= & \left|\sum_{k=1}^{2}(-1)^k \frac{e^{\phi_k(s,X_k(s))}}{(1+|P_k(s)|^2)^{\frac{1}{2}}}(\partial_t\phi_k-\partial_x\phi_k)\int_{X_k(t)}^{X_k(s)+t-s}\psi(u)du\right| \\
\leq & \|\psi\|_{L^1}\left|\sum_{k=1}^{2}(-1)^k \frac{e^{\phi_k(s,X_k(s))}}{(1+|P_k(s)|^2)^{\frac{1}{2}}}(\partial_t\phi_k-\partial_x\phi_k)\right| + C\left|\int_{X_1(s)+t-s}^{X_2(s)+t-s}\psi(u)du\right| \\
+ & C\left|\int_{X_1(t)}^{X_2(t)}\psi(u)du\right|. \qquad (70)
\end{aligned}
$$

By (67), (68), (70) one gets

$$
\begin{aligned}
|I_4| \leq & C\int_{\mathbb{R}}\int_{\mathbb{R}} f_0(x,p)(1+|p|)\left|\int_{X_1(t)}^{X_2(t)}\psi(u)du\right| dp\,dx \\
+ & C\|\psi\|_{L^1}\int_{\mathbb{R}}\int_{\mathbb{R}} f_0\int_0^t\left|\sum_{k=1}^{2}(-1)^k\frac{e^{\phi_k(s,X_k(s))}}{(1+|P_k(s)|^2)^{\frac{1}{2}}}(\partial_t\phi_k-\partial_x\phi_k)\right| ds\;dp\,dx \\
+ & C\int_{\mathbb{R}}\int_{\mathbb{R}} f_0\int_0^t\left|\int_{X_1(s)+t-s}^{X_2(s)+t-s}\psi(u)du\right| ds\;dp\,dx \\
= & T_1+T_2+T_3. \qquad (71)
\end{aligned}
$$

We estimate now one by one the three terms $(T_i)_{1\leq i\leq 3}$. It is easily seen, by using (13), that $1+|p|\leq C(1+|P_1(t)|)$. By performing similar computations as those in (62) and by using Proposition 2.2, Lemma 3.1, one gets

$$
\begin{aligned}
T_1 \leq & C\int_{\mathbb{R}}\int_{\mathbb{R}} f_0(x,p)(1+|P_1(t)|)\int_{\mathbb{R}}|\psi(y)|\mathbf{1}_{\{|y-X_1(t)|\leq|X_1(t)-X_2(t)|\}}\,dy\;dp\,dx \\
\leq & C\int_{\mathbb{R}}|\psi(y)|\int_{\mathbb{R}}\int_{\mathbb{R}} f_1(t,X_1(t),P_1(t))e^{\varphi_0(x)-\phi_1(t,X_1(t))}(1+|P_1(t)|) \\
\times & \mathbf{1}_{\{|y-X_1(t)|\leq C\int_0^t\|\phi_1(s)-\phi_2(s)\|_{1,\infty}ds\}}\det J_1(t)\;dp\,dx\;dy \\
\leq & 2e^{\|\varphi_0\|_\infty+\|\phi_1\|_\infty}C^2\left\|\int_{\mathbb{R}} f_1(t,\cdot,p)(1+|p|)\;dp\right\|_\infty\|\psi\|_{L^1}\int_0^t\|\phi_1(s)-\phi_2(s)\|_{1,\infty}ds \\
\leq & C'\|\psi\|_{L^1}\int_0^t\|\phi_1(s)-\phi_2(s)\|_{1,\infty}ds, \qquad (72)
\end{aligned}
$$

for some constant C' depending only on the initial conditions and T. By using now Remark 3.2 and Corollary 2.1 we have

$$\begin{aligned} T_2 &\leq C\|\psi\|_{L^1} \int_{\mathbb{R}}\int_{\mathbb{R}} f_0(x,p) \int_0^t \{\|\phi_1(s) - \phi_2(s)\|_{1,\infty} + \|\partial_t\phi_1(s) - \partial_t\phi_2(s)\|_\infty \\ &+ \int_0^s \|\phi_1(\tau) - \phi_2(\tau)\|_{1,\infty}\, d\tau\} ds\, dp\, dx \\ &\leq C''\|\psi\|_{L^1}\|f_0\|_{L^1} \int_0^t \{\|\phi_1(s) - \phi_2(s)\|_{1,\infty} + \|\partial_t\phi_1(s) - \partial_t\phi_2(s)\|_\infty\} ds. \quad (73) \end{aligned}$$

The term T_3 can be estimated as before (see (61), (62), (63)). We obtain

$$T_3 \leq C\|\psi\|_{L^1} \int_0^t \|\phi_1(s) - \phi_2(s)\|_{1,\infty} ds. \quad (74)$$

Combining (71), (72), (73), (74) we deduce

$$|I_4| \leq C\|\psi\|_{L^1} \int_0^t \{\|\phi_1(s) - \phi_2(s)\|_{1,\infty} + \|\partial_t\phi_1(s) - \partial_t\phi_2(s)\|_\infty\} ds,$$

for any $t \in [0,T]$, $\psi \in L^1$ and for some constant C depending only on the initial conditions and T. A similar inequality holds for I_5 and thus (66) implies

$$\begin{aligned} \left| \int_{\mathbb{R}} (\partial_x\tilde{\phi}_1(t,x) - \partial_x\tilde{\phi}_2(t,x))\psi(x)\, dx \right| &\leq C\|\psi\|_{L^1} \int_0^t \{\|\phi_1(s) - \phi_2(s)\|_{1,\infty} \\ &+ \|\partial_t\phi_1(s) - \partial_t\phi_2(s)\|_\infty\} ds, \quad (75) \end{aligned}$$

saying that for any $t \in [0,T]$ and any $\phi_1, \phi_2 \in \mathcal{D}_4$ we have

$$\|\partial_x\mathcal{F}\phi_1(t) - \partial_x\mathcal{F}\phi_2(t)\|_\infty \leq C \int_0^t \{\|\phi_1(s) - \phi_2(s)\|_{1,\infty} + \|\partial_t\phi_1(s) - \partial_t\phi_2(s)\|_\infty\} ds. \quad (76)$$

3.6 Estimate of $\partial_t\mathcal{F}\phi_1 - \partial_t\mathcal{F}\phi_2$

By (19) we know that for any $\phi_1, \phi_2 \in \mathcal{D}_4$ we have

$$\partial_t(\tilde{\phi}_1 - \tilde{\phi}_2)(t,x) = -\frac{1}{2}\int_0^t \{(\mu_1 - \mu_2)(s, x-(t-s)) + (\mu_1 - \mu_2)(s, x+(t-s))\} ds.$$

With the notations of the previous paragraph we obtain for any function $\psi \in L^1(\mathbb{R})$

$$\int_{\mathbb{R}} (\partial_t\tilde{\phi}_1(t,x) - \partial_t\tilde{\phi}_2(t,x))\psi(x)\, dx = -\frac{1}{2}I_4(t) - \frac{1}{2}I_5(t).$$

By the computations of paragraph 3.5 one gets for any $\psi \in L^1(\mathbb{R})$, $\phi_1, \phi_2 \in \mathcal{D}_4$

$$\left| \int_{\mathbb{R}} \partial_t(\tilde{\phi}_1 - \tilde{\phi}_2)\psi(x)\, dx \right| \leq C\|\psi\|_{L^1} \int_0^t \{\|\phi_1(s) - \phi_2(s)\|_{1,\infty} + \|\partial_t\phi_1(s) - \partial_t\phi_2(s)\|_\infty\} ds,$$

implying that

$$\|\partial_t \mathcal{F}\phi_1(t) - \partial_t \mathcal{F}\phi_2(t)\|_\infty \leq C \int_0^t \{\|\phi_1(s) - \phi_2(s)\|_{1,\infty} + \|\partial_t\phi_1(s) - \partial_t\phi_2(s)\|_\infty\} ds. \quad (77)$$

4 Existence and uniqueness for the one dimensional Nordström-Vlasov equations

Assume that $\varphi_0 \in W^{2,\infty}(\mathbb{R}), \varphi_1 \in W^{1,\infty}(\mathbb{R})$, $f_0 \in L^1(\mathbb{R}^2)$, $f_0 \geq 0$ and that (21), (44) hold. Take an arbitrary $T > 0$. The application $\mathcal{F}$ satisfies $\mathcal{F}(\mathcal{D}_4) \subset \mathcal{D}_4$ where the set $\mathcal{D}_4$ contains all functions ϕ such that

$$\phi \in C^1([0,T] \times \mathbb{R}),\ \phi(0,\cdot) = \varphi_0,\ \partial_t\phi(0,\cdot) = \varphi_1,$$

$$-\|\varphi_0\|_\infty - T\|\varphi_1\|_\infty - \frac{T}{2} e^{2\|\varphi_0\|_\infty + T\|\varphi_1\|_\infty} \|f_0\|_{L^1} \leq \phi(t,x) \leq \|\varphi_0\|_\infty + T\|\varphi_1\|_\infty,$$

$$\max\{\|\partial_x\phi(t)\|_\infty, \|\partial_t\phi(t)\|_\infty\} \leq (\|\varphi_0'\|_\infty + \|\varphi_1\|_\infty + C_1 T) e^{C_2 T\, t},\ \forall\, t \in [0,T],$$

$$\max\{\|\partial_x^2\phi(t)\|_\infty, \|\partial_{xt}^2\phi(t)\|_\infty\} \leq C_5,\ \ \|\partial_t^2\phi\|_\infty \leq C_6,\ \forall\, t \in [0,T],$$

where the constants C_1, C_2 are defined in Proposition 3.1 and the constants C_5, C_6 are defined in (53), (54). For any function $u \in W^{2,\infty}(]0,T[\times\mathbb{R})$ we introduce the notation $|||u(t)||| := \|u(t)\|_\infty + \|\partial_x u(t)\|_\infty + \|\partial_t u(t)\|_\infty$, $\forall\, t \in [0,T]$. From (65), (76), (77) we know that there is a constant C depending only on the initial conditions and T such that

$$|||\mathcal{F}\phi_1(t) - \mathcal{F}\phi_2(t)||| \leq C \int_0^t |||\phi_1(s) - \phi_2(s)|||\, ds,\ \forall\, \phi_1, \phi_2 \in \mathcal{D}_4. \quad (78)$$

We have the uniqueness result

Proposition 4.1 *Assume that* $\varphi_0 \in W^{2,\infty}(\mathbb{R}), \varphi_1 \in W^{1,\infty}(\mathbb{R})$, $f_0 \in L^1(\mathbb{R}^2)$, $f_0 \geq 0$ *and that* (21), (44) *hold. Then there is at most one mild solution* (f, ϕ) *(i.e.,* $\phi \in C^1([0,T] \times \mathbb{R})$, $\partial_t\phi, \partial_x\phi \in L^\infty(]0,T[; W^{1,\infty}(\mathbb{R}))$ *and* f *is solution by characteristics) for the Nordström-Vlasov system.*

Proof. Consider two mild solutions $(f_k, \phi_k)_{k\in\{1,2\}}$ for the Nordström-Vlasov system, which means that $(\phi_k)_{k\in\{1,2\}}$ are fixed points for $\mathcal{F}$. Repeating the computations of paragraphs 3.1, 3.2, 3.3 one gets that $(\phi_k)_{k\in\{1,2\}} \subset \mathcal{D}_4$ and therefore (78) implies

$$|||\phi_1(t) - \phi_2(t)||| = |||\mathcal{F}\phi_1(t) - \mathcal{F}\phi_2(t)||| \le C \int_0^t |||\phi_1(s) - \phi_2(s)|||ds, \ \forall\, t \in [0,T].$$

We conclude by Gronwall lemma. □

We prove now the existence of mild solution for the Nordström-Vlasov system.

Proof. (of Theorem 1.1) Consider the sequence $(\phi_n)_{n\ge 1} \subset \mathcal{D}_4$ defined by

$$\phi_0(t,x) = \frac{1}{2}\{\varphi_0(x+t) + \varphi_0(x-t)\} + \frac{1}{2}\int_{x-t}^{x+t} \varphi_1(y)\, dy, \ (t,x) \in [0,T] \times \mathbb{R},$$

and $\phi_{n+1} = \mathcal{F}\phi_n, \forall\, n \ge 0$. By using (78) we deduce easily that $(\phi_n)_{n\ge1}$ converges in $C^1([0,T]\times\mathbb{R})$ towards a fixed point of $\mathcal{F}$. Actually $\phi \in \mathcal{D}_4$. Denote by f the mild solution of (5), (6) corresponding to ϕ and let $\mu(t,x) = \int_{\mathbb{R}} f(t,x,p)(1+p^2)^{-\frac{1}{2}}\, dp$, $(t,x) \in [0,T]\times\mathbb{R}$. By the definition of the application $\mathcal{F}$ we deduce easily that (f,ϕ) solves the Nordström-Vlasov system. □

In the following we show that the global solution constructed above preserves the total energy. We introduce the notations

$$\rho(t,x) := e^{-\phi(t,x)} \int_{\mathbb{R}} f(t,x,p)\, dp, \ \ j(t,x) := e^{-\phi(t,x)} \int_{\mathbb{R}} v(p) f(t,x,p)\, dp,$$

$$e(t,x) := \int_{\mathbb{R}} (1+p^2)^{\frac{1}{2}} f(t,x,p)\, dp + \frac{1}{2}|\partial_x\phi(t,x)|^2 + \frac{1}{2}|\partial_t\phi(t,x)|^2,$$

$$\pi(t,x) := \int_{\mathbb{R}} p f(t,x,p)\, dp - \partial_x\phi(t,x)\partial_t\phi(t,x).$$

Proposition 4.2 *Assume that $\varphi_0 \in W^{2,\infty}(\mathbb{R})$, $\varphi_0' \in L^2(\mathbb{R})$, $\varphi_1 \in W^{1,\infty}(\mathbb{R})\cap L^2(\mathbb{R})$, $(1+|p|)f_0 \in L^1(\mathbb{R}^2)$ and that (21), (44) hold. Then the solution of the one dimensional Nordström-Vlasov system satisfies*

$$\partial_t\rho + \partial_x j = 0, \ \text{ in } \mathcal{D}'([0,+\infty[\times\mathbb{R}), \tag{79}$$

$$\partial_t e + \partial_x \pi = 0, \ \text{ in } \mathcal{D}'([0,+\infty[\times\mathbb{R}). \tag{80}$$

In particular the total mass and energy are preserved for any $t > 0$

$$\int_{\mathbb{R}} \rho(t,x)\ dx = \int_{\mathbb{R}} \rho(0,x)\ dx, \quad \int_{\mathbb{R}} e(t,x)\ dx = \int_{\mathbb{R}} e(0,x)\ dx.$$

Proof. Note that by Theorem 1.1 we know that $\int_{\mathbb{R}}(1+|p|)f(\cdot,\cdot,p)\ dp$ is bounded on $[0,T]\times\mathbb{R}, \forall\, T > 0$ and therefore ρ, j, e and π are well defined for $(t,x) \in [0,+\infty[\times\mathbb{R}$. Let us prove now the continuity equation. For any $\theta \in C_c^1([0,+\infty[\times\mathbb{R})$ we have by Proposition 2.1

$$\begin{aligned}\int_0^{+\infty}\!\!\int_{\mathbb{R}} \{\rho(t,x)\partial_t\theta + j(t,x)\partial_x\theta\}\ dx\, dt &= \int_{\mathbb{R}}\int_{\mathbb{R}} f_0 e^{-\varphi_0(x)} \int_0^{+\infty} \frac{d}{dt}\theta(t,X(t))\ dt\ dp\, dx \\ &= -\int_{\mathbb{R}} \rho(0,x)\theta(0,x)\ dx,\end{aligned}$$

and thus $\partial_t\rho + \partial_x j = 0$ holds in $\mathcal{D}'([0,+\infty[\times\mathbb{R})$. The conservation of the total mass has been already checked. In order to prove (80) we intend to apply the weak formulation (11) with the test function $\psi(t,x,p) = e^{\phi(t,x)}\theta(t,x)(1+p^2)^{\frac{1}{2}}$, for any $\theta \in C_c^1([0,+\infty[\times\mathbb{R})$. Actually ψ is not a C_b^1 function but we can apply first the weak formulation with $\psi_R = \psi\ \chi_R(p)$ and then let $R \to +\infty$. We skip these standard arguments. We find that

$$\begin{aligned}-\int_{\mathbb{R}}\int_{\mathbb{R}} (1+p^2)^{\frac{1}{2}}\theta(0,x) f_0(x,p)\ dp\, dx &= \int_0^{+\infty}\!\!\int_{\mathbb{R}}\int_{\mathbb{R}} (1+p^2)^{\frac{1}{2}}(\partial_t\theta + v(p)\partial_x\theta) f\ dp\, dx\, dt \\ &+ \int_0^{+\infty}\!\!\int_{\mathbb{R}}\int_{\mathbb{R}} \{(1+p^2)^{\frac{1}{2}}(\partial_t\phi + v(p)\partial_x\phi) \\ &- v(p)(pS\phi + (1+p^2)^{-\frac{1}{2}}\partial_x\phi)\}\theta f\ dp\, dx\, dt \\ &= \int_0^{+\infty}\!\!\int_{\mathbb{R}}\int_{\mathbb{R}} ((1+p^2)^{\frac{1}{2}}\partial_t\theta + p\partial_x\theta) f\ dp\, dx\, dt \\ &+ \int_0^{+\infty}\!\!\int_{\mathbb{R}} \mu(t,x)\theta(t,x)\partial_t\phi\ dx\, dt. \qquad (81)\end{aligned}$$

By multiplying the wave equation by $\theta(t,x)\partial_t\phi$ one gets

$$\int_0^{+\infty}\!\!\int_{\mathbb{R}} \theta\left\{\frac{1}{2}\partial_t|\partial_t\phi|^2 - \partial_x^2\phi\ \partial_t\phi\right\}\ dx\, dt + \int_0^{+\infty}\!\!\int_{\mathbb{R}} \mu(t,x)\theta(t,x)\partial_t\phi\ dx\, dt = 0.$$

After integration by parts we obtain

$$\begin{aligned}&- \frac{1}{2}\int_0^{+\infty}\!\!\int_{\mathbb{R}} \{|\partial_t\phi|^2 + |\partial_x\phi|^2\}\partial_t\theta\ dx\, dt - \frac{1}{2}\int_{\mathbb{R}} \{|\varphi_1(x)|^2 + |\varphi_0'(x)|^2\}\theta(0,x)\ dx \\ &+ \int_0^{+\infty}\!\!\int_{\mathbb{R}} \partial_x\phi\partial_t\phi\partial_x\theta\ dx\, dt + \int_0^{+\infty}\!\!\int_{\mathbb{R}} \mu(t,x)\theta(t,x)\partial_t\phi\ dx\, dt = 0. \qquad (82)\end{aligned}$$

Combining (81), (82) yields

$$-\int_0^{+\infty}\!\!\int_{\mathbb{R}} \{e(t,x)\partial_t\theta + \pi(t,x)\partial_x\theta\}\;dx\,dt - \int_{\mathbb{R}} e(0,x)\theta(0,x)\;dx = 0,$$

and thus (80) holds. In order to prove that the total energy is conserved, observe that $\partial_t\phi, \partial_x\phi \in L^\infty(]0,T[;L^2(\mathbb{R}))$, $\forall\, T>0$. Indeed, since φ_0', $\varphi_1 \in L^2(\mathbb{R})$, by using the representation formula for $\partial_x\phi, \partial_t\phi$ (see (20), (19)) it is sufficient to check that $D^\pm(t,x) = \int_0^t \mu(s, x\pm(t-s))\;ds$ belong to $L^2(\mathbb{R})$ for any $t\in[0,T]$, $T>0$. This is obvious since we have for any $t\in[0,T]$

$$\|D^\pm(t)\|_\infty \le t \left\| \int_{\mathbb{R}} f(\cdot,\cdot,p)\;dp \right\|_{L^\infty(]0,T[\times\mathbb{R})}, \|D^\pm(t)\|_{L^1} \le t \left\| \int_{\mathbb{R}}\int_{\mathbb{R}} f(\cdot,x,p)\;dp\,dx \right\|_{L^\infty(]0,T[)}.$$

In particular $\partial_x\phi\;\partial_t\phi \in L^\infty(]0,T[;L^1(\mathbb{R}))$, $T>0$. By Proposition 2.1 we know also that $(1+p^2)^{\frac{1}{2}} f \in L^\infty(]0,T[;L^1(\mathbb{R}^2))$ for any $T>0$. Take now $\eta \in C_c^1([0,+\infty[)$ and apply (80) with $\theta(t,x) = \eta(t)\chi_R(x)$, $R>0$. We obtain

$$-\int_0^{+\infty}\!\!\int_{\mathbb{R}} e(t,x)\eta'\chi_R\;dx\,dt - \int_0^{+\infty}\!\!\int_{\mathbb{R}} \pi(t,x)\frac{\eta(t)}{R}\chi\prime\left(\frac{x}{R}\right)\;dx\,dt - \int_{\mathbb{R}} e(0,x)\eta(0)\chi_R\;dx = 0.$$

After letting $R\to+\infty$ one gets $-\int_0^{+\infty}\frac{d\eta}{dt}\int_{\mathbb{R}} e(t,x)\;dx\;dt - \eta(0)\int_{\mathbb{R}} e(0,x)\;dx = 0$ and thus $\int_{\mathbb{R}} e(t,x)\;dx = \int_{\mathbb{R}} e(0,x)\;dx$, $\forall\, t\ge 0$. □

5 Finite speed propagation

Assume that $(f_0^k, \varphi_0^k, \varphi_1^k)_{k\in\{1,2\}}$ satisfy the hypotheses of Theorem 1.2 and denote by $(f_k,\phi_k)_{k\in\{1,2\}}$ the global solutions of the one dimensional Nordström-Vlasov system. We intend to estimate $(\phi_1-\phi_2)$ with respect to $(f_0^1 - f_0^2, \varphi_0^1 - \varphi_0^2, \varphi_1^1-\varphi_1^2)$. In particular we deduce that the solution of the Nordström-Vlasov system propagates with finite speed, which coincides with the waves speed in (2). Surely, the key point here is that we are dealing with relativistic particles and therefore the characteristics propagate with finite speed $|X'(s)| = |v(P(s))| < 1$. Let us explain how we can adapt the computations in the paragraphs 3.4, 3.5, 3.6 for the above purposes. Take $0\le t\le R$ and $\psi\in L^1(\mathbb{R})$ compactly supported in $[-(R-t), R-t]$. From now on the notation C_R stands for various constants depending on R, $\|\varphi_0^k\|_{W^{2,\infty}(\mathbb{R})}$, $\|\varphi_1^k\|_{W^{1,\infty}(\mathbb{R})}$, $\|f_0^k\|_{L^1(\mathbb{R}^2)}$, $\sum_{m=0}^2 \| \, |p|^m g_0^k\|_{L^1(\mathbb{R})}$, $k \in \{1,2\}$. As in formula (55) one gets

$$\left|\int_{\mathbb{R}}(\phi_1(t,y)-\phi_2(t,y))\psi \, dy\right| \leq (\|\varphi_0^1-\varphi_0^2\|_{L^\infty([-R,R])} + R\,\|\varphi_1^1-\varphi_1^2\|_{L^\infty([-R,R])})\|\psi\|_{L^1} + \frac{1}{2}|I_7(t)|, \tag{83}$$

where

$$I_7(t) = \int_0^t\int_{\mathbb{R}}\int_{\mathbb{R}}(f_1-f_2)(s,x,p)\frac{\theta(s,x)}{(1+p^2)^{\frac{1}{2}}}\;dp\,dx\,ds, \quad \theta(s,x) = \int_{x-(t-s)}^{x+(t-s)}\psi(y)\;dy.$$

By Proposition 2.1 one have

$$\int_0^t\int_{\mathbb{R}}\int_{\mathbb{R}} f_k(s,x,p)\frac{\theta(s,x)}{(1+p^2)^{\frac{1}{2}}}\;dp\,dxds = \int_{\mathbb{R}}\int_{\mathbb{R}} f_0^k(x,p)e^{-\varphi_0^k(x)}\int_0^t \frac{\theta(s,X_k(s))}{(1+|P_k(s)|^2)^{\frac{1}{2}}} \times e^{\phi_k(s,X_k(s))}\;ds\;dp\,dx,$$

where (X_k,P_k) are the characteristics corresponding to ϕ_k, $k\in\{1,2\}$. Observe that for any $x>R$ and $s\in[0,t]$ we have $X_k(s;0,x,p)-(t-s)>R-t$ implying that $\theta(s,X_k(s))=0$. Similarly, for any $x<-R$ and $s\in[0,t]$ we have $X_k(s;0,x,p)+(t-s)<-R+t$ implying that $\theta(s,X_k(s))=0$. Therefore we have $\theta(s,X_k(s)) = \theta(s,X_k(s))\,\mathbf{1}_{\{|x|\leq R\}}$ and thus we obtain

$$\begin{aligned} |I_7(t)| &\leq C_R\int_{-R}^{R}\int_{\mathbb{R}}\left|\sum_{k=1}^{2}(-1)^k f_0^k(x,p)e^{-\varphi_0^k(x)}\right|\;dp\,dx\;\|\psi\|_{L^1} \\ &+ \left|\int_{-R}^{R}\int_{\mathbb{R}} f_0^1(x,p)e^{-\varphi_0^1(x)}\int_0^t\sum_{k=1}^{2}(-1)^k\frac{\theta(s,X_k(s))}{(1+|P_k(s)|^2)^{\frac{1}{2}}}e^{\phi_k(s,X_k(s))}\;ds\;dp\,dx\right| \\ &= |I_8|+|I_9(t)|. \end{aligned} \tag{84}$$

For any $(x,p)\in[-R,R]\times\mathbb{R}$, $k\in\{1,2\}$ we define

$$s_k(t,x,p) = \sup\{s\in[0,t]\;:\;|X_k(s;0,x,p)|\leq R-s\}.$$

Observing that the function $s\to|X_k(s;0,x,p)|-(R-s)$ is strictly increasing we deduce that $|X_k(s;0,x,p)|\leq R-s, s\in[0,s_k(t,x,p)]$ and $|X_k(s;0,x,p)|> R-s, s\in]s_k(t,x,p),t]$. Notice that for any $(x,p)\in[-R,R]\times\mathbb{R}$ the characteristics

$(X_k(\cdot;0,x,p), P_k(\cdot;0,x,p))$, $k \in \{1,2\}$ satisfy the property (102) on $[0, s(t,x,p)]$, where $s(t,x,p) = \max_{k\in\{1,2\}} s_k(t,x,p)$. If $s \in]s(t,x,p), t]$ we claim that $\theta(s, X_k(s)) = 0$, $k \in \{1,2\}$. Indeed, assuming that $X_k(s) > R - s$ implies $X_k(s) - (t-s) > R - t$ and if $X_k(s) < -(R-s)$ implies $X_k(s) + (t-s) < -(R-t)$. Therefore in both cases $[X_k(s) - (t-s), X_k(s) + (t-s)] \cap \operatorname{supp} \psi = \emptyset$ and thus $\theta(s, X_k(s)) = 0$, $k \in \{1,2\}$. By the above considerations and the properties of characteristics satisfying (102) (see the comments in the Appendix) we obtain as in (59)

$$
\begin{aligned}
|I_9(t)| &= \left| \int_{-R}^{R}\int_{\mathbb{R}} f_0^1 e^{-\varphi_0^1} \int_0^{s(t,x,p)} \sum_{k=1}^{2} (-1)^k \frac{\theta(s, X_k(s))}{(1+|P_k(s)|^2)^{\frac{1}{2}}} e^{\phi_k(s, X_k(s))} \, ds \, dp \, dx \right| \\
&\leq C_R \|\psi\|_{L^1} \int_{-R}^{R}\int_{\mathbb{R}} f_0^1 e^{-\varphi_0^1} \int_0^{s(t,x,p)} (\|\phi_1(s) - \phi_2(s)\|_{L^\infty([-(R-s), R-s])} \\
&+ \int_0^s \|\phi_1(\tau) - \phi_2(\tau)\|_{W^{1,\infty}([-(R-\tau), R-\tau])} \, d\tau + |p(\varphi_0^1 - \varphi_0^2)(x)|) \, ds \, dp \, dx \\
&+ C_R \int_{-R}^{R}\int_{\mathbb{R}} f_0^1 e^{-\varphi_0^1} \int_0^{s(t,x,p)} \left| \int_{X_1(s)-(t-s)}^{X_2(s)-(t-s)} \psi(y) \, dy \right| \, ds \, dp \, dx \\
&+ C_R \int_{-R}^{R}\int_{\mathbb{R}} f_0^1 e^{-\varphi_0^1} \int_0^{s(t,x,p)} \left| \int_{X_1(s)+(t-s)}^{X_2(s)+(t-s)} \psi(y) \, dy \right| \, ds \, dp \, dx \\
&= I_{10}(t) + I_{11}(t) + I_{12}(t). \qquad (85)
\end{aligned}
$$

We introduce the notations $D^R(s) = \|\phi_1(s) - \phi_2(s)\|_{L^\infty([-(R-s), R-s])}$, $D^R_\alpha(s) = \|\partial_\alpha \phi_1(s) - \partial_\alpha \phi_2(s)\|_{L^\infty([-(R-s), R-s])}$, $\alpha \in \{x, t\}$. The term $I_{10}(t)$ can be estimated as follows

$$
I_{10}(t) \leq C_R \|\psi\|_{L^1} \left(\|\varphi_0^1 - \varphi_0^2\|_{L^\infty([-R,R])} + \int_0^t \{D^R(s) + D^R_x(s)\} \, ds \right). \qquad (86)
$$

For the term $I_{11}(t)$ we proceed as in (61), (62)

$$
\begin{aligned}
I_{11}(t) &\leq C_R \int_{\mathbb{R}} |\psi| \int_{-R}^{R}\int_{\mathbb{R}} f_0^1 \int_0^{s(t,x,p)} \mathbf{1}_{\{|y-(X_1(s)-(t-s))| \leq C_R(|p| \, \|\varphi_0^1 - \varphi_0^2\|_{L^\infty([-R,R])} + \int_0^s (D^R + D^R_x) d\tau)\}} ds \, dp \, dx \, dy \\
&\leq C_R \int_{\mathbb{R}} |\psi(y)| \int_0^t \int_{\mathbb{R}}\int_{\mathbb{R}} f_1(s, X, P) r(s, y, X, P) \, dP \, dX \, ds \, dy,
\end{aligned}
$$

where $r(s,y,X,P) = \mathbf{1}_{\{|y-(X-(t-s))| \leq C_R((1+|P|)\|\varphi_0^1 - \varphi_0^2\|_{L^\infty([-R,R])} + \int_0^s (D^R(\tau) + D^R_x(\tau)) d\tau)\}}$. Observe that for any $(s, y, P) \in [0,t] \times \mathbb{R}^2$ we have $\int_{\mathbb{R}} r(s,y,X,P) \, dX \leq \alpha |P| + \beta$ with

$$
\alpha = 2C_R \|\varphi_0^1 - \varphi_0^2\|_{L^\infty([-R,R])}, \quad \beta = 2C_R \left\{ \|\varphi_0^1 - \varphi_0^2\|_{L^\infty([-R,R])} + \int_0^t \{D^R(\tau) + D^R_x(\tau)\} \, d\tau \right\},
$$

and therefore, by Lemma 3.2

$$I_{11}(t) \;\le\; C_R\|\psi\|_{L^1}\left(\|\varphi_0^1-\varphi_0^2\|_{L^\infty([-R,R])}+\int_0^t\{D^R(s)+D_x^R(s)\}\,ds\right). \quad (87)$$

A similar estimate holds for $I_{12}(t)$ and finally collecting (83), (84), (85), (86), (87) implies

$$\begin{aligned} D^R(t) \;&\le\; C_R\left(\|\varphi_0^1-\varphi_0^2\|_{L^\infty([-R,R])}+\|\varphi_1^1-\varphi_1^2\|_{L^\infty([-R,R])}+\int_{-R}^R\int_{\mathbb{R}}|f_0^1-f_0^2|\,dp\,dx\right)\\ &+\; C_R\int_0^t\{D^R(s)+D_x^R(s)\}\,ds. \end{aligned} \quad (88)$$

The next step is to estimate $D_x^R(t)$. As in (66) we have

$$\begin{aligned} \left|\int_{\mathbb{R}}(\partial_x\phi_1-\partial_x\phi_2)\psi\,dy\right| \;&\le\; \|\psi\|_{L^1}(\|{\varphi_0^1}'-{\varphi_0^2}'\|_{L^\infty([-R,R])}+\|\varphi_1^1-\varphi_1^2\|_{L^\infty([-R,R])})\\ &+\; \frac{1}{2}|I_{13}(t)|+\frac{1}{2}|I_{14}(t)|, \end{aligned} \quad (89)$$

where

$$I_{13}(t)=\int_0^t\int_{\mathbb{R}}\int_{\mathbb{R}}(f_1-f_2)(s,x-(t-s),p)\frac{\psi(x)}{(1+p^2)^{\frac{1}{2}}}\,dp\,dx\,ds,$$

and

$$I_{14}(t)=\int_0^t\int_{\mathbb{R}}\int_{\mathbb{R}}(f_1-f_2)(s,x+(t-s),p)\frac{\psi(x)}{(1+p^2)^{\frac{1}{2}}}\,dp\,dx\,ds.$$

Performing the same steps as in (67), (68), (69) one gets

$$\begin{aligned} I_{13}(t)=&\sum_{k=1}^2(-1)^k\int_{\mathbb{R}}\int_{\mathbb{R}}f_0^k((1+p^2)^{\frac{1}{2}}+p)\int_{x+t}^{X_k(t)}\psi(u)\,du\,dp\,dx \\ &-\sum_{k=1}^2(-1)^k\int_{\mathbb{R}}\int_{\mathbb{R}}f_0^k e^{-\varphi_0^k}\int_0^t\frac{e^{\phi_k(s,X_k(s))}}{(1+|P_k(s)|^2)^{\frac{1}{2}}}(\partial_t\phi_k-\partial_x\phi_k)\int_{X_k(t)}^{X_k(s)+(t-s)}\psi(u)\,du\,ds\,dp\,dx. \end{aligned} \quad (90)$$

Notice that for any $(s,x,p)\in[0,t]\times\mathbb{R}^2$ such that $|x|>R$ we have $\int_{x+t}^{X_k(t)}\psi(u)\,du=\int_{X_k(t)}^{X_k(s)+(t-s)}\psi(u)\,du=0$. Therefore in the above equality the integrations with respect to x can be restricted to $[-R,R]$. Observe also that for any $s\in]s_k(t,x,p),t]$ we have $\int_{X_k(t)}^{X_k(s)+(t-s)}\psi(u)\,du=0$ and therefore we obtain

$$\begin{aligned} |I_{13}(t)| \;&\le\; C_R\|\psi\|_{L^1}\int_{-R}^R\int_{\mathbb{R}}\left\{(1+|p|)|f_0^1-f_0^2|+|f_0^1e^{-\varphi_0^1}-f_0^2e^{-\varphi_0^2}|\right\}\,dp\,dx\\ &+\; I_{15}(t), \end{aligned} \quad (91)$$

where

$$\begin{aligned}
I_{15}(t) &\leq C_R \int_{-R}^{R}\int_{\mathbb{R}} f_0^1(x,p)(1+|P_1(t)|)\left|\int_{X_1(t)}^{X_2(t)} \psi(u)\; du\right|\; dp\,dx \\
&+ C_R\|\psi\|_{L^1}\int_{-R}^{R}\int_{\mathbb{R}} f_0^1 \int_0^{s(t,x,p)} \left|\sum_{k=1}^{2}(-1)^k \frac{e^{\phi_k(s,X_k(s))}}{(1+|P_k(s)|^2)^{\frac{1}{2}}}(\partial_t\phi_k - \partial_x\phi_k)\right| ds\; dp\,dx \\
&+ C_R \int_{-R}^{R}\int_{\mathbb{R}} f_0^1(x,p) \int_0^{s(t,x,p)} \left|\int_{X_1(s)+(t-s)}^{X_2(s)+(t-s)} \psi(u)\; du\right|\; ds\; dp\,dx \\
&= I_{16}(t) + I_{17}(t) + I_{18}(t). \qquad (92)
\end{aligned}$$

Notice that $\int_{X_1(t)}^{X_2(t)} \psi(u)\; du$ vanishes on $([-R,R]\times\mathbb{R}) - A(t)$, where $A(t) = \{(x,p) \in [-R,R]\times\mathbb{R} \;:\; \exists\, \lambda \in [0,1], |\lambda X_1(t) + (1-\lambda)X_2(t)| \leq R - t\}$ and if $(x,p) \in A(t)$, then $(X_k, P_k)_{k\in\{1,2\}}$ verify the property (102) on $[0,t]$. Therefore for any $(x,p) \in A(t)$ we can apply (106) and we obtain as in (72)

$$\begin{aligned}
I_{16}(t) &\leq C_R \int_{-R}^{R}\int_{\mathbb{R}} f_0^1(x,p)(1+|P_1(t)|)\mathbf{1}_{\{(x,p)\in A(t)\}}\left|\int_{X_1(t)}^{X_2(t)} \psi(u)\; du\right|\; dp\,dx \\
&\leq C_R \int_{\mathbb{R}} |\psi(u)| \int_{\mathbb{R}}\int_{\mathbb{R}} (1+|P|) f_1(t,X,P)\tilde{r}(u,X,P)\; dP\; dX\; du,
\end{aligned}$$

where $\tilde{r}(u,X,P) = \mathbf{1}_{\{|X-u| \leq C_R((1+|P|)\; \|\varphi_0^1 - \varphi_0^2\|_{L^\infty([-R,R])} + \int_0^t \{D^R(s) + D_x^R(s)\} ds)\}}$. Applying now Lemma 3.2 yields

$$I_{16}(t) \leq C_R \|\psi\|_{L^1}\left(\|\varphi_0^1 - \varphi_0^2\|_{L^\infty([-R,R])} + \int_0^t \{D^R(s) + D_x^R(s)\}\; ds\right). \qquad (93)$$

Observe that at this step we have used the assumption $\int_{\mathbb{R}} |p|^2 g_0^k(p)\; dp < +\infty$. Taking into account that (106) applies on $[0, s(t,x,p)]$ for $(X_k, P_k)(\cdot; 0, x, p)$, $k \in \{1,2\}$ we deduce as in (73), (74) (use also Lemma 3.2 for estimating $I_{18}(t)$)

$$I_{17}(t) \leq C_R\|\psi\|_{L^1}\left(\|\varphi_0^1 - \varphi_0^2\|_{L^\infty([-R,R])} + \int_0^t \{D^R(s) + D_x^R(s) + D_t^R(s)\}\; ds\right), \quad (94)$$

and

$$I_{18}(t) \leq C_R\|\psi\|_{L^1}\left(\|\varphi_0^1 - \varphi_0^2\|_{L^\infty([-R,R])} + \int_0^t \{D^R(s) + D_x^R(s)\}\; ds\right). \qquad (95)$$

Collecting all the partial results (89), (91), (93), (94), (95) yields

$$D_x^R(t) \leq C_R\left(D_0^R + \int_0^t \{D^R(s) + D_x^R(s) + D_t^R(s)\}\; ds\right). \qquad (96)$$

A similar estimate holds for D_t^R and therefore (88), (96) implies

$$D^R(t) + D_x^R(t) + D_t^R(t) \le C_R \left(D_0^R + \int_0^t \{D^R(s) + D_x^R(s) + D_t^R(s)\}\ ds \right).$$

The statement in Theorem 1.2 follows by Gronwall lemma.

Proof. (of Corollary 1.1) Notice that under the hypotheses of Corollary 1.1 we can apply Theorem 1.2. Indeed, the only step in the proof of Theorem 1.2 where we need the additional hypothesis $\int_{\mathbb{R}} |p|^2 g_0^k(p)\ dp < +\infty$ is in (93). But in our case $\|\varphi_0^1 - \varphi_0^2\|_{L^\infty([-R,R])} = 0$ and thus, by Remark 3.1, the inequality (93) still holds true by assuming only $\int_{\mathbb{R}} |p| g_0^k(p)\ dp < +\infty$. We deduce that $\phi_1(t,x) = \phi_2(t,x)$, $t \in [0,R], |x| \le R - t$. Observe also that for any $t \in [0,R], x \in [-(R-t), R-t], p \in \mathbb{R}$ we have $|X_k(s;t,x,p)| \le |x| + (t-s) \le R - s, \ \ s \in [0,t],\ k \in \{1,2\}$. By (106) one gets for any $t \in [0,R], |x| \le R - t, p \in \mathbb{R}$

$$(|X_1 - X_2| + |Q_1 - Q_2|)(0;t,x,p) \le C_R \left(|p(\phi_1 - \phi_2)(t,x)| + \int_0^t \{D^R(s) + D_x^R(s)\}\ ds \right) = 0,$$

implying that $(X_1, P_1)(0;t,x,p) = (X_2, P_2)(0;t,x,p)$. We deduce easily that $f_1(t,x,p) = f_2(t,x,p)$ for any $t \in [0,R], |x| \le R - t, p \in \mathbb{R}$. □

6 Appendix

We give here the details for the proofs of Proposition 2.2 and Corollary 2.1. If ϕ is smooth and $(X(s), P(s))$ is a characteristic corresponding to ϕ we denote by $Q(s)$ the quantity $Q(s) = e^{\phi(s,X(s))} P(s)$. Recall that we have (see (13))

$$\frac{dQ}{ds} = -\frac{e^{\phi(s,X(s))}}{(1 + |P(s)|^2)^{\frac{1}{2}}} \partial_x \phi(s, X(s)), \ \ \forall s \in [0,T]. \tag{97}$$

We start with the following easy lemma (the proof is left to the reader).

Lemma 6.1 *Consider* $\phi_k \in C^1([0,T]\times\mathbb{R})\cap L^\infty(]0,T[\times\mathbb{R})$, $\partial_x\phi_k, \partial_t\phi_k \in L^\infty(]0,T[;W^{1,\infty}(\mathbb{R}))$, $k \in \{1,2\}$ *and* (X_k, P_k) *two characteristics corresponding to* ϕ_k, $k \in \{1,2\}$. *We denote* $C(s) = \max_{k\in\{1,2\}} \|\phi_k(s)\|_{L^\infty}$. *Then we have for any* $s \in [0,T]$

$$\left|\sum_{k=1}^{2}(-1)^k e^{\phi_k(s,X_k(s))}\right| \leq e^{C(s)}(\|\phi_1(s)-\phi_2(s)\|_{L^\infty}+\|\partial_x\phi_1(s)\|_{L^\infty}|X_1(s)-X_2(s)|), \quad (98)$$

$$\begin{aligned} |P_1(s) - P_2(s)| &\leq e^{C(s)}|Q_2(s)|(\|\phi_1(s) - \phi_2(s)\|_{L^\infty} + \|\partial_x\phi_1(s)\|_{L^\infty}|X_1(s) - X_2(s)|) \\ &+ e^{C(s)}|Q_1(s) - Q_2(s)|, \end{aligned} \quad (99)$$

$$\begin{aligned} \left|\sum_{k=1}^{2}\frac{(-1)^k}{(1+|P_k(s)|^2)^{\frac{1}{2}}}\right| &\leq e^{2C(s)}(\|\phi_1(s) - \phi_2(s)\|_{L^\infty} + \|\partial_x\phi_1(s)\|_{L^\infty}|X_1(s) - X_2(s)|) \\ &+ e^{C(s)}|Q_1(s) - Q_2(s)|, \end{aligned} \quad (100)$$

$$\begin{aligned} |v(P_1(s)) - v(P_2(s))| &\leq 2e^{2C(s)}(\|\phi_1(s) - \phi_2(s)\|_{L^\infty} + \|\partial_x\phi_1(s)\|_{L^\infty}|X_1(s) - X_2(s)|) \\ &+ 2e^{C(s)}|Q_1(s) - Q_2(s)|. \end{aligned} \quad (101)$$

At this point let us make some remarks which will be useful when studying the finite speed propagation property. Take $0 \leq t \leq R$ and assume that (X_k, P_k) are characteristics corresponding to ϕ_k, $k \in \{1,2\}$, which satisfy the following property

$$\forall\ s \in [0,t],\ \exists\ \lambda(s) \in [0,1]\ :\ |\lambda(s)X_1(s) + (1-\lambda(s))X_2(s)| \leq R - s, \quad (102)$$

saying that at any time $s \in [0,t]$ the segment between $X_1(s), X_2(s)$ has non void intersection with $[-(R-s), R-s]$. Then all the statements in Lemma 6.1 hold with $\|\phi_1(s)-\phi_2(s)\|_{L^\infty}$ replaced by $\|\phi_1(s)-\phi_2(s)\|_{L^\infty([-(R-s),R-s])}$ and $\|\partial_x\phi_1(s)\|_{L^\infty}$ replaced by $\max_{k\in\{1,2\}}\|\partial_x\phi_k(s)\|_{L^\infty}$. Indeed, let us check this for formula (98), the same applying for the other. Denote by $Y(s)$ the number of $[-(R-s), R-s]$ such that $Y(s) = \lambda(s)X_1(s) + (1-\lambda(s))X_2(s)$, $\lambda(s) \in [0,1]$. We have

$$\begin{aligned} \left|\sum_{k=1}^{2}(-1)^k e^{\phi_k(s,X_k(s))}\right| &\leq e^{C(s)}|\phi_1(s, X_1(s)) - \phi_2(s, X_2(s))| \\ \leq\ & e^{C(s)}\{|\phi_1(s, Y(s)) - \phi_2(s, Y(s))| + \sum_{k=1}^{2}|\phi_k(s, Y(s)) - \phi_k(s, X_k(s))|\} \\ \leq\ & e^{C(s)}\left(\|\phi_1(s) - \phi_2(s)\|_{L^\infty([-(R-s),R-s])} + \max_{k\in\{1,2\}}\|\partial_x\phi_k(s)\|_{L^\infty}|X_1(s) - X_2(s)|\right). \end{aligned}$$

Proof. (of Proposition 2.2) We have for any $s \in [0, T]$

$$\begin{aligned}\frac{1}{2}\frac{d}{ds}|X_1(s) - X_2(s)|^2 &= (v(P_1(s)) - v(P_2(s)))(X_1(s) - X_2(s)) \\ &\leq |X_1(s) - X_2(s)||v(P_1(s)) - v(P_2(s))|.\end{aligned}$$

We introduce the notations

$$C(s) = \max_{k\in\{1,2\}} \|\phi_k(s)\|_{L^\infty},\ \tilde{C}(s) = \max_{k\in\{1,2\}} \|\partial_x\phi_k(s)\|_{L^\infty},\ \tilde{\tilde{C}}(s) = \max_{k\in\{1,2\}} \|\partial_x^2\phi_k(s)\|_{L^\infty}.$$

We deduce by Lemma 6.1, inequality (101)

$$\begin{aligned}\frac{d}{ds}|X_1(s) - X_2(s)| &\leq 2e^{2C(s)}(\|\phi_1(s) - \phi_2(s)\|_{L^\infty} + \|\partial_x\phi_1(s)\|_{L^\infty}|X_1(s) - X_2(s)|) \\ &+ 2e^{C(s)}|Q_1(s) - Q_2(s)|. \qquad (103)\end{aligned}$$

Using now (97) yields

$$\frac{1}{2}\frac{d}{ds}|Q_1(s) - Q_2(s)|^2 \leq |Q_1(s) - Q_2(s)| \left|\sum_{k=1}^{2}(-1)^k \frac{e^{\phi_k(s,X_k(s))}}{(1 + |P_k(s)|^2)^{\frac{1}{2}}}\partial_x\phi_k(s, X_k(s))\right|,$$

and therefore we obtain

$$\begin{aligned}\frac{d}{ds}|Q_1(s) - Q_2(s)| &\leq \left|\sum_{k=1}^{2}(-1)^k \frac{e^{\phi_k(s,X_k(s))}}{(1 + |P_k(s)|^2)^{\frac{1}{2}}}\partial_x\phi_k(s, X_k(s))\right|, \qquad (104) \\ &\leq e^{C(s)}\tilde{C}(s)\left|\sum_{k=1}^{2}\frac{(-1)^k}{(1 + |P_k(s)|^2)^{\frac{1}{2}}}\right| + \left|\sum_{k=1}^{2}(-1)^k e^{\phi_k(s,X_k(s))}\partial_x\phi_k(s, X_k(s))\right|.\end{aligned}$$

As usual we have the estimate

$$\begin{aligned}\left|\sum_{k=1}^{2}(-1)^k e^{\phi_k}\partial_x\phi_k\right| &\leq \tilde{C}(s)\left|\sum_{k=1}^{2}(-1)^k e^{\phi_k(s,X_k(s))}\right| + e^{C(s)}\left|\sum_{k=1}^{2}(-1)^k\partial_x\phi_k(s, X_k(s))\right| \\ &\leq e^{C(s)}\tilde{C}(s)\{\|\phi_1(s) - \phi_2(s)\|_{L^\infty} + \tilde{C}(s)|X_1(s) - X_2(s)|\} \\ &+ e^{C(s)}\{\|\partial_x\phi_1(s) - \partial_x\phi_2(s)\|_{L^\infty} + \tilde{\tilde{C}}(s)|X_1(s) - X_2(s)|\}. \qquad (105)\end{aligned}$$

Combining (100), (103), (104), (105) yields for any $0 \leq s \leq t \leq T$

$$\begin{aligned}\frac{d}{ds}\{|X_1(s) - X_2(s)| &+ |Q_1(s) - Q_2(s)|\} \leq C\{|X_1(s) - X_2(s)| + |Q_1(s) - Q_2(s)|\} \\ &+ C\{\|\phi_1(s) - \phi_2(s)\|_{L^\infty} + \|\partial_x\phi_1(s) - \partial_x\phi_2(s)\|_{L^\infty}\},\end{aligned}$$

for some constant depending on $\sup_{s\in[0,t]}\{C(s)+\tilde{C}(s)+\tilde{\tilde{C}}(s)\}$. We deduce that for any $0\le s\le t\le T$ we have

$$|X_1(s)-X_2(s)|+|Q_1(s)-Q_2(s)|\le C\int_0^s\{|X_1(\tau)-X_2(\tau)|+|Q_1(\tau)-Q_2(\tau)|\}d\tau$$
$$+\quad C|p(\phi_1-\phi_2)(0,x)|+C\int_0^t\{\|\phi_1(\tau)-\phi_2(\tau)\|_{L^\infty}+\|\partial_x\phi_1(\tau)-\partial_x\phi_2(\tau)\|_{L^\infty}\}d\tau.$$

By Gronwall lemma we obtain for any $0\le s\le t\le T$

$$|X_1-X_2|(s)+|Q_1-Q_2|(s)\le Ce^{Cs}\left(|p(\phi_1-\phi_2)(0,x)|+\int_0^t\|\phi_1(\tau)-\phi_2(\tau)\|_{W^{1,\infty}}d\tau\right).$$

Notice that if $(X_k,P_k)_{k\in\{1,2\}}$ satisfy the property (102), then in formula (105) we can replace $\|\phi_1(s)-\phi_2(s)\|_{L^\infty}$ by $\|\phi_1(s)-\phi_2(s)\|_{L^\infty([-(R-s),R-s])}$ and $\|\partial_x\phi_1(s)-\partial_x\phi_2(s)\|_{L^\infty}$ by $\|\partial_x\phi_1(s)-\partial_x\phi_2(s)\|_{L^\infty([-(R-s),R-s])}$. Finally one gets for such characteristics

$$\begin{aligned}|X_1(s)-X_2(s)|+|Q_1(s)-Q_2(s)|\ &\le\ Ce^{Cs}|p(\phi_1-\phi_2)(0,x)| \\ &+\ Ce^{Cs}\int_0^t\|\phi_1(\tau)-\phi_2(\tau)\|_{W^{1,\infty}([-(R-\tau),R-\tau])}d\tau.\end{aligned}\tag{106}$$

□

Proof. (of Corollary 2.1) Using (98), (100) and Proposition 2.2 yields for any $t\in[0,T]$

$$\begin{aligned}\left|\sum_{k=1}^2\frac{(-1)^ke^{\phi_k(t,X_k(t))}}{(1+|P_k(t)|^2)^{\frac12}}\right|\ &\le\ C(\|\phi_1(t)-\phi_2(t)\|_{L^\infty}+|X_1(t)-X_2(t)|+|Q_1(t)-Q_2(t)|)\\ &\le\ C\left(\|\phi_1(t)-\phi_2(t)\|_{L^\infty}+\int_0^t\|\phi_1(s)-\phi_2(s)\|_{W^{1,\infty}}ds\right)\\ &+\ C|p(\phi_1-\phi_2)(0,x)|,\end{aligned}$$

where C depends only on $\max_{k\in\{1,2\}}\{\|\phi_k\|_{L^\infty(]0,T[;W^{2,\infty}(\mathbb{R}))}\}$. In order to check the second inequality we can write

$$\begin{aligned}&|\sum_{k=1}^2(-1)^k\frac{e^{\phi_k(t,X_k(t))}}{(1+|P_k(t)|^2)^{\frac12}}\partial_x\phi_k(t,X_k(t))|\le\|\partial_x\phi_1(t)\|_{L^\infty}\left|\sum_{k=1}^2(-1)^k\frac{e^{\phi_k(t,X_k(t))}}{(1+|P_k(t)|^2)^{\frac12}}\right|\\ &\quad+\ e^{\|\phi_2(t)\|_{L^\infty}}(\|\partial_x\phi_1(t)-\partial_x\phi_2(t)\|_{L^\infty}+\|\partial_x^2\phi_2(t)\|_{L^\infty}|X_1(t)-X_2(t)|)\\ &\quad\le\ C\left(|p(\phi_1-\phi_2)(0,x)|+\|\phi_1(t)-\phi_2(t)\|_{W^{1,\infty}}+\int_0^t\|\phi_1(s)-\phi_2(s)\|_{W^{1,\infty}}ds\right).\end{aligned}$$

The third inequality follows in similar manner. As before it is easily seen that, for characteristics satisfying (102), all the statements of Corollary 2.1 hold with $\|\phi_1(s)-\phi_2(s)\|_{L^\infty}$, $\|\partial_{(t,x)}\phi_1(s)-\partial_{(t,x)}\phi_2(s)\|_{L^\infty}$ replaced by $\|\phi_1(s)-\phi_2(s)\|_{L^\infty([-(R-s),R-s])}$, $\|\partial_{(t,x)}\phi_1(s)-\partial_{(t,x)}\phi_2(s)\|_{L^\infty([-(R-s),R-s])}$. □

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