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INDIRECT STABILIZATION OF WEAKLY COUPLED SYSTEMS WITH HYBRID BOUNDARY CONDITIONS

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Abstract. We investigate stability properties of indirectly damped systems of evolution equations in Hilbert spaces, under new compatibility assumptions. We prove polynomial decay for the energy of solutions and optimize our results by interpolation techniques, obtaining a full range of power-like decay rates. In particular, we give explicit estimates with respect to the initial data. We discuss several applications to hyperbolic systems with *hybrid* boundary conditions, including the coupling of two wave equations subject to Dirichlet and Robin type boundary conditions, respectively.

Key words: indirect stabilization, energy estimates, interpolation spaces, evolution equations, hyperbolic systems

1. Introduction

There is no doubt that the interest of the scientific community in the stabilization and control of systems of partial differential equations has remarkably increased, in recent years. This is probably due to the fact that such systems arise in several applied mathematical models, such as those used for studying the vibrations of flexible structures and networks (see [18] and references therein), or fluids and fluid-structure interactions (see, for instance, [8], [9], [16], [21], [26], [30]).

When dealing with systems involving quantities described by several components, pretending to control or observe all the state variables might be irrealistic. In applications to mathematical models for the vibrations of flexible structures (see [3] and [7]), electromagnetism (see, for instance, [20]), or fluid control (see [17] and the references therein), it may happen that only part of such components can be observed. This is why it becomes essential to study whether controlling only a reduced number of state variables suffices to ensure the stability of the full system.

It turns out that certain systems possess an internal structure that compensates for the aforementioned lack of control variables. Such a phenomenon is referred to as *indirect stabilization* or *indirect control* (see [27]). An

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example of indirect stabilization occurs with the hyperbolic system

$$\begin{cases} \partial_t^2 u - \Delta u + \partial_t u + \alpha v = 0 & \text{in} \quad \Omega \times \mathbb{R} \\ \partial_t^2 v - \Delta v + \alpha u = 0 & \text{in} \quad \Omega \times \mathbb{R} \\ u = 0 = v & \text{on} \quad \partial\Omega \times \mathbb{R}\,, \end{cases} \tag{1.1}$$

where Ω is a bounded open domain of $\mathbb{R}^N$, and the 'frictional' term $\partial_t u$ acts as a stabilizer. Indeed, a general result proved in [4] ensures that, for sufficiently smooth initial conditions and $|\alpha| > 0$ small enough, the energy of the solution (u, v) of (1.1) decays to zero at a polynomial rate as $t \to \infty$.

The above indirect stabilization property holds true for more general systems of partial differential equations, under the compatibility assumption (1.10) below, see [4]. For applications to problems in mechanical engineering, however, it is extremely important to consider also boundary conditions that fail to satisfy the assumption of [4]. This is the case of Neumann or Robin boundary conditions, which describe different physical situations such as hinged or clamped devices. For instance, let us change the boundary conditions in (1.1) as follows:

$$\begin{cases} \partial_t^2 u - \Delta u + \partial_t u + \alpha v = 0 & \text{in} \quad \Omega \times \mathbb{R} \\ \partial_t^2 v - \Delta v + \alpha u = 0 & \text{in} \quad \Omega \times \mathbb{R} \\ u + \frac{\partial u}{\partial \nu} = 0 = v & \text{on} \quad \partial\Omega \times \mathbb{R}\,. \end{cases} \tag{1.2}$$

Then, as is shown in Proposition 5.4 below, the compatibility assumption (1.10) is not satisfied. Nevertheless, in this paper we will prove polynomial stability for system (1.2), using a new hypothesis which is specially designed to handle boundary conditions as above—that we call *hybrid*.

More generally, in a real Hilbert space H, with scalar product $\langle \cdot, \cdot \rangle$ and norm $|\cdot|$, we shall study the system of evolution equations

$$\begin{cases} u''(t) + A_1 u(t) + Bu'(t) + \alpha v(t) = 0 \\ v''(t) + A_2 v(t) + \alpha u(t) = 0 \end{cases} \tag{1.3}$$

where

(H1) $A_i : D(A_i) \subset H \to H$ $(i = 1, 2)$ are densely defined closed linear operators such that

$$A_i = A_i^*\,, \qquad \langle A_i u, u \rangle \geq \omega_i |u|^2 \qquad \forall u \in D(A_i)$$

for some $\omega_1, \omega_2 > 0$,

(H2) B is a bounded linear operator on H such that

$$B = B^*\,, \qquad \langle Bu, u \rangle \geq \beta |u|^2 \qquad \forall u \in H$$

for some $\beta > 0$,

(H3) α is a real number such that

$$0 < |\alpha| < \sqrt{\o_1 \o_2}\,.$$

System (1.3), with the initial conditions

$$\begin{cases} u(0) = u^0\,, & u'(0) = u^1\,, \\ v(0) = v^0\,, & v'(0) = v^1\,, \end{cases} \tag{1.4}$$

can be formulated as a Cauchy problem for a certain first order evolution equation in the product space

$$\mathcal{H} := D(A_1^{1/2}) \times H \times D(A_2^{1/2}) \times H\,.$$

More precisely, let us define the energies associated to operators A_1, A_2 by

$$E_i(u,p) = \frac{1}{2}\left(|A_i^{1/2}u|^2 + |p|^2\right) \quad \forall (u,p) \in D(A_i^{1/2}) \times H \ (i=1,2)\,, \tag{1.5}$$

and the total energy of the system as

$$\mathcal{E}(U) := E_1(u,p) + E_2(v,q) + \alpha\langle u,v\rangle \tag{1.6}$$

for every $U = (u,p,v,q) \in \mathcal{H}$. Then, assumption (H1) yields, for $i = 1,2$,

$$|u|^2 \le \frac{2}{\omega_i}\, E_i(u,p) \qquad \forall u \in D(A_i^{1/2}),\ \forall p \in H\,. \tag{1.7}$$

Moreover, in view of $(H3)$, for all $U = (u,p,v,q) \in \mathcal{H}$

$$\mathcal{E}(U) \ge \nu(\alpha)\Big[E_1(u,p) + E_2(v,q)\Big]\,, \tag{1.8}$$

where $\nu(\alpha) = 1 - |\alpha|(\omega_1\omega_2)^{-1/2} > 0$.
We define a bilinear form on $\mathcal{H}$ by

$$(U|\widehat{U}) := \langle A_1^{1/2}u, A_1^{1/2}\widehat{u}\rangle + \langle p,\widehat{p}\rangle + \langle A_2^{1/2}v, A_2^{1/2}\widehat{v}\rangle + \langle q,\widehat{q}\rangle + \alpha\langle u,\widehat{v}\rangle + \alpha\langle v,\widehat{u}\rangle$$

for every $U, \widehat{U} \in \mathcal{H}$. Thanks to (1.8) and since

$$(U|U) = 2\mathcal{E}(U) \quad \forall U \in \mathcal{H}\,,$$

the above bilinear form is a scalar product on $\mathcal{H}$, and $\mathcal{H}$ is a Hilbert space. Let $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ be the operator defined by

$$\begin{cases} D(\mathcal{A}) = D(A_1) \times D(A_1^{1/2}) \times D(A_2) \times D(A_2^{1/2}) \\ \mathcal{A}U = (p, -A_1u - Bp - \alpha v, q, -A_2 v - \alpha u) \qquad \forall U \in D(\mathcal{A})\,. \end{cases}$$

Then, problem (1.3) takes the equivalent form

$$\begin{cases} U'(t) = \mathcal{A}U(t) \\ U(0) = U_0 := (u^0, u^1, v^0, v^1)\,. \end{cases} \tag{1.9}$$

As will be proved in Lemma 4.2, $\mathcal{A}$ is a maximal dissipative operator. Then, from classical results (see, for instance, [25]), it follows that $\mathcal{A}$ generates a C_0-semigroup, $e^{t\mathcal{A}}$, on $\mathcal{H}$. Also,

$$e^{t\mathcal{A}}U_0 = (u(t), p(t), v(t), q(t))\,,$$

where (u,v) is the solution of problem (1.3)-(1.4), and $(p,q) = (u',v')$.

In order to introduce our asymptotic analysis of system (1.3)-(1.4)—or, equivalently, (1.9)—let us observe that, as is explained in [4], no exponential stability can be expected. Therefore, weaker decay rates at infinity, such as polynomial ones, are to be sought for. Polynomial decay results for (1.3) were obtained in [4] assuming that, for some integer $j \ge 2$,

$$|A_1 u| \le c|A_2^{j/2}u| \qquad \forall u \in D(A_2^{j/2})\,. \tag{1.10}$$

Similar decay estimates for the case of boundary damping (that is, when operator B is unbounded) were derived in [2]. Also, we refer the reader to [13], [14] and [29] for indirect stabilization with localized damping, and to [6] for the study of a one-dimensional wave system coupled through velocities. Following this, in [10], resolvent estimates and spectral analysis were used to prove polynomial decay for (1.3), covering some of the examples treated in [4]. For related results that can be deduced by a Riesz basis approach, see [22]. The optimality of spectral-analysis-derived decay rates was shown in [11] and [15], taking into account the asymptotic behaviour of the resolvent on the imaginary axis. In concrete examples, however, such resolvent estimates might be hard to obtain. Finally, we refer to [5] for general results for localized or boundary dampings and noncoercive couplings.

In this paper, we will replace (1.10) by

$$D(A_2) \subset D(A_1^{1/2}) \qquad \text{and} \qquad |A_1^{1/2}u| \le c|A_2u| \quad \forall u \in D(A_2), \tag{1.11}$$

which is satisfied by a large class of systems including (1.2) as a special case (see Section 5 below). Under such a condition we will show that any solution U of (1.9) satisfies the integral inequality

$$\int_0^T \mathcal{E}(U(t))dt \le c_1 \sum_{k=0}^{4} \mathcal{E}(U^{(k)}(0)) \qquad \forall\, T > 0\,,\ U_0 \in D(\mathcal{A}^4)\,. \tag{1.12}$$

Moreover, since the energy of solutions is decreasing in time, (1.12) implies, in turn, the polynomial decay of order n of $\mathcal{E}$, that is,

$$\mathcal{E}(U(t)) \le \frac{c_n}{t^n} \sum_{k=0}^{4n} \mathcal{E}(U^{(k)}(0)) \qquad \forall t > 0 \tag{1.13}$$

for all $n \ge 1$ and $U_0 \in D(\mathcal{A}^{4n})$ (see Corollary 3.3 below). Notice that (1.13) yields, in particular, the strong stability of $e^{t\mathcal{A}}$.

Passing from polynomial to a general power-like decay estimate is quite natural, once (1.13) has been established. Indeed, in Section 4, using *interpolation theory*, we obtain the fractional decay rate

$$\mathcal{E}(U(t)) \le \frac{c_n}{t^{n/4}} \sum_{k=0}^{n} \mathcal{E}(U^{(k)}(0)) \qquad \forall t > 0 \tag{1.14}$$

for all $n \ge 1$ and $U_0 \in D(\mathcal{A}^n)$ (see Corollary 4.5 below). Moreover, taking initial data in $\big(\mathcal{H}, D(\mathcal{A}^n)\big)_{\theta,2}$ for any $0 < \theta < 1$, we deduce the continuous decay rate

$$\|U(t)\|_{\mathcal{H}}^2 \le \frac{c_{n,\theta}}{t^{n\theta/4}} \|U_0\|^2_{(\mathcal{H},D(\mathcal{A}^n))_{\theta,2}} \qquad \forall t > 0\,. \tag{1.15}$$

In particular, for $n = 1$, (1.14) implies that, for every $U_0 \in D(\mathcal{A})$, the solution U of problem (1.9) satisfies

$$E_1(u(t), u'(t)) + E_2(v(t), v'(t)) \le \frac{c}{t^{1/4}} \|U_0\|^2_{D(\mathcal{A})} \qquad \forall t > 0\,, \tag{1.16}$$

and there exists $c_1 > 0$ such that

$$\|U_0\|^2_{D(\mathcal{A})} \le c_1 \left(|A_1 u^0|^2 + |A_1^{1/2} u^1|^2 + |A_2 v^0|^2 + |A_2^{1/2} v^1|^2\right)\,.$$

Furthermore, we would like to point out that not only can interpolation techniques be applied to systems satisfying (1.11), but they yield stronger results in the framework studied in [4] as well. We describe such applications in Section 6, where we show how to deduce power-like decay rates from the energy estimates of [4], thus recovering, in a more general set-up, related asymptotic estimates that can be obtained by spectral analysis.

This paper is organized as follows. Section 2 recalls preliminary notions, mainly related to interpolation theory which is so relevant for most of this paper. Section 3 is devoted to our polynomial decay result and its proof. In Section 4, we complete the analysis with estimates in interpolation spaces. In Section 5, we describe several applications to systems of partial differential operators. Finally, in Section 6, we show how to improve the results of [4] by interpolation.

2. Preliminaries

In this section, we introduce the main tools required to deal with interpolation theory between Banach spaces. For a general exposition of this theory the reader is referred to [28] and [24]. Interesting introductions are also given in [12] from the point of view of control theory, and [23] for the specific case of analytic semigroups.

In this section $(X, |\cdot|_X)$ stands for a real Banach space. Let $(Y, |\cdot|_Y)$ be another Banach space. We say that Y is continuously embedded into X, and we write $Y \hookrightarrow X$, if $Y \subset X$ and

$$|x|_X \le c|x|_Y \qquad \forall x \in Y$$

for some constant $c > 0$.

We denote by $\mathcal{L}(Y; X)$ the Banach space of all bounded linear operators $T : Y \to X$ equipped with the standard operator norm. If $Y = X$, we refer to such a space as $\mathcal{L}(X)$. For any given subspace D of X, we denote by $T_{|D}$ the restriction of T to D.

Definition 2.1. Let $(D, |\cdot|_D)$ be a closed subspace of X. A subspace $(Y, |\cdot|_Y)$ of X is said to be an interpolation space between D and X if

(a) $D \hookrightarrow Y \hookrightarrow X$, and
(b) for every $T \in \mathcal{L}(X)$ such that $T_{|D} \in \mathcal{L}(D)$, we have that $T_{|Y} \in \mathcal{L}(Y)$.

Let X, D be Banach spaces, with D continuously embedded into X. For any $\alpha \in [0, 1]$, we denote by $J_\alpha(X, D)$ the family of all subspaces Y of X containing D such that

$$|x|_Y \le c|x|_D^\alpha \, |x|_X^{1-\alpha} \quad \forall x \in D$$

for some constant $c > 0$.

Let us introduce, for each $x \in X$ and $t > 0$, the quantity

$$K(t, x, X, D) := \inf_{\substack{x=a+b,\\ a\in X,\, b\in D}} \left(|a|_X + t|b|_D\right). \tag{2.1}$$

Let $0 < \theta < 1$ be fixed. We define

$$(X, D)_{\theta,2} := \left\{ x \in X : \int_0^{+\infty} |t^{-\theta-\frac{1}{2}} K(t, x, X, D)|^2 \, dt < +\infty \right\} \tag{2.2}$$

and

$$|x|_{\theta,2}^2 := \int_0^{+\infty} |t^{-\theta-\frac{1}{2}} K(t, x, X, D)|^2 \, dt \,.$$

The space $(X, D)_{\theta,2}$, endowed with the norm $|\cdot|_{\theta,2}$, is a Banach space.

The reader is referred to [24] for the proof of the following results.

Theorem 2.2. *Let X_1, X_2, D_1, D_2 be Banach spaces such that D_i is continuously embedded in X_i, for $i = 1, 2$. If $T \in \mathcal{L}(X_1; X_2) \cap \mathcal{L}(D_1; D_2)$, then $T \in \mathcal{L}((X_1, D_1)_{\theta,2}; (X_2, D_2)_{\theta,2})$ for every $\theta \in (0, 1)$. Moreover,*

$$\|T\|_{\mathcal{L}((X_1,D_1)_{\theta,2};(X_2,D_2)_{\theta,2})} \le \|T\|^{1-\theta}_{\mathcal{L}(X_1;X_2)} \|T\|^{\theta}_{\mathcal{L}(D_1;D_2)} .$$

Consequently, the space $(X, D)_{\theta,2}$ belongs to $J_\theta(X, D)$ for every $\theta \in (0, 1)$. Let $\alpha \in [0, 1]$ and denote by $K_\alpha(X, D)$ the family of all subspaces $(Y, |\cdot|_Y)$ of X containing D such that

$$\sup_{t>0,\, x\in Y} \frac{K(t, x, X, D)}{t^\alpha |x|_Y} < +\infty .$$

Theorem 2.3 (Reiteration Theorem)**.** *Let $0 < \theta_0 < \theta_1 < 1$. Fix $\theta \in]0, 1[$ and set $\emptyset = (1 - \theta)\theta_0 + \theta\theta_1$.*

1) *If $E_i \in K_{\theta_i}(X, D)$, $i = 0, 1$, then $(E_0, E_1)_{\theta,2} \subset (X, D)_{\emptyset,2}$.*
2) *If $E_i \in J_{\theta_i}(X, D)$, $i = 0, 1$, then $(X, D)_{\emptyset,2} \subset (E_0, E_1)_{\theta,2}$.*

Consequently, if $E_i \in J_{\theta_i}(X, D) \cap K_{\theta_i}(X, D)$, $i = 0, 1$, then we have that $(E_0, E_1)_{\theta,2} = (X, D)_{\emptyset,2}$, with equivalence between the respective norms.

Remark 2.4. Since $(X, D)_{\theta,2}$ is contained in $J_\theta(X, D) \cap K_\theta(X, D)$, for every $0 < \theta_0, \theta_1 < 1$ we have

$$((X, D)_{\theta_0,2}, (X, D)_{\theta_1,2})_{\theta,2} = (X, D)_{(1-\theta)\theta_0+\theta\theta_1,2} . \tag{2.3}$$

Since $X \in J_0(X, D) \cap K_0(X, D)$ and $D \in J_1(X, D) \cap K_1(X, D)$, we also have

$$(X, (X, D)_{\theta_1,2})_{\theta,2} = (X, D)_{\theta\theta_1,2} \quad \text{and} \tag{2.4}$$

$$((X, D)_{\theta_0,2}, D)_{\theta,2} = (X, D)_{(1-\theta)\theta_0+\theta,2} . \tag{2.5}$$

2.1. **Interpolation spaces and fractional powers of operators.** Let $(H, \langle \cdot, \cdot \rangle)$ be a real Hilbert space, with norm $|\cdot|$. Let $A : D(A) \subset H \to H$ be a densely defined closed linear operator on H such that

$$\langle Ax, x\rangle \ge \delta |x|^2 , \quad \forall\, x \in D(A) \tag{2.6}$$

for some $\delta > 0$. We denote by A^θ the fractional power of A for any $\theta \in \mathbb{R}$ (see, for instance, [12, Chapter 1 - Section 5]).
Denoting by A^* the adjoint of A, we call A self-adjoint if $D(A) = D(A^*)$ and $\langle Ax, y\rangle = \langle x, Ay\rangle$ for every x, $y \in D(A)$. For the proof of the following result we refer to [24, Theorem 4.36].

Theorem 2.5. *Let A be a self-adjoint operator satisfying* (2.6)*. Then, for every $\theta \in (0, 1)$, α, $\beta \in \mathbb{R}$ such that $\beta > \alpha \ge 0$,*

$$(D(A^\alpha), D(A^\beta))_{\theta,2} = D(A^{(1-\theta)\alpha+\theta\beta}) . \tag{2.7}$$

In particular,

$$(H, D(A^\beta))_{\theta,2} = D(A^{\beta\theta}) . \tag{2.8}$$

We say that A is an m-accretive operator if

$$\begin{cases} \langle Ax, x\rangle \ge 0 & \forall x \in D(A) \quad (\textit{accretivity}) \\ (\lambda I + A)D(A) = H & \text{for some } \lambda > 0 \quad (\textit{maximality}) \end{cases}$$

Notice that, if there exists some $\lambda > 0$ that satisfies the maximality condition, then the same condition is verified for every $\lambda > 0$. Moreover, we say that A is m-dissipative if $-A$ is m-accretive.

We refer the reader to [24, Section 4.3] for the proof of the next result.

Proposition 2.6. *Let $(A, D(A))$ be an m-accretive operator on a Hilbert space H, with A^{-1} bounded in H. Then for every $\alpha, \beta \in \mathbb{R}$, $\beta > \alpha \geq 0$, $\theta \in (0,1)$, A satisfies (2.7) and (2.8). In particular,*

$$D(A^\theta) = (H, D(A))_{\theta,2} \quad \forall\, 0 < \theta < 1\,. \tag{2.9}$$

Corollary 2.7. *If A is the infinitesimal generator of a $\mathcal{C}_0$-semigroup of contractions on H, with A^{-1} bounded in H, then $D(A^m) = (H, D(A^k))_{\theta,2}$ for every $k \in \mathbb{N}$, $\theta \in (0,1)$ such that $m = \theta k$ is an integer.*

2.2. **An abstract decay result.** We recall an abstract result obtained in [1] in a slightly different form, and in [4, Theorem 2.1] in the current version. Let $A : D(A) \subset H \to H$ be the infinitesimal generator of a $\mathcal{C}_0$-semigroup of bounded linear operators on H.

Theorem 2.8. *Let $L : H \to [0, +\infty)$ be a continuous function such that, for some integer $K \geq 0$ and some constant $c \geq 0$,*

$$\int_0^T L(e^{tA}x)dt \leq c \sum_{k=0}^{K} L(A^k x) \qquad \forall T \geq 0\,,\ \forall x \in D(A^K)\,. \tag{2.10}$$

Then, for any integer $n \geq 1$, any $x \in D(A^{nK})$ and any $0 \leq s \leq T$

$$\int_s^T L(e^{tA}x)\, \frac{(t-s)^{n-1}}{(n-1)!}\, dt \leq c^n (1+K)^{n-1} \sum_{k=0}^{nK} L(e^{sA} A^k x)\,. \tag{2.11}$$

If, in addition, $L(e^{tA}x) \leq L(e^{sA}x)$ for any $x \in H$ and any $0 \leq s \leq t$, then

$$L(e^{tA}x) \leq c^n (1+K)^{n-1}\, \frac{n!}{t^n} \sum_{k=0}^{nK} L(A^k x) \qquad \forall t > 0 \tag{2.12}$$

for any integer $n \geq 1$ and any $x \in D(A^{nK})$.

3. Main result

We are now ready to state and prove the polynomial decay of solutions to weakly coupled systems. In addition to the standing assumptions $(H1), (H2), (H3)$, we will assume that

$$D(A_2) \subset D(A_1^{1/2}) \qquad \text{and} \qquad |A_1^{1/2} u| \leq c|A_2 u| \quad \forall u \in D(A_2) \tag{3.1}$$

for some constant $c > 0$. Condition (3.1) can be formulated in the following equivalent ways.

Lemma 3.1. *Under assumption $(H1)$ the following properties are equivalent.*

(a) *Assumption* (3.1) *holds.*
(b) $A_1^{1/2} A_2^{-1} \in \mathcal{L}(H)$.

(c) *For some constant $c > 0$*

$$|\langle A_1 u, v\rangle| \le c|A_2 v|\langle A_1 u, u\rangle^{1/2} \qquad \forall u \in D(A_1)\,,\ \forall v \in D(A_2)\,. \tag{3.2}$$

Proof. The implications (a)$\Leftrightarrow$(b)$\Rightarrow$(c) being straightforward, let us proceed to show that (c)$\Rightarrow$(a). Consider the Hilbert space $V_1 = D(A_1^{1/2})$ with the scalar product

$$\langle u, v\rangle_{V_1} = \langle A_1^{1/2} u, A_1^{1/2} v\rangle$$

and recall that $D(A_1)$ is a dense subspace of V_1. Let $v \in D(A_2)$ and define the linear functional $\phi_v : D(A_1) \to \mathbb{R}$ by

$$\phi_v(u) = \langle A_1 u, v\rangle \qquad \forall u \in D(A_1)\,.$$

Owing to (c), ϕ_v can be extended to a bounded linear functional on V_1 (still denoted by ϕ_v) satisfying $\|\phi_v\| \le c|A_2 v|$. Therefore, by the Riesz Theorem, there is a unique vector $w \in V_1$ such that

$$\phi_v(u) = \langle A_1^{1/2} u, A_1^{1/2} w\rangle \qquad \forall u \in V_1\,.$$

Hence, $\langle A_1 u, (v - w)\rangle = 0$ for all $u \in D(A_1)$, and so $v = w \in V_1$ since A_1 is invertible. Moreover, $|A_1^{1/2} v| = |w|_{V_1} \le c|A_2 v|$. $\square$

The main result of this section is the following.

Theorem 3.2. *Assume $(H1), (H2), (H3)$ and (3.1). If $U_0 \in D(\mathcal{A}^4)$, then the solution U of problem (1.9) satisfies*

$$\int_0^T \mathcal{E}(U(t))dt \le c_1 \sum_{k=0}^{4} \mathcal{E}(U^{(k)}(0)) \qquad \forall\, T > 0 \tag{3.3}$$

for some constant $c_1 > 0$.

The proof of Theorem 3.2 will be given in several steps. First, let us recall that, as showed in [4, Lemma 3.3], system (1.9) is dissipative. Indeed, under the only assumptions $(H1)$ and $(H2)$, the energy of the solution $U = (u, u', v, v')$ of problem (1.9) with $U_0 \in D(\mathcal{A})$ satisfies

$$\frac{d}{dt}\mathcal{E}(U(t)) = -|B^{1/2} u'(t)|^2 \qquad \forall t \ge 0. \tag{3.4}$$

In particular, $t \mapsto \mathcal{E}(U(t))$ is nonincreasing on $[0, \infty)$. Thanks to (3.4), Theorem 3.2, and Theorem 2.8, we deduce the next result.

Corollary 3.3. *Assume $(H1), (H2), (H3)$ and (3.1).*

(a) *If $U_0 \in D(\mathcal{A}^{4n})$ for some integer $n \ge 1$, then the solution U of problem (1.9) satisfies*

$$\mathcal{E}(U(t)) \le \frac{c_n}{t^n} \sum_{k=0}^{4n} \mathcal{E}(U^{(k)}(0)) \qquad \forall t > 0 \tag{3.5}$$

for some constant $c_n > 0$.

(b) *For every $U_0 \in \mathcal{H}$ we have*

$$\mathcal{E}(U(t)) \to 0 \quad \textit{as} \quad t \to \infty.$$

We now proceed with the proof of Theorem 3.2. Hereafter, C will denote a generic positive constant, independent of α. To begin with, let us recall that, thanks to [4, Lemma 3.4], the solution of (1.9) with $U_0 \in D(\mathcal{A})$ verifies

$$\int_0^T \mathcal{E}(U(t))dt \leq \int_0^T |v'(t)|^2 dt + C\,\mathcal{E}(U(0)) \tag{3.6}$$

for some constant $C \geq 0$ and every $T \geq 0$. Hence, the main technical point of the proof is to bound the right-hand side of (3.6) by the total energy of U (and a finite number of its derivatives) at 0.

Lemma 3.4. *Let $U = (u, u', v, v')$ be the solution of problem (1.9) with $U_0 \in D(\mathcal{A})$. Then*

$$\int_0^T |A_1^{-1/2} v|^2 dt \leq C \int_0^T |A_2^{-1/2} u|^2 dt + \frac{C}{\alpha^2}\Big[\mathcal{E}(U(0)) + \mathcal{E}(U'(0))\Big]. \tag{3.7}$$

Proof. Rewrite (1.9) as system (1.3) to obtain

$$\int_0^T \langle u'' + A_1 u + Bu' + \alpha v, A_1^{-1} v\rangle dt - \int_0^T \langle v'' + A_2 v + \alpha u, A_2^{-1} u\rangle dt = 0\,.$$

Hence, by straightforward computations,

$$\begin{aligned}\alpha \int_0^T |A_1^{-1/2} v|^2 dt &\leq \alpha \int_0^T |A_2^{-1/2} u|^2 dt \\ &\quad - \int_0^T \langle Bu', A_1^{-1} v\rangle dt + \int_0^T \big[\langle v'', A_2^{-1} u\rangle - \langle u'', A_1^{-1} v\rangle\big] dt\,.\end{aligned}$$

Integration by parts transforms the last inequality into

$$\begin{aligned}\alpha \int_0^T |A_1^{-1/2} v|^2 dt &\leq \alpha \int_0^T |A_2^{-1/2} u|^2 dt - \int_0^T \langle A_1^{-1/2} Bu', A_1^{-1/2} v\rangle dt \\ &\quad + \int_0^T \big[\langle A_1^{-1/2} v, A_1^{1/2} A_2^{-1} u''\rangle - \langle A_1^{-1/2} u'', A_1^{-1/2} v\rangle\big] dt \\ &\quad + \Big[\langle v', A_2^{-1} u\rangle - \langle v, A_2^{-1} u'\rangle\Big]_0^T.\end{aligned} \tag{3.8}$$

We now proceed to bound the right-hand side of (3.8). We have

$$\left|\int_0^T \langle A_1^{-1/2} Bu', A_1^{-1/2} v\rangle dt\right| \leq \frac{\alpha}{4} \int_0^T |A_1^{-1/2} v|^2 dt + \frac{C}{\alpha} \int_0^T |B^{1/2} u'|^2 dt\,.$$

Similarly, thanks to assumption (3.1) and the fact that B is positive definite,

$$\left|\int_0^T \langle A_1^{-1/2} v, A_1^{1/2} A_2^{-1} u''\rangle dt\right| \leq \frac{\alpha}{4} \int_0^T |A_1^{-1/2} v|^2 dt + \frac{C}{\alpha} \int_0^T |B^{1/2} u''|^2 dt\,.$$

Also,

$$\left|\int_0^T \langle A_1^{-1/2} u'', A_1^{-1/2} v\rangle dt\right| \leq \frac{\alpha}{4} \int_0^T |A_1^{-1/2} v|^2 dt + \frac{C}{\alpha} \int_0^T |B^{1/2} u''|^2 dt\,.$$

Finally, observe that the last term in (3.8) can be bounded as follows

$$\left|\Big[\langle v', A_2^{-1} u\rangle - \langle v, A_2^{-1} u'\rangle\Big]_0^T\right| \leq C\mathcal{E}(U(0))\,.$$

Combining the above estimates with (3.8), we obtain

$$\int_0^T |A_1^{-1/2}v|^2dt \le C\int_0^T |A_2^{-1/2}u|^2dt + \frac{C}{\alpha}\mathcal{E}(U(0)) + \frac{C}{\alpha^2}\int_0^T \big[|B^{1/2}u'|^2 + |B^{1/2}u''|^2\big]dt\,.$$

The conclusion follows from the above inequality and the dissipation identity (3.4) applied to u and u'. □

Lemma 3.5. *Let $U = (u, u', v, v')$ be the solution of problem* (1.9) *with $U_0 \in D(\mathcal{A})$. Then*

$$\int_0^T |v|^2dt \le C\alpha^2\int_0^T |u|^2dt + \frac{C}{\alpha^2}\sum_{k=1}^{3}\mathcal{E}(U^{(k)}(0))\,. \tag{3.9}$$

Proof. Since $\langle v'' + A_2v + \alpha u, A_2^{-1}v\rangle = 0$, integrating over $[0,T]$ we have

$$\int_0^T |v|^2dt = -\alpha\int_0^T \langle v, A_2^{-1}u\rangle dt - \int_0^T \langle v'', A_2^{-1}v\rangle dt\,. \tag{3.10}$$

The last term in the above identity can be bounded using assumption (3.1) and Lemma 3.1 as follows

$$\left|\int_0^T \langle v'', A_2^{-1}v\rangle dt\right| = \left|\int_0^T \langle A_1^{-1/2}v'', A_1^{1/2}A_2^{-1}v\rangle dt\right| \le \frac{1}{4}\int_0^T |v|^2dt + C\int_0^T |A_1^{-1/2}v''|^2dt\,. \tag{3.11}$$

Now, applying (3.7) to v'' and (3.4) to u', we obtain

$$\begin{aligned}\int_0^T |A_1^{-1/2}v''|^2dt &\le C\int_0^T |A_1^{-1/2}u''|^2dt + \frac{C}{\alpha^2}\big[\mathcal{E}(U''(0)) + \mathcal{E}(U'''(0))\big]\\ &\le C\mathcal{E}(U'(0)) + \frac{C}{\alpha^2}\big[\mathcal{E}(U''(0)) + \mathcal{E}(U'''(0))\big]\,.\end{aligned} \tag{3.12}$$

On the other hand,

$$\left|\alpha\int_0^T \langle v, A_2^{-1}u\rangle dt\right| \le \frac{1}{4}\int_0^T |v|^2dt + C\alpha^2\int_0^T |u|^2dt\,. \tag{3.13}$$

The conclusion follows combining (3.10),...,(3.13). □

Let us now complete the proof of Theorem 3.2.

Proof of Theorem 3.2. To prove (3.3) it suffices to apply (3.9) to v' and use the resulting estimate to bound the right-hand side of (3.6). Since B is positive definite, the conclusion follows by the dissipation identity (3.4). □

Remark 3.6. Similar results can be obtained for systems of equations coupled with different coefficients, such as

$$\begin{cases} u''(t) + A_1u(t) + Bu'(t) + \alpha_1 v(t) = 0\\ v''(t) + A_2v(t) + \alpha_2 u(t) = 0\,. \end{cases} \tag{3.14}$$

In this case, $(H3)$ should be replaced with

(H3') α_1, α_2 are two real numbers such that $0 < \alpha_1\alpha_2 < \o_1\o_2$.

Let us explain how to adapt our approach to the case of $\alpha_1 \neq \alpha_2$, when α_1, $\alpha_2 > 0$. The total energy is defined by

$$\mathcal{E}(U) := \alpha_2 E_1(u,p) + \alpha_1 E_2(v,q) + \alpha_1\alpha_2\langle u, v\rangle\,,$$

where E_1 and E_2 are the energies of the two components, defined in (1.5). Moreover, for each $U \in \mathcal{H}$,

$$\mathcal{E}(U) \geq \nu(\alpha_1,\alpha_2)\Big[\alpha_2 E_1(u,p) + \alpha_1 E_2(v,q)\Big]\,,$$

with $\nu(\alpha_1,\alpha_2) = 1 - (\alpha_1\alpha_2)^{1/2}(\o_1\o_2)^{-1/2} > 0$. Finally, for each $U_0 \in D(\mathcal{A})$, the solution $U(t) = (u(t), p(t), v(t), q(t))$ of the first order evolution equation associated with system (3.14) satisfies

$$\frac{d}{dt}\mathcal{E}(U(t)) = -\alpha_2|B^{1/2}u'(t)|^2 \qquad \forall t \geq 0. \tag{3.15}$$

In particular, $t \mapsto \mathcal{E}(U(t))$ is nonincreasing on $[0,\infty)$. From this point, reasoning as in the above proof, the reader can easily derive the conclusion of Theorem 3.2.

4. Results with data in interpolation spaces

We now show how to improve Corollary 3.3 when the initial data belong to an interpolation space between $\mathcal{H}$ and the domain of a power of $\mathcal{A}$.

Theorem 4.1. *Assume* $(H1), (H2), (H3)$ *and* (3.1). *If* $U_0 \in (\mathcal{H}, D(\mathcal{A}^{4n}))_{\theta,2}$ *for some* $n \geq 1$ *and* $0 < \theta < 1$, *then the solution* U *of problem* (1.9) *satisfies*

$$\|U(t)\|^2_{\mathcal{H}} \leq \frac{c_{n,\theta}}{t^{n\theta}}\|U_0\|^2_{(\mathcal{H},D(\mathcal{A}^{4n}))_{\theta,2}} \qquad \forall t > 0 \tag{4.1}$$

for some constant $c_{n,\theta} > 0$.

Proof. Let $n \geq 1$ and $t > 0$ be fixed, and consider the operator $\Lambda_t : \mathcal{H} \to \mathcal{H}$ defined by

$$\Lambda_t(U_0) = U(t) \in \mathcal{H} \quad \text{where} \quad \begin{cases} U' = \mathcal{A}U \\ U(0) = U_0 \end{cases}$$

for each $U_0 \in \mathcal{H}$. Since $\Lambda_t \in \mathcal{L}(\mathcal{H}) \cap \mathcal{L}(D(\mathcal{A}^{4n}); \mathcal{H})$, we can apply Theorem 2.2 to conclude that $\Lambda_t \in \mathcal{L}((\mathcal{H}, D(\mathcal{A}^{4n}))_{\theta,2}; \mathcal{H})$ for every $\theta \in (0,1)$. Moreover,

$$\|\Lambda_t\|_{\mathcal{L}((\mathcal{H},D(\mathcal{A}^{4n}))_{\theta,2};\mathcal{H})} \leq \|\Lambda_t\|^{1-\theta}_{\mathcal{L}(\mathcal{H})}\,\|\Lambda_t\|^{\theta}_{\mathcal{L}(D(\mathcal{A}^{4n});\mathcal{H})}\,. \tag{4.2}$$

Since $\|U(t)\|_{\mathcal{H}} \leq \|U_0\|_{\mathcal{H}}$ for each $U_0 \in \mathcal{H}$, we have $\|\Lambda_t\|_{\mathcal{L}(\mathcal{H})} \leq 1$. Moreover, by (3.5), $\|\Lambda_t\|_{\mathcal{L}(D(\mathcal{A}^{4n}),\mathcal{H})} \leq \sqrt{c_n}/t^{n/2}$ for all $t > 0$. Then, (4.2) yields the conclusion. □

Although $(\mathcal{H}, D(\mathcal{A}^{4n}))_{\theta,2}$ is usually difficult to identify explicitly, we can single out important special cases where such an identification is possible. We need a preliminary result.

Lemma 4.2. *The operator* $\mathcal{A} : D(\mathcal{A}) \to \mathcal{H}$ *is invertible, with* $\mathcal{A}^{-1}$ *bounded. Moreover,* $\mathcal{A}$ *is m-dissipative (thus,* $\mathcal{A}$ *generates a* $\mathcal{C}_0$*-semigroup of contractions on* $\mathcal{H}$*).*

Proof. For any $U = (u, p, v, q)$, $\widehat{U} = (\hat{u}, \hat{p}, \hat{v}, \hat{q}) \in \mathcal{H}$, the identity $\mathcal{A}U = \widehat{U}$ is equivalent to

$$p = \hat{u}\,, \quad -A_1 u - Bp - \alpha v = \hat{p}\,, \quad q = \hat{v}\,, \quad -A_2 v - \alpha u = \hat{q}\,.$$

Hence, $p = \hat{u} \in D(A_1^{1/2})$, $q = \hat{v} \in D(A_2^{1/2})$. So, in order to compute $\mathcal{A}^{-1}$ it suffices to solve the system

$$\begin{cases} A_1 u + \alpha v = f \\ A_2 v + \alpha u = g\,, \end{cases} \tag{4.3}$$

for suitably chosen f, $g \in H$. Since $I - \alpha^2 A_1^{-1} A_2^{-1}$ is invertible thanks to $(H3)$, it is easy to check that (4.3) admits the solution

$$\begin{cases} \bar{u} = \left(I - \alpha^2 A_1^{-1} A_2^{-1}\right)^{-1} A_1^{-1}(f - \alpha A_2^{-1} g) \in D(A_1) \\ \bar{v} = A_2^{-1}(g - \alpha \bar{u}) \in D(A_2)\,. \end{cases}$$

Thus, $\mathcal{A}$ is invertible, and $\mathcal{A}^{-1}$ is bounded.
Moreover, $\mathcal{A}$ is dissipative, since

$$(\mathcal{A}U|U) \leq -\langle Bp, p\rangle_H \leq -\beta |p|_H^2 \leq 0 \quad \forall\, U \in D(\mathcal{A})\,.$$

In addition, it is easy to check that there exists $\lambda > 0$ such that the range of $\lambda I - \mathcal{A}$ equals $\mathcal{H}$. Thus, from Lumer-Phillips Theorem (see [25, Theorem 4.3]), $\mathcal{A}$ generates a $\mathcal{C}_0$-semigroup of contractions on $\mathcal{H}$. □

Applying Corollary 2.7, we obtain the following result.

Corollary 4.3. *If $\theta k = m$, for some $0 < \theta < 1$ and k, $m \in \mathbb{N}$, then*

$$D(\mathcal{A}^m) = (\mathcal{H}, D(\mathcal{A}^k))_{\theta,2}\,. \tag{4.4}$$

Remark 4.4. In particular, for $k = 4n$, $n \geq 1$, we obtain $(\mathcal{H}, D(\mathcal{A}^{4n}))_{\theta_j,2} = D(\mathcal{A}^j)$ for $j = 1, \ldots, 4n-1$, choosing $\theta_j = \frac{j}{4n}$. Applying Theorem 4.1 to these values of θ_j, one can show that, if $U_0 \in D(\mathcal{A}^j)$, then the associated solution $U(t)$ of problem (1.9) satisfies

$$\|U(t)\|_{\mathcal{H}}^2 \leq \frac{c_{n,j}}{t^{j/4}} \|U_0\|_{D(\mathcal{A}^j)}^2 \qquad \forall t > 0$$

for some constant $c_{n,j} > 0$. On the other hand, we claim that $c_{n,j}$ can be chosen independent of n. Indeed, if $j \neq 4n$, then one can choose the smallest $n_j \geq 1$ such that $j < 4n_j$, and use the identity $D(\mathcal{A}^j) = (\mathcal{H}, D(\mathcal{A}^{4n_j}))_{\theta_j,2}$ with $\theta_j = j/(4n_j)$. Hence, $c_{n_j,j} = c_j$.

Corollary 4.5. *Assume $(H1), (H2), (H3)$ and (3.1).*

i) *If $U_0 \in D(\mathcal{A}^n)$ for some $n \geq 1$, then the solution of (1.9) satisfies*

$$\mathcal{E}(U(t)) \leq \frac{c_n}{t^{n/4}} \sum_{k=0}^{n} \mathcal{E}(U^{(k)}(0)) \qquad \forall t > 0 \tag{4.5}$$

for some constant $c_n > 0$.

ii) *If $U_0 \in (\mathcal{H}, D(\mathcal{A}^n))_{\theta,2}$ for some $n \geq 1$ where $0 < \theta < 1$, then the solution of (1.9) satisfies*

$$\|U(t)\|_{\mathcal{H}}^2 \leq \frac{c_{n,\theta}}{t^{n\theta/4}} \|U_0\|_{(\mathcal{H}, D(\mathcal{A}^n))_{\theta,2}}^2 \qquad \forall t > 0 \tag{4.6}$$

for some constant $c_{n,\theta} > 0$.

iii) *If* $U_0 \in D((-\mathcal{A})^\theta)$ *for some* $0 < \theta < 1$, *then the solution of problem* (1.9) *satisfies*

$$\|U(t)\|^2_{\mathcal{H}} \le \frac{c_\theta}{t^{\theta/4}} \|U_0\|^2_{D((-\mathcal{A})^\theta)} \qquad \forall t > 0 \tag{4.7}$$

for some constant $c_\theta > 0$.

Proof. Points *i)* and *ii)* derive from Corollary 3.3 and following the proof of Theorem 4.1, thanks to Remark 4.4. In order to prove point *iii)*, first we deduce from Lemma 4.2 that $-\mathcal{A}$ is invertible with bounded inverse. Moreover, it is m-accretive on $\mathcal{H}$, hence (2.9) yields

$$(\mathcal{H}, D(\mathcal{A}))_{\theta,2} = (\mathcal{H}, D(-\mathcal{A}))_{\theta,2} = D((-\mathcal{A})^\theta)$$

for every $0 < \theta < 1$. Thus, the conclusion follows applying point *ii)* with $n = 1$. □

Under further assumptions, the norm in $(\mathcal{H}, D(\mathcal{A}))_{\theta,2}$ can be given a more explicit form. For this purpose, for each $k \ge 0$ consider the space

$$\mathcal{H}_k = D(A_1^{(k+1)/2}) \times D(A_1^{k/2}) \times D(A_2^{(k+1)/2}) \times D(A_2^{k/2}) .$$

We recall the following result (see [4, Lemma 3.1]).

Lemma 4.6. *Assume* $(H1)$, *and* $(H2)$. *Let* $n \ge 1$ *be such that*

$$BD(A_1^{(k+1)/2}) \subset D(A_1^{k/2}) \tag{4.8}$$

$$D(A_1^{(k/2)+1}) \subset D(A_2^{k/2}) \tag{4.9}$$

$$D(A_2^{(k/2)+1}) \subset D(A_1^{k/2}) \tag{4.10}$$

for every integer k *satisfying* $0 < k \le n-1$. *(no assumption is made if* $n = 1$*). Then* $\mathcal{H}_k \subset D(\mathcal{A}^k)$ *for every* $0 \le k \le n$.

It is also shown in [4] that $\mathcal{H}_k = D(\mathcal{A}^k)$ for every $0 \le k \le n$, provided (4.9) and (4.10) are replaced by the stronger assumptions

$$\begin{aligned} &D(A_1^{(k+1)/2}) \subset D(A_2^{k/2}) \\ &D(A_2^{(k+1)/2}) \subset D(A_1^{k/2}) \end{aligned} \qquad \text{for every} \quad 0 < k \le n-1 .$$

Let $0 < \theta < 1$ and $k \ge 1$ be fixed. As a direct consequence of Theorem 2.2, choosing appropriate spaces and operator T, one can show that, if $\mathcal{H}_k$ is contained in $D(\mathcal{A}^k)$, then $(\mathcal{H}, \mathcal{H}_k)_{\theta,2}$ is contained in $(\mathcal{H}, D(\mathcal{A}^k))_{\theta,2}$. Moreover, $(\mathcal{H}, \mathcal{H}_k)_{\theta,2}$ equals

$$\begin{aligned} \mathcal{H}_{k,\theta} \;:=\; & (D(A_1^{1/2}), D(A_1^{(k+1)/2}))_{\theta,2} \times (H, D(A_1^{k/2}))_{\theta,2} \\ & \times (D(A_2^{1/2}), D(A_2^{(k+1)/2}))_{\theta,2} \times (H, D(A_2^{k/2}))_{\theta,2} . \end{aligned}$$

Notice that, since A_i is self-adjoint and (2.6) holds for $i = 1, 2$, applying Theorem 2.5 we have for every $0 \le \alpha < \beta$, $i = 1, 2$,

$$(D(A_i^\alpha), D(A_i^\beta))_{\theta,2} = D(A_i^{(1-\theta)\alpha + \theta\beta}) ,$$

Therefore, $\mathcal{H}_{k,\theta}$ equals $D(A_1^{\frac{1}{2}+\frac{k}{2}\theta}) \times D(A_1^{\frac{k}{2}\theta}) \times D(A_2^{\frac{1}{2}+\frac{k}{2}\theta}) \times D(A_2^{\frac{k}{2}\theta})$.

Observing that, for initial data in $\mathcal{H}_{n,\theta}$, we can bound (above and below) the norm of U_0 by the norms of its components, we have the following.

Corollary 4.7. *Assume* $(H1), (H2), (H3)$ *and* (3.1).

1) *If* $\mathcal{H}_n \subset D(\mathcal{A}^n)$ *for some* $n \geq 2$*, then for each* $U_0 \in \mathcal{H}_n$ *the solution* U *of problem* (1.9) *satisfies*

$$\|U(t)\|^2_{\mathcal{H}} \leq \frac{c_n}{t^{n/4}} \|U_0\|^2_{\mathcal{H}_n} \qquad \forall t > 0 \tag{4.11}$$

for some constant $c_n > 0$*, where*

$$\|U_0\|^2_{\mathcal{H}_n} = |u^0|^2_{D(A_1^{(n+1)/2})} + |u^1|^2_{D(A_1^{n/2})} + |v^0|^2_{D(A_2^{(n+1)/2})} + |v^1|^2_{D(A_2^{n/2})} .$$

2) *Let* $n \geq 1$ *and* $0 < \theta < 1$ *be fixed. If* $\mathcal{H}_n \subset D(\mathcal{A}^n)$*, then for every* $U_0 \in \mathcal{H}_{n,\theta}$ *the solution* U *of* (1.9) *satisfies*

$$\|U(t)\|^2_{\mathcal{H}} \leq \frac{c_{n,\theta}}{t^{n\theta/4}} \|U_0\|^2_{\mathcal{H}_{n,\theta}} \qquad \forall t > 0 \tag{4.12}$$

for some constant $c_{n,\theta} > 0$*, where*

$$\|U_0\|^2_{\mathcal{H}_{n,\theta}} \asymp |u^0|^2_{D(A_1^{(1+n\theta)/2})} + |u^1|^2_{D(A_1^{n\theta/2})} + |v^0|^2_{D(A_2^{(1+n\theta)/2})} + |v^1|^2_{D(A_2^{n\theta/2})} ,$$

where $\asymp$ *stands for the equivalence between norms.*

5. Applications to PDEs

In this section we describe some examples of systems of partial differential equations that can be studied by the results of this paper, but fail to satisfy the compatibility condition (1.10). We will hereafter denote by Ω a bounded domain in $\mathbb{R}^N$ with a sufficiently smooth boundary Γ. For $i = 1, \dots, N$ we will denote by ∂_i the partial derivative with respect to x_i and by ∂_t the derivative with respect to the time variable. We will also use the notation $H^k(\Omega), H^k_0(\Omega)$ for the usual Sobolev spaces with norm

$$\|u\|_{k,\Omega} = \Big[\int_\Omega \sum_{|p| \leq k} |D^p u|^2 dx \Big]^{\frac{1}{2}} ,$$

where we have set $D^p = \partial_1^{p_1} \cdots \partial_N^{p_N}$ for any multi-index $p = (p_1, \dots, p_N)$. Finally, we will denote by $C_\Omega > 0$ the largest constant such that Poincaré's inequality

$$C_\Omega \|u\|^2_{0,\Omega} \leq \|\nabla u\|^2_{0,\Omega} \tag{5.1}$$

holds true for any $u \in H^1_0(\Omega)$. In the following examples we take

$$H = L^2(\Omega) , \ B = \beta I .$$

Example 5.1. Let β, $\lambda > 0$, $\alpha \in \mathbb{R}$, and consider the problem

$$\begin{cases} \partial_t^2 u - \Delta u + \beta \partial_t u + \lambda u + \alpha v = 0 \\ \partial_t^2 v - \Delta v + \alpha u = 0 \end{cases} \quad \text{in } \Omega \times (0, +\infty) \tag{5.2}$$

with boundary conditions

$$\frac{\partial u}{\partial \nu}(\cdot, t) = 0 \text{ on } \Gamma , \qquad v(\cdot, t) = 0 \text{ on } \Gamma \quad \forall t > 0 \tag{5.3}$$

and initial conditions

$$\begin{cases} u(x,0) = u^0(x) & u'(x,0) = u^1(x) \\ v(x,0) = v^0(x) & v'(x,0) = v^1(x) \end{cases} \quad x \in \Omega . \tag{5.4}$$

The above system can be rewritten in abstract form taking

$$\begin{gathered} D(A_1)=\left\{u\in H^2(\Omega):\frac{\partial u}{\partial\nu}=0 \text{ on } \Gamma\right\}, \quad A_1u=-\Delta u+\lambda u\,, \\ D(A_2)=H^2(\Omega)\cap H^1_0(\Omega)\,, \quad A_2v=-\Delta v\,. \end{gathered} \tag{5.5}$$

Notice that, in order to verify assumption $(H3)$, we shall choose α such that $0<|\alpha|<(C_\Omega(C_\Omega+\lambda))^{1/2}$. Then,

$$\begin{aligned} &|\langle A_1u,v\rangle| = \left|\int_\Omega \nabla u\nabla v\,dx+\lambda\int_\Omega uv\,dx\right| \\ &\le \left(\int_\Omega|\nabla u|^2\,dx\right)^{1/2}\left(\int_\Omega|\nabla v|^2\,dx\right)^{1/2}+\lambda\left(\int_\Omega u^2dx\right)^{1/2}\left(\int_\Omega v^2dx\right)^{1/2} \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \le c\,\langle A_1u,u\rangle^{1/2}\,|A_2v|\,, \end{aligned}$$

where we have used the coercivity of A_2 and the well-known inequality

$$\int_\Omega v^2+|\nabla v|^2\,dx\le c\int_\Omega|\Delta v|^2\,dx \quad \forall\, v\in H^2(\Omega)\cap H^1_0(\Omega)\,.$$

Since condition (3.2) is fulfilled, we get the following conclusions.

i) If $(u^0,u^1,v^0,v^1)\in D(A_1)\times D(A_1^{1/2})\times D(A_2)\times D(A_2^{1/2})$, then the solution U of problem (5.2)-(5.3)-(5.4) satisfies

$$E_1(u(t),u'(t))+E_2(v(t),v'(t))\le\frac{c}{t^{1/4}}\|U_0\|^2_{D(\mathcal{A})} \qquad \forall t>0 \tag{5.6}$$

for some constant $c>0$. Moreover, there exists $c_1>0$ such that

$$\|U_0\|^2_{D(\mathcal{A})}\le c_1\left(\|u^0\|^2_{2,\Omega}+\|u^1\|^2_{1,\Omega}+\|v^0\|^2_{2,\Omega}+\|v^1\|^2_{1,\Omega}\right)\,.$$

ii) By point *ii)* of Corollary 4.5, if $U_0\in(\mathcal{H},D(\mathcal{A}^n))_{\theta,2}$ for some $0<\theta<1$, $n\ge1$, then the solution of (5.2)-(5.3)-(5.4) satisfies

$$E_1(u(t),u'(t))+E_2(v(t),v'(t))\le\frac{c_{n,\theta}}{t^{n\theta/4}}\|U_0\|^2_{(\mathcal{H},D(\mathcal{A}^n))_{\theta,2}} \tag{5.7}$$

for every $t>0$ and some constant $c_{n,\theta}>0$. Moreover, point *iii)* of Corollary 4.5 ensures that, if $U_0\in D((-\mathcal{A})^\theta)$ for some $0<\theta<1$, then

$$E_1(u(t),u'(t))+E_2(v(t),v'(t))\le\frac{c_\theta}{t^{\theta/4}}\|U_0\|^2_{D((-\mathcal{A})^\theta)} \qquad \forall t>0 \tag{5.8}$$

for some constant $c_\theta>0$.

Example 5.2. Another interesting situation occurs while coupling two equations of different orders. Let $\beta>0$, $\alpha\in\mathbb{R}$, and consider the system

$$\begin{cases} \partial_t^2u+\Delta^2u+\beta\partial_tu+\alpha v=0 \\ \partial_t^2v-\Delta v+\alpha u=0 \end{cases} \quad \text{in } \Omega\times(0,+\infty) \tag{5.9}$$

with boundary conditions

$$\Delta u(\cdot,t)=0=\frac{\partial\Delta u}{\partial\nu}(\cdot,t) \text{ on } \Gamma\,, \qquad v(\cdot,t)=0 \text{ on } \Gamma \quad \forall t>0 \tag{5.10}$$

and initial conditions (5.4). Define

$$\begin{gathered} D(A_1)=\left\{u\in H^4(\Omega):\Delta u=0=\frac{\partial\Delta u}{\partial\nu} \text{ on } \Gamma\right\}, \quad A_1u=\Delta^2u\,, \\ D(A_2)=H^2(\Omega)\cap H^1_0(\Omega)\,, \quad A_2v=-\Delta v\,. \end{gathered}$$

Suppose $0 < |\alpha| < C_\Omega C_\Omega^{1/2}$, as required by $(H3)$. Observing that

$$\begin{aligned}|\langle A_1 u, v\rangle| &= \left|\int_\Omega \Delta u \Delta v\, dx\right| \\ &\le \left(\int_\Omega |\Delta u|^2\, dx\right)^{1/2} \left(\int_\Omega |\Delta v|^2\, dx\right)^{1/2} \le c\,\langle A_1 u, u\rangle^{1/2} |A_2 v|\,,\end{aligned}$$

we conclude that condition (3.2) is fulfilled. So, for every $U_0 \in D(\mathcal{A})$, the solution U of problem (5.9)-(5.10)-(5.4) satisfies

$$E_1(u(t), u'(t)) + E_2(v(t), v'(t)) \le \frac{c}{t^{1/4}} \|U_0\|^2_{D(\mathcal{A})} \qquad \forall t > 0 \tag{5.11}$$

for some constant $c > 0$. Moreover, there exists $c_1 > 0$ such that

$$\|U_0\|^2_{D(\mathcal{A})} \le c_1 \left(\|u^0\|^2_{4,\Omega} + \|u^1\|^2_{2,\Omega} + \|v^0\|^2_{2,\Omega} + \|v^1\|^2_{1,\Omega}\right)\,.$$

Example 5.3. Let $\beta > 0$, $\alpha \in \mathbb{R}$, and consider the problem

$$\begin{cases}\partial_t^2 u - \Delta u + \beta \partial_t u + \alpha v = 0 \\ \partial_t^2 v - \Delta v + \alpha u = 0\end{cases} \qquad \text{in } \Omega \times (0, +\infty) \tag{5.12}$$

with boundary conditions

$$\begin{aligned}\left(\frac{\partial u}{\partial \nu} + u\right)(\cdot, t) &= 0 \text{ on } \Gamma \\ v(\cdot, t) &= 0 \text{ on } \Gamma\end{aligned} \qquad \forall t > 0 \tag{5.13}$$

and initial conditions (5.4). Let us define

$$\begin{aligned}&D(A_1) = \left\{u \in H^2(\Omega) : \frac{\partial u}{\partial \nu} + u = 0 \text{ on } \Gamma\right\}\,,\ A_1 u = -\Delta u\,, \\ &D(A_2) = H^2(\Omega) \cap H_0^1(\Omega)\,,\ A_2 v = -\Delta v\,,\end{aligned} \tag{5.14}$$

and assume $0 < |\alpha| < C_\Omega$. Observe that

$$\begin{aligned}|\langle A_1 u, v\rangle| &= \left|\int_\Omega \nabla u \nabla v\, dx\right| \\ &\le \left(\int_\Omega |\nabla u|^2\, dx\right)^{1/2} \left(\int_\Omega |\nabla v|^2\, dx\right)^{1/2} \le c\,\langle A_1 u, u\rangle^{1/2} |A_2 v|\,,\end{aligned}$$

since

$$\langle A_1 u, u\rangle = \int_\Omega |\nabla u|^2\, dx + \int_\Gamma |u|^2\, dS\,, \qquad \int_\Omega |\nabla v|^2\, dx \le c \int_\Omega |\Delta v|^2\, dx\,.$$

Thus, condition (3.1) is fulfilled. So, the energy of the solution of problem (5.12)-(5.13)-(5.4) satisfies

$$E_1(u(t), u'(t)) + E_2(v(t), v'(t)) \le \frac{c}{t^{1/4}} \|U_0\|^2_{D(\mathcal{A})} \qquad \forall t > 0 \tag{5.15}$$

for some constant $c > 0$. Moreover, there exists $c_1 > 0$ such that

$$\|U_0\|^2_{D(\mathcal{A})} \le c_1 \left(|A_1 u^0|^2 + |A_1^{1/2} u^1|^2 + |A_2 v^0|^2 + |A_2^{1/2} v^1|^2\right)\,.$$

Our next result show that the operators in Example 5.3 do not fulfill the compatibility condition (1.10).

Proposition 5.4. *Let A_1, A_2 be defined as in* (5.14)*. Then for every $k \in \mathbb{N}$, $k \geq 2$, $D(A_2^{k/2})$ is not included in $D(A_1)$.*

Proof. Since $D(A_2^k) \subset D(A_2^{k/2})$ for every $k \in \mathbb{N}$, it is sufficient to prove that $D(A_2^k)$ is not included in $D(A_1)$ for every $k \in \mathbb{N}$, $k \geq 1$. For this purpose, let us fix $k \in \mathbb{N}$, $k \geq 1$, and consider the problem

$$\begin{cases} (-\Delta)^k v_0 = 1 \\ v_0 = 0 = \Delta v_0 = \cdots = \Delta^{k-1} v_0 \qquad \text{on } \Gamma \,. \end{cases} \tag{5.16}$$

Define the sequence $v_1, v_2, \ldots, v_{k-1}$ by

$$\begin{cases} -\Delta v_0 = v_1 \\ v_{0|\Gamma} = 0 \end{cases} \quad \cdots \quad \begin{cases} -\Delta v_{k-2} = v_{k-1} \\ v_{k-2|\Gamma} = 0 \end{cases} \quad \begin{cases} -\Delta v_{k-1} = 1 \\ v_{k-1|\Gamma} = 0 \,. \end{cases} \tag{5.17}$$

We will argue by contradiction, assuming $D(A_2^k) \subset D(A_1)$. Since v_0 belongs to $D(A_2) \cap D(A_1)$, we have $v_{0|\Gamma} = 0 = \frac{\partial v_0}{\partial \nu}_{|\Gamma}$. Moreover, from the first system in (5.17), it follows that

$$\int_\Omega v_1 dx = \int_\Omega (-\Delta v_0) dx = -\int_\Gamma \frac{\partial v_0}{\partial \nu} dS = 0 \,.$$

Hence, $\displaystyle\int_\Omega v_1 dx = 0$. Let us prove by induction that

$$\int_\Omega \nabla v_{k-i} \nabla v_i dx = 0 \quad \forall\, i = 1, 2, \ldots, k-1 \,. \tag{5.18}$$

For $i = 1$ we have

$$\int_\Omega \nabla v_{k-1} \nabla v_1 dx = \int_\Omega (-\Delta v_{k-1}) v_1 dx = \int_\Omega v_1 dx = 0 \,,$$

since $v_{k-1|\Gamma} = 0 = v_{1|\Gamma}$. Now, let $i > 1$ and suppose

$$\int_\Omega \nabla v_{k-i} \nabla v_i dx = 0 \,.$$

Then,

$$\begin{aligned} 0 = & \int_\Omega v_{k-i} (-\Delta v_i) dx = \int_\Omega v_{k-i} v_{i+1} dx \\ = & \int_\Omega (-\Delta v_{k-i-1}) v_{i+1} dx = \int_\Omega \nabla v_{k-(i+1)} \nabla v_{i+1} dx \,. \end{aligned}$$

Thus, (5.18) holds for $i + 1$. Moreover, from (5.18) follows that

$$\int_\Omega v_{k-i} v_{i+1} dx = 0 \quad \forall\, i = 1, 2, \ldots, k-1 \,, \tag{5.19}$$

since

$$\int_\Omega v_{k-i} v_{i+1} dx = \int_\Omega v_{k-i} (-\Delta v_i) dx = \int_\Omega \nabla v_{k-i} \nabla v_i dx = 0 \,.$$

Now, let k be even, say $k = 2p$, $p \in \mathbb{N}^*$. Then, by (5.18) with $i = p$ we obtain

$$\int_\Omega |\nabla v_p|^2 dx = 0 \,, \text{ whence } v_p = 0 \,.$$

So, by a cascade effect,

$$v_{p+1} = -\Delta v_p = 0 \Rightarrow v_{p+2} = -\Delta v_{p+1} = 0 \Rightarrow \cdots \Rightarrow v_{k-1} = -\Delta v_{k-2} = 0\,.$$

Since $-\Delta v_{k-1} = 1$, we get a contradiction. If, on the contrary, k is odd, i.e. $k = 2p+1$, then, applying (5.19) with $i = p$, we conclude that

$$\int_\Omega |v_{p+1}|^2 dx = 0\,, \text{ whence } v_{p+1} = 0\,.$$

Finally, we have that $v_{p+1} = v_{p+2} = \cdots = v_{k-1} = 0$. Since $-\Delta v_{k-1} = 1$, we get a contradiction again. Therefore, $D(A_2^k)$ is not included in $D(A_1)$. □

Example 5.5. Given $\beta > 0$, $\alpha \in \mathbb{R}$, let us now consider the undamped Petrowsky equation coupled with the damped wave equation,

$$\begin{cases} \partial_t^2 u - \Delta u + \beta \partial_t u + \alpha v = 0 \\ \partial_t^2 v + \Delta^2 v + \alpha u = 0 \end{cases} \quad \text{in } \Omega \times (0, +\infty) \tag{5.20}$$

with boundary conditions

$$\begin{aligned} \left(\frac{\partial u}{\partial \nu} + u\right)(\cdot, t) &= 0 \text{ on } \Gamma \\ v(\cdot, t) = \Delta v(\cdot, t) &= 0 \text{ on } \Gamma \end{aligned} \qquad \forall t > 0 \tag{5.21}$$

and initial conditions (5.4). Define

$$\begin{aligned} D(A_1) &= \left\{ u \in H^2(\Omega) : \frac{\partial u}{\partial \nu} + u = 0 \text{ on } \Gamma \right\}\,, \quad A_1 u = -\Delta u\,, \\ D(A_2) &= \left\{ v \in H^4(\Omega) : v = \Delta v = 0 \text{ on } \Gamma \right\}\,, \quad A_2 v = \Delta^2 v\,. \end{aligned}$$

Once again, we have

$$\begin{aligned} |\langle A_1 u, v \rangle| &= \left| \int_\Omega \nabla u \nabla v \, dx \right| \\ &\le \left(\int_\Omega |\nabla u|^2 \, dx \right)^{1/2} \left(\int_\Omega |\nabla v|^2 \, dx \right)^{1/2} \le c \, \langle A_1 u, u \rangle^{1/2} |A_2 v|\,. \end{aligned}$$

Thus, condition (3.1) is fulfilled and, for $0 < |\alpha| < C_\Omega^{3/2}$, the polynomial decay of the solution to (5.20)-(5.21)-(5.4) follows as in Example 5.1.

Example 5.6. Let $\beta > 0$, $\alpha \in \mathbb{R}$, and consider the system

$$\begin{cases} \partial_t^2 u - \Delta u + \beta \partial_t u + \alpha v = 0 \\ \partial_t^2 v + \Delta^2 v + \alpha u = 0 \end{cases} \quad \text{in } \Omega \times (0, +\infty) \tag{5.22}$$

with boundary conditions

$$\begin{aligned} \left(\frac{\partial u}{\partial \nu} + u\right)(\cdot, t) &= 0 \text{ on } \Gamma \\ v(\cdot, t) = \frac{\partial v}{\partial \nu}(\cdot, t) &= 0 \text{ on } \Gamma \end{aligned} \qquad \forall t > 0 \tag{5.23}$$

and initial conditions (5.4). Let us define

$$\begin{aligned} D(A_1) &= \left\{ u \in H^2(\Omega) : \frac{\partial u}{\partial \nu} + u = 0 \text{ on } \Gamma \right\}\,, \quad A_1 u = -\Delta u\,, \\ D(A_2) &= \left\{ v \in H^4(\Omega) : v = \frac{\partial v}{\partial \nu} = 0 \text{ on } \Gamma \right\}\,, \quad A_2 v = \Delta^2 v\,. \end{aligned}$$

We have

$$
\begin{aligned}
|\langle A_1 u, v\rangle| &= \left|\int_\Omega \nabla u \nabla v \, dx\right| \\
&\le \left(\int_\Omega |\nabla u|^2 \, dx\right)^{1/2} \left(\int_\Omega |\nabla v|^2 \, dx\right)^{1/2} \le c \, \langle A_1 u, u\rangle^{1/2} |A_2 v| .
\end{aligned}
$$

Thus, condition (3.1) is verified and, for $0 < |\alpha| < C_\Omega^{3/2}$, the solution U of problem (5.22)-(5.23)-(5.4) satisfies the conclusions as in Example 5.1, so that $\mathcal{E}$ decays polynomially at ∞.

Example 5.7. Another interesting situation occurs when an operator fulfills different boundary conditions on proper subsets of Γ. For instance, let Γ_0 be an open subset of Γ (with respect to the topology of Γ) and set $\Gamma_1 = \Gamma \backslash \Gamma_0$. We assume that $\overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset$. Let $\beta > 0$, $\alpha \in \mathbb{R}$, and consider the problem

$$
\begin{cases}
\partial_t^2 u - \Delta u + \beta \partial_t u + \alpha v = 0 \\
\partial_t^2 v - \Delta v + \alpha u = 0
\end{cases}
\qquad \text{in } \Omega \times (0, +\infty) \tag{5.24}
$$

with boundary conditions

$$
\begin{aligned}
u(\cdot, t) = 0 \text{ on } \Gamma_0 , \quad \frac{\partial u}{\partial \nu}(\cdot, t) &= 0 \text{ on } \Gamma \setminus \Gamma_0 \\
v(\cdot, t) &= 0 \text{ on } \Gamma
\end{aligned}
\qquad \forall t > 0 \tag{5.25}
$$

and initial conditions (5.4). Let us set

$$
\begin{gathered}
D(A_1) = \left\{ u \in H^2(\Omega) : u = 0 \text{ on } \Gamma_0 , \ \frac{\partial u}{\partial \nu} = 0 \text{ on } \Gamma \setminus \Gamma_0 \right\} , \\
A_1 u = -\Delta u , \\
D(A_2) = H^2(\Omega) \cap H_0^1(\Omega) , \quad A_2 v = -\Delta v .
\end{gathered}
$$

Then,

$$
\begin{aligned}
|\langle A_1 u, v\rangle| &= \left|\int_\Omega \nabla u \nabla v \, dx\right| \\
&\le \left(\int_\Omega |\nabla u|^2 \, dx\right)^{1/2} \left(\int_\Omega |\nabla v|^2 \, dx\right)^{1/2} \le c \, \langle A_1 u, u\rangle^{1/2} |A_2 v| .
\end{aligned}
$$

So, for $0 < |\alpha| < C_\Omega$, condition (3.1) is fulfilled, and the same conclusions as in Example 5.1 hold for problem (5.24)-(5.25)-(5.4).

6. Improvement of previous results

In this section we apply interpolation theory to extend the polynomial stability result of [4] to a larger class of initial data. We will denote by $j \ge 2$ the integer for which (1.10) is satisfied. As is shown in [4, Theorem 4.2], under assumptions $(H1), (H2), (H3)$ and (1.10), if $U_0 \in D(\mathcal{A}^{nj})$ for some integer $n \ge 1$, the solution U of problem (1.9) satisfies

$$
\mathcal{E}(U(t)) \le \frac{c_n}{t^n} \sum_{k=0}^{nj} \mathcal{E}(U^{(k)}(0)) \qquad \forall t > 0 \tag{6.1}
$$

for some constant $c_n > 0$. We recall that assumption (1.10) covers many situations of interest for applications to systems of evolution equations. Indeed (see [4, Section 5] for further details), this is the case for

i) $(A_1, D(A_1)) = (A_2, D(A_2))$, where (1.10) is fulfilled with $j = 2$;
ii) $D(A_1) = D(A_2)$, with $j = 2$;
iii) $(A_2, D(A_2)) = (A_1^2, D(A_1^2))$, again with $j = 2$;
iv) $(A_1, D(A_1)) = (A_2^2, D(A_2^2))$, with $j = 4$.

The following result completes the analysis of [4], taking the initial data in suitable interpolation spaces.

Theorem 6.1. *Assume* $(H1), (H2), (H3)$ *and* (1.10), *and let* $0 < \theta < 1$, $n \geq 1$. *Then for every* U_0 *in* $(\mathcal{H}, D(\mathcal{A}^{nj}))_{\theta,2}$, *the solution* U *of* (1.9) *satisfies*

$$\|U(t)\|_{\mathcal{H}}^2 \leq \frac{c_{n,\theta}}{t^{n\theta}} \|U_0\|_{(\mathcal{H}, D(\mathcal{A}^{nj}))_{\theta,2}}^2 \qquad \forall t > 0 \tag{6.2}$$

for some constant $c_{n,\theta} > 0$.

Reasoning as in Remark 4.4, one can derive estimate (6.1) also for $U_0 \in D(\mathcal{A}^k)$, for every $k = 1, \ldots, nj - 1$, with decay rate k/j.

Corollary 6.2. *Assume* $(H1), (H2), (H3)$ *and (1.10).*

i) *If* $U_0 \in D(\mathcal{A}^n)$ *for some* $n \geq 1$, *then the solution of (1.9) satisfies*
$$\|U(t)\|_{\mathcal{H}}^2 \leq \frac{c_n}{t^{n/j}} \|U_0\|_{D(\mathcal{A}^n)}^2 \qquad \forall t > 0 \tag{6.3}$$
for some constant $c_n > 0$.

ii) *If* $U_0 \in (\mathcal{H}, D(\mathcal{A}^n))_{\theta,2}$ *for some* $n \geq 1$ *and* $0 < \theta < 1$, *then the solution of (1.9) satisfies*
$$\|U(t)\|_{\mathcal{H}}^2 \leq \frac{c_{n,\theta}}{t^{n\theta/j}} \|U_0\|_{(\mathcal{H}, D(\mathcal{A}^n))_{\theta,2}}^2 \qquad \forall t > 0 \tag{6.4}$$
for some constant $c_{n,\theta} > 0$.

iii) *If* $U_0 \in D((-\mathcal{A})^\theta)$ *for some* $0 < \theta < 1$, *then the solution of problem* (1.9) *satisfies*
$$\|U(t)\|_{\mathcal{H}}^2 \leq \frac{c_\theta}{t^{\theta/j}} \|U_0\|_{D((-\mathcal{A})^\theta)}^2 \qquad \forall t > 0 \tag{6.5}$$
for some constant $c_\theta > 0$.

In particular, the previous fractional decay rates can be achieved for initial data in $\mathcal{H}_n$ or in $\mathcal{H}_{n,\theta}$, whenever $\mathcal{H}_n \subset D(\mathcal{A}^n)$, as in Corollary 4.7. This happens, for instance, if any of the following conditions is satisfied:

i) $(A_1, D(A_1)) = (A_2, D(A_2))$;
ii) $D(A_1) = D(A_2)$;
iii) $(A_2, D(A_2)) = (A_1^2, D(A_1^2))$.

Let us apply Corollary 6.2 to two examples from [4].

Example 6.3. Given $\beta > 0$, $\kappa > 0$, $\alpha \in \mathbb{R}$, let us study the problem

$$\begin{cases} \partial_t^2 u - \Delta u + \beta \partial_t u + \kappa u + \alpha v = 0 \\ \partial_t^2 v - \Delta v + \kappa v + \alpha u = 0 \end{cases} \qquad \text{in} \qquad \Omega \times (0, +\infty) \tag{6.6}$$

with boundary conditions

$$u(\cdot, t) = 0 = v(\cdot, t) \qquad \text{on} \quad \Gamma \quad \forall t > 0 \tag{6.7}$$

and initial conditions

$$\begin{cases} u(x,0) = u^0(x)\,, & u'(x,0) = u^1(x) \\ v(x,0) = v^0(x)\,, & v'(x,0) = v^1(x) \end{cases} \qquad x \in \Omega\,. \tag{6.8}$$

Let $H = L^2(\Omega)$, $B = \beta I$, and $A_1 = A_2 = A$ be defined by

$$D(A) = H^2(\Omega) \cap H_0^1(\Omega)\,, \qquad Au = -\Delta u + \kappa u \quad \forall u \in D(A)\,.$$

Notice that (1.10) is fulfilled with $j = 2$, and condition $0 < |\alpha| < C_\Omega + \kappa =: \omega$ is required in order to fulfill $(H3)$.

As showed in [4, Example 6.1], if $u^0, v^0 \in H^2(\Omega) \cap H_0^1(\Omega)$, $u^1, v^1 \in H_0^1(\Omega)$, then

$$\begin{aligned} &\int_\Omega \Big(|\partial_t u|^2 + |\nabla u|^2 + |\partial_t v|^2 + |\nabla v|^2 \Big) dx \\ &\qquad \le \frac{c}{t} \Big(\|u^0\|_{2,\Omega}^2 + \|u^1\|_{1,\Omega}^2 + \|v^0\|_{2,\Omega}^2 + \|v^1\|_{1,\Omega}^2 \Big) \qquad \forall t > 0\,. \end{aligned}$$

Moreover, if $u^0, v^0 \in H^{n+1}(\Omega)$ and $u^1, v^1 \in H^n(\Omega)$ are such that

$$u^0 = \cdots = \Delta^{[\frac{n}{2}]} u^0 = 0 = v^0 = \cdots = \Delta^{[\frac{n}{2}]} v^0 \quad \text{on} \quad \Gamma,$$

$$u^1 = \cdots = \Delta^{[\frac{n-1}{2}]} u^1 = v^1 = \cdots = \Delta^{[\frac{n-1}{2}]} v^1 = 0 \quad \text{on} \quad \Gamma,$$

then

$$\begin{aligned} &\int_\Omega \Big(|\partial_t u|^2 + |\nabla u|^2 + |\partial_t v|^2 + |\nabla v|^2 \Big) dx \\ &\qquad \le \frac{c_n}{t^n} \Big(\|u^0\|_{n+1,\Omega}^2 + \|u^1\|_{n,\Omega}^2 + \|v^0\|_{n+1,\Omega}^2 + \|v^1\|_{n,\Omega}^2 \Big) \qquad \forall t > 0\,. \end{aligned}$$

Furthermore, applying Corollary 6.2, we conclude that if U_0 belongs to $\mathcal{H}_{n,\theta} = (\mathcal{H}, D(\mathcal{A}^n))_{\theta,2}$ for some $0 < \theta < 1$, $n \ge 1$, then the solution to (6.6)-(6.7)-(6.8) satisfies

$$\int_\Omega \Big(|\partial_t u|^2 + |\nabla u|^2 + |\partial_t v|^2 + |\nabla v|^2 \Big) dx \le \frac{c_{n,\theta}}{t^{n\theta/2}} \|U_0\|_{\mathcal{H}_{n,\theta}}^2 \qquad \forall t > 0 \tag{6.9}$$

for some constant $c_{n,\theta} > 0$, with

$$\|U_0\|_{\mathcal{H}_{n,\theta}}^2 \asymp |u^0|^2_{D(A_1^{\frac{1}{2}+\frac{n}{2}\theta})} + |u^1|^2_{D(A_1^{\frac{n}{2}\theta})} + |v^0|^2_{D(A_2^{\frac{1}{2}+\frac{n}{2}\theta})} + |v^1|^2_{D(A_2^{\frac{n}{2}\theta})}\,.$$

Example 6.4. Taking $\beta > 0$, $0 < |\alpha| < C_\Omega^{3/2}$, and the same operators A_1 and A_2 as in Example 5.2, but with different boundary conditions, we can consider the system

$$\begin{cases} \partial_t^2 u + \Delta^2 u + \beta \partial_t u + \alpha v = 0 \\ \partial_t^2 v - \Delta v + \alpha u = 0 \end{cases} \qquad \text{in} \qquad \Omega \times (0, +\infty) \tag{6.10}$$

with boundary conditions

$$v(\cdot, t) = u(\cdot, t) = \Delta u(\cdot, t) = 0 \qquad \text{on} \quad \Gamma \quad \forall t > 0 \tag{6.11}$$

and initial conditions as in (6.8). Let us set $H = L^2(\Omega)$, $B = \beta I$, and

$$\begin{gathered} D(A_1) = \big\{ u \in H^4(\Omega) : \Delta u = 0 = u \text{ on } \Gamma \big\}\,, \quad A_1 u = \Delta^2 u\,, \\ D(A_2) = H^2(\Omega) \cap H_0^1(\Omega)\,, \quad A_2 v = -\Delta v\,. \end{gathered}$$

In this case, since $A_1 = A_2^2$, condition (1.10) holds with $j = 4$. Consequently, as is shown in [4, Example 6.4], for initial condition $U_0 \in D(\mathcal{A}^4)$

$$\int_\Omega \Big(|\partial_t u|^2 + |\Delta u|^2 + |\partial_t v|^2 + |\nabla v|^2\Big)dx \leq \frac{C}{t}\|U_0\|^2_{D(\mathcal{A}^4)} \qquad \forall t > 0\,,$$

for some constant $C > 0$. By point $i)$ of Corollary 6.2, we can generalize this result to initial data $U_0 \in D(\mathcal{A}^n)$ for some $n \geq 1$. Indeed, in this case the solution to (6.10)-(6.11)-(6.8) satisfies

$$\int_\Omega \Big(|\partial_t u|^2 + |\Delta u|^2 + |\partial_t v|^2 + |\nabla v|^2\Big)dx \leq \frac{c_n}{t^{n/4}}\|U_0\|^2_{D(\mathcal{A}^n)} \qquad \forall t > 0\,,$$

for some constant $c_n > 0$. Moreover, thanks to point $ii)$ of Corollary 6.2, if $U_0 \in (\mathcal{H}, D(\mathcal{A}^n))_{\theta,2}$ for some $n \geq 1$ and $0 < \theta < 1$, then

$$\int_\Omega \Big(|\partial_t u|^2 + |\Delta u|^2 + |\partial_t v|^2 + |\nabla v|^2\Big)dx \leq \frac{c_{n,\theta}}{t^{n\theta/4}}\|U_0\|^2_{(\mathcal{H},D(\mathcal{A}^n))_{\theta,2}} \qquad \forall t > 0$$

for some constant $c_{n,\theta} > 0$. Furthermore, thanks to point $iii)$ of Corollary 6.2, if U_0 belongs to $\mathcal{H}_{1,\theta} = D((-\mathcal{A})^\theta)$ for some $0 < \theta < 1$, then the solution to (6.10)-(6.11)-(6.8) satisfies

$$\int_\Omega \Big(|\partial_t u|^2+|\Delta u|^2+|\partial_t v|^2+|\nabla v|^2\Big)dx \leq \frac{c_\theta}{t^{\theta/4}}\|U_0\|^2_{D((-\mathcal{A})^\theta)} \qquad \forall t > 0 \quad (6.12)$$

for some constant $c_\theta > 0$, with

$$\|U_0\|^2_{D((-\mathcal{A})^\theta)} \asymp |u^0|^2_{D(A_1^{\frac{1}{2}+\frac{1}{2}\theta})} + |u^1|^2_{D(A_1^{\frac{1}{2}\theta})} + |v^0|^2_{D(A_2^{\frac{1}{2}+\frac{1}{2}\theta})} + |v^1|^2_{D(A_2^{\frac{1}{2}\theta})}\,.$$

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